Combinatorics, Probability and Computing (2018) 27, 774–793.
doi:10.1017/S0963548318000147

A Stability Theorem for Matchings in Tripartite 3-Graphs

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Received 19 January 2017; revised 1 Feb 2018; first published online 2 April 2018

It follows from known results that every regular tripartite hypergraph of positive degree, with n vertices in each class, has matching number at least $n/2$. This bound is best possible, and the extremal configuration is unique. Here we prove a stability version of this statement, establishing that every regular tripartite hypergraph with matching number at most $(1+\varepsilon)n/2$ is close in structure to the extremal configuration, where 'closeness' is measured by an explicit function of ε.

2010 *Mathematics subject classification*: Primary 05C65
Secondary 05C70, 05D15

1. Introduction

One of the simplest statements about matchings in bipartite graphs is the following corollary of Hall's theorem.

Theorem 1.1. *Let G be a bipartite regular multigraph of positive degree. Then G has a perfect matching.*

Our principal aim in this paper is to study the hypergraph analogue of this result. A k-uniform multihypergraph (in which multiple edges are allowed), which we will call a k-graph for short, is *k-partite* if its vertices can be partitioned into k classes $V_1,\dots,V_k$ such that every edge has exactly one vertex from each class V_i.

In this paper, we will limit our interests to 3-partite 3-graphs. For these, we have the following version of Theorem 1.1.

Theorem 1.2. *Let $\mathcal{H}$ be a regular* 3*-partite* 3*-graph of positive degree, with n vertices in each class. Then $\mathcal{H}$ has a matching of size at least $n/2$.*

† Partially supported by NSERC. This author also thanks the Mittag-Leffler Institute in Djursholm, Sweden, where part of this work was done.

This is an immediate consequence of a theorem of Aharoni [2], which verified the 3-partite case of a famous old conjecture due to Ryser [12] relating the minimum size $\tau(\mathcal{H})$ of a vertex cover of $\mathcal{H}$ (a set of vertices meeting all edges) to the maximum size $v(\mathcal{H})$ of a matching in $\mathcal{H}$.

Theorem 1.3 (Aharoni's theorem). *Let $\mathcal{H}$ be a 3-partite 3-graph. Then $\tau(\mathcal{H}) \leqslant 2v(\mathcal{H})$.*

Proof of Theorem 1.2. Let $\mathcal{H}$ be an r-regular 3-partite 3-graph with n vertices in each class. Then $\mathcal{H}$ has rn edges, but each vertex only intersects r of them, hence any vertex cover must have at least $rn/r = n$ vertices, so $\tau(\mathcal{H}) \geqslant n$. By Aharoni's theorem, we have $v(\mathcal{H}) \geqslant \tau(\mathcal{H})/2 \geqslant n/2$, which proves the theorem. □

Theorem 1.2 is best possible, as can be seen by the following example. The *truncated Fano plane* $\mathcal{F}$ (also called the *Pasch configuration*) is the 3-partite 3-graph with six vertices $x_1, x_2, x_3, y_1, y_2, y_3$ and four edges $x_1x_2x_3$, $x_1y_2y_3$, $y_1x_2y_3$, $y_1y_2x_3$, where the sets $\{x_i, y_i\}$ are the vertex classes. It is easy to check that $\mathcal{F}$ is 2-regular and $v(\mathcal{F}) = 1$. For a hypergraph $\mathcal{H}$ and an integer s, we let $s \cdot \mathcal{H}$ denote the hypergraph with the same vertices as $\mathcal{H}$ and with each edge replaced by s parallel copies.

If $\mathcal{H}$ consists of $n/2$ disjoint copies of $(r/2) \cdot \mathcal{F}$, then $v(\mathcal{H}) = n/2$, illustrating the tightness of Theorem 1.2 for every even r and every even n. This is the unique extremal configuration, a fact which follows from [8] in which the extremal hypergraphs for Aharoni's theorem are characterized.

Our main aim in this paper is to prove the following stability version of Theorem 1.2.

Theorem 1.4. *Let $r \geqslant 2$. Let $\mathcal{H}$ be an r-regular 3-partite 3-graph with n vertices in each class, and let $\varepsilon \geqslant 0$. If $v(\mathcal{H}) \leqslant (1+\varepsilon)n/2$, then $\mathcal{H}$ has at least $(1-(22r-77/3)\varepsilon)n/2$ components that are copies of $(r/2) \cdot \mathcal{F}$.*

It is also possible to weaken the condition that the hypergraph is regular, resulting in a weaker conclusion.

Theorem 1.5. *Let $r \geqslant 2$. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes A, B and C, such that $|A| = n$, and let $\varepsilon \geqslant 0$. Suppose that every vertex of A has degree at least r, and that every vertex in $B \cup C$ has degree at most r. If $v(\mathcal{H}) \leqslant (1+\varepsilon)n/2$, then $\mathcal{H}$ contains at least $(1-(72r^2-150r+77)\varepsilon)n/2$ disjoint copies of $(r/2) \cdot \mathcal{F}$.*

Theorem 1.5 may be viewed as a direct hypergraph analogue of the corresponding weakening of Theorem 1.1, with the condition that the minimum degree of vertices in vertex class A is at least the maximum degree of vertices in class B, and which concludes that the bipartite graph has a matching of size $|A|$.

To prove Theorems 1.4 and 1.5 we rely on a version of Hall's theorem for hypergraphs, that uses a graph parameter η whose definition is topological (the connectedness of the independence complex). However, the only properties of η we will need come from known theorems which can be stated in purely graph-theoretical terms. Thus none of our proofs will make any explicit reference to topology. This background material is described in Section 2. In Section 3 we prove a new lower bound on η for line graphs of bipartite multigraphs, which will form the basis of our work in this paper. Section 4 contains the proofs of Theorems 1.4 and 1.5, and in Section 5 we describe some constructions that show a limit on the amount by which our theorems could be improved. We close by mentioning a few open problems.

2. Tools

We begin by describing the version of Hall's theorem for k-partite k-graphs that we will need. In this setting, the analogue of the neighbourhood of a vertex subset S (which in the bipartite graph case is just an independent set of vertices) is a $(k-1)$-partite $(k-1)$-graph called the *link* of S.

Definition. Let $\mathcal{H}$ be a k-partite k-graph with vertex classes $V_1,\dots,V_k$, and let $S \subseteq V_i$. The *link* of S is the $(k-1)$-partite $(k-1)$-graph $\mathrm{lk}S$ whose vertex classes are the sets $\{V_1,\dots,V_k\} \setminus \{V_i\}$, and whose edges are $\{e - v : v \in S, v \in e \in E(\mathcal{H})\}$.

The generalization of Hall's theorem to k-partite k-graphs [4, 6] can be stated in terms of a number of parameters of the link hypergraphs, for instance their matching numbers, or, as in its original formulation [6], their *matching width* (the maximum among all matchings of the size of the smallest matching intersecting each of its edges). The formulation we use here is based on the parameter $\eta(J)$, which is defined to be the topological connectedness of the independence complex of the graph J plus 2. (We add 2 in order to make η additive under disjoint union, which makes practically every formula involving it simpler. See *e.g.* [5] for a discussion of this parameter.) Our graphs J will usually be subgraphs of the line graph $L(G)$ of a bipartite graph G. The relevant version of Hall's theorem for hypergraphs is as follows.

Theorem 2.1 (Hall's theorem for hypergraphs). *Let $\mathcal{H}$ be a k-partite k-graph with vertex classes $V_1,\dots,V_k$, and let $d \geqslant 0$. If $\eta(L(\mathrm{lk}S)) \geqslant |S| - d$ for every subset $S \subseteq V_i$, then $\mathcal{H}$ has a matching of size at least $|V_i| - d$.*

The only properties of η we will need for our purposes are contained in the next three statements (and in fact the third follows easily from the second).

The first lemma is derived from basic properties of connectedness that can be found in any textbook on topology.

Lemma 2.2.

(1) *If the graph J has no vertices, then $\eta(J) = 0$.*
(2) *If the graph J contains an isolated vertex, then $\eta(J) = \infty$.*
(3) *If J and K are disjoint graphs, then*

$$\eta(J \cup K) \geqslant \eta(J) + \eta(K).$$

Note that the last part implies in particular that adding any non-empty component to a graph increases the value of η by at least 1.

The next statement is Meshulam's theorem [9], which relates $\eta(J)$ to that of two subgraphs of J, obtained by deleting an edge, or by what we call 'exploding' an edge. If J is a graph and $e \in E(J)$ is an edge, then we denote the edge deletion of e by $J - e$. We denote the edge *explosion* of e by $J \divideontimes e$, which is the subgraph of J that remains after deleting both endpoints of e and all their neighbours.

Theorem 2.3 (Meshulam's theorem). *If J is a graph and $e \in E(J)$, then*

$$\eta(J) \geqslant \min(\eta(J - e), \eta(J \divideontimes e) + 1).$$

This result (in a different formulation) is proved in [9]. For more on Meshulam's theorem see *e.g.* [1] and [11, Section 5.3].

Various lower bounds on $\eta(J)$ in terms of other graph parameters have been proved: see *e.g.* [5, 9]. Of particular interest to us is the following bound for line graphs (which was used for example in [6] but also follows easily from Theorem 2.3).

Theorem 2.4. *If G is a multigraph, then*

$$\eta(L(G)) \geqslant \frac{\nu(G)}{2}.$$

In the next section, we will apply Meshulam's theorem to obtain an alternate version of the above bound for bipartite graphs, which takes into account the maximum degree as well as the matching number.

3. The connectedness of line graphs of bipartite multigraphs

In order to state and prove our results, we will need some definitions first.

If G is a multigraph and $J \subseteq L(G)$ is a subgraph of the line graph of G, we let G_J denote the subgraph of G with $V(G_J) = V(G)$ and $E(G_J) = V(J)$. Note that this makes sense, as the vertices of J are a subset of the edges of G.

An *r-regular C_4* is a bipartite multigraph consisting of a cycle of length 4 and edges parallel to the edges of the cycle so that every vertex has degree r.

An edge $e \in E(J)$ is called *decouplable* if $\eta(J - e) \leqslant \eta(J)$. It is called *explodable* if $\eta(J \divideontimes e) \leqslant \eta(J) - 1$. Note that by Theorem 2.3, every edge is either decouplable or explodable.

A graph is called *reduced* if no edge is decouplable (hence every edge is explodable). A subgraph $J' \subseteq J$ is called a *reduction* of J if J' is reduced, $V(J') = V(J)$, and $\eta(J') \leqslant \eta(J)$. Note that one may obtain a reduction of a graph J by iteratively deleting decouplable edges until there are none left.

In the proof of our theorem, we will be applying Meshulam's theorem to edges of the line graph, but will be regularly referring back to the original bipartite graph, whose edges are vertices of the line graph. To help eliminate confusion among vertices of the graph G,

https://doi.org/10.1017/S0963548318000147 Published online by Cambridge University Press

vertices of the line graph $L(G)$, edges of the graph, and edges of the line graph, we will use different terminology. Vertices and edges will always refer to vertices and edges of the original graph, while edges of the line graph will be called *adjacencies*, or *J-adjacencies* for J a subgraph of the line graph. If a pair of edges of the graph intersect, they will be adjacent in the line graph, but not necessarily J-adjacent.

When talking about decouplable or explodable edges of the line graph, rather than say something like 'decouplable adjacency', we will often refer to these as decouplable (explodable) pairs of edges (of the original graph).

Our main aim in this section is to prove the following theorem.

Theorem 3.1. *Let G be a bipartite multigraph with maximum degree $r \geqslant 2$ that does not contain an r-regular C_4 component, and let $J \subseteq L(G)$. Then*

$$\eta(J) \geqslant \frac{(2r-3)\nu(G_J) + |V(J)|}{6r-7}.$$

Note that this is an improvement over the bound on $\eta(L(G))$ given in Theorem 2.4 whenever $|E(G)| \geqslant ((2r-1)/2)\nu(G)$, and agrees with the bound when equality holds. In order to prove it, we will need the following lemma.

Lemma 3.2. *Let G be a bipartite multigraph with maximum degree $r \geqslant 2$ that does not contain an r-regular C_4 component, and let $J \subseteq L(G)$ be reduced and non-empty. Then if $\eta(J) \neq \infty$, J contains an explodable pair me of one of the following types:*

(1) $\nu(G_{J \circledast me}) \geqslant \nu(G_J) - 1$ *and* $|V(J \circledast me)| \geqslant |V(J)| - (3r-2)$,
(2) $\nu(G_{J \circledast me}) \geqslant \nu(G_J) - 2$ *and* $|V(J \circledast me)| \geqslant |V(J)| - (2r-1)$, *or*
(3) *every reduction J' of $J \circledast me$ contains an explodable pair $m'e'$ such that* $\nu(G_{J' \circledast m'e'}) \geqslant \nu(G_J) - 3$, *and* $|V(J' \circledast m'e')| \geqslant |V(J)| - (6r-5)$.

Proof of Theorem 3.1 from Lemma 3.2. Let G be a bipartite multigraph with maximum degree $r \geqslant 2$ that does not contain an r-regular C_4 component, and let $J \subseteq L(G)$. Also, suppose that $|V(J)| \geqslant ((2r-1)/2)\nu(G_J)$ (otherwise we may simply apply Theorem 2.4 to prove our theorem).

We construct a sequence of subgraphs $J_0, \ldots, J_n$ with $J_0 = J$ and J_n having no edges, in which J_i is obtained from J_{i-1} by either deleting a decouplable J_i-adjacency or exploding an explodable pair of edges in G_{J_i}. This means that $\eta(J_{i-1}) \geqslant \eta(J_i)$, with strict inequality whenever we perform an explosion.

We start by iteratively deleting decouplable adjacencies until we have a reduced subgraph $J_k \subseteq J$. Applying Lemma 3.2, we find that there is an explodable pair of type (1), (2) or (3). We explode this pair to arrive at J_{k+1}. In the case of an explosion of type (3), we then iteratively decouple decouplable pairs to arrive at a reduction J' of J_{k+1} and then explode $m'e'$. We continue in this fashion until J_n has no edges.

In the end, we will get a bound $\eta(J) \geqslant t + \eta(J_n)$, where t is the number of explosions we perform in the sequence. Let x_i denote the number of explosions of type (i). Note that for every explosion of type (3), we perform another explosion, so the total number of explosions is $t = x_1 + x_2 + 2x_3$. If J_n has a vertex, it is isolated, which would show $\eta(J) = \infty$, so we may assume that J_n is the empty graph, and so $\nu(G_{J_n}) = 0$ and $\eta(J_n) = 0$. Since the matching number is only affected by explosions, we thus obtain a bound

$$x_1 + 2x_2 + 3x_3 \geqslant \nu(G_J),$$

since explosions of type (i) decrease the matching number by at most i. Similarly, these explosions must reduce the vertex number to $|V(J_n)| = 0$, giving us the bound

$$(3r-2)x_1 + (2r-1)x_2 + (6r-5)x_3 \geqslant |V(J)|.$$

Since we do not assume any control over the values of x_i, we suppose that we obtain the worst bound, where $t = x_1 + x_2 + 2x_3$ is minimized among all triples of non-negative integers (x_1, x_2, x_3) satisfying the above two constraints. Relaxing the integer program to a linear program gives us the bound in the theorem, since for $|V(J)| \geqslant ((2r-1)/2)\nu(G_J)$, the minimum is obtained at

$$x_1 = 0, \quad x_2 = \frac{(6r-5)\nu(G_J) - 3|V(J)|}{6r-7}, \quad x_3 = \frac{2|V(J)| - (2r-1)\nu(G_J)}{6r-7},$$

with a value of

$$t_{\min} = \frac{(2r-3)\nu(G_J) + |V(J)|}{6r-7}.$$

This can be confirmed by considering the dual linear program, which is to maximize $\nu(G_J)y_1 + |V(J)|y_2$ among positive real pairs (y_1, y_2) subject to the constraints

$$\begin{aligned} y_1 + (3r-2)y_2 &\leqslant 1, \\ 2y_1 + (2r-1)y_2 &\leqslant 1, \\ 3y_1 + (6r-5)y_2 &\leqslant 2. \end{aligned}$$

It is enough to note that

$$y_1 = \frac{2r-3}{6r-7}, \quad y_2 = \frac{1}{6r-7}$$

is feasible for the dual program, and its value is $\nu(G_J)y_1 + |V(J)|y_2 = t_{\min}$. □

Proof of Lemma 3.2. Let G be a bipartite multigraph with maximum degree $r \geqslant 2$, and let $J \subseteq L(G)$ be reduced and contain an edge. Suppose that there are no explodable pairs of any of the types (1), (2) and (3). We aim to show that G contains an r-regular C_4 component.

Lemma 3.3. *Each explosion in J destroys at most $3r-2$ edges of G.*

Proof. Any pair of intersecting edges have at most three vertices in which to meet other edges. Thus, as G has maximum degree r, there are at most $3r-2$ edges incident to those vertices, because the two edges in question count towards the degree of two of these vertices each. Thus, every explosion that reduces the matching number by at most 1 is automatically an explosion of type (1). □

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Lemma 3.4. *No two edges that are parallel are J-adjacent.*

Proof. If e and f are parallel, then $\nu(G_{J*ef}) \geqslant \nu(G_J) - 2$, and $|V(J * ef)| \geqslant |V(J)| - (2r-2)$, so this would be an explosion of type (2), which does not exist. Hence e and f cannot be J-adjacent, as J is reduced. □

Lemma 3.5. *If $M \subseteq V(J)$ is a maximum matching of G_J, and $e \in V(J) \setminus M$ is J-adjacent to an edge of M, then e is J-adjacent to two edges of M (one at each endpoint of e).*

Proof. Suppose e is J-adjacent to only $m \in M$, but no other edge of M. Then exploding me would destroy only one edge of M, which reduces the matching number by at most 1. Thus by Lemma 3.3 this would be an explosion of type (1), which we assume not to exist. Thus, e must be J-adjacent to a second edge of M. □

We now make a few definitions, which will provide the setup for the two upcoming Lemmas 3.6 and 3.7.

For a maximum matching $M \subseteq V(J)$ and two edges $m \in M$, and $e \in V(J) \setminus M$ with $me \in E(J)$, define $\mathcal{P}(M, m, e)$ to be the set of edges in $V(J)$ contained in some M-alternating path in G_J starting with m, e. Let A be the vertex class of G containing the starting point of these paths, and let B be the other. Let $Y \subseteq A$ be the set of vertices in edges of $\mathcal{P}(M, m, e)$ contained in A, but not including the vertices of m and e. Let $X \subseteq B$ be the set of vertices in edges of $\mathcal{P}(M, m, e)$ contained in B, this time including the vertex in $m \cap e$.

Let $m' \in M$ be the other edge of M besides m that is J-adjacent to e, which is guaranteed to exist by Lemma 3.5.

Lemma 3.6. *All vertices of Y are M-saturated.*

Proof. Suppose $y \in Y$ is M-unsaturated. By the definition of Y, there is an M-alternating path in G_J starting m, e, and ending in vertex y. Exploding me destroys two edges m and m' of M, since it is not of type (1). However, for $M' = M \setminus \{m, m'\}$, we have that the rest of the path ending in y is an M'-augmenting path in G_{J*me}, which means that in fact $\nu(G_{J*me}) \geqslant \nu(G_J) - 1$, and therefore the explosion of me is of type (1) after all. This is a contradiction, thus no $y \in Y$ can be M-unsaturated. □

Lemma 3.7. *Every edge of M with a vertex in Y is J-adjacent in Y to an edge whose other endpoint is not in X.*

Proof. Consider what happens when we explode me. This destroys m and m', and destroys at most $3r-2$ edges in total by Lemma 3.3. Let d be the vertex of G_J in $m' \cap X$. Let J' be a reduction of $J * me$, and let $M' = M \setminus \{m, m'\}$. We will make use of the fact that me is not an explosion of type (3). This means that J' does not contain a pair of J'-adjacent edges whose explosion would reduce the matching number by at most 1 and destroy at most $3r-3$ edges.

To prove Lemma 3.7 we establish two claims.

Claim 3.8. *All edges of M' with a vertex in Y are not J'-adjacent to any edge preceding or succeeding them in an M'-alternating path in $G_{J'}$ starting at d.*

Proof. Consider any M'-alternating path P in $G_{J'}$ starting at d. Since these are all parts of the M-alternating paths in G_J starting with m, e, we see that every edge of M' incident to X is in one of these paths. Note that d has degree at most $r-1$ in $G_{J'}$, since m' was incident to it and was destroyed in the explosion of me. Denote the edges of the path P by $e_1, m_1, e_2, m_2, \ldots$, so that $m_i \in M'$ and e_1 is incident to d. Recalling that J' is reduced, we claim that none of the pairs in the path are J'-adjacent. Indeed, e_1 and m_1 are not, because if they were explodable, this would make me an explosion of type (3). To see this, note that since we only destroy one edge of M' in the second explosion, we reduce $\nu(G'_J)$ by at most 1, and since d has degree at most $r-1$, we destroy at most $3r-3$ edges in the second explosion. This kind of explosion has been ruled out. Neither are m_1 and e_2 J'-adjacent, since exploding this pair would not destroy e_1, which means we could add it to $M' \setminus \{m_1, m_2\}$ to have a matching of size $\nu(G'_J)-1$ after the second explosion, and again we destroy at most $3r-3$ edges incident to $e_1 \cap m_1$, since we do not destroy e_1. This would again make me an explosion of type (3), which contradicts our assumptions.

Continuing in this fashion along the path, we see that e_i and m_i are not J'-adjacent, because exploding this pair would reduce the matching number by at most 1, as e_i is not J'-adjacent to m_{i-1}, and for the same reason, we only destroy $3r-3$ edges in the second explosion, which would make me an explosion of type (3). Next, we see that m_i and e_{i+1} are not J'-adjacent, because exploding this pair would leave an $(M' \setminus \{m_i, m_{i+1}\})$-augmenting path $e_1, m_1, \ldots, e_i$, so even though two edges of M' are destroyed, the matching number decreases only by 1, if at all, and again, we only destroy $3r-3$ edges in this second explosion because e_i is not destroyed. This proves the claim. □

Claim 3.9. *No edge of M' incident to Y is J'-adjacent to an edge between X and Y.*

Proof. Suppose $e' \in V(J') \setminus M'$ that joins X and Y is J'-adjacent to some $m'' \in M'$ incident to Y. We claim that if $m''e'$ were explodable, then me would be an explosion of type (3), and hence m'' and e' are not J'-adjacent, as J' is reduced.

If e' is incident to $b \in m \cap e$, then exploding $m''e'$ reduces $\nu(G_{J'})$ by only 1 and destroys at most $3r-4$ edges, since m and e are already gone. This would make m an explosion of type (3). If e' is incident to d, then it is the predecessor of m'' on some M'-alternating path, so they are not J'-adjacent by Claim 3.8. Hence we may assume e' is incident to a vertex of $X \setminus \{b, d\}$. If it is parallel to m'', then exploding $m''e'$ would destroy one edge of M' and at most $2r-2$ edges, which would again make me a type (3) explosion.

Therefore e' meets an edge $m''' \in M'$ in a vertex of $X \setminus \{b, d\}$. If e' is not J'-adjacent to m''' then $m''' \neq m''$ and exploding $m''e'$ would reduce the matching number by at most one and destroy at most $3r-3$ edges, making me a type (3) explosion. Hence $m'''e' \in E(J')$. By Lemma 3.6 we know e' shares its Y-vertex with some $m^* \in M'$. If e' is not J'-adjacent to m^* then in the same way exploding $m'''e'$ shows me is a type (3) explosion. Hence (by renaming if necessary) we may assume e' is J'-adjacent to $m'' \in M'$ at Y, and to $m''' \neq m''$ at X.

If there is an M'-alternating path from d to m''' that does not use m'', appending e' and m'' to this path shows by Claim 3.8 that e' and m'' are not J'-adjacent, which is a contradiction. If there is no such path, then e' together with the part between m'' and m''', inclusive, of an M'-alternating path P from d to m''' forms an M'-alternating cycle. In this case, let M'' be obtained from M' by switching on that M'-alternating cycle. By Claim 3.8, exploding $m''e'$ only destroys one edge of M'' (namely e'), so the resulting graph has a matching of size at least $\nu(G_{J'}) - 1$. The explosion also fails to destroy the successor of m'' on P, so we lose at most $3r - 3$ edges in the second explosion, which makes me of type (3). □

Thus every edge of M' incident to Y is not J'-adjacent to any edge between X and Y. However, none of these edges of M' are isolated in J', since we have $\eta(J') \leqslant \eta(J) - 1 < \infty$ (recalling that J' was obtained from J by performing one explosion and reducing). This means that they each must be J'-adjacent to some edge that is not between X and Y. If such an edge e^* is incident to X, we would have an M'-alternating path going from d to the matching edge then to e^*, making the other endpoint of e^* a vertex of Y by definition. Thus e^* is not incident to X, which proves Lemma 3.7, since J'-adjacent implies J-adjacent. □

We now complete the proof of Lemma 3.2.

Choose the triple (M, m, e) consisting of a maximum matching M of G_J and a pair of J-adjacent edges $m \in M$ and $e \in V(J) \setminus M$ so that $|\mathcal{P}(M, m, e)|$ is maximized among all such triples. We claim that m and e are in fact part of an r-regular C_4 component of G_J. Let m' be the other edge of M that is J-adjacent to e, which exists by Lemma 3.5, and let the vertices of m, e and m' be a, b, c and d, with $m = ab$, $e = bc$ and $m' = cd$.

Claim 3.10. *Every edge that is J-adjacent to m at a is incident to d.*

Proof. Suppose that e' were such an edge. By Lemma 3.5, it is J-adjacent to another edge $\hat{m} \in M$. If $e' \cap \hat{m} \notin X$, then we have a contradiction, as any edge e^* in $\mathcal{P}(M, m, e)$ can be reached by an M-alternating path starting with $\hat{m}$, e', then continuing with m, e, and the rest of the path that shows e^* is in $\mathcal{P}(M, m, e)$. But $\hat{m} \notin \mathcal{P}(M, m, e)$, since it is not incident to X, which runs contrary to the assumption that $|\mathcal{P}(M, m, e)|$ is maximum. Therefore, $\hat{m}$ must be incident to X. If $\hat{m} \neq m'$, then $\hat{m}$ is also incident to Y, and so by Lemma 3.7, it has an edge e'' J-adjacent to it in Y, which is not incident to X, and by Lemma 3.5, e'' is J-adjacent to another edge $\hat{m}' \in M$. But then $\mathcal{P}(M, \hat{m}', e'')$ would strictly contain $\mathcal{P}(M, m, e)$. This is because for any edge in $\mathcal{P}(M, m, e)$, if the path from m, e containing it passes through $\hat{m}$, we can start with $\hat{m}'$, e'', $\hat{m}$ and continue along the path to reach it from $\hat{m}'$, e''. If on the other hand the path from m, e does not include $\hat{m}$, we can reach it by starting with $\hat{m}'$, e'', $\hat{m}$, e', m, e, and continuing along the path. This also contradicts our choice of (M, m, e). This means the only option is $\hat{m} = m'$. Thus e' is incident to d. □

Next, we establish that there is an edge $f = ad$, which is J-adjacent to m. If there were no such edge, then exploding me would destroy only edges incident to b and c, of which

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there are at most $2r-1$, since e is counted twice. Since also $v(G_J)$ would be reduced by at most 2, this would be an explosion of type (2), which we assume not to exist. Thus there must be an edge incident to a that is J-adjacent to m, and by Claim 3.10, we have seen that such an edge must be incident to d.

Now consider the matching $M^\times = M \cup \{e,f\} \setminus \{m,m'\}$, obtained by switching M along the C_4 on $abcd$. Note that $\mathcal{P}(M^\times,e,m) = \mathcal{P}(M,m,e) \cup \{f\} \setminus \{m'\}$, since any M-alternating path starting m, e, m' can be converted to an $M^\times$-alternating path by starting with e, m, f, and continuing the same way. Therefore this triple is also maximizing, so Claim 3.10 applies with M replaced by $M^\times$ to show that the only edges J-adjacent to e at c are parallel to m'.

We now show that m' and f have no J-neighbours at c or a, respectively, except those parallel to e and m, respectively. Suppose m' has a J-neighbour g contradicting this statement. Then by Lemma 3.5 we see that g has another J-neighbour $h \in M$ not among $\{m,m'\}$. But now applying Lemma 3.5 again to g with M replaced by $M^\times$ tells us that since g is J-adjacent to $h \in M^\times$ it must also be J-adjacent to $e \in M^\times$. This contradicts our finding in the previous paragraph. Thus such a g cannot exist. Similarly, if f has a J-neighbour g at a that is not parallel to m, then Lemma 3.5 applied first with $M^\times$ and then with M leads to a contradiction.

We have therefore shown that none of m, m', e and f have any J-neighbours incident to $\{a,c\}$ that leave the C_4 on $abdc$.

Now suppose that there is an edge incident to d that is not incident to a or c. Such an edge is disjoint from m and e, so it survives the explosion of me. By what we have proved above, the explosion of me only destroys edges incident to b and d, of which there are at most $2r$. But since at least one edge incident to d survives, the explosion would destroy at most $2r-1$ edges, and it clearly only destroys 2 edges of M, hence this would be an explosion of type (2). Therefore, there are no edges incident to d, except those that go to a or c. A similar argument, by threatening to explode $m'f$, shows that there are no edges incident to b, except those that go to a or c. If any of b or d is not of degree r, then me would again be an explosion of type (2), so they are both maximum degree vertices. This forces all edges incident to a and c to be those from b and d by a simple counting argument. Therefore, $abcd$ form the vertices of an r-regular C_4-component of G_J. This proves the lemma by contraposition. □

Corollary 3.11. *Let G be a bipartite multigraph with maximum degree $r \geqslant 2$ that contains at most k components that are r-regular C_4. Then*

$$\eta(L(G)) \geqslant \frac{(2r-3)v(G) + |E(G)| - k}{6r-7}.$$

Proof. Assume, without loss of generality, that G has exactly k components that are r-regular C_4. Let G' be equal to G with all its r-regular C_4 components removed. We have $|E(G')| = |E(G)| - 2rk$ and $v(G') = v(G) - 2k$. Applying Theorem 3.1 to G', we have

$$\eta(L(G')) \geqslant \frac{(2r-3)v(G') + |E(G')|}{6r-7}.$$

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Adding k non-empty components to $L(G')$ will increase its value of η by at least k by Lemma 2.2, so $\eta(L(G)) \geqslant \eta(L(G')) + k$, and this gives the desired bound via a straightforward calculation. □

We remark that Theorem 3.1 is tight when $r = 2$, as can be seen by taking G to be the disjoint union of any number of paths P_4 of length 3 and cycles of length 10 (since $\eta(P_4) = 1$, and $\eta(C_{10}) = 3$).

4. Stability

We have two versions of our stability theorem, which for convenience we restate here. One is for r-regular 3-partite 3-graphs, and the other has slightly less stringent degree conditions, which of course results in a weaker bound.

Theorem 4.1. *Let $r \geqslant 2$. Let $\mathcal{H}$ be an r-regular 3-partite 3-graph with n vertices in each class, and let $\varepsilon \geqslant 0$. If $\nu(\mathcal{H}) \leqslant (1+\varepsilon)n/2$, then $\mathcal{H}$ has at least $(1-(22r-77/3)\varepsilon)n/2$ components that are $(r/2)\cdot\mathcal{F}$.*

Theorem 4.2. *Let $r \geqslant 2$. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes A, B and C, such that $|A| = n$, and let $\varepsilon \geqslant 0$. Suppose that every vertex of A has degree at least r, and that every vertex in $B \cup C$ has degree at most r. If $\nu(\mathcal{H}) \leqslant (1+\varepsilon)n/2$, then $\mathcal{H}$ contains at least $(1-(72r^2-150r+77)\varepsilon)n/2$ disjoint copies of $(r/2)\cdot\mathcal{F}$.*

Our strategy is to use the low matching number to find a subset of each vertex class whose links have low connectedness. From this, we deduce that each link must have many r-regular C_4 components. We analyse how these can interact and deduce that a number of them must extend to $(r/2)\cdot\mathcal{F}$. We break the proofs down into several lemmas that apply in both situations.

Lemma 4.3. *Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes A, B and C, such that $|A| = n$, and let $\varepsilon \geqslant 0$. Suppose that every vertex of A has degree at least r, and that every vertex in $B \cup C$ has degree at most r. If $\nu(\mathcal{H}) \leqslant (1+\varepsilon)n/2$, then $\mathrm{lk}A$ contains at least $(1-(6r-7)\varepsilon)n/2$ components that are r-regular C_4.*

Proof. We know that there must be some $S \subseteq A$ such that $\eta(L(\mathrm{lk}S)) \leqslant |S| - (n - \nu(\mathcal{H}))$, otherwise $\mathcal{H}$ would have a matching larger than $\nu(\mathcal{H})$ by Theorem 2.1. Now $\mathrm{lk}S$ has at least $r|S|$ edges and maximum degree at most r, so $\tau(\mathrm{lk}S) \geqslant |S|$, and so by König's theorem it follows from this that $\nu(\mathrm{lk}S) \geqslant |S|$.

Let k be the number of r-regular C_4 components of $\mathrm{lk}S$. By Corollary 3.11, we have

$$\begin{aligned}\eta(L(\mathrm{lk}S)) &\geqslant \frac{(2r-3)\nu(\mathrm{lk}S) + |E(\mathrm{lk}S)| - k}{6r-7} \\ &\geqslant \frac{(2r-3)|S| + r|S| - k}{6r-7} \\ &= \frac{(3r-3)|S| - k}{6r-7}.\end{aligned}$$

Combining this with our upper bound, we find

$$\begin{aligned} k &\geqslant (6r-7)(n-\nu(\mathcal{H})) - (3r-4)|S| \\ &\geqslant (6r-7)\left(n-(1+\varepsilon)\frac{n}{2}\right) - (3r-4)n \\ &= (1-(6r-7)\varepsilon)\frac{n}{2}. \end{aligned}$$

Since the vertices of an r-regular C_4 have degree r, which is the maximum degree of any vertex in $B \cup C$, no additional edges of lk$A$ intersect any of these components of lk$S$, hence these are indeed components of lk$A$, which proves our lemma. □

We say a subgraph of a link of $\mathcal{H}$ *hosts* an edge e of $\mathcal{H}$ if the edge of the link corresponding to e is present in the subgraph.

Lemma 4.4. *Let $\mathcal{H}$ be a 3-partite 3-graph, let A be one of its vertex classes, and suppose that every vertex in A has degree at most r. If an r-regular C_4 in* lk$A$ *does not host two disjoint edges of $\mathcal{H}$, then the edges it hosts form a copy of $(r/2)\cdot\mathcal{F}$.*

Proof. Let e, f, g and h be pairwise non-parallel edges of the r-regular C_4 in lk$A$, so that e, f and g, h form matchings. Since no pair of edges extend to disjoint edges of $\mathcal{H}$, all e-parallel and f-parallel edges must meet in the same vertex, and similarly, all g-parallel and h-parallel edges meet in the same vertex. These, however, must be two different vertices, since they are incident to $2r$ edges altogether. Thus, each of these vertices is incident to r edges, and so there are r total e-parallel and f-parallel edges, and r total g-parallel and h-parallel edges. To form an r-regular C_4, there must be the same number of e-parallel edges as f-parallel ones, and similarly the same number of g-parallel and h-parallel edges. Thus there must be $r/2$ of each, and this forms an $(r/2)\cdot\mathcal{F}$, as desired. □

Lemma 4.5. *Let $\mathcal{H}$ be a 3-partite 3-graph. If an r-regular C_4 component K of a link of a vertex class of $\mathcal{H}$ is host to two disjoint edges of $\mathcal{H}$, and all of the vertices of K are part of r-regular C_4 components of the links of the other vertex classes, then K belongs to a component of $\mathcal{H}$ that either*

(1) *has two vertices in each class and a matching of size 2, or*
(2) *has four vertices in each class and a matching of size 4.*

In particular, K belongs to a component of $\mathcal{H}$ with a perfect matching.

Proof. Let V_1, V_2 and V_3 be the vertex classes of $\mathcal{H}$, and suppose that the r-regular C_4 component K in question is a component of lkV_1.

Let $a_1a_2a_3$ and $b_1b_2b_3$ be two disjoint edges of $\mathcal{H}$ with $a_i, b_i \in V_i$ and a_2, b_2, a_3 and b_3 being the vertices of K. We consider two cases.

Case 1. a_1a_2 and b_1b_2 belong to the same r-regular C_4 component of lkV_3.

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In this case, all edges incident to a_1 or b_1 are incident to a_2 or b_2, hence incident to a_3 or b_3, and *vice versa*. Thus the a_i and b_i are the vertices of a component of type (1).

Case 2. a_1a_2 and b_1b_2 belong to two different r-regular C_4 components of $\text{lk}V_3$.

In this case, let the vertices of the components be a_1, c_1, a_2, c_2, and b_1, d_1, b_2, d_2, respectively. Now consider $\text{lk}V_2$. It has edges a_1a_3 and b_1b_3. If a_1b_3 were an edge of $\text{lk}V_2$, then $a_3a_1b_3b_1$ would be a path in $\text{lk}V_2$. By our assumption on the vertices of K, we find that $\{a_3, a_1, b_3, b_1\}$ is the vertex set of an r-regular C_4 component in $\text{lk}V_2$. This would preclude the existence of any edge between a_3 or b_3 and c_1. But any edge of $\mathcal{H}$ corresponding to c_1a_2 in $\text{lk}V_3$ must be incident to a_3 or b_3, as seen by looking at $\text{lk}V_1$. This contradiction implies that a_1a_3 and b_1b_3 are in separate components of $\text{lk}V_2$, and thus the edges of $\mathcal{H}$ corresponding to a_2b_3 in $\text{lk}V_1$ must extend to c_1, rather than a_1 (these being the only two options given by $\text{lk}V_3$). A similar argument shows that edges corresponding to b_2a_3 extend to d_1. Now by assumption, a_3 and b_3 are each part of an r-regular C_4 component of $\text{lk}V_2$, and given the edges we already have shown to exist, we know that these are two distinct components, and we know three vertices of each. Denote the remaining vertices by d_3 and c_3, respectively, so that a_1, d_1, a_3, d_3 are the vertices of one component, and b_1, c_1, b_3, c_3 the vertices of the other component.

Since a_3 and c_1 are in distinct components of $\text{lk}V_2$, we see that all edges of $\mathcal{H}$ corresponding to a_2a_3 extend to a_1 (note that looking at $\text{lk}V_3$ shows a_1 and c_1 were the only options). Similarly, all edges corresponding to b_2b_3 extend to b_1, all the ones corresponding to a_2b_3 extend to c_1, and b_2a_3 to d_1. Now in $\text{lk}V_2$ there are the edges a_1d_3 and b_1c_3. These do not extend to a_2 or b_2 as seen in $\text{lk}V_1$, and hence must extend to c_2 and d_2, respectively, by considering $\text{lk}V_3$. Similarly, the edges c_1c_3 and d_1d_3 in $\text{lk}V_2$ must extend to c_2 and d_2, respectively.

Thus, we have deduced the structure of the subgraph $\mathcal{G}$ of $\mathcal{H}$ induced by these twelve vertices. It has four vertices in each class and a matching $a_1a_2a_3$, $b_1b_2b_3$, $c_1c_2c_3$, $d_1d_2d_3$ of size 4. All that remains to complete the proof is to show that this is a component of $\mathcal{H}$, which would make it a component of type (2).

Suppose there were an edge e of $\mathcal{H}$ containing a vertex u of $\mathcal{G}$ and a vertex v not in $\mathcal{G}$. Let V_i be the vertex class of u, let V_j be the vertex class of v, and let V_k be the third vertex class of $\mathcal{H}$. The presence of e would mean that there is an edge uv in $\text{lk}V_k$. But since the parts of $\mathcal{G}$ present in the links $\text{lk}V_2$ and $\text{lk}V_3$ are components of those links, uv cannot be part of these links, and hence $k = 1$. Now consider the third vertex w of e, which is in V_1. If w is a vertex of $\mathcal{G}$, then vw is an edge of $\text{lk}V_i$ of the type we just excluded, and if w is a vertex not in $\mathcal{G}$, then uw is an edge of $\text{lk}V_j$ giving us a similar contradiction. Thus no such edge e can exist, and $\mathcal{G}$ is indeed a component of $\mathcal{H}$.

As these cases were exhaustive, the claim follows. □

We remark that with the previous three lemmas in hand, it would be a short step to conclude that any 3-partite 3-graph satisfying the conditions of Theorem 4.1 contains at least $(1-(30r-35)\varepsilon)n/2$ components that are $(r/2)\cdot\mathcal{F}$ (see the proof of Theorem 4.1). In order to get the improved bound stated in the theorem, we will need one more lemma.

Definition. We say a vertex is V_i-*bad* if it is part of a component of $\mathrm{lk}V_i$ that is not an r-regular C_4. Call a vertex *bad* if it is V_i-bad for some i, and call a vertex *good* otherwise.

Lemma 4.6. *Let $\mathcal{H}$ be a 3-partite 3-graph of maximum degree r with vertex classes V_1, V_2 and V_3. Let x denote the number of r-regular C_4 components of $\mathcal{H}$ that contain exactly one bad vertex, and let y denote the number containing at least two bad vertices. Then $y \geqslant x$.*

Proof. We will prove the lemma by establishing the following technical statement.

Claim 4.7. *Let $\mathcal{H}$ be a 3-partite 3-graph of maximum degree r with vertex classes V_1, V_2 and V_3. Let $\{i, j, k\} = \{1, 2, 3\}$. If an r-regular C_4 component of $\mathrm{lk}V_i$ is such that all of its vertices are good except one V_k-bad vertex in V_j, then it shares vertices of V_k with two r-regular C_4 components of $\mathrm{lk}V_j$ that each have two bad vertices (one V_i-bad, and one V_k-bad), and shares one vertex of V_j with an r-regular C_4 component of $\mathrm{lk}V_k$ that has exactly one V_i-bad vertex in V_j. Furthermore, these four r-regular C_4 components do not share vertices with any r-regular C_4 component outside of these four.*

This claim proves the lemma, since every r-regular C_4 component with only one bad vertex appears together with another r-regular C_4 component with only one bad vertex and two r-regular C_4 components with two bad vertices each, and these four form a unit that does not touch any other such unit (hence there is no overlap in our counting). This implies that there must be at least as many r-regular C_4 components with two bad vertices as there are ones with only one bad vertex, hence $y \geqslant x$.

Thus it remains to prove Claim 4.7. We know by Lemma 4.4 that such a C_4 component must be host to two disjoint edges of $\mathcal{H}$, otherwise it would extend to an $(r/2) \cdot \mathcal{F}$ and all of its links would be r-regular C_4. Thus, let $a_1a_2a_3$ and $b_1b_2b_3$ be two disjoint edges of $\mathcal{H}$ with $a_i, b_i \in V_i$ and a_2, b_2, a_3 and b_3 being the vertices of an r-regular C_4 component of $\mathrm{lk}V_1$, all of whose vertices are part of r-regular C_4 components in the other links except for b_3. We consider two cases.

Case 1. a_1a_2 and b_1b_2 belong to the same r-regular C_4 component of $\mathrm{lk}V_3$.

In this case, all edges incident to a_1 or b_1 are incident to a_2 or b_2, hence incident to a_3 or b_3, and *vice versa*. But this means that the r-regular C_4 component of $\mathrm{lk}V_2$ that a_3 participates in must have $\{a_1, b_1, a_3, b_3\}$ as its vertex set, which contradicts the fact that b_3 is not in an r-regular C_4 component of $\mathrm{lk}V_2$. Therefore, this case is impossible.

Case 2. a_1a_2 and b_1b_2 belong to two different r-regular C_4 components of $\mathrm{lk}V_3$.

In this case, let the vertices of the components be a_1, c_1, a_2, c_2, and b_1, d_1, b_2, d_2, respectively. Now consider $\mathrm{lk}V_2$. It has edges a_1a_3 and b_1b_3. Note that these edges are in separate components of $\mathrm{lk}V_2$, since a_3 participates in an r-regular C_4, while b_3 does not. Therefore, there are no edges a_1b_3 or b_1a_3 in $\mathrm{lk}V_2$, which implies that all edges parallel to a_2b_3 in $\mathrm{lk}V_1$ extend to c_1, rather than a_1 (these being the only two options given by $\mathrm{lk}V_3$), and similarly all edges parallel to b_2a_3 in $\mathrm{lk}V_1$ extend to d_1 (not b_1). These edges of $\mathcal{H}$ correspond to edges c_1b_3 and d_1a_3, respectively, in $\mathrm{lk}V_2$. Now by assumption, a_3 is part of

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an r-regular C_4 component of $\mathrm{lk}V_2$, and given the edges we have already shown to exist, we know three of its vertices. Denote the remaining vertex by d_3 so that $\{a_1, d_1, a_3, d_3\}$ is the vertex set of that component.

Since a_3 and c_1 are in distinct components of $\mathrm{lk}V_2$, we see that all edges of $\mathcal{H}$ corresponding to a_2a_3 extend to a_1. Similarly, all edges corresponding to b_2b_3 extend to b_1, all the ones corresponding to a_2b_3 extend to c_1, and b_2a_3 to d_1. Now in $\mathrm{lk}V_2$ there is at least one edge a_1d_3. Any such edge does not extend to a_2 as seen in $\mathrm{lk}V_1$, and hence must extend to c_2 by considering $\mathrm{lk}V_3$. Similarly, the edges parallel to d_1d_3 in $\mathrm{lk}V_2$ must extend to d_2.

Since b_1b_3 and c_1b_3 are edges of $\mathrm{lk}V_2$ in the component of b_3, which is not an r-regular C_4, we have that b_1 and c_1 are both V_2-bad vertices (recall Definition 4). We claim that c_2 and d_2 are V_1-bad vertices. Suppose to the contrary that they were good. Then by the existence of edges c_2d_3 and d_2d_3 in $\mathrm{lk}V_1$, these are part of the same r-regular C_4 component of $\mathrm{lk}V_1$. Call its fourth vertex c_3. Now any edge parallel to c_2c_3 in $\mathrm{lk}V_1$ extends to c_1, since it may only extend to c_1 or a_1 by $\mathrm{lk}V_3$, and cannot extend to a_1 by $\mathrm{lk}V_2$. Similarly, any edge parallel to d_2c_3 in $\mathrm{lk}V_1$ extends to b_1. We just showed that all edges on c_3 go to c_1 or b_1 in $\mathrm{lk}V_2$. What we showed earlier is that all edges on b_3 go to c_1 or b_1 in $\mathrm{lk}V_2$. These account for all edges on c_3 and b_3, putting b_3 in an r-regular C_4 component, which is a contradiction, because b_3 was assumed not to participate in one of those in $\mathrm{lk}V_2$. Therefore, the component of $\mathrm{lk}V_1$ including c_2 and d_2 is not an r-regular C_4, hence these are V_1-bad vertices.

Thus we have two r-regular C_4 components of $\mathrm{lk}V_3$ each with two bad vertices: $\{a_1, c_1, a_2, c_2\}$ harbours an r-regular C_4 with bad vertices c_1 and c_2, while $\{b_1, d_1, b_2, d_2\}$ harbours an r-regular C_4 with bad vertices b_1 and d_2. We also have an r-regular C_4 in $\mathrm{lk}V_2$ on $\{a_1, d_1, a_3, d_3\}$ with a single V_1-bad vertex d_3. Since all of the good vertices of these four r-regular C_4 components are shared among themselves, this proves the claim and hence the lemma. □

Proof of Theorem 4.1. Let $\mathcal{H}$ be an r-regular 3-partite 3-graph with n vertices in each class, and assume $\nu(\mathcal{H}) \leqslant (1+\varepsilon)n/2$. Let V_1, V_2 and V_3 be the vertex classes of $\mathcal{H}$.

First, we modify $\mathcal{H}$ by replacing each component of $\mathcal{H}$ that has a perfect matching with r parallel copies of the perfect matching. Note that this does not change $\nu(\mathcal{H})$ nor the number of vertices in each class, and keeps $\mathcal{H}$ r-regular. This change also clearly does not create any new copies of $(r/2)\cdot\mathcal{F}$, so if we prove that the modified hypergraph has some number of $(r/2)\cdot\mathcal{F}$ components, these must have been present in $\mathcal{H}$ to begin with. Thus, we may assume that every perfect matching component of $\mathcal{H}$ is just r parallel copies of an edge.

For each i, by applying Lemma 4.3 with $A = V_i$, we have that $\mathrm{lk}V_i$ contains at least $(1-(6r-7)\varepsilon)n/2$ components that are r-regular C_4. Call an r-regular C_4 component of a link *good* if it contains no bad vertices (see Definition 4), and *ruined* otherwise. We aim to show that at least one of the links has at least $(1-(22r-77/3)\varepsilon)n/2$ good r-regular C_4 components.

Claim 4.8. *Each link contributes at most $(6r-7)\varepsilon n$ bad vertices to any vertex class.*

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Proof. This is immediate since each link has in each vertex class at least $(1-(6r-7)\varepsilon)n$ vertices belonging to r-regular C_4 components. □

The idea for completing the proof is as follows. If the bad vertices in each vertex class each ruin a different r-regular C_4 component of one link, then we may have as many as $(12r-14)\varepsilon n$ ruined r-regular C_4 components in that link, leaving us with only $(1-(30r-35)\varepsilon)n/2$ good components. But then that link has many r-regular C_4 components with only one bad vertex, so by Lemma 4.6, the other links must have many such components with at least two bad vertices, and so these links will have more good components.

To make this precise, we count the total number of bad vertices in all three links. By Claim 4.8, each link contributes at most $(6r-7)\varepsilon n$ bad vertices to each vertex class. Since there are two vertex classes per link and three links in total, we have at most $6(6r-7)\varepsilon n$ bad vertices in all. Now let x_i count the number of r-regular C_4 components of $\mathrm{lk}\, V_i$ with exactly one bad vertex, and let y_i count the number of r-regular C_4 components of $\mathrm{lk}\, V_i$ with at least two bad vertices. Let $x = x_1 + x_2 + x_3$ and let $y = y_1 + y_2 + y_3$. Note that any bad vertex contributes to at most one of x_1, x_2, x_3, y_1, y_2 and y_3, since in one of the two links containing that vertex, it is in an r-regular C_4 component. Therefore, we find that $x + 2y \leqslant 6(6r-7)\varepsilon n$, as there must be at least $x + 2y$ bad vertices.

Now let V_i be the vertex class such that x_i is the least among x_1, x_2 and x_3. We thus have $x_i \leqslant x/3$. By Lemma 4.6 we know that $y \geqslant x$. Therefore $3x \leqslant x + 2y \leqslant 6(6r-7)\varepsilon n$, and so we have $x_i \leqslant \frac{2}{3}(6r-7)\varepsilon n$. By Claim 4.8 we know that $\mathrm{lk}\, V_i$ has at most $2(6r-7)\varepsilon n$ bad vertices that were contributed from the other two links, which leaves at most $2(6r-7)\varepsilon n - x_i$ bad vertices to ruin the r-regular C_4 components counted by y_i. Since these each use at least two of these vertices, we have $y_i \leqslant \frac{1}{2}(2(6r-7)\varepsilon n - x_i)$. Combining our inequalities we find that $\mathrm{lk}\, V_i$ therefore has

$$x_i + y_i \leqslant (6r-7)\varepsilon n + \frac{1}{2}x_i \leqslant \frac{4}{3}(6r-7)\varepsilon n$$

ruined r-regular C_4 components. The rest must be good, so we have at least

$$(1-(6r-7)\varepsilon)\frac{n}{2} - \frac{4}{3}(6r-7)\varepsilon n = \left(1 - \left(22r - \frac{77}{3}\right)\varepsilon\right)\frac{n}{2}$$

good r-regular C_4 components in $\mathrm{lk}\, V_i$.

If any good r-regular C_4 component hosts two disjoint edges of $\mathcal{H}$, then by Lemma 4.5 it is part of a perfect matching component of $\mathcal{H}$, which is a contradiction, since we replaced these with parallel copies of a matching (so their links do not contain any r-regular C_4 components). Therefore, all good r-regular C_4 components extend to copies of $(r/2)\cdot\mathcal{F}$ by Lemma 4.4, so we have found the desired number of those in $\mathcal{H}$, completing the proof. □

Proof of Theorem 4.2. This follows along very similar lines to the proof of Theorem 4.1. Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes A, B and C, such that $|A| = n$, and suppose that every vertex of A has degree at least r, and that every vertex in $B \cup C$ has degree at most r. Assume that $\nu(\mathcal{H}) \leqslant (1+\varepsilon)n/2$.

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First, we modify $\mathcal{H}$ by removing edges from vertices of A that have degree strictly larger than r until every vertex of A has degree exactly r. Note that this does not hurt any of our assumptions and cannot create copies of $(r/2) \cdot \mathcal{F}$. After this modification, $\mathcal{H}$ has maximum degree r.

Next, we again modify $\mathcal{H}$ (as in the proof of Theorem 4.1) by replacing each component of $\mathcal{H}$ that has a perfect matching with r parallel copies of the perfect matching. Note that again this change does not affect our assumptions, and also clearly does not create any new copies of $(r/2) \cdot \mathcal{F}$. Thus, we may assume that every perfect matching component of $\mathcal{H}$ is just r parallel copies of an edge.

Now apply Lemma 4.3 to $\mathcal{H}$ to find that lk$A$ contains at least $(1-(6r-7)\varepsilon)n/2$-many r-regular C_4 components. Now delete from $\mathcal{H}$ all vertices of B and C that are not in one of the r-regular C_4 components. This leaves at least $n' = (1-(6r-7)\varepsilon)n$ vertices in each of these classes. Note that all vertices of B and C now have degree r.

Next, we follow along the lines of the proof of Lemma 4.3 to find out about r-regular C_4 components of lk$B$ and lk$C$. There must be some $S \subseteq B$ such that $\eta(L(\text{lk}S)) \leqslant |S| - (|B| - \nu(\mathcal{H}))$, otherwise $\mathcal{H}$ would have a matching larger than $\nu(\mathcal{H})$ by Theorem 2.1. We have $\nu(\text{lk}S) \geqslant |S|$, so by Corollary 3.11, if lk$S$ has k-many r-regular C_4 components, then

$$\eta(L(\text{lk}S)) \geqslant \frac{(2r-3)|S| + r|S| - k}{6r-7}.$$

Combining this with our upper bound, we find

$$\begin{aligned} k &\geqslant (6r-7)(|B| - \nu(\mathcal{H})) - (3r-4)|S| \\ &\geqslant (6r-7)\left(n' - (1+\varepsilon)\frac{n}{2}\right) - (3r-4)n' \\ &= (1-(36r^2-72r+35)\varepsilon)\frac{n}{2}. \end{aligned}$$

Since lk$B$ has maximum degree r, these components of lk$S$ are all components of lk$B$, hence we have found at least $(1-(36r^2-72r+35)\varepsilon)n/2$-many r-regular C_4 components in lk$B$. The same holds for lk$C$.

Call an r-regular C_4 component of a link *good* if it contains no bad vertices (see Definition 4), and *ruined* otherwise. We claim that lk$A$ has at least $(1-(72r^2-150r+77)\varepsilon)n/2$ good r-regular C_4 components.

Note that there are no A-bad vertices, since we deleted them all before considering lk$B$ and lk$C$. This means that all ruined r-regular C_4 components of lk$A$ have at least two bad vertices, since if they only had one, Claim 4.7 of Lemma 4.6 would imply the existence of an A-bad vertex (in fact, three of them). There are at most

$$n' - (1-(36r^2-72r+35)\varepsilon)n = (36r^2-78r+42)\varepsilon n$$

B-bad vertices in C, and also no more than that many C-bad vertices in B. Since the ruined r-regular C_4 components of lk$A$ each have two bad vertices, this means that there are in fact at most $(36r^2-78r+42)\varepsilon n$ ruined r-regular C_4 components in lk$A$. Therefore, since the rest are good, there are indeed at least $(1-(72r^2-150r+77)\varepsilon)n/2$ good r-regular C_4 components in lk$A$.

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If any good r-regular C_4 component hosts two disjoint edges of $\mathcal{H}$, then by Lemma 4.5 it is part of a perfect matching component of $\mathcal{H}$, which is a contradiction, since we replaced these with parallel copies of a matching (so their links do not contain any r-regular C_4 components). Therefore, all good r-regular C_4 components extend to copies of $(r/2)\cdot\mathcal{F}$ by Lemma 4.4, so we have found the desired number of those in $\mathcal{H}$, completing the proof. □

5. $(r/2)\cdot\mathcal{F}$-free 3-graphs

Theorems 4.1 and 4.2 have the following easy corollaries, respectively.

Corollary 5.1. *Let $\mathcal{H}$ be an r-regular 3-partite 3-graph with n vertices in each vertex class. If $\mathcal{H}$ does not contain a copy of $(r/2)\cdot\mathcal{F}$, then*

$$\nu(\mathcal{H}) \geqslant \left(1+\frac{1}{22r-77/3}\right)\frac{n}{2}.$$

Corollary 5.2. *Let $\mathcal{H}$ be a 3-partite 3-graph with vertex classes A, B and C, such that $|A|=n$. Suppose that every vertex of A has degree at least r, and that every vertex in $B\cup C$ has degree at most r. If $\mathcal{H}$ contains no subgraph isomorphic to $(r/2)\cdot\mathcal{F}$, then*

$$\nu(\mathcal{H}) \geqslant \left(1+\frac{1}{72r^2-150r+77}\right)\frac{n}{2}.$$

It would be interesting to determine the correct function $\alpha(r)$ for which $\nu(\mathcal{H})\geqslant(1+\alpha(r))n/2$ for every $\mathcal{H}$ satisfying the conditions of Corollary 5.1. The following constructions give upper bounds on $\alpha(r)$.

Theorem 5.3. *For every even $r\geqslant 2$ there exists an r-regular 3-partite 3-graph $\mathcal{H}$ with n vertices per vertex class, not containing a copy of $(r/2)\cdot\mathcal{F}$, such that*

$$\nu(\mathcal{H}) \leqslant \left(1+\frac{1}{r+1}\right)\frac{n}{2}.$$

For every odd $r\geqslant 3$ there exists an r-regular 3-partite 3-graph $\mathcal{H}$ with n vertices per vertex class (obviously not containing a copy of $(r/2)\cdot\mathcal{F}$) such that

$$\nu(\mathcal{H}) \leqslant \left(1+\frac{1}{r}\right)\frac{n}{2}.$$

Proof. First suppose $r\geqslant 2$ is even. Let $(r/2)\cdot\mathcal{F}^-$ denote the 3-partite 3-graph obtained by removing a single edge from $(r/2)\cdot\mathcal{F}$. Note that it has three vertices of degree $r-1$ and three vertices of degree r. Take $r/2$ disjoint copies of $(r/2)\cdot\mathcal{F}^-$ together with three vertices a, b and c, one in each class. For each copy F of $(r/2)\cdot\mathcal{F}^-$, add three edges, each using two of a, b and c and one of the three degree-$(r-1)$ vertices of F. Each group of three edges contributes 2 to the degree of a, b and c, and 1 to the degree of the

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degree-$(r-1)$ vertices, hence after all $r/2$ such groups are added, the resulting 3-graph is r-regular and clearly $(r/2)\cdot\mathcal{F}$-free. It has $n = r+1$ vertices per vertex class, and its largest matching is of size at most $r/2+1$, since in any matching we can pick at most one edge from each copy of $(r/2)\cdot\mathcal{F}^-$, and all of the edges we added intersect in one of a, b or c. This gives the desired bound for even r.

If $r \geqslant 3$ is odd, we can use a very similar construction as above. Instead of $(r/2)\cdot\mathcal{F}^-$, which does not exist for odd r, let $((r-1)/2)\cdot\mathcal{F}^+$ denote the 3-partite 3-graph obtained from $((r-1)/2)\cdot\mathcal{F}$ by adding an extra copy of one of its edges. Note that it has three vertices of degree $r-1$ and three vertices of degree r. Taking $(r-1)/2$ disjoint copies of $((r-1)/2)\cdot\mathcal{F}^+$ together with three vertices a, b and c, one in each class, we add edges containing two of these vertices and one degree-$(r-1)$ vertex of an $((r-1)/2)\cdot\mathcal{F}^+$ as in the previous construction. We also add the edge abc. The resulting 3-graph is r-regular and clearly $(r/2)\cdot\mathcal{F}$-free (since this 3-graph does not exist for odd r). It has $n = r$ vertices per vertex class, and its largest matching is of size at most $(r-1)/2+1$, since we can pick at most one edge from each copy of $((r-1)/2)\cdot\mathcal{F}^+$, and all of the edges we added intersect in one of the three extra vertices a, b and c. This gives the desired bound for odd r. □

All of these examples have high edge multiplicity, and one may expect substantially better lower bounds on the matching number for *simple* hypergraphs. We close with the following (special case of a) conjecture of Alon and Kim from [7] about this case.

Conjecture 5.4 (Alon and Kim [7]). *The edges of any* 3*-uniform simple hypergraph with maximum degree* r *can be covered by* $(3/2+o(1))r$ *matchings.*

If true, this conjecture would imply that every simple r-regular 3-partite 3-graph with n vertices in each class has a matching of size at least $(2/3-o(1))n$. The constant $2/3$ here is best possible (as shown in [3]).

Acknowledgement

The authors are grateful to an anonymous referee for many helpful suggestions that improved the presentation of the paper.

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