COMPLETE SETS OF OBSERVABLES AND PURE STATES

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1. Introduction. It was shown in (**1**) that a complete set of bounded observables is metrically complete. However, an extra axiom was needed to prove this result (**1**, footnote, p. 436). In this note we prove the above-mentioned result without the extra axiom. We also show that there is an abundance of pure states if M is closed in the weak topology and give a necessary and sufficient condition for the latter to be the case.

2. Complete sets of observables. In this paper we shall assume that L is an orthocomplemented partially ordered set or logic in which the following axiom holds: if a, b, c mutually split, then $a \leftrightarrow b \vee c$. It can be shown (see **2**) that this axiom holds if L happens to be a lattice and that the results in (**4**) hold in logics satisfying this axiom. We draw freely from the definitions and theorems in (**1**).

THEOREM 2.1. *A complete set K of bounded observables on L is a commutative real Banach algebra with unity satisfying*:

(i) $|x^2| = |x|^2$ *for all x in K,*

(ii) x^2 *is a continuous function of x,*

(iii) $|x^2 - y^2| \leqq \max(|x^2|, |y^2|)$.

Proof. It has been shown in (**1**) that K is a commutative normed algebra with unity. The proofs of (i), (ii), and (iii) are straightforward and left to the reader. We now show that K is metrically complete. Let x_n be a Cauchy sequence in K, let $R(x_n)$ denote the range of x_n, $n = 1, 2, \ldots$, and let B be the smallest Boolean sub σ-algebra containing $\bigcup R(x_n)$. Notice that B exists by Theorem 3.1 of (**4**). Since B is separable, there is an observable z such that the range $R(z) = B$ (**4**, Proposition 3.15). We now show that $z \in K$. Otherwise, there is an $x \in K$ and $x \not\leftrightarrow z$. But since $x \leftrightarrow x_n$, there is a Boolean sub σ- algebra B_1 which contains $R(x) \cup (\bigcup R(x_n))$. Then B_1 contains $\bigcup R(x_n)$ but cannot contain B, which contradicts the minimality of B. Now, applying Proposition 3.16 of (**4**), there exist real Borel functions u_n such that $x_n = u_n(z)$ and since x_n is Cauchy there are positive integers $n(p)$, $p = 1, 2, \ldots$, such that $n, m \geqq n(p)$ implies $|u_n(z) - u_m(z)| \leqq p^{-1}$. Letting $\Delta(\epsilon) = \{\lambda: |\lambda| \leqq \epsilon\}$, we have $\sigma[u_n(z) - u_m(z)] \subset \Delta(p^{-1})$ and

$$0 = [u_n(z) - u_m(z)](\Delta(p^{-1})') = z\{\omega: |u_n(\omega) - u_m(\omega)| > p^{-1}\}$$

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for all $n, m \geqq n(p)$. Letting

$$N(p) = \bigcup\{\omega: |u_n(\omega) - u_m(\omega)| > p^{-1}, \quad n, m \geqq n(p)\},$$

we have $|u_n(\omega) - u_m(\omega)| \leqq p^{-1}$ on $N(p)'$ for all $n, m \geqq n(p)$ and $z[N(p)]=0$. Now if $N = \bigcup N(p)$, then $z(N) = \bigvee z[N(p)] = 0$. We assert that u_n is uniformly Cauchy on N'. Indeed, if $\epsilon > 0$, then there is an integer q such that $q^{-1} < \epsilon$ and if $n, m > n(q)$, we have $|u_n(\omega) - u_m(\omega)| < q^{-1} < \epsilon$ on $N(q)'$ and hence on N'. Therefore, u_n converges uniformly on N' to a Borel function u. We now show that $u_n(z) \to u(z)$. For any $\epsilon > 0$, if n is sufficiently large, we have $\{\omega: |u_n(\omega) - u(\omega)| > \epsilon\} \subset N$. Hence,

$$\begin{aligned}[u_n(z) - u(z)](\Delta(\epsilon)') &= z\{\omega: u_n(\omega) - u(\omega) \in \Delta(\epsilon)'\} \\ &= z\{\omega: |u_n(\omega) - u(\omega)| > \epsilon\} = 0.\end{aligned}$$

So for n sufficiently large, $\sigma[u_n(z) - u(z)] \subset \Delta(\epsilon)$ and $|u_n(z) - u(z)| < \epsilon$. Since $z \in K$, we have, of course, that $u(z) \in K$ and thus $x_n \to u(z) \in K$ and K is metrically complete.

COROLLARY. *A complete set of bounded observables on L is isometrically isomorphic to the continuous real-valued functions on a compact Hausdorff space.*

Proof. This follows from a theorem due to Segal (see **3**, p. 933).

3. Pure states and the closure of M. Let M denote the set of all states on L. A set of states $M_1 \subset M$ is said to be *full* in the following cases:

(i) if $a \neq 0$, there is an $m \in M_1$ such that $m(a) = 1$;

(ii) if $a \neq b$, there is an $m \in M_1$ such that $m(a) \neq m(b)$.

A set of states $M_2 \subset M$ is said to be *quite full* if $m(b) = 1$ whenever $m(a) = 1$ for all $m \in M_2$ implies $a \leqq b$. The following theorem was proved in (**1**).

THEOREM 3.1. *If M is weakly closed, then M is the weakly closed convex hull of its pure states.*

Since the pure states are physically those in which we have a maximum amount of information concerning the condition of the system, it is important to show that there are a lot of pure states.

THEOREM 3.2. *Suppose that M is weakly closed and M_p is the set of pure states. If M is full [quite full], then M_p is full [quite full].**

Proof. Suppose that M is full and $a \neq b$. If $m(a) = m(b)$ for every $m \in M_p$, then convex combinations of pure states and limits of nets of convex combinations of pure states agree on a and b. It follows from Theorem 3.1 that

*The author is indebted to Harry Mullikin for the proof of part of this theorem.

$m(a) = m(b)$ for all $m \in M$, which is a contradiction. Now suppose that $a \neq 0$ and let $M_a = \{m \in M: m(a) = 1\}$. Then M_a is a non-empty subset of M which is weakly closed. Thus M_a is compact and convex and by the Kreĭn-Milman theorem it is the weakly closed convex hull of its extreme points. Let m_0 be an extreme point of M_a. To show that $m_0 \in M_p$, suppose that

$$m_0 = \lambda m_1 + (1 - \lambda)m_2 \quad \text{for } m_1, m_2 \in M, \quad 0 < \lambda < 1.$$

Then

$$1 = m_0(a) = \lambda m_1(a) + (1 - \lambda)m_2(a).$$

Hence $m_1(a) = m_2(a) = 1$ and $m_1, m_2 \in M_a$. Therefore $m_1 = m_2 = m_0$. Thus $M_a \cap M_p \neq \emptyset$, and M_p is full. Now suppose that M is quite full and that every $m \in M_p$ which satisfies $m(a) = 1$ also satisfies $m(b) = 1$. Let $M_a = \{m \in M: m(a) = 1\}$ and $M_b = \{m \in M: m(b) = 1\}$. As before, M_a $[M_b]$ is the weakly closed convex hull of $M_a \cap M_p$ $[M_b \cap M_p]$. Now since $M_a \cap M_p \subset M_b \cap M_p$ we must have $M_a \subset M_b$ and hence $a \leqq b$. Thus, M_p is quite full.

We now give an example which shows that M need not be weakly closed. Let (Ω, A) be a measurable space on which there is a finitely additive measure μ which is not countably additive. Now, bounded observables on A may be identified with bounded measurable functions on (Ω, A) (cf. **4**, Proposition 3.3). Denote the set of bounded observables on A by X and the dual of X by X'. If $x \in X$ and f is the corresponding measurable function, we define $\mu(x) = \int f\, d\mu$, where the integral is defined in the same way as the Lebesgue integral except that μ is only finitely additive. It is easy to check that μ, defined in this way, is in X'. We now show that there is a net of states m_α such that $m_\alpha(a) \to \mu(a)$ for every $a \in A$. Hence $m_\alpha \to \mu$ weakly and since $\mu \notin M$, this would show that M is not weakly closed in X'. A finite collection $(a_1, \ldots, a_n)$ of disjoint sets in A is called a *partition* if $\Omega = \bigcup a_i$. If p_1, p_2 are partitions, we write $p_1 \geqq p_2$ if every set of p_1 is contained in some set of p_2. It is easy to see that the collection P of partitions is a directed set. Now, associate with each partition $p = (a_1, \ldots, a_k)$ a set of points $\{q_1, \ldots, q_k\} \subset \Omega$ such that $q_i \in a_i$, $i = 1, \ldots, k$. If m_{q_i} denotes the measure concentrated at q_i, then we associate with every partition $p = (a_1, \ldots, a_k)$ a measure

$$m_p = \sum_{i=1}^{k} \mu(a_i) m_{q_i}.$$

Since the m_{q_i} are states and

$$\sum_{i=1}^{k} \mu(a_i) = 1,$$

we see that m_p is a state. We claim that $\{m_p: p \in P\}$ is a net which converges weakly to μ. Indeed, if $a = \Omega$ or $a = \phi$, then clearly $m_p(a) \to \mu(a)$. Now, if $a \neq \Omega, \phi$, define the partition $p_0 = (a, a')$. If $p = (a_1, \ldots, a_k) \geqq p_0$ reorder the a_i's if necessary, so that

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$$a = \bigcup_{j=1}^{l} a_j \quad \text{and} \quad a' = \bigcup_{j=l+1}^{k} a_j.$$

Then

$$m_p(a) = \sum_{i=1}^{k} \mu(a_i) m_{q_i}(a) = \sum_{i=1}^{l} \mu(a_i) = \mu(a).$$

Therefore $m_p(a) = \mu(a)$ if $p \geqq p_0$ and hence $m_p(a) \to \mu(a)$ for every $a \in A$. We shall use some of the techniques in this example to prove Theorem 3.4.

Let X be the bounded observables on a logic L and X' the dual of X. If for $f \in X'$, $m \in M$, and a real number c, we have $f(x) = cm(x)$ for all $x \in X$, we write $f = cm$. A linear functional f on X is *positive* if $f(x) \geqq 0$ whenever $\sigma(x) \geqq 0$.

LEMMA 3.3. *Let f be a positive linear functional on X. Then* (a) $f \in X'$ *and* (b) $|f| = 1$ *if and only if* $f(1) = 1$.

Proof. (a) If $|x| \leqq 1$, we have $|\sigma(x)| \leqq 1$ and thus $\sigma(1 \pm x) \geqq 0$. Thus $f(1) \pm f(x) = f(1 \pm x) \geqq 0$ and $|f(x)| \leqq f(1)$. (b) Suppose that $|f| = 1$. Since $|1| = 1$, we have $f(1) \leqq 1$. But by part (a), $|f(x)| \leqq f(1)$ for all x with $|x| \leqq 1$. Thus $f(1) = 1$. The converse is similar.

A finite set of disjoint non-zero propositions $\{a_1, \ldots, a_n\}$ is a *partition* of a logic if $\bigvee a_i = 1$. If p_1 and p_2 are partitions, we write $p_1 \leqq p_2$ if every proposition in p_2 is $\leqq$ some proposition of p_1. It is easily checked that the partitions form a partially ordered set. We say that a logic L is *directed* if the partitions form a directed set; that is, for any two partitions p_1, p_2 there is a partition p_3 such that $p_1 \leqq p_3$, $p_2 \leqq p_3$.

THEOREM 3.4. *Let L be a directed logic with a full set of states M. Then M is weakly closed if and only if every positive linear functional f on X has the form $f = |f|m$ for some $m \in M$.*

Proof. To prove sufficiency, let m_α be a net of states converging weakly to $f \in X'$. Since the m_α's are positive, it follows that f is positive and hence $f = |f|m$ for some $m \in M$. Since $m_\alpha(1) = 1$, we have $f(1) = 1$ and by Lemma 3.3 (b), $|f| = 1$. Hence $f = m$ and M is weakly closed. Conversely, suppose M is weakly closed and f is a positive linear functional on X. If $|f| = 0$, the proof is complete; so suppose $|f| \neq 0$ and let $g = f/|f|$. Now g is a positive linear functional with $|g| = 1$ so by Lemma 3.3, $g(1) = 1$. Define $m(a) = g(x_a)$ for all $a \in L$. It is easy to see, as in our previous example, that m generates a continuous linear functional. We now show that m is a state. Clearly, $m(1) = 1$. For every partition $p = \{a_1, \ldots, a_n\}$ of L define a set of states $\{m_{a_1}, \ldots, m_{a_n}\}$ such that $m_{a_i}(a_i) = 1$ (here we use the fullness of M), and define

$$m_p = \sum_{i=1}^{n} g(x_{a_i}) m_{a_i}.$$

Notice that m_p is a state. Indeed,

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$$\sum_{i=1}^{n} g(x_{a_i}) = g\left(\sum_{i=1}^{n} x_{a_i}\right) = g(1) = 1,$$

and $0 \leqq g(x_{a_i}) \leqq 1$ since g is positive. Since the partitions form a directed set, m_p is a net of states. To show that m_p converges weakly to m, let $0 \neq a \in L$ and let $p_0 = \{a, a'\}$. If $\{a_1, \ldots, a_n\} = p \geqq p_0$, reorder the a_i's if necessary so that $a_i \leqq a$, $i = 1, \ldots, j$, $a_i \leqq a'$, $i = j + 1, \ldots, n$. It easily follows that $\bigvee\{a_i, i = 1, \ldots, j\} = a$ and $\bigvee\{a_i, i = j + 1, \ldots, n\} = a'$. Then

$$m_p(a) = \sum_{i=1}^{j} g(x_{a_i}) m_{a_i}(a) = \sum_{i=1}^{j} g(x_{a_i}) = g\left(\sum_{i=1}^{j} x_{a_i}\right) = g(x_a) = m(a).$$

Thus $m_p \to m$ weakly, and since M is weakly closed, $m \in M$. Now

$$g(x_a) = m(a) = \int \lambda m[x_a(d\lambda)]$$

for all $a \in L$. If $s = \sum_{i=1}^{n} c_i x_{a_i}$ is a simple observable, we have

$$g(s) = \sum_{i=1}^{n} c_i g(x_{a_i}) = \sum_{i=1}^{n} c_i \int \lambda m[x_{a_i}(d\lambda)] = \sum_{i=1}^{n} c_i m(a_i).$$

Now there is an observable z and Borel sets E_i such that $x_{a_i} = \chi_{E_i}(z)$, $i = 1, \ldots, n$. Thus

$$\begin{aligned}\int \lambda m[s(d\lambda)] &= \int \lambda m[(\sum c_i \chi_{E_i})(z)(d\lambda)] \\ &= \int \sum c_i \chi_{E_i}(\lambda) m[z(d\lambda)] = \sum c_i \int \chi_{E_i}(\lambda) m[z(d\lambda)] \\ &= \sum c_i m[z(E_i)] = \sum c_i m(a_i).\end{aligned}$$

Hence, $g(s) = \int \lambda m[s(d\lambda)]$ for simple observables. Now, if x is any bounded observable, there is a sequence $s_i(x)$ of simple functions of x converging to x in norm, where the s_i are defined as in Lemma 7.1 of (**1**). Thus

$$\int s_i(\lambda) m[x(d\lambda)] = g(s_i) \to g(x) \quad \text{as} \quad i \to \infty.$$

But by definition of s_i and the monotone convergence theorem,

$$\int s_i(\lambda) m[x(d\lambda)] \to \int \lambda m[x(d\lambda)] \quad \text{as} \quad i \to \infty.$$

Thus $g(x) = \int \lambda m[x(d\lambda)] = m(x)$ for all $x \in X$. Hence $f/|f| = g = m$ and $f = |f| m$.

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