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Nonlinear magnetohydrodynamic stability for axisymmetric incompressible ideal flows

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Abstract. In this paper, the general theory developed by Vladimirov and Moffatt [*J. Fluid Mech.* **283**, 125–139 (1995)] and Vladimirov et al. [*J. Fluid Mech.* **329**, 187–205 (1996); *J. Plasma Phys.* **57**, 89–120 (1997)] is extended to nonlinear (Lyapunov) stability for axisymmetric (invariant under rotations around fixed axis) solutions of the ideal incompressible magnetohydrodynamic flows for a situation of arbitrary flow and a poloidal field. The appropriate norm is the sum of magnetic and kinetic energies and the mean square vector potential of the magnetic field.

1. Introduction

In this paper we extend the approach initiated in Vladimirov and Moffatt (1995, Part I) and Vladimirov et al. (1996, Part II) and then continued in Vladimirov et al. (1997, Part III). In Part I, new variational principles for incompressible magnetohydrodynamic (MHD) flows were established and a frozen-in field (generalized vorticity) was constructed. The existence of this frozen-in field has consequences for the construction of Casimirs, the integral invariants that play an essential role in the derivation of sufficient conditions for stability (or stability criteria) for steady solutions $\{U, H\}$ of the governing equations. In Part II, stability criteria for two-dimensional flows were established. In Part III, a helpful analogy between axisymmetric MHD flows and flows of a stratified fluid in the Boussinesq approximation was demonstrated. They used Arnold's (1965a, b) theory to obtain linear and nonlinear stability criteria. Arnold's approach to problems of stability is summarized by Saffman (1990, Sec. 14.2). Similar variational principles have been developed for the treatment of the stability of magnetostatic equilibria of perfectly conducting fluids (Bernstein et al. 1958). These variational principles were applied to the stability of force-free magnetic fields by Voslamber and Callebaut (1962). Earlier theories (Frieman & Rotenberg 1960; Moffatt 1989; Friedlander and Vishik 1990) have considered virtual displacements under which the magnetic field is frozen, but have not addressed the problem of identifying the second frozen field that (together with the magnetic field) determines the structure and evolution of perturbations from a given steady state. Vladimirov et al. (1997) considered first 'isomagnetic' perturbations under which the magnetic field is a 'frozen field', and they obtained sufficient conditions for nonlinear stability of axisymmetric ideal incompressible MHD flows. They then obtained sufficient conditions for nonlinear stability to arbitrary axisymmetric perturbations (that are not 'isomagnetic') for a particular class of steady MHD flows, namely for the purely poloidal field and flow (where toroidal components of both the velocity and the magnetic field are absent).

In this paper, we consider a more general situation, namely arbitrary flow and a poloidal field (all the components of the velocity field are non-zero, while the toroidal component of the magnetic field is absent) and we establish nonlinear stability criteria for this class of MHD flows. The difficulty here centers on the problem of appropriate continuation of functions describing the steady state beyond their initial range of the definition. This difficulty is addressed in detail and successfully overcome. The paper is organized as follows.

In this section, the governing equations of ideal incompressible MHD flows are introduced. In Sec. 2, we consider nonlinear stability of the steady state, i.e. Lyapunov stability with respect to a norm based on the total energy and the mean square vector potential of the perturbed magnetic field. We establish nonlinear stability criterion for arbitrary axisymmetric perturbations for arbitrary flow and a poloidal field. In Sec. 3, we establish further nonlinear stability criteria for arbitrary flow and poloidal field. For the axisymmetric situation, the components of the fields $\{U, H\}$ are functions of r and z (in cylindrical polar coordinates (r, ϕ, z)). This case is of a particular importance in the context of plasma confinement devices (e.g. tokamak and reversed-field pinch). Section 4 concludes with a summary.

We conclude this introduction with a statement of the governing equations. Consider an incompressible, homogenous, inviscid and perfectly conducting fluid contained in a domain $\mathbf{\Omega}$ with fixed boundary $\partial\mathbf{\Omega}$. Let $\mathbf{u}(\mathbf{x}, t)$ be the velocity field, $\mathbf{h}(\mathbf{x}, t)$ the magnetic field (in Alfvén velocity units), $\mathbf{p}(\mathbf{x}, t)$ the pressure field (divided by density) and $\mathbf{j}(\mathbf{x}, t) = \boldsymbol{\nabla} \wedge \mathbf{h}$ the current density in the fluid. Then the governing equations are

$$D\mathbf{u} \equiv \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla}\right) u = -\boldsymbol{\nabla} p + \mathbf{j} \wedge \mathbf{h}, \tag{1.1}$$

$$L\mathbf{h} \equiv \frac{\partial \mathbf{h}}{\partial t} - \boldsymbol{\nabla} \wedge (\mathbf{u} \wedge \mathbf{h}) = 0, \tag{1.2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \boldsymbol{\nabla} \cdot \mathbf{h} = 0. \tag{1.3}$$

The operator L is a form of Lie derivative, and the equation $L\mathbf{h} = 0$ means that the $\mathbf{h}$-field is frozen in the fluid, the flux of $\mathbf{h}$ through any closed material circuit being conserved. We suppose that the boundary $\partial\mathbf{\Omega}$ is perfectly conducting, and the magnetic field $\mathbf{h}$ does not penetrate through $\mathbf{\Omega}$. The boundary conditions are then

$$\mathbf{n} \cdot \mathbf{u} = 0, \qquad \mathbf{n} \cdot \mathbf{h} = 0 \qquad \text{on } \partial\mathbf{\Omega}, \tag{1.4}$$

where $\mathbf{n}$ is the outward unit normal to $\partial\mathbf{\Omega}$.

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2. General perturbations and stability criterion for arbitrary flow and a poloidal field

Suppose that $\mathbf{u}$ and $\mathbf{h}$ are invariant under rotations around a fixed axis. It is natural to use cylindrical polar coordinates (r, ϕ, z), z being a coordinate along the axis symmetry. Let $\boldsymbol{\Omega}$ denote the meridional section of this domain, with boundary $\partial\boldsymbol{\Omega}$ on which $\mathbf{n} = (n_1, 0, n_2)$ is the normal. For simplicity, we suppose that $\boldsymbol{\Omega}$ is simply connected.

We now decompose $\mathbf{u}$ and $\mathbf{h}$ into poloidal and toroidal parts:

$$\mathbf{h}(r, z, t) = \mathbf{b} + r\rho_2\mathbf{e}_\phi, \qquad \mathbf{b} = \frac{1}{r}\frac{\partial\rho}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\rho}{\partial z}\mathbf{e}_r, \tag{2.1a}$$

$$\mathbf{u}(r, z, t) = \mathbf{v} + \frac{\rho_1}{r}\mathbf{e}_\phi, \qquad \mathbf{v} = \frac{1}{r}\frac{\partial\psi}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\psi}{\partial z}\mathbf{e}_r, \tag{2.1b}$$

where $\psi(r, z, t)$ is the stream function of the (r, z) components of $\mathbf{u}$, $\rho(r, z, t)$ is the magnetic flux function of the (r, z) components of $\mathbf{h}$, $\rho_1(r, z; t)$ is the toroidal component of $\mathbf{u}$ multiplied by r, and $\rho_2(r, z; t)$ is the toroidal component of $\mathbf{h}$ divided by r.

Consider now steady-state solutions of (1.1)–(1.3) in the form

$$\mathbf{v} = \mathbf{V}(r, z) = \frac{1}{r}\frac{\partial\boldsymbol{\Psi}}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial\boldsymbol{\Psi}}{\partial z}\mathbf{e}_r, \qquad \rho_1 = R_1(r, z), \qquad p = P(r, z), \tag{2.2a}$$

$$\mathbf{b} = \mathbf{B}(r, z) = \frac{1}{r}\frac{\partial A}{\partial r}\mathbf{e}_z - \frac{1}{r}\frac{\partial A}{\partial z}\mathbf{e}_z - \frac{1}{r}\frac{\partial A}{\partial z}\mathbf{e}_r, \qquad \rho_2 = R_2(r, z), \qquad \rho = A(r, z), \tag{2.2b}$$

Capital letters will be used throughout for properties of the steady state whose stability is to be investigated.

An analogy between axisymmetric MHD flows and flows of a stratified fluid in the Boussinesq approximation was noted by Vladimirov et al. (1997). They proved that

$$\mathbf{V} = \boldsymbol{\Psi}'(A)\mathbf{B}, \qquad \boldsymbol{\Psi}'(A) \equiv \frac{\mathrm{d}\boldsymbol{\Psi}}{\mathrm{d}A}, \tag{2.3}$$

$$\boldsymbol{\Psi}'(A)R_2 - \frac{R_1}{r^2} = G_1(A), \qquad \boldsymbol{\Psi}'(A)R_1 - r^2R_2 = G_2(A), \tag{2.4}$$

and the generalized Grad–Shafranov equation

$$\boldsymbol{\Psi}'(A)Q - J - G_1'(A)R_1 - G_2'(A)R_2 + \boldsymbol{\Psi}''(A)R_1R_2 = G(A), \tag{2.5}$$

where $G_1(A)$, $G_2(A)$ and $G(A)$ are some functions, and

$$Q \equiv -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\boldsymbol{\Psi}}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\boldsymbol{\Psi}}{\partial z^2}\right], \qquad J \equiv -\left[\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial A}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 A}{\partial z^2}\right]. \tag{2.6}$$

Consider the steady state (2.2) when all components of the velocity are non-zero while the toroidal component of the magnetic field is identically zero:

$$\mathbf{V} = \boldsymbol{\Psi}'(A)\mathbf{B}, \qquad \rho_1 = R_1(r, z), \qquad \rho_2 = R_2(r, z) = 0, \qquad \rho = A(r, z). \tag{2.7}$$

The functions $G_1(A)$, $G_2(A)$ and $G(A)$ (given by (2.4) and (2.5)) are

$$\left.\begin{aligned} G_1(A) &= -\frac{R_1}{r^2}, \qquad G_2(A) = \boldsymbol{\Psi}'(A)R_1, \\ G(A) &= \boldsymbol{\Psi}'(A)Q - J - G_1'(A)R_1 \end{aligned}\right\}. \tag{2.8}$$

Now consider finite-amplitude perturbations of the steady solution (2.7), given by

$$\left.\begin{array}{ll}\psi(r,z,t)=\mathbf{\Psi}(r,z)+\tilde{\psi}(r,z,t), & \mathbf{v}(r,z,t)=\mathbf{V}(r,z)+\tilde{\mathbf{v}}(r,z,t),\\ \rho(r,z,t)=A(r,z)+\tilde{\rho}(r,z,t), & \mathbf{b}(r,z,t)=\mathbf{B}(r,z,t)+\tilde{\mathbf{b}}(r,z,t),\end{array}\right\}\quad (r,z)\in\mathbf{\Omega}, \tag{2.9}$$

with

$$\tilde{\mathbf{v}}\cdot\mathbf{n}=0,\qquad \tilde{\rho}=0\qquad \text{on}\ \ \partial\mathbf{\Omega}. \tag{2.10}$$

Let

$$A^{-}\equiv\min_{\mathbf{\Omega}}A(r,z),\qquad A^{+}\equiv\max_{\mathbf{\Omega}}A(r,z),$$

and let Λ be the closed interval $[A^{-},A^{+}]$. Let us now introduce the notation

$$\nu=(\nu_1,\nu_2,\nu_3,\nu_4,\nu_5,\nu_6,\nu_7)\equiv\left(\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial r},\frac{1}{r}\frac{\partial\tilde{\psi}}{\partial z},\frac{1}{r}\frac{\partial\tilde{\rho}}{\partial r},\frac{1}{r}\frac{\partial\tilde{\rho}}{\partial z},\tilde{\rho},\frac{\tilde{\rho}_1}{r},r\tilde{\rho}_2\right). \tag{2.11}$$

To measure the deviation of the perturbed solution (2.9) from the unperturbed one, we shall exploit the norm (or, more accurately, the seminorm) given by

$$\|\nu\|^2\equiv\int_{\Omega}\nu_i\nu_i\,d\tau=\int_{\Omega}\left[\frac{1}{r^2}(\mathbf{\nabla}\tilde{\psi})^2+\frac{1}{r^2}(\mathbf{\nabla}\tilde{\rho})^2+\tilde{\rho}^2+\frac{1}{r^2}\tilde{\rho}_1^2+r^2\tilde{\rho}_2^2\right]d\tau. \tag{2.12}$$

We adopt the standard Lyapunov definition of stability: the steady state (2.2) is stable if for any $\varepsilon>0$ there exists $\delta>0$ such that $\|\nu(0)\|<\delta\Rightarrow\|\nu(t)\|<\varepsilon$. For the subsequent analysis, it is convenient to define the following functions: for $(r,z)\in\mathbf{\Omega}$ and $a\in\mathbf{\Lambda}$

$$\sigma(a)\equiv\mathbf{\Psi}'(a),\qquad \sigma'(a)\equiv\mathbf{\Psi}''(a),\qquad \sigma''(a)\equiv\mathbf{\Psi}'''(a), \tag{2.13a}$$

$$\alpha(r,z;a)\equiv-Q(r,z)\sigma'(a)+G'(a)+R_1(r,z)G_1''(a), \tag{2.13b}$$

$$\beta(a)\equiv G_1'(a), \tag{2.13c}$$

$$\gamma(r,z;a)\equiv-R_1(r,z)\sigma'(a)+G_2'(a), \tag{2.13d}$$

$$\mu_1(\xi,\sigma)\equiv\frac{1-\xi}{(1-\xi)^2-\sigma^2},\qquad \mu_2(\xi,\sigma)\equiv\frac{\sigma}{(1-\xi)^2-\sigma^2}, \tag{2.13e}$$

with $G_1(A)$, $G_2(A)$ and $G(A)$ given by (2.8). It is useful to introduce the notation, related to these functions,

$$\left.\begin{array}{ll}\alpha^{-}(r,z)\equiv\min\limits_{a\in\Lambda}\{\alpha(r,z;a)\}, & \alpha^{+}(r,z)\equiv\max\limits_{a\in\Lambda}\{\alpha(r,z;a)\},\\ \beta_0\equiv\max\limits_{a\in\Lambda}|\beta(a)|, & \gamma_0(r,z)\equiv\max\limits_{a\in\Lambda}|\gamma(r,z;a)|,\end{array}\right\}\quad (r,z)\in\mathbf{\Omega}, \tag{2.14a}$$

$$\sigma_0\equiv\max_{a\in\Lambda}|\mathbf{\Psi}'(a)|,\qquad \sigma_0'\equiv\max_{a\in\Lambda}|\mathbf{\Psi}''(a)|. \tag{2.14b}$$

Vladimirov et al. (1997) considered first a particular class of finite-amplitude perturbations ('isomagnetic' perturbations) with general initial data for the stream function $\tilde{\psi}(r,z;0)$ and with initial data for the flux function such that

$$A^{-}\leqslant\rho(r,z;0)=A(r,z)+\tilde{\rho}(r,z;0)\leqslant A^{+}, \tag{2.15}$$

and they obtained a criterion for nonlinear stability (see Vladimirov et al. (1997), p. 104). Then they obtained sufficient conditions for nonlinear stability with respect to arbitrary initial perturbations (without restriction (2.15) on the initial data) for the purely poloidal field and flow ($R_1 = R_2 = 0$; see Vladimirov et al. 1997, p. 109). In our case here, the toroidal component of the velocity field is non-zero ($R_1 \neq 0$).

Let α_0^- and α_0^+ be the minimum and maximum values of the function $\alpha(r, z; a)$ and let $\bar{\gamma}_0$ be the maximum value of the function $|\gamma(r, z; a)|$ for all $(r, z) \in \Omega$ and $a \in \Lambda$, i.e.

$$\left.\begin{aligned} \alpha_0^- &\equiv \min_{(r,z)\in\Omega, a\in\Lambda} \alpha(r, z; a) = \min_{(r,z)\in\Omega} \alpha^-(r, z), \\ \alpha_0^+ &\equiv \max_{(r,z)\in\Omega, a\in\Lambda} \alpha(r, z; a) = \max_{(r,z)\in\Omega} \alpha^+(r, z), \end{aligned}\right\} \tag{2.16a}$$

$$\bar{\gamma}_0 \equiv \max_{(r,z)\in\Omega, a\in\Lambda} |\gamma(r, z; a)|. \tag{2.16b}$$

We obtain the following nonlinear stability criterion.

Criterion 2.1. *Suppose that:*

(i) the function $\Psi(A)$ defined by (2.7) is a twice continuously differentiable function for all $A \in \Lambda$;

(ii) the functions $G(A)$ and $G_2(A)$ (defined by (2.8)) are continuously differentiable, while $G_1(A)$ (defined by (2.8)) is twice continuously differentiable for all $A \in \Lambda$;

(iii) there exist constants ε^- and ε^+ such that for $a \in \Lambda$ and $(r, z) \in \Omega$,

$$0 < \varepsilon^- < 1, \qquad \varepsilon^+ > 2 - \varepsilon^-, \qquad |\Psi'(a)| < 1 - \varepsilon^-, \tag{2.17a}$$

$$\alpha_0^- > \varepsilon^- + \mu_1(\varepsilon^-, \sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) + 2\mu_2(\varepsilon^-, \sigma_0)\bar{\gamma}_0\beta_0, \tag{2.17b}$$

$$\alpha_0^+ < \varepsilon^+ + \mu_1(\varepsilon^+, \sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) - 2\mu_2(\varepsilon^+, \sigma_0)\bar{\gamma}_0\beta_0; \tag{2.17c}$$

(iv) either

$$\max_{a\in\Lambda} |\Psi'(a)| = |\Psi'(a^*)|, \qquad A^- < a^* < A^+, \tag{2.18a}$$

or

$$\max_{a\in\Lambda} |\Psi'(a)| = |\Psi'(a^*)|, \qquad \Psi''(a^*) = 0, \qquad a^* = A^- \quad \text{or} \quad a^* = A^+; \tag{2.18b}$$

(v) either

$$\max_{a\in\Lambda} |G_1'(a)| = |G'(b^*)|, \qquad A^- < b^* < A^+, \tag{2.19a}$$

or

$$\max_{a\in\Lambda} |G_1'(a)| = |G_1'(b^*)|, \qquad G_1''(b^*) = 0, \qquad b^* = A^- \quad \text{or} \quad b^* = A^+. \tag{2.19b}$$

Then the steady state (2.7) is nonlinearly stable to arbitrary finite-amplitude perturbations. Moreover, the following a priori estimate holds:

$$\varepsilon^- \|\nu(t)\| \leqslant \varepsilon^+ \|\nu(0)\|. \tag{2.20}$$

Proof. In the case of 'isomagnetic' perturbations, Vladimirov et al. (1997, pp. 107, 108) proved that the steady state (2.7) is nonlinearly stable to perturbations with initial data satisfying (2.15) and the a priori estimate (2.20) holds true if the following conditions are satisfied:

$$\varepsilon^- < 1, \qquad \sigma^2 < (1-\varepsilon^-)^2, \tag{2.21a}$$

$$\alpha > \varepsilon^- + \mu_1(\varepsilon^-,\sigma)\left(\sigma'^2|B|^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) + 2\mu_2(\varepsilon^-,\sigma)\beta\gamma, \tag{2.21b}$$

$$\varepsilon^+ > 1, \qquad \sigma^2 > (1-\varepsilon^+)^2, \tag{2.22a}$$

$$\alpha < \varepsilon^+ + \mu_1(\varepsilon^+,\sigma)\left(\sigma'^2|B|^2 + r^2\beta^2 + \frac{\gamma^2}{r^2}\right) + 2\mu_2(\varepsilon^+,\sigma)\beta\gamma, \tag{2.22b}$$

for all $(r,z) \in \mathbf{\Omega}$ and $a_0, a_1, a_2, a_3, a_4 \in \mathbf{\Lambda}$, where the functions $\sigma(A)$, $\sigma'(A)$, $\alpha(r,z;a)$, $\beta(r,z;a)$ and $\gamma(r,z;a)$ appearing in the inequalities (2.21) and (2.22) are taken at $a=a_0$, $a=a_1$, $a=a_2$, $a=a_3$ and $a=a_4$ respectively, with

$$a_0 = A + \tilde{\rho}, \qquad a_i = A + \theta_i\tilde{\rho}, \qquad 0 < \theta_i < 1 \qquad (i=1,2,3,4). \tag{2.23}$$

They used Arnold's (1965a) technique and the conserved functional $\mathscr{R} = E + \pi + \Pi_M + \Pi_C + \Gamma$ (a functional similar to $\mathscr{R}$ was used by Almaguer et al. (1988) to obtain sufficient conditions for linear stability of compressible MHD flows), where

$$E = \frac{1}{2}\int_\Omega \left\{\frac{1}{r^2}[(\nabla\psi)^2 + (\nabla\rho)^2 + \rho_1^2] + r^2\rho_2^2\right\} d\tau \tag{2.24}$$

is the conserved energy of the system,

$$\Pi_C = \int_\Omega N_1(\rho)(\mathbf{v}\cdot\mathbf{b} + \rho_1\rho_2)\, d\tau \tag{2.25}$$

is the conserved 'generalized' cross-helicity,

$$\Gamma = \int_\Omega L(\rho)\rho_1\, d\tau, \tag{2.26}$$

is the conserved 'generalized' angular momentum,

$$\Pi_M = \int_\Omega S(\rho)\rho_2\, d\tau, \tag{2.27}$$

is the conserved 'generalized' magnetic helicity and the functional

$$\pi = \int_\Omega N_2(\rho)\, d\tau. \tag{2.28}$$

The functions $N_1(\rho)$, $N_2(\rho)$, $L(\rho)$ and $S(\rho)$ are arbitrary. Vladimirov et al. (1997) chose these functions such that

$$N_1(A) = -\mathbf{\Psi}'(A), \quad N_2'(A) = G(A), \quad L(A) = G_1(A), \quad S(A) = G_2(A). \tag{2.29}$$

Therefore, Criterion 2.1 is proved by showing that the conditions (2.21) and (2.22) hold provided that the conditions (2.17)–(2.19) are satisfied. Then we consider arbitrary perturbations for which the quantities a_0, a_1, a_2, a_3 and a_4 defined by (3.23) may be outside $\mathbf{\Lambda}$. The inequalities (2.21) and (2.22) must therefore be satisfied for all real a_0, a_1, a_2, a_3 and a_4.

For the steady state (2.7), the functions σ, α, β and γ in (2.21) and (2.22) are now defined by

$$\left.\begin{array}{ll}\sigma(a) = -N_1(a), & \alpha(r,z;a) = Q(r,z)N_1'(a) + N_2''(a) + R_1(r,z)L''(a),\\ \beta(a) = L'(a), & \gamma(r,z;a) = R_1(r,z)N_1'(a) + S'(a).\end{array}\right\} \quad (2.30)$$

Thus, according to (2.29), for all $a \in \Lambda$, the new definitions coincide with the old ones (2.13a–d). Initially, $N_1(a), N_2(a), L(a)$ and $S(a)$ were arbitrary, and then were defined only for $a \in \Lambda$ by (2.29). Hence $N_1(a), N_2(a), L(a)$ and $S(a)$ are still arbitrary for $a \notin \Lambda$. We can therefore extend the definitions of $N_1(a), N_2(a), L(a)$ and $S(a)$ to all real a in any way we need, and then extend the definition of $\sigma(a), \alpha(r,z;a), \beta(r,z;a)$ and $\gamma(a)$ by using (2.30).

We shall extend (continue) $N_1(a), N_2(a), L(a)$ and $S(a)$ to all $a \notin \Lambda$ in such a way that, first, $N_1(a)$ and $S(a)$ remain continuously differentiable and $N_2(a)$ and $L(a)$ twice continuously differentiable, and, secondly, the inequalities

$$\alpha_0^- \leqslant \alpha(r,z;a) \leqslant \alpha_0^+, \quad (2.31a)$$

$$|\sigma(a)| = |N_1(a)| \leqslant \sigma_0, \quad (2.31b)$$

$$|\sigma'(a)| = |N_1'(a)| \leqslant \sigma_0', \quad (2.31c)$$

$$|\beta(a)| = |L'(a)| \leqslant \beta_0, \quad (2.31d)$$

$$|\gamma(r,z;a)| \leqslant \bar{\gamma}_0 \quad (2.31e)$$

remain valid for all $a \notin \Lambda$. If such a continuation is possible then Criterion 2.1 will be proved as follows.

We shall prove that (2.21) and (2.22) are satisfied under the conditions (2.17)–(2.19). First we take ε^- and ε^+ such that

$$0 < \varepsilon^- < 1 - \sigma_0, \qquad 2 - \varepsilon^- < \varepsilon^+ < \infty; \quad (2.32)$$

then the inequalities (2.21a) and (2.22a) are satisfied. Secondly, according to the definitions (2.13e) and (2.14) and the inequalities (2.31), we have

$$\left.\begin{array}{l}\mu_1(\varepsilon^-,\sigma)\left(\sigma'^2|B|^2 + r^2\beta^2 + \dfrac{\gamma^2}{r^2}\right) \leqslant \mu_1(\varepsilon^-,\sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \dfrac{\bar{\gamma}_0^2}{r^2}\right),\\ \mu_2(\varepsilon^-,\sigma)\beta\gamma \leqslant \mu_2(\varepsilon^-,\sigma_0)\beta_0\bar{\gamma}_0,\end{array}\right\} \quad (2.33a)$$

$$\left.\begin{array}{l}\mu_1(\varepsilon^+,\sigma)\left(\sigma'^2|B|^2 + r^2\beta^2 + \dfrac{\gamma^2}{r^2}\right) \geqslant \mu_1(\varepsilon^+,\sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \dfrac{\bar{\gamma}_0^2}{r^2}\right),\\ \mu_2(\varepsilon^+,\sigma)\beta\gamma \geqslant -\mu_2(\varepsilon^+,\sigma_0)\beta_0\bar{\gamma}_0.\end{array}\right\} \quad (2.33b)$$

Thirdly, we suppose that there exist ε^- and ε^+ satisfying (2.32) such that

$$\varepsilon^- + \mu_1(\varepsilon^-,\sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) + 2\mu_2(\varepsilon^-,\sigma_0)\beta_0\bar{\gamma}_0 < \alpha_0^-, \quad (2.34a)$$

$$\varepsilon^+ + \mu_1(\varepsilon^+,\sigma_0)\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) - 2\mu_2(\varepsilon^+,\sigma_0)\beta_0\bar{\gamma}_0 > \alpha_0^+. \quad (2.34b)$$

If (2.34a,b) are satisfied then, in view of the inequalities (2.33a,b), the conditions (2.21b) and (2.22b) are satisfied too. Hence we have shown that the conditions (2.32) and (2.34) are in fact sufficient for all six inequalities (2.21) and (2.22) to be satisfied.

Finally, comparing (2.32) and (2.34) with (2.17), we see that they coincide. Criterion 2.1 is thus established. □

Extension of the definitions of the functions $N_1(a), N_2(a), L(a)$ and $S(a)$

It will be sufficient to construct explicitly continuation of $N_1(a), N_2(a), L(a)$ and $S(a)$ to all $a > A^+$. Continuation to $a < A^-$ can be achieved in a similar way (see Appendices A and B). For the function $N_1(a)$, suppose that (2.18a) is true; then three different situations are possible for $N_1'(A^+)$:

(i) $N_1'(A^+) > 0$;

(ii) $N_1'(A^+) < 0$;

(iii) $N_1'(A^+) = 0$.

(i) If $N_1'(A^+) > 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\psi_1 z}, \tag{2.35a}$$

where

$$z \equiv a - A^+, \qquad \psi_1 \equiv \frac{N_1'(A^+)}{\sigma_0 - N_1(A^+)}. \tag{2.35b}$$

We note from (2.18a) that $\psi_1 > 0$. From (2.35a,b), we have

$$N_1(a) = N_1(A^+) + \frac{N_1'(A^+)}{\psi_1}\frac{\psi_1 z}{1+\psi_1 z} < \sigma_0, \qquad a \geqslant A^+. \tag{2.36a}$$

Also,

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\psi_1 z} \geqslant N_1(A^+) > -\sigma_0, \qquad a \geqslant A^+. \tag{2.36b}$$

From (2.36a,b), we obtain

$$|N_1(a)| < \sigma_0, \qquad a \geqslant A^+. \tag{2.37}$$

Hence the inequality (2.31b) is satisfied. Note that, with the definition (2.35), the function $N_1(a)$ is continuously differentiable for all $a \geqslant A^+$.

$$N_1'(a) = N_1'(A^+)\frac{1}{(1+\psi_1 z)^2} \leqslant N_1'(A^+), \qquad a \geqslant A^+, \tag{2.38a}$$

$$|N_1'(a)| = N_1'(a) \leqslant N_1'(A^+) \leqslant \sigma_0', \qquad a \geqslant A^+. \tag{2.38b}$$

Hence (2.31c) is also satisfied.

(ii) If $N_1'(A^+) < 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\psi_2 z}, \tag{2.39a}$$

where

$$z \equiv a - A^+, \qquad \psi_2 \equiv -\frac{N_1'(A^+)}{\sigma_0 + N_1(A^+)}. \tag{2.39b}$$

It is easy to verify that with this choice, $N_1(a)$ is continuously differentiable for all $a \geqslant A^+$, and satisfies (2.31b,c):

$$N_1'(a) = N_1'(A^+)\frac{1}{(1+\psi_2)^2} \geqslant N_1'(A^+), \quad a \geqslant A^+. \tag{2.40}$$

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(iii) If $N_1'(A^+) = 0$ then we define $N_1(a)$ for $a > A^+$ such that $N_1(a) = N_1(A^+)$, so that (2.31b,c) are satisfied. Suppose now that (2.18b) is true; then, as in case (ii), we take $N_1(a) = N_1(A^+)$, and the inequalities (2.31b,c) are satisfied. If neither (2.18a) nor (2.18b) is satisfied then the function $|\boldsymbol{\Psi}'(a)|$ attains its maximum value at one of the endpoints of Λ, and at that point $\boldsymbol{\Psi}''(a) \neq 0$, i.e.

$$\max_{a\in\Lambda} |\boldsymbol{\Psi}'(a)| = |\boldsymbol{\Psi}'(a^*)|, \qquad \boldsymbol{\Psi}''(a^*) \neq 0, \qquad a^* = A^- \quad \text{or} \quad a^* = A^+.$$

In this case, it is impossible to make a sufficiently smooth continuation of $N_1(a)$ for all real a such that the inequality (2.31b) holds true.

Now we shall extend the definition of the function $L(a)$. Suppose that (2.19a) is true; then three different situations are possible for $L''(A^+)$:

(i) $L''(A^+) > 0$;

(ii) $L''(A^+) < 0$;

(iii) $L''(A^+) = 0$.

(i) If $L''(A^+) > 0$ then we define $L(a)$ for $a > A^+$ such that

$$L'(a) = L'(A^+) + L''(A^+)\frac{z}{1+\psi_3 z}, \tag{2.41a}$$

where

$$z \equiv a - A^+, \quad \psi_3 \equiv \frac{L''(A^+)}{\beta_0 - L'(A^+)}. \tag{2.41b}$$

Note from (2.19a) that $\psi_3 > 0$.

It is easy to see that, with this definition, $L(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and satisfies (2.31d):

$$L''(a) = L''(A^+)\frac{1}{(1+\psi_3 z)^2} \leqslant L''(A^+), \qquad a \geqslant A^+. \tag{2.42}$$

(ii) If $L''(A^+) < 0$ then we define $L(a)$ for $a > A^+$ such that

$$L'(a) = L'(A^+) + L''(A^+)\frac{z}{1+\psi_4 z}, \tag{2.43a}$$

where

$$z \equiv a - A^+, \qquad \psi_4 \equiv -\frac{L''(A^+)}{\beta_0 + L'(A^+)}. \tag{2.43b}$$

Note from (2.19a) that $\psi_4 > 0$. It is easy to verify that, with this definition, $L(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and satisfies (2.31d):

$$L''(a) = L''(A^+)\frac{1}{(1+\psi_4 z)^2} \geqslant L''(A^+), \qquad a \geqslant A^+. \tag{2.44}$$

(iii) If $L''(A^+) = 0$ then we define $L(a)$ for $a > A^+$ such that $L'(a) = L'(A^+)$, so that (2.31d) is satisfied. Suppose now that (2.19b) is true; then, as in case (ii), we take $L'(a) = L'(A^+)$, and the inequality (2.31d) is satisfied. If neither (2.19a) nor (2.19b) is satisfied then the function $|L'(a)|$ attains its maximum value at one of the endpoints of Λ, and at that point $L''(a) \neq 0$, i.e.

$$\max_{a\in\Lambda} |L'(a)| = |L'(a^*)|, \qquad L''(a^*) \neq 0, \qquad a^* = A^- \quad \text{or} \quad a^* = A^+.$$

In this case, it is impossible to make a sufficiently smooth continuation of $L'(a)$ for all real a such that the inequality (2.31d) holds true.

Now we choose the functions $S(a)$ and $N_2(a)$ for $a > A^+$ in such a way that (2.31a) and (2.31e) are satisfied. Before doing this, let us introduce the functions $\hat{\gamma}^-(a)$, $\hat{\gamma}^+(a)$, $\hat{\alpha}^-(a)$ and $\hat{\alpha}^+(a)$ such that

$$\hat{\gamma}^-(a) \equiv \min_{(r,z)\in\Omega} \gamma(r,z;a), \tag{2.45a}$$

$$\hat{\gamma}^+(a) \equiv \max_{(r,z)\in\Omega} \gamma(r,z;a), \tag{2.45b}$$

$$\hat{\alpha}^-(a) \equiv \min_{(r,z)\in\Omega} \alpha(r,z;a), \tag{2.45c}$$

$$\hat{\alpha}^+(a) \equiv \max_{(r,z)\in\Omega} \alpha(r,z;a), \tag{2.45d}$$

where $\gamma(r,z;a)$ and $\alpha(r,z;a)$ are defined by (2.30). It follows from these definitions that

$$\hat{\gamma}^-(a) = S'(a) + \begin{cases} N_1'(a)\,\min\limits_{\Omega}\,R_1(r,z) & \text{if } N_1'(a) > 0, \\ 0 & \text{if } N_1'(a) = 0, \\ N_1'(a)\,\max\limits_{\Omega}\,R_1(r,z) & \text{if } N_1'(a) < 0, \end{cases} \tag{2.46a}$$

$$\hat{\gamma}^+(a) = S'(a) + \begin{cases} N_1'(a)\,\max\limits_{\Omega}\,R_1(r,z) & \text{if } N_1'(a) > 0, \\ 0 & \text{if } N_1'(a) = 0, \\ N_1'(a)\,\min\limits_{\Omega}\,R_1(r,z) & \text{if } N_1'(a) < 0, \end{cases} \tag{2.46b}$$

$$\begin{aligned}\hat{\alpha}^-(a) = N_2''(a)&\\ + &\begin{cases} N_1'(a)\min\limits_{\Omega} Q(r,z) + L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) > 0,\; L''(a) > 0, \\ N_1'(a)\min\limits_{\Omega} Q(r,z) + L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) > 0,\; L''(a) < 0, \\ N_1'(a)\min\limits_{\Omega} Q(r,z) & \text{if } N_1'(a) > 0,\; L''(a) = 0, \\ N_1'(a)\max\limits_{\Omega} Q(r,z) + L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) < 0,\; L''(a) > 0, \\ N_1'(a)\max\limits_{\Omega} Q(r,z) + L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) < 0,\; L''(a) < 0, \\ N_1'(a)\max\limits_{\Omega} Q(r,z) & \text{if } N_1'(a) < 0,\; L''(a) = 0, \\ L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) = 0,\; L''(a) > 0, \\ L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) = 0,\; L''(a) < 0, \\ 0 & \text{if } N_1'(a) = 0,\; L''(a) = 0, \end{cases}\end{aligned} \tag{2.47a}$$

$$\begin{aligned}\hat{\alpha}^+(a) = N_2''(a)&\\ + &\begin{cases} N_1'(a)\max\limits_{\Omega} Q(r,z) + L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) > 0,\; L''(a) > 0, \\ N_1'(a)\max\limits_{\Omega} Q(r,z) + L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) > 0,\; L''(a) < 0, \\ N_1'(a)\max\limits_{\Omega} Q(r,z) & \text{if } N_1'(a) > 0,\; L''(a) = 0, \\ N_1'(a)\min\limits_{\Omega} Q(r,z) + L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) < 0,\; L''(a) > 0, \\ N_1'(a)\min\limits_{\Omega} Q(r,z) + L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) < 0,\; L''(a) < 0, \\ N_1'(a)\min\limits_{\Omega} Q(r,z) & \text{if } N_1'(a) < 0,\; L''(a) = 0, \\ L''(a)\max\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) = 0,\; L''(a) > 0, \\ L''(a)\min\limits_{\Omega} R_1(r,z) & \text{if } N_1'(a) = 0,\; L''(a) < 0, \\ 0 & \text{if } N_1'(a) = 0 \quad L''(a) = 0. \end{cases}\end{aligned} \tag{2.47b}$$

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1. If $N_1'(A^+) > 0$ and $L''(A^+) > 0$ then

$$\hat{\alpha}^-(a) = N_2''(a) + N_1'(a) \min_\Omega Q(r,z) + L''(a) \min_\Omega R_1(r,z), \tag{2.48a}$$

$$\hat{\alpha}^+(a) = N_2''(a) + N_1'(a) \max_\Omega Q(r,z) + L''(a) \max_\Omega R_1(r,z), \tag{2.48b}$$

$$\hat{\gamma}^-(a) = S'(a) + N_1'(a) \min_\Omega R_1(r,z), \tag{2.48c}$$

$$\hat{\gamma}^+(a) = S'(a) + N_1'(a) \max_\Omega R_1(r,z). \tag{2.48d}$$

Now for all $a > A^+$, we take $N_2(a)$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \min_\Omega Q(r,z) - L''(a) \min_\Omega R_1(r,z), \tag{2.49}$$

where $N_1(a)$ and $L(a)$ are given by (2.35a) and (2.41a) respectively. With this choice, $N_2(a)$ is twice continuously differentiable for all $a \geqslant A^+$, and, from (2.48a),

$$\hat{\alpha}^-(a) = \hat{\alpha}^-(A^+) \geqslant \alpha_0^-, \qquad a \geqslant A^+. \tag{2.50}$$

Also, from (2.48b), we obtain

$$\begin{aligned}\hat{\alpha}^+(a) = \hat{\alpha}^-(A^+) + N_1'(a)\left[\max_\Omega Q(r,z) - \min_\Omega Q(r,z)\right]\\ +L''(a)\left[\max_\Omega R_1(r,z) - \min_\Omega R_1(r,z)\right],\end{aligned} \tag{2.51}$$

$$\begin{aligned}\hat{\alpha}^+(A^+) = \hat{\alpha}^-(A^+) + N_1'(A^+)\left[\max_\Omega Q(r,z) - \min_\Omega Q(r,z)\right]\\ +S''(A^+)\left[\max_\Omega R_2(r,z) - \min_\Omega R_2(r,z)\right].\end{aligned} \tag{2.52}$$

Eliminating $\hat{\alpha}^-(A^+)$ from (2.51) and (2.52), we find

$$\begin{aligned}\hat{\alpha}^+(a) = \hat{\alpha}^+(A^+) + [N_1'(a) - N_1'(A^+)]\left[\max_\Omega Q(r,z) - \min_\Omega Q(r,z)\right]\\ +[L''(a) - L''(A^+)]\left[\max_\Omega R_1(r,z) - \min_\Omega R_1(r,z)\right].\end{aligned} \tag{2.53}$$

From (2.38a) and (2.42), we obtain

$$\hat{\alpha}^+(a) \leqslant \hat{\alpha}^+(A^+) \leqslant \alpha_0^+, \qquad a \geqslant A^+. \tag{2.54}$$

From (2.50) and (2.54), we obtain

$$\alpha_0^- \leqslant \hat{\alpha}^-(a) \leqslant \alpha(r,z;a) \leqslant \hat{\alpha}^+(a) \leqslant \alpha_0^+, \qquad a \geqslant A^+, \tag{2.55}$$

and then (2.31a) is satisfied. What remains now to extend (continue) the function $S(a)$ to all $a > A^+$. We define $S(a)$ for $a > A^+$ such that

$$S'(a) = \hat{\gamma}^-(A^+) - N_1'(a) \min_\Omega R_1(r,z), \tag{2.56}$$

where $N_1(a)$ is given by (2.35a). With this choice, $S(a)$ is continuously differentiable for all $a \geqslant A^+$, and, from (2.48c),

$$\hat{\gamma}^-(a) = \hat{\gamma}^-(A^+) \geqslant -\bar{\gamma}_0, \qquad a \geqslant A^+. \tag{2.57}$$

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Also, from (2.48d), we obtain

$$\hat{\gamma}^{+}(a) = \hat{\gamma}^{-}(A^{+}) + N_1'(a)\left[\max_{\Omega} R_1(r,z) - \min_{\Omega} R_1(r,z)\right], \tag{2.58}$$

$$\hat{\gamma}^{+}(A^{+}) = \hat{\gamma}^{-}(A^{+}) + N_1'(A^{+})\left[\max_{\Omega} R_1(r,z) - \min_{\Omega} R_1(r,z)\right]. \tag{2.59}$$

Eliminating $\hat{\gamma}^{-}(A^{+})$ from (2.58) and (2.59), we get

$$\hat{\gamma}^{+}(a) = \hat{\gamma}^{+}(A^{+}) + [N_1'(a) - N_1'(A^{+})]\left[\max_{\Omega} R_1(r,z) - \min_{\Omega} R_1(r,z)\right]. \tag{2.60}$$

From (2.38a) and (2.60), we obtain

$$\hat{\gamma}^{+}(a) \leqslant \hat{\gamma}^{+}(A^{+}) \leqslant \bar{\gamma}_0, \qquad a \geqslant A^{+}. \tag{2.61}$$

From (2.57) and (2.61), we have

$$-\bar{\gamma}_0 \leqslant \hat{\gamma}^{-}(a) \leqslant \gamma(r,z;a) \leqslant \hat{\gamma}^{+}(a) \leqslant \bar{\gamma}_0, \qquad a \geqslant A^{+}. \tag{2.62}$$

and then the inequality (2.31e) is satisfied. So our continuation satisfies all of the conditions (2.31).

2. If $N_1'(A^{+}) > 0$ and $S''(A^{+}) < 0$ then

$$\hat{\alpha}^{-}(a) = N_2''(a) + N_1'(a)\min_{\Omega} Q(r,z) + L''(a)\max_{\Omega} R_1(r,z), \tag{2.63a}$$

$$\hat{\alpha}^{+}(a) = N_2''(a) + N_1'(a)\max_{\Omega} Q(r,z) + L''(a)\min_{\Omega} R_1(r,z). \tag{2.63b}$$

We choose the function $N_2(a)$ for $a > A^{+}$ such that

$$N_2''(a) = \hat{\alpha}^{-}(A^{+}) - N_1'(a)\min_{\Omega} Q(r,z) - L''(a)\max_{\Omega} R_1(r,z), \tag{2.64}$$

where $N_1(a)$ and $L(a)$ are given by (2.35a) and (2.43a) respectively. It may be shown that $N_2(a)$ defined by (2.64) is twice continuously differentiable for all $a \geqslant A^{+}$ and that (2.31a) is satisfied.

3. If $N_1'(A^{+}) > 0$ and $L''(A^{+}) = 0$ then

$$\hat{\alpha}^{-}(a) = N_2''(a) + N_1'(a)\min_{\Omega} Q(r,z), \tag{2.65a}$$

$$\hat{\alpha}^{+}(a) = N_2''(a) + N_1'(a)\max_{\Omega} Q(r,z). \tag{2.65b}$$

We choose the function $N_2(a)$ for $a > A^{+}$ such that

$$N_2''(a) = \hat{\alpha}^{-}(A^{+}) - N_1'(a)\min_{\Omega} Q(r,z), \tag{2.66}$$

where $N_1(a)$ is given by (2.35a).

In cases 2 and 3, the functions $\hat{\gamma}^{-}(a)$ and $\hat{\gamma}^{+}(a)$ are given by (2.48c) and (2.48d) respectively, and hence the function $S(a)$ is defined by (2.56), so (2.31e) is satisfied.

4. If $N_1'(A^{+}) < 0$ and $L''(A^{+}) > 0$ then

$$\hat{\alpha}^{-}(a) = N_2''(a) + N_1'(a)\max_{\Omega} Q(r,z) + L''(a)\min_{\Omega} R_1(r,z), \tag{2.67a}$$

$$\hat{\alpha}^{+}(a) = N_2''(a) + N_1'(a)\min_{\Omega} Q(r,z) + L''(a)\max_{\Omega} R_1(r,z), \tag{2.67b}$$

$$\hat{\gamma}^-(a) = L'(a) + N_1'(a) \max_{\Omega} R_1(r,z), \tag{2.67c}$$

$$\hat{\gamma}^+(a) = L'(a) + N_1'(a) \min_{\Omega} R_1(r,z). \tag{2.67d}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \max_{\Omega} Q(r,z) - L''(a) \min_{\Omega} R_1(r,z), \tag{2.68}$$

where $N_1(a)$ and $L(a)$ are given by (2.39a) and (2.41a) respectively. In this case, we choose the function $S(a)$ for $a > A^+$ such that

$$S'(a) = \hat{\gamma}^-(A^+) - N_1'(a) \max_{\Omega} R_1(r,z), \tag{2.69}$$

where $N_1(a)$ is given by (2.39a). It may be shown that $S(a)$ defined by (2.69) is continuously differentiable for all $a \geqslant A^+$, and (2.31e) is satisfied.

5. If $N_1'(A^+) < 0$ and $L''(A^+) < 0$ then

$$\hat{\alpha}^-(a) = N_2''(a) + N_1'(a) \max_{\Omega} Q(r,z) + L''(a) \max_{\Omega} R_1(r,z), \tag{2.70a}$$

$$\hat{\alpha}^+(a) = N_2''(a) + N_1'(a) \min_{\Omega} Q(r,z) + L''(a) \min_{\Omega} R_1(r,z). \tag{2.70b}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \max_{\Omega} Q(r,z) - L''(a) \max_{\Omega} R_1(r,z), \tag{2.71}$$

where $N_1(a)$ and $L(a)$ are given by (2.39a) and (2.43a) respectively.

6. If $N_1'(A^+) < 0$ and $L''(A^+) = 0$ then

$$\hat{\alpha}^-(a) = N_2''(a) + N_1'(a) \max_{\Omega} Q(r,z), \tag{2.72a}$$

$$\hat{\alpha}^+(a) = N_2''(a) + N_1'(a) \min_{\Omega} Q(r,z). \tag{2.72b}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - N_1'(a) \max_{\Omega} Q(r,z), \tag{2.73}$$

where $N_1(a)$ is given by (2.39a).

In cases 5 and 6, the functions $\hat{\gamma}^-(a)$ and $\hat{\gamma}^+(a)$ are given by (2.67c) and (2.67d) respectively, and hence the function $S(a)$ is defined by (2.69), so that (2.31e) is satisfied.

7. If $N_1'(A^+) = 0$ and $L''(A^+) > 0$ then

$$\hat{\alpha}^-(a) = N_2''(a) + L''(a) \min_{\Omega} R_1(r,z), \tag{2.74a}$$

$$\hat{\alpha}^+(a) = N_2''(a) + L''(a) \max_{\Omega} Q(r,z), \tag{2.74b}$$

$$\hat{\gamma}^-(a) = \hat{\gamma}^+(a) = S'(a). \tag{2.74c}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - L''(a) \min_{\Omega} R_1(r,z), \tag{2.75}$$

where $L(a)$ is given by (2.41a). We choose the function $S(a)$ for $a > A^+$ such that

$$S'(a) = S'(A^+). \tag{2.76}$$

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8. If $N_1'(A^+) = 0$ and $L''(A^+) < 0$ then

$$\hat{\alpha}^-(a) = N_2''(a) + L''(a) \max_{\Omega} R_1(r,z), \tag{2.77a}$$

$$\hat{\alpha}^+(a) = N_2''(a) + L''(a) \min_{\Omega} Q(r,z). \tag{2.77b}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = \hat{\alpha}^-(A^+) - L''(a) \max_{\Omega} R_1(r,z), \tag{2.78}$$

where $L(a)$ is given by (2.43a).

9. If $N_1'(A^+) = 0$ and $L''(A^+) = 0$ then

$$\hat{\alpha}^-(a) = \hat{\alpha}^+(a) = N_2''(a). \tag{2.79}$$

We choose the function $N_2(a)$ for $a > A^+$ such that

$$N_2''(a) = N_2''(A^+). \tag{2.80}$$

In cases 8 and 9, the functions $\hat{\gamma}^-(a)$ and $\hat{\gamma}^+(a)$ are given by (2.74c), and hence the function $S(a)$ is defined by (2.76).

Thus we have shown that a smooth continuation of the functions $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all real a is possible such that the conditions (2.31) remain satisfied. □

Let us now discuss the existence of constants ε^- and ε^+ satisfying the conditions (2.17) of Criterion 2.1. It is convenient to rewrite the inequalities (2.17b,c) in the form

$$\alpha_0^- > f_1(\varepsilon^-), \qquad \alpha_0^+ < f_2(\varepsilon^+), \tag{2.81}$$

where

$$f_i(\varepsilon) \equiv \varepsilon + \frac{1-\varepsilon}{(1-\varepsilon)^2 - \sigma_0^2}\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) + (-1)^{i-1}\frac{2\sigma_0\bar{\gamma}_0\beta_0}{(1-\varepsilon)^2 - \sigma_0^2}, \tag{2.82}$$

$i = 1, 2$.

It follows from (2.17a) that

$$0 < \varepsilon^- < 1 - \sigma_0, \qquad \varepsilon^+ > 1 + \sigma_0. \tag{2.83}$$

We note that the function $f_1(\varepsilon)$ is an increasing function for all $0 < \varepsilon < 1 - \sigma_0$, and the function $f_2(\varepsilon)$ is an increasing function for all $\varepsilon > 1 + \sigma_0$.

The condition (2.81a) is satisfied for any $\varepsilon < \varepsilon_0^-$, and (2.81b) is valid for $\varepsilon > \varepsilon_0^+$, where ε_0^- and ε_0^+ are points on the two regions I and II of Fig. 1 corresponding to $f_1(\varepsilon_0^-) = \alpha^-$ and $f_2(\varepsilon_0^+) = \alpha^+$ respectively. Figure 1 also shows that a constant ε^- satisfying the conditions (2.17) does exist, provided that

$$\alpha_0^- > f_1(0) = \frac{1}{1-\sigma_0^2}\left(\alpha_0'^2|B|^2(r,z) + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2} + 2\alpha_0\bar{\gamma}_0\beta_0\right), \tag{2.84}$$

for all $(r,z) \in \Omega$, while a constant ε^+ always exists for any given ε^- and α^+. We can now formulate the following corollary.

Corollary 2.1. *The steady state (2.7) is nonlinearly stable to arbitrary axisymmetric perturbations provided that:*

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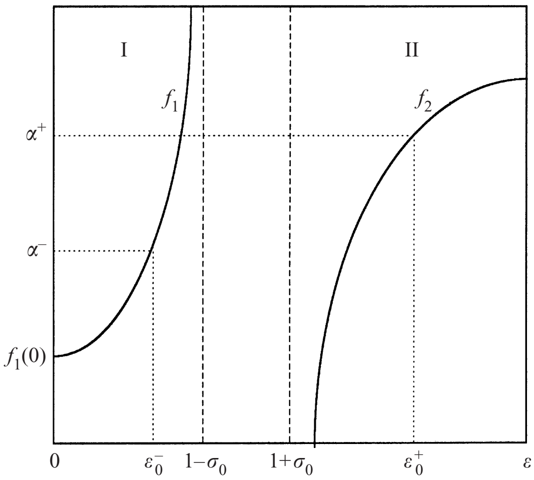

Figure 1. The functions $f_1(\varepsilon)$ and $f_2(\varepsilon)$ defined by (2.82).

(i)

$$|V|^2(r,z) < |B|^2(r,z) < \frac{1}{\sigma_0'^2}\left[(1-\sigma_0^2)\alpha_0^- - 2\sigma_0\gamma_0\beta_0 - r^2\beta_0^2 - \frac{\bar{\gamma}_0^2}{r^2}\right] \tag{2.85}$$

for all $(r,z) \in \Omega$;

(ii) the conditions (2.18) and (2.19) are satisfied.

3. Nonlinear stability criteria

In Sec. 2, we obtained nonlinear stability criterion for arbitrary axisymmetric perturbations for arbitrary flow and poloidal field. In this section, we shall obtain another criteria for the same problem.

Consider the steady state (2.7); we shall obtain the following nonlinear stability criterion.

Criterion 3.1. *Suppose that:*

(i) the same conditions as in Criterion 2.1 hold concerning the smoothness of the functions $\Psi(A)$, $G_1(A)$, $G_2(A)$ *and* $G(A)$;

(ii) there exist constants ε^-, ε^+ *and* ε^* *such that, for* $a \in \Lambda$ *and* $(r,z) \in \Omega$,

$$\varepsilon^* > 0, \qquad 0 < \varepsilon^- < 1, \qquad \varepsilon^+ > 2 - \varepsilon^-, \qquad |\Psi'(a)| < 1 - \varepsilon^-, \tag{3.1a}$$

$$\alpha_0^- > \varepsilon^- + \mu_1(\varepsilon^-,\sigma_0)\left[\alpha_0'^2|B|^2 + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2}\right] + 2\mu_2(\varepsilon^-,\sigma_0)\bar{\gamma}_0\beta_0(1+\varepsilon^*), \tag{3.1b}$$

$$\alpha_0^+ < \varepsilon^+ + \mu_1(\varepsilon^+,\sigma_0)\left[\alpha_0'^2|B|^2 + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2}\right] - 2\mu_2(\varepsilon^+,\sigma_0)\bar{\gamma}_0\beta_0(1+\varepsilon^*); \tag{3.1c}$$

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(iii) either

$$\max_{a\in\Lambda} |\Psi'(a)| = |\Psi'(a^*)|, \qquad A^- < a^* < A^+, \tag{3.2a}$$

or

$$\max_{a\in\Lambda} |\Psi'(a)| = |\Psi'(a^*)|, \qquad \Psi''(a^*) = 0, \qquad a^* = A^- \quad \text{or} \quad a^* = A^+. \tag{3.2b}$$

Then the a priori estimate (2.20) holds, and the steady state (2.7) is nonlinearly stable to arbitrary finite-amplitude perturbations.

Proof. As with Criterion 2.1, to prove this proposition, it is sufficient to show that the inequalities (2.21) and (2.22) are satisfied for all real a_0, a_1, a_2, a_3 and a_4. We shall extend (continue) the functions $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all $a \notin \Lambda$ in such a way that, first, these functions remain smooth enough and, secondly, the inequalities

$$\alpha_0^- \leqslant \alpha(r, z; a) \leqslant \alpha_0^+, \tag{3.3a}$$

$$|\sigma(a)| = |N_1(a) \leqslant \sigma_0, \tag{3.3b}$$

$$|\sigma'(a)| = |N_1'(a)| \leqslant \sigma_0', \tag{3.3c}$$

$$|\beta(a)| = |L'(a)| < \beta_0(1 + \varepsilon^*), \tag{3.3d}$$

$$|\gamma(r, z; a)| \leqslant \bar{\gamma}_0, \tag{3.3e}$$

remain valid for all $a \notin \Lambda$. If such a continuation is possible then the proof of Criterion 3.1 reduces effectively to the proof of Criterion 2.1.

Extension of the definitions of $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$

It will be sufficient to construct explicitly a continuation of $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all $a > A^+$. Continuation to all $a < A^-$ can be achieved in a similar way (see Appendix C).

The extension of the definition of $N_1(a)$ and the proof of (3.3b,c) are as in Criterion 2.1 (see (2.35)–(2.40) and Appendix A).

Now we extended the definition of the function $L(a)$. Three different situations are possible:

(i) $L''(A^+) > 0$;

(ii) $L''(A^+) < 0$;

(iii) $L''(A^+) = 0$.

(i) If $L''(A^+) > 0$ then we define $L(a)$ for $a > A^+$ such that

$$L'(a) = L'(A^+) + L''(A^+)\frac{z}{1 + \lambda_1 z}, \tag{3.4a}$$

where

$$z \equiv a - A^+, \qquad \lambda_1 \equiv \frac{L''(A^+)}{\beta_0(1 + \varepsilon^*) - L'(A^+)} > 0. \tag{3.4b}$$

It is easy to see that, with this definition, $L(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and satisfies (3.3d):

$$L''(a) = L''(A^+)\frac{1}{(1 + \lambda_1 z)^2} \leqslant L''(A^+), \qquad a \geqslant A^+. \tag{3.5}$$

(ii) If $L''(A^+) < 0$ then we define $L(a)$ for $a > A^+$ such that

$$L'(a) = L'(A^+) + L''(A^+)\frac{z}{1+\lambda_2 z}, \tag{3.6a}$$

where

$$z \equiv a - A^+, \qquad \lambda_2 \equiv -\frac{L''(A^+)}{\beta_0(1+\varepsilon^*) + L'(A^+)} > 0. \tag{3.6b}$$

Also it is easy to verify that, with this choice, $L(a)$ is twice continuously differentiable for all $a \geqslant A^+$ and satisfies (3.3d):

$$L''(a) = L''(A^+)\frac{1}{(1+\lambda_2 z)^2} \geqslant L''(A^+), \qquad a \geqslant A^+. \tag{3.7}$$

(iii) If $L''(A^+) = 0$ then we take $L(a)$ for $a > A^+$ such that $L'(a) = L'(A^+)$, so that (3.3d) is satisfied.

The extension of the definition of $N_2(a)$ and $S(a)$, and the proof of the inequalities (3.3a,e), are as in Criterion 2.1, with the new form of $L(a)$ (see (2.45)–(2.80)). □

Similarly to Criterion 2.1, the analysis of existence of positive constants ε^-, ε^+ and ε^* satisfying the conditions (3.1) shows that ε^- and ε^+ always exist for a given ε^* provided that

$$\alpha_0^- > \frac{1}{1-\alpha_0^2}\left[\sigma_0'^2|B|^2(r,z) + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2} + 2\sigma_0\bar{\gamma}_0\beta_0(1+\varepsilon^*)\right], \tag{3.8}$$

for all $(r,z) \in \mathbf{\Omega}$.

We can now formulate the following corollary.

Corollary 3.1. *The steady state (2.7) is nonlinearly stable to arbitrary axisymmetric perturbations, provided that:*

(i) there exists a constant $\varepsilon^* > 0$ such that

$$|v|^2(r,z) < |B|^2(r,z) < \frac{1}{\sigma_0'^2}\left[(1-\alpha_0^2)\alpha_0^- 2\alpha_0\gamma_0\beta_0(1+\varepsilon^*) - r^2\beta_0^2(1+\varepsilon^*)^2 - \frac{\bar{\gamma}_0^2}{r^2}\right] \tag{3.9}$$

for all $(r,z) \in \mathbf{\Omega}$;

(ii) either (3.2a) or (3.2b) is satisfied.

Criterion 3.2. *Suppose that:*

(i) the same conditions as in Criterion 2.1 hold concerning the smoothness of functions $\Psi(A)$, $G_1(A)$, $G_2(A)$ and $G(A)$;

(ii) there exist constants ε^-, ε^+ and ε^* such that, for $a \in \Lambda$ and $(r,z) \in \mathbf{\Omega}$,

$$0 < \varepsilon^- < \varepsilon^* < 1, \qquad \varepsilon^+ > 2 - \varepsilon^-, \qquad |\Psi'(a)| < 1 - \varepsilon^*, \tag{3.10a}$$

$$\alpha_0^- > \varepsilon^- + \mu_1(\varepsilon^-, 1-\varepsilon^*)\left(\alpha_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) + 2\mu_2(\varepsilon^-, 1-\varepsilon^*)\beta_0\bar{\gamma}_0, \tag{3.10b}$$

$$\alpha_0^+ < \varepsilon^+ + \mu_1(\varepsilon^+, 1-\varepsilon^*)\left(\alpha_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) - 2\mu_2(\varepsilon^+, 1-\varepsilon^*)\beta_0\bar{\gamma}_0; \tag{3.10c}$$

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(iii) either

$$\max_{a\in\Lambda}|L'(a)| = |L'(a^*)|, \qquad A^- < a^* < A^+, \tag{3.11a}$$

or

$$\max_{a\in\Lambda}|L'(a)| = |L'(a^*)|, \qquad L''(a^*) = 0, \qquad a^* = A^- \quad \text{or} \quad a^* = A^+. \tag{3.11b}$$

Then the a priori estimate (2.20) holds, and the steady state (2.7) is nonlinearly stable to arbitrary finite-amplitude perturbations.

Proof. As with Criterion 2.1, to prove this proposition, it is sufficient to show that the inequalities (2.21) and (2.22) are satisfied for all real a_0, a_1, a_2, a_3 and a_4. We therefore need to extend (continue) the functions $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all real a in such a way that these functions remain smooth enough and the inequalities

$$\alpha_0^- \leqslant \alpha(r,z;a) \leqslant \alpha_0^+, \tag{3.12a}$$

$$|\sigma(a)| = |N_1(a)| < 1-\varepsilon^*, \tag{3.12b}$$

$$|\sigma'(a)| = N_1'(a)| \leqslant \alpha_0', \tag{3.12c}$$

$$|\beta(a)| = |L'(a)| \leqslant \beta_0, \tag{3.12d}$$

$$|\gamma(r,z;a)| \leqslant \bar{\gamma}_0 \tag{3.12e}$$

remain valid for all $a \notin \Lambda$. If such a continuation is possible then the proof of Criterion 3.2 reduces effectively to the proof of Criterion 2.1.

Extension of the definition of $N_1(a), N_2(a), L(a)$ *and* $S(a)$

It will be sufficient to construct explicitly a continuation of $N_1(a), N_2(a), L(a)$ and $S(a)$ to all $a > A^+$. Continuation to $a < A^-$ can be achieved in a similar way (see Appendix D).

Three different situations are possible for $N_1'(A^+)$:

(i) $N_1'(A^+) > 0$;

(ii) $N_1'(A^+) < 0$;

(iii) $N_1'(A^+) = 0$.

(i) If $N_1'(A^+) > 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\xi_1 z}, \tag{3.13a}$$

where

$$z \equiv a - A^+, \qquad \xi \equiv \frac{N_1'(A^+)}{1-\varepsilon^* - N_1(A^+)}. \tag{3.13b}$$

Note from (3.10a) that $\xi_1 > 0$. It is easy to see that, with this definition, $N_1(a)$ is continuously differentiable for all $a \geqslant A^+$ and satisfies (3.12b,c):

$$N_1'(a) = N_1'(A^+)\frac{1}{(1+\xi_1 z)^2} \leqslant N_1'(A^+), \qquad a \geqslant A^+. \tag{3.14}$$

(ii) If $N_1'(A^+) < 0$ then we define $N_1(a)$ for $a > A^+$ such that

$$N_1(a) = N_1(A^+) + N_1'(A^+)\frac{z}{1+\xi_2 z}, \tag{3.15a}$$

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where

$$z \equiv a - A^+, \qquad \xi_2 \equiv -\frac{N_1'(A^+)}{1 - \varepsilon^* + N_1(A^+)}. \tag{3.15b}$$

It is easy to verify that, with this choice, $N_1(a)$ is continuously differentiable for all $a \geqslant A^+$ and satisfies (3.12b,c):

$$N_1'(a) = N_1'(A^+)\frac{1}{(1+\xi_2)^2} \geqslant N_1'(A^+), \qquad a \geqslant A^+. \tag{3.16}$$

(iii) If $N_1'(A^+) = 0$ then we take $N_1(a)$ for $a > A^+$ such that $N_1(a) = N_1(A^+)$, so that (3.12b,c) are satisfied.

The extension of the definition of $L(a)$ and the proof of (3.12d) are as in Criterion 2.1 (see (2.41)–(2.44) and Appendix B). The extension of the definition of $N_2(a)$ and $S(a)$ and the proof of the inequalities (2.3a,e) are as in Criterion 2.1 (see (2.45)–(2.80)). Thus we have shown that a smooth continuation of the functions $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all real a is possible such that the conditions (3.12) remain satisfied. □

Let us now discuss the existence of constants ε^-, ε^+ and ε^* satisfying the conditions (3.10) of Criterion 3.2. It is convenient to rewrite the inequalities (3.10b,c) in the form

$$\alpha_0^- > g_1(\varepsilon^-), \qquad \alpha_0^+ < g_2(\varepsilon^+), \tag{3.17}$$

where

$$g_i(\varepsilon) \equiv \varepsilon + \frac{1-\varepsilon}{(1-\varepsilon)^2 - (1-\varepsilon^*)^2}\left(\sigma_0'^2|B|^2 + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2}\right) + (-1)^{i-1}\frac{2(1-\varepsilon^*)\bar{\gamma}_0\beta_0}{(1-\varepsilon)^2-(1-\varepsilon^*)^2}, \quad i = 1, 2. \tag{3.18}$$

It follows from (3.10a) that

$$0 < \varepsilon^- < \varepsilon^*, \qquad \varepsilon^+ > 2 - \varepsilon^*. \tag{3.19}$$

The condition (3.17a) is satisfied for any $\varepsilon < \varepsilon_c^-$, and (3.17b) is valid for $\varepsilon > \varepsilon_c^+$, where ε_c^- and ε_c^+ are points on the two regions I and II of Fig. 2 corresponding to $g_1(\varepsilon_c^-) = \alpha^-$ and $g_2(\varepsilon_c^+) = \alpha^+$ respectively. Figure 2 also shows that a constant ε^- satisfying the conditions (3.10) does exist, provided that

$$\alpha_0^- > g_1(0) = \frac{1}{1-(1-\varepsilon^*)^2}\left[\sigma_0'^2|B|^2(r,z) + r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2} + 2(1-\varepsilon^*)\bar{\gamma}_0\beta_0\right], \tag{3.20}$$

for all $(r, z) \in \Omega$, while a constant ε^+ always exists for any given ε^- and α^+.

Let $\delta \equiv 1 - \varepsilon^*$; then we can formulate the following corollary.

Corollary 3.2. *The steady state (2.7) is nonlinearly stable to arbitrary axisymmetric perturbations, provided that:*

(i) there exists a constant δ *such that*

$$0 < \delta < 1, \; \frac{1}{\delta^2}|V|^2(r,z) < |B|^2(r,z) < \frac{1}{\sigma_0'^2}\left[(1-\delta^2)\alpha_0^- - \left(r^2\beta_0^2 + \frac{\bar{\gamma}_0^2}{r^2} + 2\delta\bar{\gamma}_0\beta_0\right)\right] \tag{3.21}$$

for all $(r, z) \in \Omega$,

(ii) either (3.11a) or (3.11b) is satisfied.

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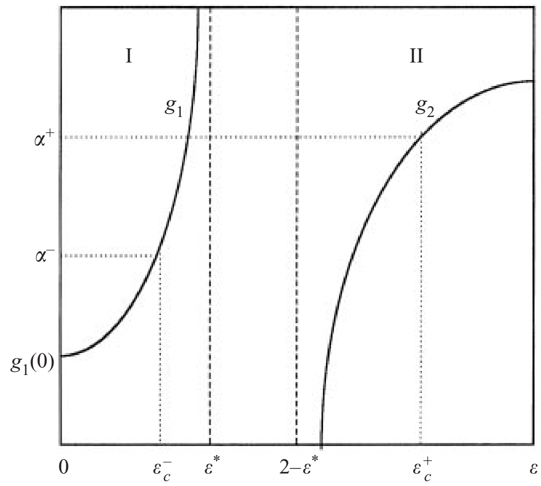

Figure 2. The functions $g_1(\varepsilon)$ and $g_2(\varepsilon)$ defined by (3.18).

Criterion 3.3. *Suppose that:*

(i) the same conditions as in Criterion 2.1 hold concerning the smoothness of functions $\Psi(A)$, $G_1(A)$, $G_2(A)$ *and* $G(A)$*;*

(ii) there exist constants ε^-, ε^+ *and* ε^* *such that, for* $a \in \Lambda$ *and* $(r, z) \in \Omega$,

$$0 < \varepsilon^- < \varepsilon^* < 1, \qquad \varepsilon^+ > 2 - \varepsilon^-, \qquad |\Psi'(a)| < 1 - \varepsilon^*, \tag{3.22a}$$

$$\alpha_0^- > \varepsilon^- + \mu_1(\varepsilon^-, 1-\varepsilon^*)\left[\sigma_0'^2|B|^2 + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2}\right] + 2\mu_2(\varepsilon^-, 1-\varepsilon^*)\bar{\gamma}_0\beta_0(1+\varepsilon^*), \tag{3.22b}$$

$$\alpha_0^+ < \varepsilon^+ + \mu_1(\varepsilon^+, 1-\varepsilon^*)\left[\sigma_0'^2|B|^2 + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2}\right] - 2\mu_2(\varepsilon^+, 1-\varepsilon^*)\bar{\gamma}_0\beta_0(1+\varepsilon^*), \tag{3.22c}$$

Then the a priori estimate (2.20) holds, and the steady state (2.7) is nonlinearly stable to arbitrary finite-amplitude perturbations.

Proof. As with Criteria 2.1, 3.1 and 3.2, it is sufficient to show that the inequalities (2.21) and (2.22) are satisfied for all real a_0, a_1, a_2, a_3 and a_4. We therefore need to extend (continue) the functions $N_1(a)$, $N_2(a)$, $L(a)$ and $S(a)$ to all $a \notin \Lambda$ in such a way that these functions remain smooth enough and the inequalities

$$\alpha_0^- \leqslant \alpha(r, z; a) \leqslant \alpha_0^+, \tag{3.23a}$$

$$|\sigma(a)| = |N_1(a)| < 1 - \varepsilon^*, \tag{3.23b}$$

$$|\sigma'(a)| = |N_1'(a)| \leqslant \sigma_0', \tag{3.23c}$$

$$|\beta(a)| = |L'(a)| < \beta_0(1+\varepsilon^*), \tag{3.23d}$$

$$|\gamma(r, z; a)| \leqslant \bar{\gamma}_0, \tag{3.23e}$$

remain valid for all $a \notin \Lambda$.

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The extension of the definition of $N_1(a)$ and the proof of the inequalities (3.23b,c) are as in Criterion 3.2 (see (3.13)–(3.16) and Appendix D). The extension of the definition of $L(a)$ and the proof of (3.23d) are as in Criterion 3.1 (see (3.4)–(3.7) and Appendix C). The extension of the definitions of $N_2(a)$ and $L(a)$ and the proof of (3.23a,e) are as in Criterion 2.1 (see (2.45)–(2.80)). □

Similarly to Criterion 3.2, the analysis of the existence of positive constants ε^-, ε^+ and ε^* satisfying the conditions (3.22) shows that ε^- and ε^+ always exists for a given ε^*, provided that

$$\alpha_0^- > f_1(0) = \frac{1}{1-(1-\varepsilon^*)^2}\left[\sigma_0'^2|B|^2(r,z) + r^2\beta_0^2(1+\varepsilon^*)^2 + \frac{\bar{\gamma}_0^2}{r^2} + 2(1-\varepsilon^{*^2})\beta_0\bar{\gamma}_0\right] \tag{3.24}$$

for all $(r,z) \in \mathbf{\Omega}$.

Let $\delta \equiv 1-\varepsilon^*$; then we can formulate the following corollary.

Corollary 3.3. *The steady state (2.7) is nonlinearly stable to arbitrary axisymmetric perturbations provided that there exists a constant* $0 < \delta < 1$ *such that*

$$\frac{1}{\delta^2}|V|^2(r,z) < |B|^2(r,z) < \frac{1}{\sigma_0'^2}\left[(1-\delta^2)\alpha_0^- - 2\delta(2-\delta)\bar{\gamma}_0\beta_0 - r^2\beta_0^2(2-\delta)^2 - \frac{\bar{\gamma}_0^2}{r^2}\right], \tag{3.25}$$

for all $(r,z) \in \mathbf{\Omega}$.

3.1. *Purely poloidal field and flow*

Consider the steady state (2.7) when the toroidal component of the velocity field is absent ($R_1 = 0$). In this case,

$$V = \Psi'(a)B, \qquad R_1 = R_2 = 0. \tag{3.26}$$

The functions $G_1(A)$, $G_2(A)$ and $G(A)$ (given by (2.8)) reduce to

$$G_1(A) = G_2(A) = 0, \qquad G(A) = \Psi'(A)Q - J. \tag{3.27}$$

The functions α, β and γ (given by (2.13b–d) reduce to

$$\left.\begin{aligned}&\alpha(r,z;a) = -Q(r,z)\sigma'(a) + G'(a),\\&\beta = \gamma = 0.\end{aligned}\right\} \tag{3.28}$$

In this case, Criterion 3.3 simplifies to the following.

Criterion 3.4. *Suppose that:*

(i) the function $\Psi(A)$ *defined by (3.26) is a twice continuously differentiable function for all* $a \in \Lambda$;

(ii) the function $G(A)$ *defined by (3.27) is continuously differentiable for all* $a \in \Lambda$;

(iii) there exist constants ε^-, ε^+ *and* ε^* *such that, for* $a \in \Lambda$ *and* $(r,z) \in \mathbf{\Omega}$,

$$0 < \varepsilon^- < \varepsilon^* < 1, \quad \varepsilon^+ > 2-\varepsilon^-, \quad |\Psi'(a)| < 1-\varepsilon^*, \tag{3.29a}$$

$$\alpha_0^- > \varepsilon^- + \frac{1-\varepsilon^-}{(1-\varepsilon^-)^2-(1-\varepsilon^*)^2}\sigma_0'^2|B^2|(r,z), \tag{3.29b}$$

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$$\alpha_0^+ < \varepsilon^+ + \frac{1-\varepsilon^+}{(1-\varepsilon^+)^2 - (1-\varepsilon^*)^2}\sigma_0'^2|B^2|(r,z), \tag{3.29c}$$

Then the a priori estimate (2.20) holds, and the steady state (3.26) is nonlinearly stable to arbitrary finite-amplitude perturbations.

Criterion 3.4 coincides with Criterion 5.1 that was obtained by Valdimirov et al. (1997, p. 109). Note that the latter is a special case of Criterion 3.3.

4. Conclusions

In this paper we have used the general theory developed by Vladimirov and Moffatt (1995) to obtain nonlinear stability criteria for steady axisymmetric MHD flows of an ideal incompressible fluid (with respect to axisymmetric perturbations). In the unperturbed state, both velocity and magnetic field are non-zero and have in general both poloidal and toroidal components. Stability properties of such a steady state is of particular importance in the context of plasma confinement devices (e.g. tokamak or reversed-field pinch).

In Secs 2 and 3 we have used Arnold's (1965a,b) theory to obtain nonlinear stability criteria. Vladimirov et al. (1997) considered 'isomagnetic' perturbations, i.e. perturbations from the steady state under which the magnetic field is frozen and the vector potential of a material fluid particles is therefore conserved. They obtained sufficient conditions for nonlinear stability with respect to arbitrary initial perturbations (unconstrained by an isomagnetic condition) in a situation, when in the steady state only the poloidal parts of both velocity and magnetic field are non-zero.

In this paper, we have considered a situation when, in the steady state, all components of the velocity field are non-zero, while only the toroidal component of the magnetic field is absent. For this situation, we have succeeded in obtaining sufficient conditions for nonlinear stability. A criterion obtained by Vladimirov et al. (1997) can be deduced again as a special case of our Criterion 3.4. Our nonlinear stability Criteria 2.1 and 3.1–3.3 evidently cover a wider class of steady MHD states than that obtained by Vladimirov et al. (1997).

Appendix A. Extension of the definition of the function $N_1(a)$ to all $a < A^-$

Suppose that (2.18a) is true; then three different situations are possible for $N_1'(A^-)$:

(i) $N_1'(A^-) > 0$;

(ii) $N_1'(A^-) < 0$;

(iii) $N_1'(A^-) = 0$.

(i) If $N_1'(A^-) > 0$ then we define $N_1(a)$ for $a < A^-$ such that

$$N_1(a) = N_1(A^-) - N_1'(A^-)\frac{z}{1+\chi_1 z}, \tag{A 1}$$

where

$$z \equiv A^- - a, \qquad \chi_1 \equiv \frac{N_1'(A^-)}{\sigma_0 + N_1(A^-)}. \tag{A 2}$$

Note from (2.18a) that $\chi_1 > 0$. It is easy to see that, with this definition, $N_1(a)$ is continuously differentiable for all $a \leqslant A^-$ and satisfies (2.31b,c):

$$N_1'(a) = N_1'(A^-)\frac{1}{(1+\chi_1 z)^2} \leqslant N_1'(A^-), \qquad a \leqslant A^-. \tag{A 3}$$

(ii) If $N_1'(A^-) < 0$ then we define $N_1(a)$ for $a < A^-$ such that

$$N_1(a) = N_1(A^-) - N_1'(A^-)\frac{z}{1+\chi_2 z}, \tag{A 4}$$

where

$$z \equiv A^- - a, \qquad \chi_2 \equiv -\frac{N_1'(A^-)}{\sigma_0 - N_1(A^-)} > 0. \tag{A 5}$$

it is also easy to verify that, with this choice, $N_1(a)$ is continuously differentiable for all $a \leqslant A^-$, and satisfies (2.31b,c):

$$N_1'(a) = N_1'(A^-)\frac{1}{(1+\chi_2)^2} \geqslant N_1'(A^-), \qquad a \leqslant A^-. \tag{A 6}$$

(iii) If $N_1'(A^-) = 0$ then we take $N_1(a)$ for $a < A^-$ such that $N_1(a) = N_1(A^+)$, so that (2.31b,c) are satisfied.

Suppose now that (2.18b) is true; then, as in case (iii), we take $N_1(a) = N_1(A^-)$, so that (2.31b,c) are satisfied.

Appendix B. Extension of the definition of the function $L(a)$ to all $a < A^-$

Suppose that (2.19a) is true; then three different situations are possible for $L''(A^-)$:

(i) $L''(A^-) > 0$;

(ii) $L''(A^-) < 0$;

(iii) $L''(A^-) = 0$.

(i) If $L''(A^-) > 0$ then we define $L(a)$ for $a < A^-$ such that

$$L'(a) = L'(A^-) - L''(A^-)\frac{z}{1+\chi_3 z}, \tag{B 1}$$

where

$$z \equiv A^- - a, \qquad \chi_3 \equiv \frac{L''(A^-)}{\beta_0 + L'(A^-)}. \tag{B 2}$$

Note from (2.19a) that $\chi_3 > 0$.

(ii) If $L''(A^-) < 0$ then we define $L(a)$ for $a < A^-$ such that

$$L'(a) = L'(A^-) - L''(A^-)\frac{z}{1+\chi_4 z}, \tag{B 3}$$

where

$$z \equiv A^- - a, \qquad \chi_4 \equiv -\frac{L''(A^-)}{\beta_0 - L'(A^-)} > 0. \tag{B 4}$$

(iii) If $L''(A^+) = 0$ then we take $L(a)$ for $a < A^-$ such that $L'(a) = L'(A^-)$.

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In the three cases (i)–(iii), it is easy to verify that $L(a)$ is twice continuously differentiable for all $a \leqslant A^-$ and satisfies (2.31d). Suppose now that (2.19b) is true; then, as in case (ii), we take $L'(a) = L'(A^+)$, and the inequality (2.31d) is satisfied.

Appendix C. Extension of the definition of the function $L(a)$ to all $a < A^-$

Three different situations are possible for $L''(A^-)$:

(i) $L''(A^-) > 0$;

(ii) $L''(A^-) < 0$;

(iii) $L''(A^-) = 0$.

(i) If $L''(A^-) > 0$ then we define $L(a)$ for $a < A^-$ such that

$$L'(a) = L'(A^-) - L''(A^-)\frac{z}{1+\eta_1 z}, \tag{C 1}$$

where

$$z \equiv A^- - a, \qquad \eta_1 \equiv \frac{L''(A^-)}{\beta_0(1+\varepsilon^*) + L'(A^-)} > 0. \tag{C 2}$$

(ii) If $L''(A^-) < 0$ then we define $L(a)$ for $a < A^-$ such that

$$L'(a) = L'(A^-) - L''(A^-)\frac{z}{1+\eta_2 z}, \tag{C 3}$$

where

$$z \equiv A^- - a, \qquad \eta_2 \equiv -\frac{L''(A^-)}{\beta_0(1+\varepsilon^*) - L'(A^-)} > 0. \tag{C 4}$$

(iii) If $L''(A^-) = 0$ then we take $L(a)$ for $a < A^-$ such that $L'(a) = L'(A^-)$,

In the three cases (i)–(iii), it is easy to verify that $L(a)$ is twice continuously differentiable for all $a \leqslant A^-$ and satisfies (3.3d).

Appendix D. Extension of the definition of the function $N_1(a)$ to all $a < A^-$

Three different situations are possible for $N_1'(A^-)$:

(i) $N_1'(A^-) > 0$;

(ii) $N_1'(A^-) < 0$;

(iii) $N_1'(A^-) = 0$.

(i) If $N_1'(A^-) > 0$ then we define $N_1(a)$ for $a < A^-$ such that

$$N_1(a) = N_1(A^-) - N_1'(A^-)\frac{z}{1+\zeta_1 z}, \tag{D 1}$$

where

$$z \equiv A^- - a, \qquad \zeta_1 \equiv \frac{N_1'(A^-)}{1-\varepsilon^* + N_1(A^-)} > 0. \tag{D 2}$$

(ii) If $N_1'(A^-) < 0$ then we define $N_1(a)$ for $a < A^-$ such that

$$N_1(a) = N_1(A^-) - N_1'(A^-)\frac{z}{1+\zeta_2 z}, \tag{D 3}$$

where

$$z \equiv A^- - a, \qquad \zeta_2 \equiv -\frac{N_1'(A^-)}{1-\varepsilon^* - N_1(A^-)} > 0. \tag{D 4}$$

(iii) If $N_1'(A^-) = 0$ then we take $N_1(a)$ for $a < A^-$ such that $N_1(a) = N_1(A^-)$,

References

Almaguer, J. A., Hameiri, E., Herrera, J. and Hohn, D. D. 1988 Lyapunov stability analysis of magnetohydrodynamic plasma equilibria with axisymmetric toroidal flow. *Phys. Fluids* **31**, 1930–1939.

Arnold, V. I. 1965a On the conditions of nonlinear stability of plane curvilinear flows of an ideal fluid. *Dokl. Acad. Nauk SSSR* **162**, 975–978 [English translation: *Soviet Maths* **6**, 773–777].

Arnold, V. I. 1965b A variational principle for three dimensional steady flows of an ideal fluid. *J. Appl. Maths Mech.* **29**, 1002–1008.

Bernstein, I. B., Frieman, E. A., Kruskal, M. D. and Kulsrud, R. M. 1958 An energy principle for hydromagnetic stability problems. *Proc. R. Soc. Lond.* **A244**, 17–40.

Friedlander, S. and Vishik, M. M. 1990 Nonlinear stability for stratified magnetohydrodynamics. *Geophys. Astrophys. Fluid Dyn.* **55**, 19–45.

Frieman, F. and Rotenberg, M. 1960 On hydromagnetic stability of stationary equilibria. *Rev. Mod. Phys.* **32**, 898–902.

Moffatt, H. K. 1989 On the existence, structure and stability of MHD equilibrium states. In: *Turbulence and Nonlinear Dynamics in MHD Flows* (ed. M. Meneguzzi, A. Pouquet and P. L. Sulem). Amsterdam: Elsevier, pp. 185–195.

Saffman, P. G. 1992 *Vortex Dynamics*. Cambridge University Press.

Vladimirov, V. A. and Moffatt, H. K. 1995 On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part I. Fundamental principles. *J. Fluid Mech.* **283**, 125–139.

Vladimirov, V. A., Moffatt, H. K. and Ilin, K. I. 1996 On general transformations and variational principles for the magnetohydrodynamics of ideal fluids. Part II. Stability criteria for two-dimensional flows. *J. Fluid Mech.* **329**, 187–205.

Vladimirov, V. A., Moffatt, H. K. and Ilin, K. I. 1997 On general transformations and variational principles for the magnetohydrodynamic of ideal fluids. Part III. Stability criteria for axisymmetric flows. *J. Plasma Phys.* **57**, 89–120.

Voslamber D. and Callebaut, D. K. 1962 Stability of force-free magnetic fields. *Phys. Rev.* **128**, 2016.