

## ON OCCUPATION TIMES OF THE FIRST AND THIRD QUADRANTS FOR PLANAR BROWNIAN MOTION

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### Abstract

In Bingham and Doney (1988) the authors presented the applied probability community with a question which is very simply stated, yet is extremely difficult to solve: what is the distribution of the quadrant occupation time of planar Brownian motion? In this paper we study an alternate formulation of this long-standing open problem: let  $X(t)$ ,  $Y(t)$ ,  $t \geq 0$ , be standard Brownian motions starting at  $x$ ,  $y$ , respectively. Find the distribution of the total time  $T = \text{Leb}\{t \in [0, 1]: X(t) \times Y(t) > 0\}$ , when  $x = y = 0$ , i.e. the occupation time of the union of the first and third quadrants. If two adjacent quadrants are used, the problem becomes much easier and the distribution of  $T$  follows the arcsine law.

*Keywords:* Planar Brownian motion; occupation time; Kontorovich–Lebedev

2010 Mathematics Subject Classification: Primary 60J65  
Secondary 60J60

### 1. Introduction

In this paper we consider a long-standing open problem in the applied probability literature: what is the quadrant occupation time of planar Brownian motion? Formally, let  $T$  be the total time that the vector process  $X(t) = (W_1(t), W_2(t))$  on  $0 \leq t \leq 1$  is in the first quadrant; the task is to find the distribution of  $T$ . In 1988, Bingham and Doney remarked [3, p. 121] that ‘in no case to our knowledge is the law of  $T$  known explicitly’. Using independence of coordinate processes, the authors obtained the first two moments of  $T$  and provided a solution for the third moment, the latter of which was corrected in [4]. Desbois [4] generalized the aforementioned quadrant problem by considering the occupation time spent in a wedge of apex  $O$  and angle  $\theta$ . Analytical results for general  $\theta$  were provided for both the second and third moments, and the fourth moment for the quadrant problem ( $\theta = \pi/2$ ) was obtained.

Despite these new additions to the literature, Desbois [4] concluded that ‘our feeling is that the occupation time problem for Brownian motion is far from being understood as soon as we leave one-dimensional or quasi-one-dimensional (graphs) situations. Clearly, new ideas are needed if we want to tackle this problem’. Our work, through its use of Kontorovich–Lebedev transforms and pasting of solutions, offers a new, but ultimately incomplete, approach on this long-standing problem. For references on the Kontorovich–Lebedev transform, we refer the reader to [1], [2], [6], and [7].

### 2. Main results

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Received 12 February 2016; revision received 27 June 2016.

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\*\* 9 September 1936–23 April 2013

**2.1. Setup**

We consider the following alternative formulation of Bingham and Doney’s [3] quadrant occupation of planar Brownian motion problem. Let

$$X(t), \quad Y(t), \quad t \geq 0,$$

be standard Brownian motions starting at  $x, y$ , respectively. We wish to find the distribution of the total time  $T = \text{Leb}\{t \in [0, 1]: X(t) \times Y(t) > 0\}$ , when  $x = y = 0$ , i.e. the occupation time of the union of the first and third quadrants. If two adjacent quadrants are used, the problem becomes much easier and the distribution follows the arcsine law [8].

The Feynman–Kac theorem states (see [5]) that

$$U(x, y) = \mathbb{E}_{x,y} \left[ \int_0^\infty \exp\left(-\alpha t - \lambda \int_0^t \mathbf{1}_{\{X(u) \times Y(u) > 0\}} du\right) dt \right]$$

is the bounded solution of the Helmholtz partial differential equation in each quadrant,

$$\frac{1}{2}(U_{xx} + U_{yy})(x, y) - \beta(x, y)U(x, y) + 1 \equiv 0,$$

where  $\beta(x, y) = \beta_1 = \alpha + \lambda$  for  $(x, y) \in Q_1, Q_3$  (the first and third quadrants, respectively) and  $\beta(x, y) = \beta_2 = \alpha$ ,  $(x, y) \in Q_2, Q_4$  (the second and fourth quadrants, respectively). The function  $U$  must be twice differentiable interior to each quadrant, continuously differentiable overall, and uniformly bounded. If we can find  $U$ , then we know  $U(0, 0)$ , and then we will have

$$U(0, 0) = \mathbb{E} \left[ \int_0^\infty e^{-\alpha t - \lambda t T} dt \right] = \mathbb{E} \left[ \frac{1}{\alpha + \lambda T} \right].$$

We turn to finding the Kontorovich–Lebedev solution in each quadrant. Let  $x = r \cos \theta$ ,  $y = r \sin(\theta)$ , and set  $V(r, \theta) = U(x, y)$ . As is well known,

$$U_{xx} + U_{yy} = V_{rr} + \frac{1}{r}V_r + \frac{1}{r^2}V_{\theta\theta}.$$

The modified Bessel function  $v(r) = \kappa_{i\nu}(r)$  satisfies the ordinary differential equation (see [9])

$$r^2v''(r) + rv'(r) - (r^2 - \nu^2)v(r) = 0.$$

It now can be easily checked that, for any functions  $f(\nu), g(\nu)$ ,

$$V(r, \theta) = \frac{1}{\beta} + \int_0^\infty f(\nu)\kappa_{i\nu}(r\sqrt{2\beta}) \sinh(\nu\theta) d\nu + \int_0^\infty g(\nu)\kappa_{i\nu}(r\sqrt{2\beta}) \cosh(\nu\theta) d\nu$$

solves the differential equation

$$\frac{1}{2} \left( V_{rr}(r, \theta) + \frac{1}{r}V_r(r, \theta) + \frac{1}{r^2}V_{\theta\theta}(r, \theta) \right) - \beta V(r, \theta) + 1 = 0, \tag{1}$$

with a different choice of  $f, g, \beta$  in each quadrant, which we must paste together to satisfy the needed smoothness. For convenience, we can use any two linearly independent combinations of  $\sinh, \cosh$ , etc. We now proceed to do so.

### 2.2. Pasting

Our strategy is to use ‘pasting’ of the solutions. We denote  $f_i$  and  $g_i$  as the densities for each of the two linear combinations of  $\sinh$ ,  $\cosh$ , respectively, in the  $i$ th quadrant. In

$$Q_1 = \left\{ (r, \theta) : r > 0, 0 < \theta < \frac{1}{2}\pi \right\},$$

we set

$$V(r, \theta) = \frac{1}{\beta_1} + \int_0^\infty f_1(v) \kappa_{iv}(r\sqrt{2\beta_1}) \sinh\left(v\left(\frac{\pi}{2} - \theta\right)\right) dv + \int_0^\infty g_1(v) \kappa_{iv}(r\sqrt{2\beta_1}) \sinh(v\theta) dv.$$

In

$$Q_2 = \left\{ (r, \theta) : r > 0, \frac{1}{2}\pi < \theta < \pi \right\},$$

we set

$$V(r, \theta) = \frac{1}{\beta_2} + \int_0^\infty f_2(v) \kappa_{iv}(r\sqrt{2\beta_2}) \sinh(v(\pi - \theta)) dv + \int_0^\infty g_2(v) \kappa_{iv}(r\sqrt{2\beta_2}) \sinh\left(v\left(\theta - \frac{\pi}{2}\right)\right) dv.$$

By symmetry,  $U(x, y) = U(y, x) = U(-x, -y)$ , and so  $g_j \equiv f_j$  for  $j = 1, 2, 3, 4$ ,  $f_3 = f_1$ ,  $f_4 = f_2$ .

### 2.3. Consequences of continuity and continuous differentiability on the axes

Note that

$$V\left(r, \frac{\pi}{2} - 0\right) = \frac{1}{\beta_1} + \int_0^\infty f_1(v) \sinh\left(\frac{v\pi}{2}\right) \kappa_{iv}(r\sqrt{2\beta_1}) dv, \\ V\left(r, \frac{\pi}{2} + 0\right) = \frac{1}{\beta_2} + \int_0^\infty f_2(v) \sinh\left(\frac{v\pi}{2}\right) \kappa_{iv}(r\sqrt{2\beta_2}) dv,$$

and the right-hand sides of these equations are equal. The derivatives, taken with respect to  $\theta$ , are

$$V_\theta\left(r, \frac{\pi}{2} - 0\right) = \int_0^\infty f_1(v) v \left(\cosh\left(\frac{v\pi}{2}\right) - 1\right) \kappa_{iv}(r\sqrt{\beta_1}) dv, \\ V_\theta\left(r, \frac{\pi}{2} + 0\right) = \int_0^\infty f_2(v) v \left(-\cosh\left(\frac{v\pi}{2}\right) + 1\right) \kappa_{iv}(r\sqrt{\beta_2}) dv,$$

and the right-hand sides of these equations are equal.

2.3.1. *Solving the above smoothness equations.* We assume that there are signed measures,  $\mu_j(dz)$ ,  $j = 1, 2$ , such that

$$f_j(v) = \frac{2 \cosh(v\pi/2)}{\pi \sinh(v\pi/2)} \left(\frac{-1}{\beta_j}\right) + \int_0^\infty \mu_j \sin(vz) dz. \tag{2}$$

We can then substitute  $f_j$  into the first two equations expressing continuity on the  $y$ -axis. The first term in  $f_j$  kills the term  $1/\beta_j$  because the Kontorovich–Lebedev transform of  $\cosh(v\pi/2)$  is, for every  $y$ ,

$$\frac{2}{\pi} \int_0^\infty \cosh\left(v\frac{\pi}{2}\right) \kappa_{iv}(y) dv = e^{-y \cos \pi/2} \equiv 1,$$

(see [9, p. 242]). This leaves the following equation for the sine transforms of  $\mu_j$  for  $V(r, \theta)$  to be continuous at each  $r$  when  $\theta = \pi/2$ :

$$\begin{aligned} V\left(r, \frac{\pi}{2}\right) &= \int_0^\infty \mu_1 dz \int_0^\infty \sin(vz) \sinh\left(\frac{v\pi}{2}\right) \kappa_{iv}(r\sqrt{2\beta_1}) dv \\ &= \int_0^\infty \mu_2 dz \int_0^\infty \sin(vz) \sinh\left(\frac{v\pi}{2}\right) \kappa_{iv}(r\sqrt{2\beta_2}) dv. \end{aligned} \tag{3}$$

Equation (8) from [9, p. 244] states that

$$\int_0^\infty \sin(vz) \sinh\left(\frac{v\pi}{2}\right) \kappa_{iv}(y) dv = \frac{\pi}{2} \sin(y \sinh(z)).$$

This gives us the continuity equation linking  $\mu_1, \mu_2$  as

$$\int_0^\infty \sin(r\sqrt{2\beta_1} \sinh(z)) \mu_1 dz = \int_0^\infty \sin(r\sqrt{2\beta_2} \sinh(z)) \mu_2 dz.$$

We now define the change of variables  $z'(z) = \phi(z), z \geq 0$ , such that

$$\sqrt{2\beta_1} \sinh(z) = \sqrt{2\beta_2} \sinh(\phi(z)).$$

Explicitly,

$$\phi(z) = \log\left(\sqrt{\frac{\beta_1}{\beta_2}} \sinh z + \sqrt{\frac{\beta_1}{\beta_2} \sinh^2(z) + 1}\right).$$

We then have, for every  $r \geq 0$ ,

$$\int_0^\infty \sin(r\sqrt{2\beta_1} \sinh(z)) \mu_1 dz = \int_0^\infty \sin(r\sqrt{2\beta_1} \sinh(z)) \mu_2 d\phi(z).$$

Since the sine transform is unique on a half interval, we have

$$\mu_1(dz) = \mu_2(d\phi(z)), \quad z \geq 0.$$

Thus, we have expressed one relationship between  $\mu_1$  and  $\mu_2$ . We now need a second equation linking  $\mu_1$  and  $\mu_2$ , and we use the equation obtained by using the continuity of the derivative on the positive  $y$ -axis.

2.3.2. *Continuity of the derivative of  $V$  on  $\theta$  at  $\theta = \pi/2$ .* Noting that

$$\begin{aligned} V_\theta\left(r, \frac{\pi}{2} - 0\right) &= \int_0^\infty f_1(v)v \left(\cosh\left(\frac{v\pi}{2}\right) - 1\right) \kappa_{iv}(r\sqrt{2\beta_1}) dv, \\ V_\theta\left(r, \frac{\pi}{2} + 0\right) &= - \int_0^\infty f_2(v)v \left(\cosh\left(\frac{v\pi}{2}\right) - 1\right) \kappa_{iv}(r\sqrt{2\beta_2}) dv, \end{aligned}$$

we immediately see that the right-hand sides are equal for all  $r \geq 0$  (the derivative of  $V$  on  $\theta$  is continuous at  $\theta = \pi/2$ ). Placing the expressions for the  $f_j$ , given in (2), in terms of the unknown  $\mu_j$ , into the right-hand side of each of the above equations gives

$$\begin{aligned} 0 &= \sum_{j=1}^2 \int_0^\infty \left[ \frac{2 \cosh(v\pi/2)}{\pi \sinh(v\pi/2)} \left(\frac{-1}{\beta_j}\right) + \int_0^\infty \mu_j \sin(vz) dz \right] \\ &\quad \times v \left(\cosh\left(\frac{v\pi}{2}\right) - 1\right) \kappa_{iv}(r\sqrt{2\beta_j}) dv. \end{aligned}$$

From [9, p. 244, Equation (7)], for  $a$  real or  $|a| \leq \pi/2$ ,

$$\frac{2}{\pi} \int_0^\infty v \sin(av) \kappa_{iv}(y) \, dv = ye^{-y \cosh a} \sinh a.$$

We then use the identities

$$\begin{aligned} \frac{\cosh(v\pi/2)}{\sinh(v\pi/2)} \left( \cosh\left(\frac{v\pi}{2}\right) - 1 \right) &= \sinh\left(\frac{v\pi}{2}\right) - \tanh\left(\frac{v\pi}{4}\right), \\ \sin(vz) \left( \cosh\left(\frac{v\pi}{2}\right) - 1 \right) &= \frac{1}{2} \sin\left(v\left(z + \frac{i\pi}{2}\right)\right) + \frac{1}{2} \sin\left(v\left(z - \frac{i\pi}{2}\right)\right) - \sin(vz), \end{aligned}$$

and link  $\mu_1$  and  $\mu_2$  as follows:

$$\begin{aligned} 0 &= \sum_{j=1}^2 \int_0^\infty \frac{2}{\pi} \left(\frac{-1}{\beta_j}\right) \left( \sinh\left(\frac{v\pi}{2}\right) - \tanh\left(\frac{v\pi}{4}\right) \right) v \kappa_{iv}(r\sqrt{2\beta_j}) \, dv \\ &\quad + \sum_{j=1}^2 \int_0^\infty \mu_j \, dz \int_0^\infty \left[ \frac{1}{2} \sin\left(v\left(z + \frac{i\pi}{2}\right)\right) + \frac{1}{2} \sin\left(v\left(z - \frac{i\pi}{2}\right)\right) \right. \\ &\quad \left. - \sin(vz) \right] v \kappa_{iv}(r\sqrt{2\beta_j}) \, dv. \end{aligned} \tag{4}$$

Since  $\sinh(z + 2i\sqrt{\pi}) = i \cosh z$  and  $\cosh(z + i\pi/2) = -i \sinh z$ , we obtain, from (4), with  $a = i\pi/2$ ,  $a = z \pm i\pi/2$ , respectively,

$$\begin{aligned} 0 &= \sum_{j=1}^2 \left( -r \sqrt{\frac{2}{\beta_j}} \right) + \frac{2}{\pi\beta_j} \int_0^\infty v \tanh\left(\frac{v\pi}{4}\right) \kappa_{iv}(r\sqrt{2\beta_j}) \, dv \\ &\quad + \sum_{j=1}^2 \frac{\pi}{2} \int_0^\infty \mu_j \frac{r\sqrt{2\beta_j}}{2} [e^{r\sqrt{2\beta_j}i \sinh z} (i \cosh z) + e^{-r\sqrt{2\beta_j}i \sinh z} (-i \cosh z) \\ &\quad - 2e^{-r\sqrt{2\beta_j} \cosh z} \sinh z] \, dz. \end{aligned}$$

We now use the substitution  $z'(z) = \phi(z)$ , implicitly defined by

$$\sqrt{2\beta_1} \sinh z = \sqrt{2\beta_2} \sinh z'$$

in the second integral of the last display, with  $j = 2$ . Combined with the fact that

$$\mu_2(d\phi(z)) = \mu_1(dz),$$

we obtain

$$\begin{aligned} &\frac{\pi}{2} \int_0^\infty \mu_1 r \sqrt{2\beta_1} [\sin(r\sqrt{2\beta_1} \sinh z) \cosh z + e^{-r\sqrt{2\beta_1} \cosh z} \sinh z] \, dz \\ &\quad + \frac{\pi}{2} \int_0^\infty \mu_1 r \sqrt{2\beta_2} [\sin(r\sqrt{2\beta_1} \sinh z) \cosh \phi(z) + e^{-r\sqrt{2\beta_2} \cosh \phi(z)} \sinh \phi(z)] \, dz \\ &= \sum_{j=1}^2 \left( -r \sqrt{\frac{2}{\beta_j}} \right) + \frac{2}{\pi\beta_j} \int_0^\infty \tanh\left(\frac{v\pi}{4}\right) v \kappa_{iv}(r\sqrt{2\beta_j}) \, dv. \end{aligned} \tag{5}$$

Despite much effort, further explicit calculations beyond (5) quickly become intractable. It is our opinion that an explicit solution to this long-standing problem may not be possible. Nonetheless, we have successfully reduced the problem to that where an analyst of special functions could pick up where we have left off. The problem now becomes one of finding the relationship between functions  $f$  and  $g$  if their Kontorovich–Lebedev transforms  $F$  and  $G$  satisfy  $F(r) = G(cr)$  for all  $r$  with  $c$  given.

### 3. Remarks

Professor Terry Lyons is credited (personal communication with Professor Nick Bingham) with saying that the simplest case beyond the half-plane (which reduces to the arc-sine law in one dimension) is the third plane  $0 < \theta < \frac{2}{3}\pi$ . This could help one to compare the differential equations we obtained in (5).

One can consider the random occupation measure generated on the unit circle (or a sphere in higher dimensions) by the angular part of a Brownian motion starting at 0 and running for time 1. Some results regarding this random measure, and its relation to the angle of the Brownian motion at time 1, were obtained in [10].

Another natural problem in two dimensions is to find the law of the occupation time  $A_u$  of an interval of length  $2\pi u$  around the unit circle, for  $0 < u < 1$ . Some problems involving cyclically stationary local-time processes were treated in [11].

### Acknowledgements

We thank Professor Nick Bingham, Professor Jim Pitman, and Professor Robin Pemantle for helpful discussions. We are extremely appreciative of the work of an anonymous referee, whose very helpful report greatly strengthened the quality of this work.

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