

AN INTEGRABLE MODEL FOR HELENE

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Abstract. The goal of our work is to determine the influence of Enceladus in the motion of Helene. We constructed a model considering an oblated central body (Saturn) and three satellites (Enceladus, Dione and Helene) under the action of central forces. The development of the potential was made assuming small eccentricities, null inclinations and two resonances present in this system (2:1 between Enceladus and Dione and 1:1 between Dione and Helene). The mean Hamiltonian preserves terms derived of the oblateness of the central body and also terms of both resonant arguments. The auxiliary Hori's system generated by this Hamiltonian is completely integrable. In the solution for Helene we included the perturbative effects of Dione and Enceladus which extend the usual treatment.

Key words: Resonance - Oblateness - Hori's auxiliary system

1. Hypothesis of the problem

Considering an isolated four bodies system: Saturn, Enceladus, Dione and Helene with masses M, m_1, m_2, m_3 and the following hypothesis:

oblateness of Saturn (J_2);

small eccentricities (e_i);

null inclinations;

resonances:

2:1 between Enceladus and Dione,

1:1 between Dione and Helene;

only gravitational forces acts on the system;

coordinate system ($G; x, y, z$), where G is the mass center of the system.

The potential was developed in function of this coordinate system ($G; x, y, z$) using the classical theory (Brouwer and Clemence (1961)) except for the term involving the mutual attraction of Dione and Helene, when the ratio (a_1/a_2) approaches to zero. For this case see Hori (1985).

2. Equations of motion

Introducing Delaunay's variables the canonical equations of motion are:

$$\frac{d(X_i, Y_i)}{dt} = \frac{\partial F}{\partial(\lambda_i, \varpi_i)}, \quad \frac{d(\lambda_i, \varpi_i)}{dt} = -\frac{\partial F}{\partial(X_i, Y_i)} \quad i = 1, 2, 3$$

and F is the Hamiltonian given by

$$F = F_0 + R$$

with

$$F_0 = \sum_{i=1}^3 \frac{\mu^2 m_i^3}{2X_i^2},$$

$$R = F_{1/3} + F_{2/3} + F_1.$$

F_0 generates Keplerian motion of Enceladus, Dione and Helene with focus G ; and the disturbing function R is developed in terms of Delaunay's elements. Taking $(\frac{m_2}{M})$ as the small parameter ε and assuming that J_2 and e_i^2 (square eccentricities) are of order $\varepsilon^{\frac{1}{2}}$ and $J_4, (\frac{m_2}{M}), (\frac{m_2}{M})$ of order $\varepsilon^{\frac{2}{3}}$, the expressions in R were developed up to ε and rearranged.

3. Intermediary orbit

The short period terms were eliminated by Hori's method considering the existence of the resonances 2:1 and 1:1. Substituting in the Hamiltonian a new set of canonical variables:

$$x_1 = X_1^* + \frac{X_2^*}{2} + \frac{X_3^*}{2}, \quad x_2 = -\left(\frac{X_2^*}{2} + \frac{X_3^*}{2}\right), \quad x_3 = \frac{X_3^*}{2}, \quad y_i = Y_i^*,$$

$$\lambda_i = \lambda_i^*, \quad \theta_2 = \lambda_1^* - 2\lambda_2^*, \quad \theta_3 = 2(\lambda_3^* - \lambda_2^*), \quad \varpi_i = \varpi_i^*, \quad i = 1, 2, 3$$

Since λ_1 is a cyclic variable, x_1 is a first integral. The study of the resonant terms will be done in the neighbourhood of the exact resonances:

$$\left. \frac{\partial F_0^*}{\partial x_2} \right|_{x_2=0} = -\frac{\mu^2 m_1^3}{(x_1 + x_2)^3} - \frac{\mu^2 m_2^3}{4(x_2 + x_3)^3},$$

$$\left. \frac{\partial F_0^*}{\partial x_3} \right|_{x_3=0} = -\frac{\mu^2 m_2^3}{4(x_2 + x_3)^3} - \frac{\mu^2 m_3^3}{4x_3^3},$$

and the Hamiltonian after these developments is given by

$$F^{**} = F_{2/3}^{**} + F_1^{**}$$

and the auxiliary equations will be generated by $F_{2/3}^{**}$:

$$F_{2/3}^{**} = (z_2^{**})^2 + \beta_1 e_{10} \cos(\theta_2^{**} + \varpi_1^{**}) + \beta_2 e_{20} \cos(\theta_2^{**} + \varpi_2^{**}) \\ + \beta_0 (z_3^{**})^2 + \beta_3 \cos(\theta_3^{**}) + \beta_4 \operatorname{cosec}(\theta_3^{**}/2)$$

where

$$z_2^{**} = \frac{x_2}{x_{20}} - q_2 \quad z_3^{**} = \frac{x_3}{x_{30}} - q_3$$

with $q_2, q_3, \beta_k, k = 0, 1, 2, 3, 4$ are constants.

The auxiliary equations are :

$$\frac{dz_2^{**}}{d\Lambda} = -\beta_1 e_{10} \sin(\theta_2^{**} + \varpi_1^{**}) - \beta_2 e_{20} \sin(\theta_2^{**} + \varpi_2^{**})$$

$$\frac{dy_1^{**}}{d\Lambda} = -\beta_1 e_{10} \sin(\theta_2^{**} + \varpi_1^{**})$$

$$\frac{dy_2^{**}}{d\Lambda} = -\beta_2 e_{20} \sin(\theta_2^{**} + \varpi_2^{**})$$

$$\frac{dz_3^{**}}{d\Lambda} = -\beta_3 \sin(\theta_3^{**}) - \frac{\beta_4}{2} \frac{\cos(\theta_3^{**}/2)}{\sin^2(\theta_3^{**}/2)}$$

$$\frac{dy_3^{**}}{d\Lambda} = 0$$

$$\frac{d\theta_2^{**}}{d\Lambda} = -2z_2^{**}$$

$$\frac{d\varpi_1^{**}}{d\Lambda} = \frac{c_1 \beta_1}{e_{10}} \cos(\theta_2^{**} + \varpi_1^{**})$$

$$\frac{d\varpi_2^{**}}{d\Lambda} = \frac{c_2 \beta_2}{2e_{20}} \cos(\theta_2^{**} + \varpi_2^{**})$$

$$\frac{d\theta_3^{**}}{d\Lambda} = -2\beta_0 z_3^{**}$$

$$\frac{d\varpi_3^{**}}{d\Lambda} = \frac{3P_{03}}{2e_{30}}$$

and this system has two first integrals

$$E = F_{2/3}^{**}$$

$$G = \frac{dz_2^{**}}{d\Lambda} - \frac{dy_1^{**}}{d\Lambda} - \frac{dy_2^{**}}{d\Lambda}$$

Except by the equations in y_3^{**} and ϖ_3^{**} which are immediately integrated, the equations in z_3^{**} and θ_3^{**} involve the solution of a general ideal resonant problem:

$$E = \beta_0 (z_3^{**})^2 + \beta_3 \cos(\theta_3^{**}) + \beta_4 \operatorname{cosec}(\theta_3^{**}/2)$$

$$4\beta_0(\Lambda - \Lambda_0) = \int \frac{d\theta_3^{**}}{\sqrt{E - \beta_3 \cos(\theta_3^{**}) - \beta_4 \operatorname{cosec}(\theta_3^{**}/2)}}$$

The other equations are identical to the auxiliary system of Salgado and Sessin (1985), so the solution is obtained in terms of the Sessin's variables

$$z = z_2^{**} \quad \theta = \theta_2^{**}$$

$$H = \frac{D_1}{c_1} h_1 - 2D_2 h_2 \quad K = \frac{D_1}{c_1} k_1 - 2D_2 k_2$$

where $D_i, c_i, i = 1, 2$ are constants and

$$h_i = e_{i0} \sin(\theta_2^{**} + \varpi_i^{**}) \quad k_i = e_{i0} \cos(\theta_2^{**} + \varpi_i^{**}), \quad i = 1, 2.$$

4. Concluding remarks

Substituting initial conditions (Voyager's data for Helene and Nautical Almanac for Enceladus and Dione) we obtained 30° for the amplitude of Helene's libration with θ_3^{**} centered at 59.96° , which is very close to the observational data. However, the same do not occur with the period of Helene's libration.

References

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