Macroeconomic Dynamics, **3**, 1999, 48–68. Printed in the United States of America.

INDEX NUMBER APPROACHES TO SEASONAL ADJUSTMENT

W. Erwin Diewert
University of British Columbia

A seasonal commodity is one that either (1) is not available during certain seasons or (2) is always available but its prices or quantities fluctuate with the season or time of year. The existence of type-1 seasonal commodities in consumer preference functions means that the usual economic approach to index number theory cannot be applied to construct a short-term month-to-month or quarter-to-quarter consumer price index. We postulate various separability assumptions on intertemporal preferences that can be used to justify various seasonal index number formulas. One of our approaches leads to an index number solution to the problem of seasonal adjustment.

Keywords: Aggregation of Commodities, Consumer Theory, Index Numbers, Inflation, Seasonal Adjustment, Separability, Time Series

1. INTRODUCTION

The problem of index number construction when there are seasonal commodities has a long history.[1] However, this index-number literature generally has not been based on the economic approach to index number theory. Hence, we follow up on a previous economic approach (Diewert, 1998) and postulate separability assumptions on intertemporal preferences that can be used to justify various seasonal index-number formulas from the viewpoint of the economic approach to index number theory.[2]

We now set out the general model of consumer behavior that we specialize in subsequent sections. Suppose that there are M seasons in the year and the statistical agency has collected price and quantity data on the consumer's purchases for $1 + T$ years. Suppose further that the dimension of the commodity space in each season remains constant over the $T + 1$ years; i.e., season m has N_m commodities for $m = 1, \ldots, M$. For season m of year t, we denote the vector of positive prices facing the consumer by $\boldsymbol{p}^{tm} \equiv [p_1^{tm}, p_2^{tm}, \ldots, p_{N_m}^{tm}]$ and the vector of commodities consumed in by $\boldsymbol{q}^{tm} \equiv [q_1^{tm}, q_2^{tm}, \ldots, q_{N_m}^{tm}]$. It will prove convenient to have notation for the annual price and quantity vectors, and so, we

The author thanks Bert Balk, Jeff Bernstein, Peter Hill, and Alice Nakamura for helpful comments, William Barnett for extremely helpful editorial assistance, and Tina Corsi for her help in preparing the manuscript. This research was supported by the Canadian Donner Foundation. Address correspondence to: W.E. Diewert, Department of Economics, University of British Columbia, #997-1873 East Mall, Vancouver, BC V6T 1Z1, Canada; e-mail: diewert@econ.ubc.ca.

© 1999 Cambridge University Press 1365-1005/99 $9.50

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

define these by

$$\boldsymbol{p}^t \equiv [\boldsymbol{p}^{t1}, \boldsymbol{p}^{t2}, \ldots, \boldsymbol{p}^{tM}]; \qquad \boldsymbol{q}^t \equiv [\boldsymbol{q}^{t1}, \boldsymbol{q}^{t2}, \ldots, \boldsymbol{q}^{tM}]; \qquad t = 0, 1, \ldots, T. \tag{1}$$

To apply the economic approach to index number theory, it is necessary to assume that the observed quantities of q_n^{tm} are a solution to an optimization problem involving the observed prices p_n^{tm}. Following Fisher (1930) and Hicks (1946; pp. 121–126), assume that the intertemporal quantity vector $[\boldsymbol{q}^0, \boldsymbol{q}^1, \ldots, \boldsymbol{q}^T]$ is a solution to the following intertemporal utility maximization problem:

$$\max_{\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^T} \left\{ U(\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^T) : \sum_{t=0}^{T} \sum_{m=1}^{M} \delta_t \rho_{tm} \boldsymbol{p}^{tm} \cdot \boldsymbol{x}^{tm} = W \right\} \tag{2}$$

where $\boldsymbol{x}^t \equiv [\boldsymbol{x}^{t1}, \boldsymbol{x}^{t2}, \ldots, \boldsymbol{x}^{tm}]$ and each seasonal quantity vector $\boldsymbol{x}^{tm}$ has the dimensionality of $\boldsymbol{q}^{tm}$, $\boldsymbol{p}^{tm} \cdot \boldsymbol{x}^{tm} \equiv \sum_{n=1}^{N_m} p_n^{tm} x_n^{tm}$; U is the consumer's intertemporal preference function (assumed to be continuous and increasing); $\delta_t > 0$ is an annual discount factor; ρ_{tm} is a within-year discount rate[3] that will make a dollar at the beginning of year t equivalent to a dollar in the middle of season m of year t and wealth W is the consumer's current and expected future discounted income viewed from the perspective of the beginning of year 0. If the consumer can borrow and lend at a constant annual nominal interest rate r, then $\delta_0 \equiv 1$ and

$$\delta_t = 1/(1+r)^t, \qquad t = 1, 2, \ldots, T. \tag{3}$$

Because we are assuming that the quantity vector $[\boldsymbol{q}^0, \boldsymbol{q}^1, \cdots, \boldsymbol{q}^T]$ is a solution to (2), it must satisfy the intertemporal budget constraint in (2) so that we can replace W by

$$W \equiv \sum_{t=0}^{T} \sum_{m=1}^{M} \delta_t \rho_{tm} \boldsymbol{p}^{tm} \cdot \boldsymbol{q}^{tm}. \tag{4}$$

Having made our basic economic assumptions [namely, that the observed sequence of annual quantity vectors $[\boldsymbol{q}^0, \boldsymbol{q}^1, \ldots, \boldsymbol{q}^t]$ solves (2) with W defined by (4)], the remainder of the paper makes additional assumptions on the structure of the intertemporal utility function U.

In Section 2, we make separability assumptions on the intertemporal utility function U that are sufficient to justify year-over-year seasonal price and quantity indexes; i.e., the price and quantity data pertaining to January of the current year are compared to the January price and quantity data of a base year. In Section 3, we get into the heart of the seasonal aggregation problem and consider methods for obtaining valid season-to-season measures of price change when there are seasonal commodities. In Section 4, we consider how to extend the scope of the annual calendar-year indexes considered previously (Diewert, 1998) to moving-year comparisons. In Section 5, we indicate how the moving-year indexes of Section 4 can be centered. These centered indexes provide an index number solution to the problem of seasonal adjustment. Section 6 concludes.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

2. YEAR-OVER-YEAR SEASONAL INDEXES

To justify the Mudgett (1955, p. 97) and Stone (1956, pp. 74–75) approach to annual index numbers when there are seasonal commodities, Diewert (1998) assumed the following restrictions on the consumer's intertemporal utility function U: There exist F and f such that

$$U(\boldsymbol{x}^0, \boldsymbol{x}^1, \ldots, \boldsymbol{x}^T) = F[f(\boldsymbol{x}^0), f(\boldsymbol{x}^1), \ldots, f(\boldsymbol{x}^T)], \qquad (\mathbf{5})$$

where f was a linearly homogeneous, increasing, and concave annual utility function and F was an intertemporal utility function that was increasing and continuous in its $T+1$ annual utility arguments. The annual utility function f treats each good in each season as a separate commodity.

To justify the existence of year-over-year seasonal indexes (e.g., January of the current year is compared to January of a base year), it is necessary to make further separability assumptions on the annual utility function f: There exists an increasing continuous function h of M variables and there exist functions f^m of N_m variables, $m = 1, \ldots, M$, such that

$$f(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M) = h[f^1(\boldsymbol{x}^1), \ldots, f^M(\boldsymbol{x}^M)], \qquad (\mathbf{6})$$

where the seasonal aggregators $f^m(\boldsymbol{x}^m)$ are increasing, linearly homogeneous, and concave.

Assumption (6) says that the annual aggregator f has a more restrictive form that aggregates the seasonal vectors $\boldsymbol{x}^m$ in two stages. In the first, the commodities in season m, $\boldsymbol{x}^m \equiv [x_1^m, x_2^m, \ldots, x_{N_m}^m]$ are aggregated by the season-specific utility function $f^m(\boldsymbol{x}^m) \equiv u_m$ and then the seasonal utilities u_m are aggregated in the second stage by h to form annual utility $u \equiv h(u_1, u_2, \ldots, u_M)$.

Substituting (5) and (6) into (2) and using the assumption that h and F are increasing in their arguments yields[4]

$$\max_{\boldsymbol{x}^m}\{f^m(\boldsymbol{x}^m) : \boldsymbol{p}^{tm} \cdot \boldsymbol{x}^m = \boldsymbol{p}^{tm} \cdot \boldsymbol{q}^{tm}\} = f^m(\boldsymbol{q}^{tm});$$
$$t = 0, 1, \ldots, T; \qquad m = 1, \ldots, M. \qquad (\mathbf{7})$$

Let the unit-cost dual c^m to the seasonal aggregator function f^m be defined by

$$c^m(p^m) \equiv \min_{x^m}\{\boldsymbol{p}^m \cdot \boldsymbol{x}^m : f^m(\boldsymbol{x}^m) = 1\}; \qquad m = 1, \ldots, M. \qquad (\mathbf{8})$$

Let P^m and Q^m be price and quantity indexes that are exact for the season m aggregator function f^m. Then, under our optimizing assumptions, we have the following equalities, applying the usual theory of exact index numbers, for $0 \leq s$, $t \leq T$ and $m = 1, \ldots, M$:

$$f^m(\boldsymbol{q}^{tm})/f^m(\boldsymbol{q}^{sm}) = Q^m(\boldsymbol{p}^{sm}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{sm}, \boldsymbol{q}^{tm}); \qquad (\mathbf{9})$$

$$c^m(\boldsymbol{p}^{tm})/c^m(\boldsymbol{p}^{sm}) = P^m(\boldsymbol{p}^{sm}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{sm}, \boldsymbol{q}^{tm}). \qquad (\mathbf{10})$$

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

Equation (9) says that the ratio of seasonal utility in season m of year t to seasonal utility in the same season m of year s is equal to the quantity index $Q^m(\boldsymbol{p}^{sm}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{sm}, \boldsymbol{q}^{tm})$, which is a function of the nominal price vectors for season m of years s and t, $\boldsymbol{p}^{sm}$ and $\boldsymbol{p}^{tm}$, and the observed quantity vectors for season m of years s and t, $\boldsymbol{q}^{sm}$ and $\boldsymbol{q}^{tm}$. If the seasonal aggregator functions are chosen to be the flexible homogeneous quadratic functions $f^m(\boldsymbol{x}^m) \equiv [\boldsymbol{x}^m \cdot A^m \boldsymbol{x}^m]^{1/2}$, where A^m is a square symmetric matrix of constants for $m = 1, \ldots, M$, then the corresponding exact Q^m and P^m will be the superlative Fisher ideal indexes Q_F^m and P_F^m for $m = 1, \ldots, M$. Equation (10) tells us that the theoretical Konüs price index for season m between years s and t, $c^m(\boldsymbol{p}^{tm})/c^m(\boldsymbol{p}^{sm})$, is exactly equal to the price index $P^m(\boldsymbol{p}^{sm}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{sm}, \boldsymbol{q}^{tm})$, which in turn will equal the Fisher ideal price index $P_F^m(\boldsymbol{p}^{sm}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{sm}, \boldsymbol{q}^{tm})$ if f^m is the homogeneous quadratic aggregator function defined above.

Note that the nominal price vectors for season m in s and t, $\boldsymbol{p}^{sm}$ and $\boldsymbol{p}^{tm}$, appear in (10). Thus the index number on the right-hand side of (10) is a valid indicator of the amount of nominal price change that has occurred in going from season m of year s to the same season m in year t.

Summarizing the results of this section, we have shown how the separability assumptions (5) and (6) justify the use of the year-over-year seasonal price and quantity indexes that appeared in (9) and (10). These year-over-year seasonal indexes have been proposed by Flux (1921; p. 184), Zarnowitz (1961; p. 266), and many others[5] but explicit economic justifications for these indexes seem to be lacking.

The year-over-year seasonal indexes defined by (9) and (10) can be aggregated further into an annual index. If the Paasche or Laspeyres index number formula is used at each stage of the two-stage aggregation procedure, then the two-stage indexes will coincide with single-stage annual Mudgett-Stone indexes.[6] If a superlative index number formula is used in (9) and (10) and the same superlative formula is used in the second stage to aggregate the year-over-year monthly information into an annual index, then this two-stage approach will approximate the corresponding single-stage superlative Mudgett-Stone annual index to the second order.[7]

We turn now to the difficult problem of making index-number comparisons between seasons within the same year when there are seasonal commodities.

3. SHORT-TERM SEASON-TO-SEASON INDEXES

It is important to have reliable short-term inflation measures for indexation, wage negotiations, calculation of real rates of return, etc. Thus we need to be able to compare the price level of the current season with those of immediately preceding seasons. The annual Mudgett-Stone price indexes defined by Diewert (1998) are not suitable for this, nor are the year-over-year seasonal indexes defined by (10), since they are not comparable over seasons or months m because the commodity baskets change over the seasons due to the existence of type-1 seasonal commodities,

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

i.e., commodities that are not available in each season. To make this lack-of-comparability problem clearer, we make the separability assumption (6) on the annual utility function f. We assume that the season m aggregator f^m has the unit-cost dual c^m, and P^m is an exact bilateral price index for f^m. Setting $s = 0$, equations (10) become

$$c^m(\boldsymbol{p}^{tm})/c^m(\boldsymbol{p}^{0m}) = P^m(\boldsymbol{p}^{0m}, \boldsymbol{p}^{tm}, \boldsymbol{q}^{0m}, \boldsymbol{q}^{tm}); \quad t = 1, \ldots, T; \qquad m = 1, \ldots, M. \quad \textbf{(11)}$$

We can interpret $c^m(\boldsymbol{p}^{tm})$ as the price or unit cost of one unit of season m subutility in year t, but there is no way of comparing these subutilities across seasons. Thus, equations (11) are of no help in obtaining comparable (across-season) price indexes.

A solution to this lack-of-comparability problem is to make a different separability assumption. We now partition the year t season m price vector $\boldsymbol{p}^{tm}$ into $[\tilde{\boldsymbol{p}}^{tm}, \hat{\boldsymbol{p}}^{tm}]$, where the commodities represented in $\tilde{p}^{tm}$ are nonseasonal and the commodities represented in $\hat{\boldsymbol{p}}^{tm}$ are seasonal. We partition the quantity vectors in a similar manner: i.e., $\boldsymbol{q}^{tm} \equiv [\tilde{\boldsymbol{q}}^{tm}, \hat{\boldsymbol{q}}^{tm}]$ and $\boldsymbol{x}^{tm} \equiv [\tilde{\boldsymbol{x}}^{tm}, \hat{\boldsymbol{x}}^{tm}]$ for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$. We now assume that the intertemporal utility function U introduced in Section 1 has the following structure: There exists an increasing, continuous function G and an increasing, linearly homogeneous, and concave function ϕ such that

$$\begin{aligned} &U(\boldsymbol{x}^{01}, \ldots, \boldsymbol{x}^{0M}; \ldots; \boldsymbol{x}^{T1}, \ldots, \boldsymbol{x}^{TM}) \\ &\quad = G[\phi(\tilde{\boldsymbol{x}}^{01}), \hat{\boldsymbol{x}}^{01}, \ldots, \phi(\tilde{\boldsymbol{x}}^{0M}), \hat{\boldsymbol{x}}^{0M}; \ldots; \phi(\tilde{\boldsymbol{x}}^{T1}), \hat{\boldsymbol{x}}^{T1}, \ldots, \phi(\tilde{\boldsymbol{x}}^{TM}), \hat{\boldsymbol{x}}^{TM}]. \end{aligned} \quad \textbf{(12)}$$

Because the monthly utility function ϕ is defined over nonseasonal goods, assumption (12) allows us to justify comparable monthly indexes.

Using our new notation for $\boldsymbol{p}^{tm} \equiv [\tilde{\boldsymbol{p}}^{tm}, \hat{\boldsymbol{p}}^{tm}]$, $\boldsymbol{x}^{tm} \equiv [\tilde{\boldsymbol{x}}^{tm}, \hat{\boldsymbol{x}}^{tm}]$, and $\boldsymbol{q}^{tm} \equiv [\tilde{\boldsymbol{q}}^{tm}, \hat{\boldsymbol{q}}^{tm}]$, we can rewrite the consumer's intertemporal budget constraint as

$$\sum_{t=0}^{T}\sum_{m=1}^{M} \delta_t \rho_{tm}[\tilde{\boldsymbol{p}}^{tm} \cdot \tilde{\boldsymbol{x}}^{tm} + \hat{\boldsymbol{p}}^{tm} \cdot \hat{\boldsymbol{x}}^{tm}] = \sum_{t=0}^{T}\sum_{m=1}^{M} \delta_t \rho_{tm}[\tilde{\boldsymbol{p}}^{tm} \cdot \tilde{\boldsymbol{q}}^{tm} + \hat{\boldsymbol{p}}^{tm} \cdot \hat{\boldsymbol{q}}^{tm}]. \quad \textbf{(13)}$$

As usual, we assume that $[\boldsymbol{q}^0, \boldsymbol{q}^1, \ldots, \boldsymbol{q}^T]$ solves the intertemporal utility maximization problem when U is defined by (12) and the budget constraint is defined by (13), where the year t observed quantity vector is $\boldsymbol{q}^t \equiv [\boldsymbol{q}^{t1}, \ldots, \boldsymbol{q}^{tm}]$ and the year t season m quantity vector is $\boldsymbol{q}^{tm} \equiv [\tilde{\boldsymbol{q}}^{tm}, \hat{\boldsymbol{q}}^{tm}]$. Using the assumptions that G and ϕ are increasing in their arguments, we can deduce that[8]

$$\max_{\tilde{\boldsymbol{x}}^{tm}}\{\phi(\tilde{\boldsymbol{x}}^{tm}) : \tilde{\boldsymbol{p}}^{tm} \cdot \tilde{\boldsymbol{x}}^{tm} = \tilde{\boldsymbol{p}}^{tm} \cdot \tilde{\boldsymbol{q}}^{tm}\} = \phi(\tilde{\boldsymbol{q}}^{tm}); \quad t = 0, 1, \ldots, T; \qquad m = 1, \ldots, M. \quad \textbf{(14)}$$

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

We let $\gamma(\tilde{\boldsymbol{p}}^{tm})$ be the unit-cost function that is dual to the short-run aggregator function ϕ. We assume that the bilateral price and quantity indexes $\tilde{P}$ and $\tilde{Q}$ are exact for the aggregator function ϕ. Then, the equalities (14) imply the following equalities for $0 \le s, t \le T$; $m = 1, \ldots, M$ and $j = 1, 2, \ldots, M$:

$$\phi(\tilde{\boldsymbol{q}}^{tm})/\phi(\tilde{\boldsymbol{q}}^{sj}) = \tilde{Q}(\tilde{\boldsymbol{p}}^{sj}, \tilde{\boldsymbol{p}}^{tm}, \tilde{\boldsymbol{q}}^{sj}, \tilde{\boldsymbol{q}}^{tm}); \quad \textbf{(15)}$$

$$\gamma(\tilde{\boldsymbol{p}}^{tm})/\gamma(\tilde{\boldsymbol{p}}^{sj}) = \tilde{P}(\tilde{\boldsymbol{p}}^{sj}, \tilde{\boldsymbol{p}}^{tm}, \tilde{\boldsymbol{q}}^{sj}, \tilde{\boldsymbol{q}}^{tm}). \quad \textbf{(16)}$$

We normalize the theoretical monthly price-level function $\gamma(\tilde{\boldsymbol{p}}^{tm})$ so that the seasonal price level in season 1 of year 0 is unity; i.e., we place the following restriction on γ:

$$\gamma(\tilde{\boldsymbol{p}}^{01}) = 1. \quad \textbf{(17)}$$

Equations (16) and the normalization (17) allow us to use the exact bilateral index-number formula $\tilde{P}$ to provide estimates for the theoretical short-term seasonal price levels $\gamma(\tilde{\boldsymbol{p}}^{tm})$. The fixed base sequence of short-term inflation estimates is

$$1,\ \tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{02}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{02}), \ldots, \tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{0M}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{0M}); \ldots;$$

$$\tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{T1}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{T1}), \ldots, \tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{TM}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{TM}). \quad \textbf{(18)}$$

Using the chain principle, the sequence of short-run inflation estimates is

$$1,\ \tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{02}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{02}),\ \tilde{P}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{02}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{02})\tilde{P}(\tilde{\boldsymbol{p}}^{02}, \tilde{\boldsymbol{p}}^{03}, \tilde{\boldsymbol{q}}^{02}, \tilde{\boldsymbol{q}}^{03}), \ldots. \quad \textbf{(19)}$$

The first two numbers in the chain sequence (19) coincide with the first two numbers in the fixed base sequence (18) but then the chain estimate for a given year t and month $m+1$ is equal to the chain estimate for the immediately preceding month m times the month-to-month bilateral link, $\tilde{P}(\tilde{\boldsymbol{p}}^{m}, \tilde{\boldsymbol{p}}^{m+1}, \tilde{\boldsymbol{q}}^{m}, \tilde{\boldsymbol{q}}^{m+1})$. There are other ways of utilizing the exact index-number bilateral relationship defined by (16) to obtain estimates for the sequence of month-to-month theoretical price levels

$$\gamma(\tilde{\boldsymbol{p}}^{01}), \gamma(\tilde{\boldsymbol{p}}^{02}), \ldots, \gamma(\tilde{\boldsymbol{p}}^{0M}); \ldots; \gamma(\tilde{\boldsymbol{p}}^{T1}), \gamma(\tilde{\boldsymbol{p}}^{T2}), \ldots, \gamma(\tilde{\boldsymbol{p}}^{TM}), \quad \textbf{(20)}$$

but the fixed-base and chain methods are the most practical ones.[9]

Although our focus in this section is on measuring short-term price change using the bilateral price index $\tilde{P}$, we also can use the companion quantity index $\tilde{Q}$ to measure short-term quantity change for nonseasonal quantities. Furthermore, the exact index-number relations (15), along with a base-period normalization such as

$$\phi(\tilde{\boldsymbol{q}}^{01}) = \tilde{\boldsymbol{p}}^{01} \cdot \tilde{\boldsymbol{q}}^{01}, \quad \textbf{(21)}$$

which sets season-1 utility in the base year 0 equal to expenditure on nonseasonal goods $\tilde{\boldsymbol{p}}^{01} \cdot \tilde{\boldsymbol{q}}^{01}$, can be used to form estimates for annual sums of seasonal utilities. If we define year t aggregate utility by $\sum_{m=1}^{M} \phi(\tilde{\boldsymbol{q}}^{tm})$, then using the fixed-base principle, we can estimate this theoretical real-quantity aggregate in units of season

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

1 year 0 constant dollars by

$$\left[\sum_{m=1}^{M} \tilde{Q}(\tilde{\boldsymbol{p}}^{01}, \tilde{\boldsymbol{p}}^{tm}, \tilde{\boldsymbol{q}}^{01}, \tilde{\boldsymbol{q}}^{tm})\right] \tilde{\boldsymbol{p}}^{01} \cdot \tilde{\boldsymbol{q}}^{01}. \tag{22}$$

The reader can work out chain system or multilateral estimates for the year t utility aggregate. However, annual quantity estimates of the form (22) will be of limited interest because of the exclusion of seasonal goods. To obtain comprehensive estimates, it will be necessary to use the Mudgett-Stone indexes described earlier (Diewert, 1998).

Our discussion can be summarized as follows:

1. A month-to-month Fisher ideal chain price index of nonseasonal commodities is our preferred alternative; see (19) with $\tilde{P} \equiv \tilde{P}_F$.
2. If quantity information is not available in a timely manner, fixed-base Laspeyres price indexes will have to be used; i.e., (18) will have to be used with $\tilde{P} \equiv \tilde{P}_L$. However, the base period should be changed as frequently as possible.

Some seasonal bilateral index-number procedures that work over commodity spaces of varying dimensions have been proposed by Diewert, (1980, pp. 506–508) and Balk, (1980a, p. 27; 1981). We now review those proposals and compare them to our preferred proposal (19), which depends on the separability assumptions (12).

Diewert (1980, p. 507) attempted to deal with the problem of disappearing and then reappearing seasonal goods by utilizing Hicks' (1940, p. 114) treatment of new goods: In seasons when a good is unavailable, determine the reservation price that would just ration the consumer's demand for the good down to zero. These reservation prices, along with the associated zero quantities, then could be used as prices and quantities that could be inserted into a bilateral season-to-season index-number formula. There are two problems with this proposal: (1) statistical agencies do not have the resources required to estimate these reservation prices[10] and (2) even if appropriate reservation prices could be estimated, the assumptions required to justify the economic approach generally would not be satisfied because some seasonal commodities cannot have their prices and quantities rationalized by maximizing an underlying utility aggregator function over the seasons since custom shifts the aggregator function over the seasons.

Balk's (1980a, p. 27; 1981) proposal for dealing with type (i) seasonal commodities makes use of the Vartia II price index (Vartia, 1976) so it is necessary to define this index. First, we define the logarithmic mean of two positive numbers, x and y, as

$$L(x, y) \equiv \begin{cases} [x - y]/[\ln x - \ln y] & \text{if } x \neq y \\ x & \text{if } x = y. \end{cases} \tag{23}$$

Balk (1981, p. 73) observed that (23) could be extended to the case in which one of the numbers x or y is zero and the other is positive so that $L(x, y) = 0$ in this case. To define the Vartia II price index, we let $\boldsymbol{p}^t$ and $\boldsymbol{q}^t$ be two generic price and quantity vectors pertaining to periods $t = 0, 1$. We define the period t expenditure

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

share on commodity n as

$$w_n^t \equiv p_n^t q_n^t / \boldsymbol{p}^t \cdot \boldsymbol{q}^t; \qquad t = 0, 1; \qquad n = 1, \ldots, N. \tag{24}$$

We define the logarithmic mean average share for commodity n between periods 0 and 1 by

$$w_n^{01} \equiv \begin{cases} L\left(w_n^0, w_n^1\right) & \text{if at least one of } w_n^0, w_n^1 \text{ is positive} \\ 0 & \text{if both } w_n^0 \text{ and } w_n^1 \text{ are } 0. \end{cases} \tag{25}$$

Finally, we define the Sato (1976) and Vartia (1976) price index P_{SV} as

$$\ln P_{SV}(\boldsymbol{p}^0, \boldsymbol{p}^1, \boldsymbol{q}^0, \boldsymbol{q}^1) \equiv \sum_{n=1}^{N} w_n^{01} \ln\left(p_n^1 / p_n^0\right) \Big/ \sum_{n=1}^{N} w_n^{01}. \tag{26}$$

Balk (1995) and Reinsdorf and Dorfman (1995) have studied the axiomatic properties of the Sato-Vartia price index and their conclusion is that it almost rivals the Fisher ideal index. Furthermore, the fact that the Sato-Vartia index is exact for constant elasticity of substitution (CES) functional forms has proved useful in empirical applications; e.g., see Feenstra (1994). However, it should be pointed out that the Sato-Vartia price index P_{SV} defined by (26) is *not* superlative; i.e., it is not exact for an aggregator function that can provide a second-order approximation to an arbitrary twice-differentiable linearly homogeneous function when the number of commodities N exceeds 2.

We now return to Balk's (1980a, p. 27; 1981) proposal for dealing with seasonal commodities. His method works as follows: When comparing type-1 commodities between two seasons when both are absent from the marketplace, drop the commodity from the index-number computation; for all other cases, use the Sato-Vartia price index. This procedure will set the weight of the commodity equal to zero if it is not present in both periods. Obviously, another way of describing this method is to use what Keynes (1930, p. 94) called the highest-common-factor method and use the Sato-Vartia price index as the index-number formula.

Balk's approach to the treatment of type-1 seasonal commodities is satisfactory from the viewpoint of the test approach to index number theory but there are two problems versus the economic approach: (1) Because the Sato-Vartia index is not superlative, it would be better to apply the highest-common-factor method but use a superlative price index in place of the Sato-Vartia index[11] and (2) Balk's procedure ignores the existence of seasonal commodities whose demands cannot be rationalized by maximizing an unchanging monthly utility function. The prices and quantities corresponding to these seasonal commodities cannot be rationalized by utility-maximizing behavior where the utility function remains constant over the two periods in question because custom shifts the demand over seasons for these seasonal commodities. The above second criticism of Balk's proposed procedure can be applied to Diewert's (1980, p. 507) economic approach to seasonal indexes and the cure to this problem is the same: Restrict the season-to-season index-number comparisons to nonseasonal commodities as we have done in this section.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

To conclude this section, we note that many consumer goods are durable; i.e., they provide services beyond the initial season of purchase. Hence, from the viewpoint of the economic approach to the short-term consumer price index, seasonal rental prices or user costs should be used as the prices for durable consumer goods and the quantity weights should reflect not only the purchases made during the season but also the available stocks of consumer durables. Note that as inflation increases, the season generally will have to shrink (so that within-season price variation can be neglected) and thus the number of affected consumer durables will increase (and more user costs will have to be constructed).

In the following section, we return to the problem of aggregating the year-over-year seasonal indexes into an annual Mudgett-Stone index, but we no longer restrict ourselves to calendar years.

4. MOVING-YEAR ANNUAL INDEXES

At the beginning of Section 2, the separability assumption (5) provides a justification for constructing annual Mudgett-Stone indexes that compare the M seasons in one calendar year with the M seasons in another calendar year. However, we could choose any month (or season) as our year ending month and the prices and quantities of this new noncalendar year could be compared over years. The separability assumptions required to justify these new noncalendar-year comparisons are analogous to our earlier separability assumptions (5) but slightly different: The annual aggregator function f now is defined over the seasonal commodity vectors for a noncalendar year. These noncalendar-year comparisons can be taken a step further: We could think about comparing the prices and quantities of a *noncalendar* year with the prices and quantities of a base *calendar* year. What are the restrictions on intertemporal preferences that would justify this type of comparison, which we will call a variable year-end comparison or a moving-year[12] comparison? We provide an answer below.

Recall the seasonal aggregators $f^1(\boldsymbol{x}^1), \ldots, f^M(\boldsymbol{x}^M)$ from Section 2. We assumed the existence of an aggregator h that allowed us to define the annual utility function $f(\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M) \equiv h[f^1(\boldsymbol{x}^1), \ldots, f^M(\boldsymbol{x}^M)]$. Here, we again assume the existence of the linearly homogeneous, increasing, and concave seasonal aggregators $f^1, \ldots, f^M$ but make the following stronger assumptions on the intertemporal utility function U:

$$U(\boldsymbol{x}^{01}, \ldots, \boldsymbol{x}^{0M}; \boldsymbol{x}^{11}, \ldots, \boldsymbol{x}^{1M}; \ldots; \boldsymbol{x}^{T1}, \ldots, \boldsymbol{x}^{TM}) \equiv \psi^{-1}\left\{\sum_{t=0}^{T}\sum_{m=1}^{M}\beta_m \psi[f^m(\boldsymbol{x}^{tm})]\right\} \quad \textbf{(27)}$$

where the $\beta_m > 0$ are parameters that allow the consumer to cardinally compare the transformed seasonal utilities $\psi[f^m(\boldsymbol{x}^{tm})]$ and $\psi(z)$ is a monotonic function

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

of one positive variable z defined by

$$\psi(z) \equiv f_\alpha(z) \equiv \begin{cases} z^\alpha & \text{if} \quad \alpha \neq 0 \\ \ln z & \text{if} \quad \alpha = 0. \end{cases} \tag{28}$$

Substituting (28) into (27) reveals that the intertemporal utility function U is a CES aggregate of the seasonal utilities $f^m(\boldsymbol{x}^{tm})$. Using the assumptions that the seasonal aggregator functions $f^m(\boldsymbol{x}^{tm})$ are linearly homogeneous in the elements of $\boldsymbol{x}^{tm}$, it can be verified that U is linearly homogeneous.[13]

We assume that $\boldsymbol{q}^{01}, \ldots, \boldsymbol{q}^{0M}; \ldots; \boldsymbol{q}^{T1}, \ldots, \boldsymbol{q}^{TM}$ solve the intertemporal utility maximization problem (2) where U is defined by (27) and (28). Then, because ψ^{-1} is a monotonic function of one variable, it can be seen that, for any year t, we must have, for $t = 0, 1, \ldots, T$,

$$\sum_{m=1}^{M} \beta_m \psi[f^m(\boldsymbol{q}^{tm})] = \max_{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M} \left\{ \sum_{m=1}^{M} \beta_m \psi[f^m(\boldsymbol{x}^m)] : \sum_{m=1}^{M} \delta_t \rho_{tm} \boldsymbol{p}^{tm} \cdot \boldsymbol{x}^m = \sum_{m=1}^{M} \delta_t \rho_{tm} \boldsymbol{p}^{tm} \cdot \boldsymbol{q}^{tm} \right\}. \tag{29}$$

Recall that $\delta_t > 0$ is the discount factor that makes a dollar at the beginning of t equivalent to a dollar at the beginning of 0. Recall also that $\boldsymbol{p}^{tm}$ is the vector of prices for season m of t and ρ_{tm} is the discount factor that makes a dollar in the middle of season m of t equivalent to a dollar at the beginning of year t. It is convenient to define the normalized vector of prices in season m of year t, $\boldsymbol{p}^{tm*}$, as follows:

$$\boldsymbol{p}^{tm*} \equiv \delta_t \rho_{tm} \boldsymbol{p}^{tm}; \qquad t = 0, 1, \ldots, T; \qquad m = 1, \ldots, M; \tag{30}$$

i.e., $\boldsymbol{p}^{tm*}$ is now the nominal price vector $\boldsymbol{p}^{tm}$ discounted to the beginning of period 0.

Now we return to the equalities (29). The annual utility $\sum_{m=1}^{M} \beta_m \psi[f^m(\boldsymbol{q}^{tm})]$ can be rescaled or transformed by the monotonic function ψ^{-1} to make the resulting annual utility function linearly homogeneous. We obtain the following equalities for $t = 0, 1, \ldots, T$:

$$\psi^{-1}\left\{ \sum_{m=1}^{M} \beta_m \psi[f^m(\boldsymbol{q}^{tm})] \right\} = \max_{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M} \left\{ \psi^{-1}\left(\sum_{m=1}^{M} \beta_m \psi[f^m(\boldsymbol{x}^m)] \right) : \sum_{m=1}^{M} \boldsymbol{p}^{tm*} \cdot \boldsymbol{x}^m = \sum_{m=1}^{M} \boldsymbol{p}^{tm*} \cdot \boldsymbol{q}^{tm} \right\}. \tag{31}$$

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

As in Section 2, we assume that the seasonal aggregators $f^m(\boldsymbol{x}^m)$ are linearly homogeneous, increasing, and concave in their arguments. We assume also that the f^m have exact index-number formulas P^m and Q^m. We again can derive the equalities (9) and (10) and we also can derive the following counterparts to (9) and (10) (with $s = 0$), where normalized prices $\boldsymbol{p}^{tm*}$ replace the nominal price vectors $\boldsymbol{p}^{tm}$, for $t = 0, 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(\boldsymbol{q}^{tm})/f^m(\boldsymbol{q}^{0m}) = Q^m(\boldsymbol{p}^{0m*}, \boldsymbol{p}^{tm*}, \boldsymbol{q}^{0m}, \boldsymbol{q}^{tm}); \tag{32}$$

$$c^m(\boldsymbol{p}^{tm*})/c^m(\boldsymbol{p}^{0m*}) = P^m(\boldsymbol{p}^{0m*}, \boldsymbol{p}^{tm*}, \boldsymbol{q}^{0m}, \boldsymbol{q}^{tm}). \tag{33}$$

We choose units of measurement to measure base-period seasonal utilities $f^m(\boldsymbol{q}^{0m})$ as follows:

$$f^m(\boldsymbol{q}^{0m}) \equiv \boldsymbol{p}^{0m*} \cdot \boldsymbol{q}^{0m} \equiv Q_m^0; \qquad m = 1, \ldots, M; \tag{34}$$

i.e., we set utility in season m of year 0, $f^m(\boldsymbol{q}^{0m})$ or Q_m^0, equal to base-period expenditures in $m, \boldsymbol{p}^{0m} \cdot \boldsymbol{q}^{0m}$, times the inflation factor ρ_{0m} which converts the dollars of season m in year 0 to dollars at the beginning of year 0; (remember that $\boldsymbol{p}^{0m*} = \delta\rho_{0m}\boldsymbol{p}^{0m}$). The normalizations (34) imply that base-year seasonal unit costs, $c^m(\boldsymbol{p}^{0m*})$, are all equal to unity; i.e.,

$$c^m(\boldsymbol{p}^{0m*}) = 1 \equiv P_m^{0*}; \qquad m = 1, \ldots, M. \tag{35}$$

We have used equations (34) and (35) to define Q_m^0 and P_m^{0*} for $m = 1, \ldots, M$. Now, we substitute (34) and (35) into (32) and (33) to obtain the following computable formulas for the year t seasonal price and quantity aggregates, $c^m(\boldsymbol{p}^{tm*})$ and $f^m(\boldsymbol{q}^t)$ for $t = 1, \ldots, T$ and $m = 1, \ldots, M$:

$$f^m(\boldsymbol{q}^{tm}) = Q^m(\boldsymbol{p}^{0m*}, \boldsymbol{p}^{tm*}, \boldsymbol{q}^{0m}, \boldsymbol{q}^{tm})\,\boldsymbol{p}^{0m*} \cdot \boldsymbol{q}^{0m} \equiv Q_m^t; \tag{36}$$

$$c^m(\boldsymbol{p}^{tm*}) = P^m(\boldsymbol{p}^{0m*}, \boldsymbol{p}^{tm*}, \boldsymbol{q}^{0m}, \boldsymbol{q}^{tm}) \equiv P_m^{t*}. \tag{37}$$

Note that we have used equations (36) and (37) to define year t and season m seasonal price and quantity aggregates, P_m^t and Q_m^t.

Using (34–37), we can see that the maximization problems in (31) can be rewritten as follows for $t = 0, 1, \ldots, T$:

$$\psi^{-1}\left\{\sum_{m=1}^{M} \beta_m \psi\left(Q_m^t\right)\right\} = \max_{Q_1,\ldots,Q_m} \left\{\psi^{-1}\left(\sum_{m=1}^{M} \beta_m \psi[Q_m]\right) : \sum_{m=1}^{M} P_m^{t*} Q_m = \sum_{m=1}^{M} P_m^{t*} Q_m^t \right\}. \tag{38}$$

Using (28), it can be seen that the utility function (38) has a CES (or mean of order α) functional form. Sato (1976, p. 225) showed that the Vartia II quantity

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

index Q_{SV} is exact for this functional form. Thus, for $t = 1, 2, \ldots, T$, we have

$$\begin{aligned}
&\ln\{h[f^1(\boldsymbol{q}^{t1}), \ldots, f^M(\boldsymbol{q}^{tM})]/h[f^1(\boldsymbol{q}^{01}), \ldots, f^M(\boldsymbol{q}^{0M})]\} \\
&\quad = \ln Q_{SV}\left(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t*}, \ldots, P_M^{t*}; Q_1^0, \ldots, Q_M^0; Q_1^t, \ldots, Q_M^t\right) \\
&\quad \equiv \sum_{m=1}^{M} w_m^{0t} \ln\left(Q_m^t/Q_m^0\right) \bigg/ \sum_{j=1}^{M} w_j^{0t}, \qquad \textbf{(39)}
\end{aligned}$$

where $w_m^{0t} \equiv L(w_m^0, w_m^t)$, $w_m^t \equiv P_m^{t*} Q_m^t / \sum_{j=1}^{M} P_j^{t*} Q_j^t$ for $m = 1, \ldots, M$ and $t = 0, 1, \ldots, T$ and $L(x, y)$ is the logarithmic mean defined by (23).

With the special structure of intertemporal preferences defined by (27) and (28), the equalities (38) and (39) established for calendar years can be extended to noncalendar years, i.e., to any consecutive run of M seasons. For example, we can establish the following counterparts to (38) and (39) for $t = 0, 1, \ldots, T - 1$:

$$\begin{aligned}
&\psi^{-1}\left\{\sum_{m=2}^{M} \beta_m \psi[f^m(\boldsymbol{q}^{tm})] + \beta_1 \psi[f^1(\boldsymbol{q}^{t+1,1})]\right\} \\
&\quad = \max_{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^M} \left\{\psi^{-1}\left(\sum_{m=2}^{M} \beta_m \psi[f^m(\boldsymbol{x}^m)] + \beta_1 \psi[f^1(\boldsymbol{x}^1)]\right) : \right. \\
&\qquad \left. \sum_{m=2}^{M} \boldsymbol{p}^{tm*} \cdot \boldsymbol{x}^m + \boldsymbol{p}^{t+1,1*} \cdot \boldsymbol{x}^1 = \sum_{m=2}^{M} \boldsymbol{p}^{tm*} \cdot \boldsymbol{q}^{tm} + \boldsymbol{p}^{t+1,1*} \cdot \boldsymbol{q}^{t+1,1}\right\} \\
&\quad = \psi^{-1}\left\{\sum_{m=2}^{M} \beta_m \psi\left[Q_m^t\right] + \beta_1 \psi\left[Q_1^{t+1}\right]\right\} \\
&\quad = \max_{Q_1, \ldots, Q_M} \left\{\psi^{-1}\left[\sum_{m=2}^{M} \beta_m \psi(Q_m) + \beta_1 \psi(Q_1)\right] : \right. \\
&\qquad \left. \sum_{m=2}^{M} P_m^{t*} Q_m + P_1^{t+1*} Q_1 = \sum_{m=2}^{M} P_m^{t*} Q_m^t + P_1^{t+1*} Q_1^{t+1}\right\} \qquad \textbf{(40)}
\end{aligned}$$

where the P_m^{t*} and Q_m^t are defined by (34–37) with the $\boldsymbol{p}^{tm*}$ defined by (30). The moving-year utility maximization problems in (40) have dropped the quantities of season 1 in t and added those of season 1 in $t + 1$. Equations (38), when $t = 0$, can be combined with (40) and the fact that the Sato-Vartia quantity index Q_{SV} is exact for the CES functional form to yield the following exact relationships for

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

$t = 0, 1, \ldots, T-1$:

$$\psi^{-1}\left\{\beta_1\psi[f^1(\boldsymbol{q}^{t+1,1})] + \sum_{m=2}^{M}\beta_m\psi[f^m(\boldsymbol{q}^{tm})]\right\}\Big/\psi^{-1}\left\{\sum_{m=1}^{M}\beta_m\psi[f^m(\boldsymbol{q}^{0m})]\right\}$$
$$= \psi^{-1}\left\{\beta_1\psi\left(Q_1^{t+1}\right) + \sum_{m=2}^{M}\beta_m\psi\left(Q_m^t\right)\right\}\Big/\psi^{-1}\left\{\sum_{m=1}^{M}\beta_m\psi\left(Q_m^0\right)\right\}$$
$$= Q_{SV}\left(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t+1*}, P_2^{t*}, \ldots, P_M^{t*}; Q_1^0, \ldots, Q_M^0; Q_1^{t+1}, Q_2^t, \ldots, Q_M^t\right). \tag{41}$$

In evaluating the Sato-Vartia quantity index on the right-hand side of (41), we use the base-year aggregate discounted seasonal prices $P_1^{0*}, \ldots, P_M^{0*}$, the base-year seasonal aggregates $Q_1^0, \ldots, Q_M^0$, the year $t+1$ aggregate season 1 discounted price P_1^{t+1*} followed by the year t season 2 to M discounted prices $P_2^{t*}, \ldots, P_M^{t*}$ and the year $t+1$ season 1 quantity aggregate Q_1^{t+1} followed by the year t season 2 to M quantity aggregates $Q_2^t, \ldots, Q_M^t$.

In a similar fashion, the aggregate seasonal price and quantity data constructed using (34–37) for any run of M consecutive seasons can be rearranged and inserted into the Sato-Vartia index-number formula, and the resulting number times the (discounted) value of base-year consumption, $\sum_{j=1}^{M}\boldsymbol{p}^{0j*}\cdot\boldsymbol{q}^{0j} = \sum_{j=1}^{M}P_j^{0*}Q_j^0$,

$$Q_{tm} \equiv Q_{SV}\left(P_1^{0*}, \ldots, P_M^{0*}; P_1^{t+1*}, \ldots, P_{m-1}^{t+1*}, P_m^{t*}, \ldots, P_M^{t*};\right.$$
$$\left. Q_1^0, \ldots, Q_M^0; Q_1^{t+1}, \ldots, Q_{m-1}^{t+1}, Q_m^t, \ldots, Q_M^t\right)\sum_{j=1}^{M}\boldsymbol{p}^{0j*}\cdot\boldsymbol{q}^{0j}, \tag{42}$$

is an estimate of the consumer's real consumption in the moving year starting in season m of year t expressed in constant dollars pertaining to the beginning of calendar year 0.

We can divide the quantity index Q_{tm} into the discounted-value ratio of the moving year starting in season m of year t to the base year to obtain a price index P_{tm}:

$$P_{tm} \equiv \left[\sum_{i=m}^{M}\boldsymbol{p}^{ti*}\cdot\boldsymbol{q}^{ti} + \sum_{j=1}^{m-1}\boldsymbol{p}^{t+1,j*}\cdot\boldsymbol{q}^{t+1,j}\right]\Big/\left[\sum_{i=1}^{M}\boldsymbol{p}^{0i*}\cdot\boldsymbol{q}^{0i}\right]Q_{tm}. \tag{43}$$

Because discounted price vectors $\boldsymbol{p}^{tm*}$ appear in (42) and (43) instead of the nominal price vectors $\boldsymbol{p}^{tm}$, it is difficult to interpret the moving-year price index P_{tm} that is defined by (43). However, our focus here is on the moving-year quantity indexes Q_{tm} defined by (42). The main advantage of these indexes over the calendar-year Mudgett-Stone indexes discussed earlier (Diewert, 1998) or the two-stage calendar-year indexes discussed at the end of Section 2, is their *timeliness*: At the

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

end of *each* season of each year, a moving-year quantity index can be calculated that will enable economic policy makers to accurately determine the progress of the economy over the current noncalendar year compared to the base calendar year. A second advantage is that they are *comprehensive*; i.e., they include *all* of the seasonal commodities whereas the short-term season-to-season quantity indexes defined in the preceding section by (15) also were timely but they had to exclude most seasonal commodities. A third advantage is that they *do not have to be seasonally adjusted* because the quantities pertaining to an entire year starting at season m of year t are compared to the quantities pertaining to a base year. Thus, the moving-year quantity indexes Q_{tm} defined by (42) can be viewed as seasonally adjusted constant-dollar consumption series at annual rates and the analysis in this section provides a rigorous justification for the use of these series from the viewpoint of the economic approach.

As in Section 2, we recommend that the seasonal aggregates Q_m^t and P_m^{t*} be defined using Fisher ideal indexes for the seasonal bilateral indexes Q^m and P^m that appeared in (36) and (37). Of course, statistical agencies may have to approximate these Fisher indexes by Paasche and Laspeyres indexes and it also may be necessary to approximate the Sato-Vartia quantity and price indexes in (42) and (43) by Paasche and Laspeyres indexes as well. Provided that the base year is changed fairly frequently, these first-order approximations should be adequate. In low inflation contexts (i.e., less than 5% per year), it also may be possible to approximate adequately the moving-year quantity indexes Q_{tm} defined by (42) by replacing the discounted price vectors $\boldsymbol{p}^{tm*}$ defined by (30) with the nominal price vectors $\boldsymbol{p}^{tm}$; this replacement also will occur in (34–37). Replacing discounted prices with nominal ones in (43) means that the resulting moving-year price index P_{tm} can be regarded as a normal (seasonally adjusted) annual price index.[14]

In the following section, we regard (42) as an index-number method of seasonal adjustment and compare this method with more traditional statistical methods of seasonal adjustment.

5. ECONOMETRIC VERSUS INDEX-NUMBER METHODS OF SEASONAL ADJUSTMENT

What we have done in the preceding section is to show that if we use the Sato-Vartia quantity index, Q_{SV}, defined by (42), to aggregate the year-over-year seasonal indexes, then we can make exact index number comparisons for any consecutive string of M seasons with the base year. These moving-year indexes have no seasonal components and hence can be regarded as seasonally adjusted monthly series at annual rates.

Instead of using the Sato-Vartia index, Q_{SV}, in (45), a superlative quantity index such as the Fisher ideal Q_F could be used to approximate Q_{SV}.[15] In the general case in which the Fisher quantity index is defined for the moving year starting at season m of year t, $Q_F \equiv [Q_P Q_L]^{1/2}$, where the Paasche and Laspeyres quantity indexes, Q_P and Q_L, are evaluated at the same aggregate seasonal prices and quantities and

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

can be regarded as share-weighted moving averages of the moving-year seasonal quantity aggregates.

As a further refinement, we can center the series of moving-year quantity indexes, $Q_{t,m}$, defined by (42). If we have monthly data so that the number of seasons M equals 12, then $Q_{t,m}$ represents the aggregate quantity of a moving year starting at month m of t relative to the aggregate quantity of a base year. An estimate of the annual quantity *centered* around month m of year t compared to the quantity of the base year is

$$Q^c_{t,m} \equiv \begin{cases} \left(\frac{1}{2}\right) Q_{t,m-6} + \left(\frac{1}{2}\right) Q_{t,m-5}; & t = 0, 1, \ldots, T-1; \quad m = 7, \ldots, 12 \\ \left(\frac{1}{2}\right) Q_{t-1,m+6} + \left(\frac{1}{2}\right) Q_{t-1,m+7}; & t = 1, 2, \ldots, T; \quad m = 1, \ldots, 5 \\ \left(\frac{1}{2}\right) Q_{t-1,12} + \left(\frac{1}{2}\right) Q_{t,1}; & t = 1, \ldots, T; \quad m = 6. \end{cases} \tag{44}$$

We cannot provide centered monthly quantity estimates for the first and last six months; i.e., $Q^c_{t,m}$ is not defined for $t = 0$ and $m = 1, 2, \ldots, 6$ and for $t = T$ and $m = 7, 8, \ldots, 12$.

Our suggested index-number method of seasonal adjustment is not really a traditional seasonal adjustment method.[16] Our index numbers Q_{tm} defined by (42) simply compare a moving-year aggregate to a corresponding base-year aggregate. *Thus we have changed the question that we are trying to answer*. The centered index-number comparisons Q^c_{tm} of the form (44) are averages of the more fundamental comparisons made in (42), where the averaging is done so that the resulting centered estimates will more closely resemble a conventional seasonally adjusted series at annual rates.

In the Appendix, we compare official U.S. data seasonally adjusted at annual rates on quarterly GDP over the years 1959–1988 with moving-year centered index numbers which aggregate the quarterly unadjusted data compiled by the Bureau of Economic Analysis (1992).[17] We found that our suggested index-number method for seasonal adjustment performs as well as the official X-11 method. The turning points are basically the same. The main differences are that the index-number adjusted series is smoother and the X-11 adjusted series growth is slower.[18] The reason for the second difference is that the X-11 series is constructed by seasonally adjusting the U.S. fixed base quarterly (unadjusted) quantity series whereas the unadjusted quarterly chain data are used in the index-number formula. Our results are consistent with the fixed 1987 base-year Laspeyres and chained comparisons of U.S. real GDP over the years 1959–1987 made by Young (1992, p. 36), who found that the average rate of growth of the fixed-base GDP index numbers was 3.1% compared to 3.4% per year for the chain indexes. Users of U.S. seasonally adjusted data should be aware that it is fixed-base data that is being seasonally adjusted. When the base year is changed, fairly substantial changes in growth rates can occur in the official seasonally adjusted fixed-base data.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

The results in the Appendix are significant because the index-number method of seasonal adjustment offers a number of advantages over the X-11 method:

(1) The index-number method can be explained fairly simply.
(2) There are many significant unannounced choices that must be made by the statistican-operator of the X-11 method (e.g., multiplicative or additive seasonals, treatment of outliers), whereas the index-number method involves only two easily stated choices.[19]
(3) Final seasonal adjustment factors using the X-11 method are not available until two or three years of additional unadjusted data become available, whereas indexes of the form (42) will be available almost immediately after the last season in the moving year and the centered indexes of the form (44) will be available after an additional six months. The statistical agency will avoid the current embarrassing problem of trying to explain why the seasonally adjusted series is still being revised years after the preliminary series has been released.
(4) The index-number method of aggregation simultaneously seasonally adjusts (normalized) prices and quantities [recall (42) and (43), above] whereas statistical methods of seasonal adjustment separately adjust prices, quantities, and values without respecting the fact that only two of these three variables are independent.
(5) Statistical seasonal adjustment methods that allow for changing seasonals run into a severe identification problem and the resulting seasonal factors that these statistical methods churn out are not well defined from a theoretical point of view.[20]

The econometric methods do have the advantage that they can be applied in situations in which there is quantity but not price information; i.e., the X-11 method can seasonally adjust an unemployment series but an index-number method cannot.

It should be emphasized that the moving-year quantity indexes defined by (42) are sufficient statistics for defining the centered moving-year quantity indexes defined by (44). Thus the statistical agency should strive to provide moving-year quantity and price indexes of forms (42) and (43) on a timely basis: Users can easily perform the simple arithmetic operations inherent in forming the centered moving-year indexes of form (44).

Our specific assumptions on intertemporal preferences represented by (27) and (28) led to the specific Sato-Vartia exact index-number formula (42) where the monthly aggregates P_m^{t*} and Q_m^t were formed using superlative index-number formulas in (34–37). In many situations, it may be necessary to approximate both the monthly price indexes P^m, which appear in (37), and the Sato-Vartia price index P_{SV}, which appears in (43), by use of the Laspeyres price indexes. Then the corresponding quantity indexes in (36) and (42) will be Paasche quantity indexes. These Paasche and Laspeyres indexes will be acceptable approximations to their superlative and Sato-Vartia counterparts, provided that the base year is changed fairly frequently.

Another approximation to our recommended theoretically exact indexes defined by (34–37) and (42–43) occurs if the inflation-adjusted prices $\boldsymbol{p}^{tm*}$ defined by (30) and used in (34–37) are replaced by the corresponding unadjusted spot-price vectors $\boldsymbol{p}^{tm}$. This will make little difference to the moving-year quantity indexes defined by (42) and (44) provided that inflation is low and seasonal fluctuations

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

are not too erratic. Numerical experiments will be required before we can be more precise.

6. CONCLUSION

The assumptions on preferences that we have made provide justifications for three types of seasonal index-number comparisons that statistical agencies should provide:

(i) For measuring short-term price change, the approach outlined in Section 3 should be used; i.e., a season-to-season short-run price index using only nonseasonal commodities should be constructed. These short-term indexes could be used as deflators when constructing the annual quantity indexes in (iii) below.

(ii) The year-over-year seasonal indexes defined by (9) and (10) in Section 2 also should be constructed. The assumptions on preferences required to justify these are the least restrictive. The business community may find these indexes the most useful.

(iii) Finally, the moving-year price and quantity indexes defined by (42) and (43) in Section 4 also should be calculated.[21] These indexes will serve as seasonally adjusted price and quantity indexes (at annual rates). If there is low inflation, spot prices $\boldsymbol{p}^{tm}$ can be used in place of the normalized prices $\boldsymbol{p}^{tm*}$ in (34–37) and (42–43).

For each of the above three indexes, the statistical agency will have to decide whether to provide Paasche and Laspeyres or superlative versions. From the viewpoint of economic theory, the superlative versions are better but they will be more costly and less timely. In the long run, statistical agencies will be able to make use of electronically recorded data on the sales of commodities to produce timely superlative indexes. However, in the short run, difficult choices must be made on how to produce price and quantity indexes when there are seasonal commodities and high inflation.

NOTES

1. See Flux (1921, pp. 184–185), Bean and Stine (1924), Crump (1924, p. 185), Mudgett (1955), Stone (1956), Rothwell (1958), Zarnowitz (1961), Turvey (1979), and Balk (1980a,b,c, 1981).

2. See Diewert (1980, pp. 506–508; 1983b) on the economic approach to seasonal indexes. This paper focuses on the theory of the seasonal consumer price index. An analogous theory exists for the seasonal producer price index with separability assumptions on the producer's intertemporal production function or factor requirements functions. See Fisher and Shell (1972) and Diewert (1980; 1983a).

3. If inflation is low, the within-year discount rates ρ_{tm} can be set equal to one.

4. We also require positivity of the discount factors δ_t and ρ_{tm} to derive (7).

5. For example, see Bean and Stine's (1924, p. 31), Type D index number or Rothwell (1958, p. 70). Incidentally, Flux (1921, p. 185) also proposed (and used) Bean and Stine's Type B index and Crump (1921, p. 207), in his discussion of Flux's (1921) paper, proposed Bean and Stine's Type A index number. Finally, Bean and Stine's (1924, p. 31) Type C index number is closely related to the Rothwell (1958, p. 71) index.

6. See Diewert (1996, pp. 19–22) for the details. To obtain a two-stage procedure that is exactly equal to the single-stage annual Paasche or Laspeyres indexes, we need to assume that the within-year discount rates ρ_{tm} are all unity.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

7. This follows from Diewert (1978, p. 889).

8. We also use the positivity of the discount factors δ_t and ρ_{tm} in deriving (14).

9. Rothwell (1958, p. 71) noted that the problem of making price comparisons between seasons with different market baskets is formally identical to the problem of making international comparisons between countries with different market baskets. This suggests that the symmetric methods used in making international comparisons could be applied to the problem of aggregating up the many bilateral price comparisons in (16) into a consistent sequence of monthly price levels. Balk (1981, p. 74), in fact, implemented this idea, calculating a system of EKS [see Gini (1931, p. 12), Eltetö and Köves (1964), and Szulc (1964)] monthly purchasing-power parities for Dutch fruit and vegetables. However, Walsh (1901, p. 399) and Balk (1981, p. 77) also noted a practical disadvantage to the use of these symmetric methods: The price levels have to be recalculated each time a new observation is added.

10. Diewert (1980, pp. 502–503) suggested an econometric approach to the estimation of reservation prices but did not implement it. Hausman (1997) seems to have been the first to implement such an econometric approach.

11. Because seasonal price and quantity changes can be huge, the choice of the index-number formula makes a large difference. When Balk (1980a, p. 41) compared his Sato-Vartia indexes for Dutch fruit and vegetables with an alternative index-number formula, he found some differences in the 50% range. Reinsdorf and Dorfman (1995, table 1) also found substantial differences between the Sato-Vartia price index and the superlative Fisher and Törnqvist price indexes for some artificial data.

12. The term "moving year" is from Mendershausen (1937, p. 245). Diewert (1983b, p. 1029) earlier used the term "split-year" comparison to describe a variable year-end index-number comparison. Following the terminology used by Crump (1924, p. 185) in a slightly different context, we also could use the term "rolling-year" comparison.

13. Diewert (1983b, p. 1034) assumed that U was the simple sum of seasonal utilities, $\sum_{t=0}^{T}\sum_{m=1}^{M} f^m(x^{tm})$. This is a special case of (27) and (28) with $\beta_m = 1$ and $\alpha = 1$.

14. Making these Paasche and Laspeyres approximations and using nominal prices p^{tm} in place of the discounted prices p^{tm*} causes (43) to become the "indice sensible" that was used as a seasonally adjusted consumer price index by the French Statistical Agency Institut National de la Statistique et des Études Economiques (1976, pp. 67–68) for several years. Diewert (1983b, p. 1040), using Turvey's (1979) artificial data on seasonal consumption, also calculated some approximations to the moving-year price indexes defined by (43): Diewert used Turvey's nominal prices instead of discounted prices and compared the results of using Laspeyres, Paasche, Fisher ideal and translog, or Törnqvist price indexes in both stages of the aggregation. The choice of index-number formula did not matter very much for that data set.

15. Because superlative indexes are exact for flexible aggregators, the flexible aggregator function can approximate the CES aggregator function in (27) to the second order.

16. For material on time-series methods of seasonal adjustment, see Bell and Hillmer (1984) and Hylleberg (1992).

17. Instead of the Sato-Vartia quantity index, we used the Fisher ideal quantity index in (75). We did not deflate the quarterly prices by an index of purchasing power because inflation was "small" over this period.

18. The average quarterly rate of growth for the official X-11 adjusted series was 0.78 compared to 0.85 per quarter for our centered Fisher ideal moving-year series.

19. The two choices are variants of (42): (a) Should the inflation-adjusted normalized prices $\boldsymbol{p}^{tm*}$ defined by (30) be replaced by the unadjusted spot prices $\boldsymbol{p}^{tm}$ and (b) should the Sato-Vartia index-number formula Q_{SV}, which appears in (42), be replaced by some other index-number formula?

20. See the discussion by Anderson (1927, pp. 552–553).

21. Under conditions of high inflation, the price indexes defined by (43) will be difficult to interpret and hence the statistical agency would not have to report them. The primary focus should be on the production of the moving-year quantity indexes defined by (42).

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

REFERENCES

Anderson, O. (1927) On the logic of the decomposition of statistical series into separate components. *Journal of the Royal Statistical Society* 90, 548–569.

Balk, B.M. (1980a) *Seasonal Products in Agriculture and Horticulture and Methods for Computing Price Indices*, Statistical Studies 24. The Hague: Netherlands Central Bureau of Statistics.

Balk, B.M. (1980b) Seasonal commodities and the construction of annual and monthly price indexes. *Statistische Hefte* 21:2, 110–116.

Balk, B.M. (1980c) A method for constructing price indices for seasonal commodities. *Journal of the Royal Statistical Society* A, 143, 68–75.

Balk, B.M. (1981) A simple method for constructing price indices for seasonal commodities. *Statistische Hefte* 22:1, 72–78.

Balk, B.M. (1995) Axiomatic price index theory: A survey. *International Statistical Review* 63, 69–93.

Bean, L.H. & O.C. Stine (1924) Four types of index numbers of farm prices. *Journal of the American Statistical Association* 19, 30–35.

Bell, W.R. & S.C. Hillmer (1984) Issues involved with the seasonal adjustment of economic time series. *Journal of Business and Economic Statistics* 2, 291–320.

Bureau of Economic Analysis (1992) *National Income and Product Accounts of the United States, 1959–88*, vol. 2, Washington, DC: U.S. Government Printing Office.

Crump, N. (1921) Discussion: Mr. Flux's Paper. *Journal of the Royal Statistical Society* 84, 207–209.

Crump, N. (1924) The interrelation and distribution of prices and their incidence upon price stabilization. *Journal of the Royal Statistical Society* 87, 167–206.

Diewert, W.E. (1978) Superlative index numbers and consistency in aggregation. *Econometrica* 46, 883–900.

Diewert, W.E. (1980) Aggregation problems in the measurement of capital. In D. Usher (ed.), *The Measurement of Capital*, pp. 433–528. Chicago: University of Chicago Press.

Diewert, W.E. (1983a) The theory of the output price index and the measurement of real output change. In W.E. Diewert & C. Montmarquette (eds.), *Price Level Measurement*, pp. 1049–1113. Ottawa: Statistics Canada.

Diewert, W.E. (1983b) The treatment of seasonality in a cost-of-living index. In W.E. Diewert & C. Montmarquette (eds.), *Price Level Measurement*, pp. 1019–1045. Ottawa: Statistics Canada.

Diewert, W.E. (1996) Seasonal Commodities, High Inflation and Index Number Theory. Discussion paper 96-06, University of British Columbia.

Diewert, W.E. (1998) High inflation, seasonal commodities, and annual index numbers. *Macroeconomic Dynamics* 2, 456–471.

Eltetö, O & P. Köves (1964) On a problem of index number computation relating to international comparison. *Statisztikai Szemle* 42, 507–518.

Feenstra, R.C. (1994) New product varieties and the measurement of international prices. *American Economic Review* 84, 157–177.

Fisher, I. (1930) *The Theory of Interest*. New York: Macmillan.

Fisher, F.M. & K. Shell (1972) *The Economic Theory of Price Indexes*. New York: Academic Press.

Flux, A.W. (1921) The measurement of price changes. *Journal of the Royal Statistical Society* 84, 167–199.

Gini, C. (1931) On the circular test of index numbers. *Metron* 9:2, 3–24.

Hausman, J. (1997) Valuation of new goods under perfect and imperfect competition. In T.F. Bresnahan & R.J. Gordon (eds.), *The Economics of New Goods*. Chicago: The University of Chicago Press.

Hicks, J.R. (1940) The valuation of the social income. *Economica* 7, 105–140.

Hicks, J.R. (1946) *Value and Capital*, 2nd ed. Oxford: Clarendon Press.

Hylleberg, S. (ed.) (1992) *Modelling Seasonality*. Oxford: Oxford University Press.

Institut National de la Statistique et des Etudes Economiques (1976), *Pour Comprendre l'Indice des Prix*. Paris: INSEE.

Keynes, J.M. (1930) *Treatise on Money*, vol. 1. London: Macmillan.

Mendershausen, H. (1937) Annual survey of statistical technique: Methods of computing and eliminating changing seasonal fluctuations. *Econometrica* 5, 234–262.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

Mudgett, B.D. (1955) The measurement of seasonal movements in price and quantity indexes. *Journal of the American Statistical Association* 50, 93–98.

Reinsdorf, M.B. & A.H. Dorfman (1995) *The Sato-Vartia Index and the Monotonicity Axiom*. Washington, DC: US Bureau of Labor Statistics.

Rothwell, D.P. (1958) Use of varying seasonal weights in price index construction. *Journal of the American Statistical Association* 53, 66–77.

Sato, K. (1976) The ideal log-change index number. *The Review of Economics and Statistics* 58, 223–338.

Stone, R. (1956) *Quantity and Price Indexes in National Accounts* Paris: The Organization for European Cooperation.

Szulc, B. (1964) Indices for multiregional comparisons. *Przeglad Statystyczny* (*Statistical Review*) 3, 239–254.

Turvey, R. (1979) The treatment of seasonal items in consumer price indices. *Bulletin of Labour Statistics*, 4th quarter, 13–33.

Vartia, Y.O. (1976) Ideal log-change index numbers. *Scandanavian Journal of Statistics* 3, 121–126.

Walsh, C.M. (1901) *The Measurement of General Exchange Value*. New York: Macmillan.

Young, A.H. (1992) Alternative measures of change in real output and prices. *Survey of Current Business* 72, 32–48.

Zarnowitz, V. (1961) Index numbers and the seasonality of quantities and prices. In G.J. Stigler (Chairman), *The Price Statistics of the Federal Government*, pp. 233–304. New York: National Bureau of Economic Research.

APPENDIX

U.S. SEASONALLY ADJUSTED AND CENTERED MOVING-YEAR ESTIMATES

The raw data for our comparisons are from the Bureau of Economic Analysis (1992): seasonally unadjusted estimates of U.S. GDP from quarter 1 of 1959 to quarter 4 of 1988 (120 quarters in all) are from Table 9.1, implicit price deflators for GDP using chain-type weights are from Table 7.2, and estimates of quarterly GDP seasonally adjusted at annual

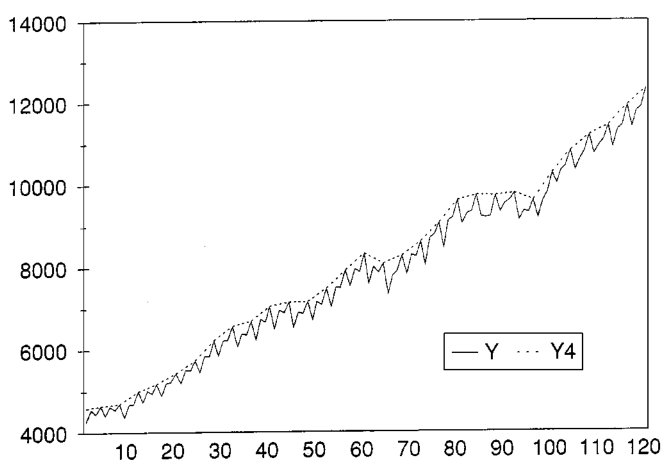

FIGURE 1. U.S. quarterly real GDP seasonally unadjusted.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press

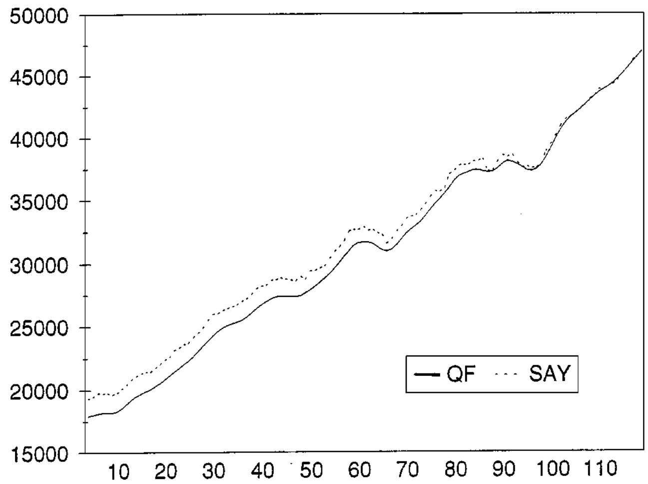

FIGURE 2. Seasonally adjusted quarterly U.S. GDP.

rates are from Table 1.2. The price index was normalized to equal 1 in the third quarter of 1987. Dividing the seasonally unadjusted GDP by the price index gave us the series Y_t, $t = 1, \ldots, 120$ (note the changed notation here, with t denoting consecutive quarters). The series Y_t is plotted in Figure 1. The units are 100 millions of 1987 third-quarter dollars. The series of fourth-quarter observations are indicated by the sharp peaks joined together by dashed lines. The seasonal fluctuations are evolving over time.

We denote the Fisher ideal fixed-base moving-year index by Q_t, where t indicates the first quarter of the moving year and the base year consists of the four quarters of 1987. (We made no adjustment for general inflation because it was not high). Q_t is defined for $t = 1, 2, \ldots, 117$. and the centered index is given by

$$Q_t^c \equiv \left(\tfrac{1}{2}\right) Q_{t-1} + \left(\tfrac{1}{2}\right) Q_{t-2}; \qquad t = 3, 4, \ldots, 118. \qquad \textbf{(A.1)}$$

These centered indexes are plotted as the solid line in Figure 2 (and denoted by QF and measured again in 100 millions of 1987 dollars). The official seasonally adjusted U.S. constant-dollar GDP series for the same 116 quarters also is plotted as the dashed line in Figure 2 (and denoted by *SAY*). *SAY* growth is slower and more erratic than that of QF but both series have roughly the same turning points and hence both can serve as guides to business-cycle movements.

https://doi.org/10.1017/S1365100599010020 Published online by Cambridge University Press