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Embedding of some classes of operators into strongly continuous semigroups

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Abstract. In this paper, we study the embedding problem of an operator into a strongly continous semigroup. We obtain characterizations for some classes of operators, namely composition operators and analytic Toeplitz operators on the Hardy space H^2. In particular, we focus on the isometric ones using the necessary and sufficient condition observed by T. Eisner.

1 Introduction

If T is a bounded linear operator on a Banach space E, then T is said to be *embeddable* into a strongly continuous semigroup if there exists $(T_t)_{t\geq 0}$ a family of bounded linear operators on E, which is a strongly continous semigroup such that $T = T_1$.

This notion has been studied by T. Eisner, especially in the monograph [10], and it turned out to be a very interesting and challenging problem. There is no known necessary and sufficient condition for any operator T on a Banach space, but T. Eisner showed that if T is embeddable, then $\dim(\ker(T))$ and $\operatorname{codim}(\overline{\operatorname{Im}}(T))$ are either 0 or ∞. This condition shows us that the forward shift operator S on H^2 and all its powers are not embeddable into a strongly continous semigroup on H^2.

One of the first easy examples is the embedding of the Volterra operator $V : f \in L^p([0,1]) \longmapsto Vf(x) = \int_0^x f(s)\,ds$ into the Riemann–Liouville semigroup on $L^p([0,1])$, for $1 \leq p < \infty$ [1, 2].

However, there exist some necessary and sufficient conditions for special classes of operators, and we highlight here the following result on isometric operators obtained by Eisner [9] and [10, Theorem V.1.19].

Theorem 1.1 *Let $V : H \to H$ be an isometry on a Hilbert space H. Then V is embeddable into a C_0-semigroup on H if and only if V is unitary or $\operatorname{codim}(VH) = \infty$.*

This theorem is very useful in the case of composition operators or analytic Toeplitz operators, where it is easy to characterize the isometric ones. The main goal of this paper is then to describe the embedding of such isometric operators and to make the associated semigroup explicit.

The paper is organized as follows. In Section 2, we recall some useful properties on (analytic) semigroup theory of operators and the main tools required on Hardy spaces. We also give a key lemma on Blaschke products for the main result on

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composition operators. Section 3 is concerned with the embedding of isometric composition operators and also the embedding of analytic Toeplitz operators. Finally, in Section 4, we show that the embedding of an isometry into a semigroup of contractions $(T_t)_{t\geq 0}$ implies that each T_t is isometric. Moreover, the embedding of an isometry into a semigroup $(T_t)_{t\geq 0}$ (not necessarily contractive) implies that, for each $t > 0$, T_t is never compact.

2 Background and preliminaries

2.1 Strongly continuous semigroups of operators

Let E be a Banach space and $(T_t)_{t\geq 0} \subset \mathcal{L}(E)$, the space of all linear and bounded operators on E endowed with its usual norm. We say that $(T_t)_{t\geq 0}$ is a *strongly continuous semigroup* or just a C_0*-semigroup* if

$$T_0 = Id, \quad T_{t+s} = T_t \circ T_s, \quad t, s > 0,$$

and for all $x \in E$,

$$t \in \mathbb{R}_+ \longmapsto T_t x \text{ is continuous i.e. } \|T_t x - x\|_E \underset{t\to 0^+}{\longrightarrow} 0.$$

See, for example, [11] for an introduction to semigroup theory of operators.

We recall here that an operator $T \in \mathcal{L}(E)$ is *embeddable* into a C_0-semigroup on E if there exists $(T_t)_{t\geq 0}$ a C_0-semigroup on E such that $T = T_1$. In this case, we write

$$T \hookrightarrow (T_t)_{t\geq 0}.$$

2.2 Analytic semiflows on $\mathbb{D}$

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disc of the complex plane $\mathbb{C}$ and $(\varphi_t)_{t\geq 0}$ be a family of analytic self-maps of $\mathbb{D}$. We say that $(\varphi_t)_{t\geq 0}$ is a *semiflow of analytic self-maps of* $\mathbb{D}$ if

$$\varphi_0 = Id, \quad \varphi_{t+s} = \varphi_t \circ \varphi_s, \quad t, s \geq 0,$$

and for all $z \in \mathbb{D}$,

$$t \in \mathbb{R}_+ \longmapsto \varphi_t(z) \text{ is continuous.}$$

Note that the pointwise continuity assumption is equivalent to the uniform continuity on all compact subsets of $\mathbb{D}$ via Montel's theorem.

It is a well known fact that when $(\varphi_t)_{t\geq 0}$ is a semiflow of analytic self-maps of $\mathbb{D}$, each function φ_t is one-to-one. See [6] for a proof using Cauchy–Lipschitz's theory or [7] for an alternative elementary proof. We recommend [6] for a very complete state of the art of analytic semiflow theory.

In the same way, we say that $\varphi \in \mathrm{Hol}(\mathbb{D})$ is *embeddable* into a semiflow of analytic self-maps of $\mathbb{D}$ if there exists $(\varphi_t)_{t\geq 0}$ a semiflow of analytic self-maps of $\mathbb{D}$ such that $\varphi = \varphi_1$. In this case, we write

$$\varphi \hookrightarrow (\varphi_t)_{t\geq 0}.$$

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As an example, let φ be an elliptic automorphism of $\mathbb{D}$, i.e., φ is an holomorphic and bijective function on $\mathbb{D}$ with a fixed point $\alpha \in \mathbb{D}$. Equivalently, there exist $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that

$$\varphi = \tau_\alpha \circ R_\theta \circ \tau_\alpha, \text{ where } R_\theta : z \longmapsto e^{i\theta} z \text{ and } \tau_\alpha : z \longmapsto \frac{\alpha - z}{1 - \overline{\alpha} z}. \tag{2.1}$$

Note that τ_α is an automorphism of $\mathbb{D}$, which coincides with its inverse. Thus, φ is embeddable into the following semiflow of analytic self-maps of $\mathbb{D}$:

$$\varphi_t = \tau_\alpha \circ R_{\theta t} \circ \tau_\alpha, \text{ with } R_{\theta t} : z \longmapsto e^{it\theta} z, \quad t \geq 0.$$

2.3 Hardy spaces and Blaschke products

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle endowed with the Lebesgue measure m. For $0 < p < \infty$, we consider the Hardy space $H^p = H^p(\mathbb{D})$, which consists of functions f holomorphic on $\mathbb{D}$ satisfying

$$\|f\|_p = \sup_{0 \leq r < 1} \left(\int_{\mathbb{T}} |f(r\zeta)|^p \, dm(\zeta) \right)^{1/p} < \infty.$$

Recall that for $p = 2$, H^2 is a reproducing kernel Hilbert space whose kernel is given by

$$k_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}, \quad \lambda, z \in \mathbb{D},$$

meaning that

$$f(\lambda) = \langle f, k_\lambda \rangle_2, \quad f \in H^2, \quad \lambda \in \mathbb{D}.$$

Moreover, we have $\operatorname{Span}_{H^2}(k_\lambda : \lambda \in \mathbb{D}) = H^2$, where $\operatorname{Span}_{\mathcal{H}}(A)$ denotes the closure in $\mathcal{H}$ of the subspace consisting of finite linear combinations of elements of A, where A is a family of vectors in a Hilbert space $\mathcal{H}$. Define by H^2_0 the space of functions $f \in H^2$ such that $f(0) = 0$.

We also define $H^\infty = H^\infty(\mathbb{D})$ to be the class of bounded analytic functions on $\mathbb{D}$, endowed with the sup norm defined by $\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$.

We also recall that a Blaschke product is a function of the form

$$B(z) = e^{i\beta} z^k \prod_{n \geq 1} \frac{|\alpha_n|}{\alpha_n} \frac{\alpha_n - z}{1 - \overline{\alpha_n} z}, \quad z \in \mathbb{D}, \tag{2.2}$$

where $\beta \in \mathbb{R}$, $k \in \mathbb{N} \cup \{0\}$ and $(\alpha_n)_{n \geq 1}$ is a finite or infinite sequence of $\mathbb{D} \backslash \{0\}$ satisfying $\sum_{n \geq 1}(1 - |\alpha_n|) < \infty$. Then, B is an inner function on $\mathbb{D}$. When $(\alpha_n)_{n \geq 1}$ is a finite sequence of $\mathbb{D}$, we say that B is a finite Blaschke product and one can easily check that such B are continuous on the closed unit disc. We refer the reader to [13] for more details about finite Blaschke products.

In the sequel, we consider the finite Blaschke product associated with a finite sequence $(\alpha_n)_{1 \leq n \leq N} \subset \mathbb{D}$ defined by

$$B(z) = \prod_{i=1}^{N} \frac{\alpha_i - z}{1 - \overline{\alpha_i} z}, \quad z \in \mathbb{D}. \tag{2.3}$$

We know that the equation $B(z) = \beta$ for $\beta \in \mathbb{D}$ has exactly N solutions in $\mathbb{D}$, taking into account the multiplicity (see [13]). The following lemma is the key to differentiate them.

Lemma 2.1 *Let $\beta \in \mathbb{D}\backslash B(Zero(B'))$. Then the equation $B(z) = \beta$ has exactly N distinct solutions in $\mathbb{D}$.*

Proof Let $\beta \in \mathbb{D}$. Then $B(z) = \beta$ is equivalent to a polynomial equation of the form

$$P(z) - \beta Q(z) = 0 \text{ with } P(z) = \prod_{i=1}^{N}(\alpha_i - z) \text{ and } Q(z) = \prod_{i=1}^{N}(1 - \overline{\alpha_i} z).$$

Since $(-1)^N \left(1 - \beta \prod_{i=1}^{N} \overline{\alpha_i}\right) \neq 0$ and since B maps $\mathbb{D}$ to $\mathbb{D}$, $\mathbb{T}$ to $\mathbb{T}$, and $\{z \in \mathbb{C} : |z| > 1\}$ to itself, there are N solutions in $\mathbb{D}$. It remains to prove that for suitable β the solutions are distinct.

Note that

$$B'(z) = \frac{P'(z)Q(z) - P(z)Q'(z)}{Q(z)^2},$$

and thus

$$P(z) - \beta Q(z) = 0 \text{ and } P'(z) - \beta Q'(z) = 0 \Longrightarrow B'(z) = 0.$$

It follows that for $\beta \in \mathbb{D}\backslash B(\text{Zero}(B'))$, the equation $B(z) = \beta$ has exactly N distinct solutions in $\mathbb{D}$. ∎

We also recall the well known and very useful Frostman's theorem [12] whose assertion is the following. Let θ be an arbitrary inner function. Then there exists a set of measure zero with respect to the area measure (even a set of capacity zero) $\Omega \subset \mathbb{D}$ such that for every $\lambda \in \mathbb{D}\backslash\Omega$, the so-called *Frostman's transform*

$$\theta_\lambda := \tau_\lambda \circ \theta = \frac{\lambda - \theta}{1 - \overline{\lambda}\theta}$$

is a Blaschke product with simple zeros. Here recall that τ_λ is the usual automorphism of $\mathbb{D}$ defined by $\tau_\lambda(z) = \frac{\lambda - z}{1 - \overline{\lambda} z}$ for $z \in \mathbb{D}$.

3 Embedding results

3.1 Composition operators on H^2

First of all, recall that the main goal of this paper is to describe classes of operators, which can be embedded into a semigroup of operators. The isometric operators are of particular interest due to the necessary and sufficient condition in Theorem 1.1 on Hilbert spaces. The choice of composition operators on H^2 is relevant since the isometric ones as well as the ones that are similar to isometries are fully characterized and, moreover, this class is quite rich. We define the *composition operator* C_φ with symbol φ on H^2 by

$$C_\varphi : f \longmapsto f \circ \varphi, \quad f \in H^2.$$

This operator is well defined and bounded on H^2. See [17] for further information about those operators on H^2.

Moreover, C_φ is an isometry on H^2 if and only if φ is inner and $\varphi(0) = 0$. In [4], F. Bayart showed that C_φ is similar to an isometry on H^2 if and only if φ is inner, and there exists $\alpha \in \mathbb{D}$ such that $\varphi(\alpha) = \alpha$. In other words, C_φ is similar to an isometry on H^2 if and only if its symbol φ is an elliptic inner function.

In Arendt et al. [3] showed the following results.

- On the Bergman space $\mathcal{A}^2$ (and even on its weighted versions $\mathcal{A}^2_\beta$ for $\beta > -1$), C_φ is similar to an isometry if and only if φ is an elliptic automorphism of $\mathbb{D}$. Thus, we will find a natural embedding according to (2.1) for C_φ, described later by Remark 3.1.
- On the classical Dirichlet space $\mathcal{D}$, C_φ is similar to an isometry if and only if φ is a univalent full map with a fixed point in $\mathbb{D}$ and the counting function n_φ associated to φ is essentially radial (see [3, Section 6]). Note that the existing criteria for the boundedness of C_φ is not that explicit and so the similarity to an isometry is much less easy to handle.

From now on, the space on which we study our operators are defined on the Hardy space H^2. Let $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic.

Remark 3.1 Observe that if $\varphi \hookrightarrow (\varphi_t)_{t\geq 0}$ where $(\varphi_t)_{t\geq 0}$ is a semiflow of analytic self-maps of $\mathbb{D}$, then $C_\varphi \hookrightarrow (C_{\varphi_t})_{t\geq 0}$ where $(C_{\varphi_t})_{t\geq 0}$ is a C_0-semigroup on H^2.

Conversely if $C_\varphi \hookrightarrow (T_t)_{t\geq 0}$ where $(T_t)_{t\geq 0}$ is a C_0-semigroup of composition operators on H^2, then applying T_t to $e_1(z) := z$ and using the fact that the convergence in H^2 implies the pointwise convergence, we get the existence of $(\varphi_t)_{t\geq 0}$ a semiflow of analytic self-maps of $\mathbb{D}$ such that $T_t = C_{\varphi_t}$.

With the reproducing kernel Hilbert space property of H^2, we can give the following first sufficient condition of embedding for isometric composition operators. This is a key to understand the strategy of the proof of the main result.

Lemma 3.2 *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an inner function such that $\varphi(0) = 0$. Assume that there exists a sequence $(z_k)_{k\geq 0}$ of distinct points in $\mathbb{D}$ such that each z_k has at least two preimages by φ. Then C_φ is embeddable into a C_0-semigroup on H^2.*

Proof Since φ is an inner function such that $\varphi(0) = 0$, it comes that C_φ is an isometry on H^2. Moreover, by hypothesis, for $K \in \mathbb{N}$, there exist $(w_1, \dots, w_K)$ and $(w_1', \dots, w_K') \in \mathbb{D}^K$ such that

$$\forall 1 \leq i \leq K, \quad w_i \neq w_i' \text{ and } \varphi(w_i) = \varphi(w_i') = z_i.$$

For each $1 \leq i \leq K$, define the function $f_i \in H^2$ by $f_i = k_{w_i} - k_{w_i'}$. Then, for all $f \in H^2$ and for every $1 \leq i \leq K$,

$$\langle C_\varphi f, f_i \rangle_2 = \langle f \circ \varphi, k_{w_i} \rangle_2 - \langle f \circ \varphi, k_{w_i'} \rangle_2 = f \circ \varphi(w_i) - f \circ \varphi(w_i') = 0.$$

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Therefore, we obtain that $(f_i)_{1\le i\le K} \subset \operatorname{Im}(C_\varphi)^\perp$. Since $(f_i)_{1\le i\le K}$ is a set of K linearly independent functions, where K is arbitrary large, we get that

$$\operatorname{codim}(\operatorname{Im}(C_\varphi)) = \dim(\operatorname{Im}(C_\varphi)^\perp) = \infty.$$

Finally, we conclude with Theorem 1.1 and then C_φ is embeddable into a C_0-semigroup on H^2. ∎

Recall that if φ is an elliptic automorphism, as in Section 2.2, then C_φ is embeddable into a semigroup of composition operators on H^2 (see (2.1) and Remark 3.1). We refer the reader to [6] for the remaining automorphism cases, for which there exist natural embeddings thanks to Remark 3.1.

Theorem 3.3 *Every composition operator C_φ which is similar to an isometry on H^2 is embeddable into a C_0-semigroup $(T_t)_{t\ge 0}$ on H^2, which is not a semigroup of composition operators, unless φ is an automorphism.*

Proof Let $\varphi : \mathbb{D} \to \mathbb{D}$ be an analytic function. Then C_φ is similar to an isometry on H^2 if and only if φ is inner and there exists $\alpha \in \mathbb{D}$ such that $\varphi(\alpha) = \alpha$. The case when φ is an elliptic automorphism is already done thanks to Remark 3.1. From now on, assume that φ is a nonautomorphic inner function with a fixed point $\alpha \in \mathbb{D}$.

Let us first remark that φ is not injective. Indeed,

- if φ is a Blaschke product as (2.2), then φ is not injective since 0 has at least two preimages;
- if φ is not a Blaschke product, then according to Frostman's theorem, the map $\tau_a \circ \varphi = \frac{a-\varphi}{1-\bar{a}\varphi}$ is a Blaschke product B with simple zeros for almost all $a \in \mathbb{D}$. Therefore, $\varphi = \tau_a^{-1}(B)$ is not injective.

In that case, there is no semiflow of analytic self-maps of $\mathbb{D}$ in which φ is embeddable. Therefore, from Remark 3.1, C_φ is not embeddable into a C_0-semigroup of composition operators on H^2. For $\varphi(\alpha) = \alpha$, let us consider the application defined as follows

$$\psi := \tau_\alpha \circ \varphi \circ \tau_\alpha.$$

Then ψ is an inner function such that $\psi(0) = 0$. Thus, C_ψ is an isometry of H^2 and we get $C_\varphi = C_{\tau_\alpha} \circ C_\psi \circ C_{\tau_\alpha}$. Since C_{τ_α} is an isomorphism of H^2, it remains to show that C_ψ is embeddable into a C_0-semigroup on H^2.

- If ψ is a finite Blaschke product B, the conclusion follows from Lemmas 2.1 and 3.2, taking $(z_k)_{k\ge 0}$ in $\mathbb{D}\backslash B(\operatorname{Zero}(B'))$.
- If ψ is inner but not a finite Blaschke product, according to Frostman's theorem, the map $\tau_\gamma \circ \psi =: B$ is a Blaschke product with simple zeros for almost all $\gamma \in \mathbb{D}$. In that case, we have

$$C_\psi H^2 = \{f \circ \psi : f \in H^2\} = \{(f \circ \tau_\gamma) \circ (\tau_\gamma \circ \psi) : f \in H^2\} = \{g \circ B : g \in H^2\} = C_B H^2. \tag{3.1}$$

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Denote by $(w_k)_{k\geq 1}$ the sequence of simple zeros of B. Considering the sequence $(k_{w_i} - k_{w_j})_{i,j\geq 1, i\neq j}$ of linearly independent functions of H^2, we deduce from (3.1) that

$$\operatorname{codim}(\operatorname{Im}(C_\psi)) = \dim(\operatorname{Im}(C_B)^\perp) = \infty.$$

The embedding of C_ψ follows from Theorem 1.1.

Finally, we conclude that C_φ is embeddable into the C_0-semigroup $(C_{\tau_\alpha} T_t C_{\tau_\alpha})_{t\geq 0}$ on H^2, where C_ψ is embedded into a C_0-semigroup denoted by $(T_t)_{t\geq 0}$ on H^2. ■

In order to describe the semigroup in which C_φ is embeddable, we need the following lemma that appears in [15, Lemma 5]. For the sake of completeness, we include a slightly different proof.

Lemma 3.4 *Let ψ be an inner function such that $\psi(0) = 0$ and such that ψ is not an automorphism. Then $\bigcap_{n\geq 0} C_\psi^n H_0^2 = \{0\}$ and thus $\bigcap_{n\geq 0} C_\psi^n H^2 = \mathbb{C}\mathbb{1}$, where $\mathbb{1}$ stands for the constant function equal to* 1.

Proof Let $g \in \bigcap_{n\geq 0} C_\psi^n H_0^2$. Then for each $n \geq 1$, there exists $f_n \in H_0^2$ such that $g(z) = f_n(\psi^{[n]}(z))$ for every $z \in \mathbb{D}$ with $\psi^{[n]} = \psi \circ \cdots \circ \psi$ (n times). Moreover, $\|g\|_2 = \|f_n\|_2$ since C_ψ is isometric. Note that if $g \neq 0$, then there exists $z_0 \in \mathbb{D}$, $z_0 \neq 0$ such that $|g(z_0)| > 0$. Since $f_n \in H_0^2$, $f_n(0) = 0$, there exists $g_n \in H^2$ such that $f_n(z) = z g_n(z)$ for every $z \in \mathbb{D}$, with $\|g_n\|_2 = \|g\|_2$. Finally, we get that

$$\begin{aligned} \left|f_n(\psi^{[n]}(z_0))\right| &= \left|\psi^{[n]}(z_0)\right| \left|g_n(\psi^{[n]}(z_0))\right| \\ &= \left|\psi^{[n]}(z_0)\right| \left|\langle g_n, k_{\psi^{[n]}(z_0)}\rangle_2\right| \\ &\leq \left|\psi^{[n]}(z_0)\right| \|g\|_2 \frac{1}{\sqrt{1 - \left|\psi^{[n]}(z_0)\right|^2}}. \end{aligned}$$

Since 0 is the Denjoy–Wolff point of ψ, we get $\psi^{[n]}(z_0) \underset{n\to\infty}{\longrightarrow} 0$. Then $\left|f_n(\psi^{[n]}(z_0))\right| \underset{n\to\infty}{\longrightarrow} 0$, and we get $g(z_0) = 0$, a contradiction. Consequently, $\bigcap_{n\geq 0} C_\psi^n H_0^2 = \{0\}$.

The second assertion of the lemma follows from the fact that if $f \in \bigcap_{n\geq 0} C_\psi^n H^2$ then $f - f(0)\mathbb{1} \in \bigcap_{n\geq 0} C_\psi^n H_0^2$. Indeed, in that case, for each $n \geq 1$, there exists $h_n \in H^2$ such that $f = C_\psi^n h_n$. Thus, $f - f(0)\mathbb{1} = C_\psi^n(h_n - f(0)\mathbb{1})$. Since $f(0) = h_n(\psi^{[n]}(0)) = h_n(0)$, we have $f - f(0)\mathbb{1} = C_\psi^n(h_n - h_n(0)\mathbb{1})$ with $h_n - h_n(0)\mathbb{1} \in H_0^2$. Therefore, $f - f(0)\mathbb{1} \in \bigcap_{n\geq 0} C_\psi^n H_0^2$. ■

Corollary 3.5 *Let φ be an inner function with a fixed point $\alpha \in \mathbb{D}$ and such that φ is not an automorphism. Then C_φ on H^2 is embeddable into the C_0-semigroup*

$$(M_{e^{it\theta}} \oplus C_{\tau_\alpha} U^* S_t U C_{\tau_\alpha})_{t\geq 0},$$

with respect to the decomposition $\mathbb{C}\mathbb{1} \oplus H_0^2$ *where* $U : H_0^2 \to L^2(\mathbb{R}_+, Im(C_\psi)^\perp)$ *is unitary,* $\theta \in \mathbb{R}$ *and* $(S_t)_{t\geq 0}$ *is the right semigroup on* $L^2(\mathbb{R}_+, Im(C_\psi)^\perp)$ *defined by*

$$(S_t f)(s) = \begin{cases} f(s-t) & s-t \geq 0 \\ 0 & s-t < 0 \end{cases}.$$

Proof First note that

$$C_\varphi = C_{\tau_\alpha} C_\psi C_{\tau_\alpha}, \tag{3.2}$$

where ψ is an inner function such that $\psi(0) = 0$ and ψ is not an automorphism. Therefore, C_ψ is isometric. Using Wold's decomposition and Lemma 3.4, we obtain that $H^2 = F \oplus^\perp G$ with $F := \bigcap_{n\geq 0} C_\psi^n H^2 = \mathbb{C}\mathbb{1}$ and, by the properties of the orthogonal direct sum, $G := \bigoplus_{n\geq 0} C_\psi^n(H^2 \ominus C_\psi H^2) = H_0^2$. In that case, $(C_\psi)_{|F}$ is unitary on a vector space of dimension 1. Therefore, $(C_\psi)_{|F} = M_{e^{i\theta}}$ for some $\theta \in \mathbb{R}$. On the other side, $(C_\psi)_{|G}$ is unitarily equivalent to the right shift S on $\ell^2(\mathbb{N}, \mathrm{Im}(C_\psi)^\perp)$. According to the embedding of S by [10, Proposition V.1.18], we obtain the embedding of C_ψ into the C_0-semigroup $(M_{e^{it\theta}} \oplus U^* S_t U)_{t\geq 0}$ where $U : H_0^2 \to L^2(\mathbb{R}_+, \mathrm{Im}(C_\psi)^\perp)$ is unitary, $\theta \in \mathbb{R}$ and $(S_t)_{t\geq 0}$ is the right semigroup on $L^2(\mathbb{R}_+, \mathrm{Im}(C_\psi)^\perp)$. We conclude the proof using (3.2). ∎

Remark 3.6 Another special case is when the symbol φ is a *linear fractional map* of the unit disc. Indeed, we have a complete characterization of the embedding of φ according to its fixed points by [5, Proposition 3.4]. Then, we have the natural embedding of C_φ by Remark 3.1. However, there are some examples where the embedding of C_φ into a C_0-semigroup of composition operators on H^2 is not possible. To that aim, it suffices to consider φ the attractive elliptic function on $\mathbb{D}$ defined by $\varphi(z) = \frac{z}{z-2}$. Indeed, φ does not satisfy the following required condition:

$$\left|\overline{\alpha} - \frac{1}{\beta}\right| l \leq |\varphi'(\alpha)| \left|1 - \frac{\alpha}{\beta}\right|, \tag{3.3}$$

where $\alpha \in \mathbb{D}$ is its Denjoy–Wolff point, $\beta \in (\mathbb{C} \cup \{\infty\}) \backslash \mathbb{D}$ its repulsive fixed point, and $l = l(\varphi'(\alpha))$ the length of the canonical spiral associated with $\varphi'(\alpha) \in \mathbb{D}\backslash\{0\}$. Consequently, φ is not embeddable into a semiflow of analytic self-maps of $\mathbb{D}$. We refer the reader to [6] for more information about Denjoy–Wolff theory.

Let us now introduce weighted composition operators. Let $w \in H^2$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. We define the *weighted composition operator* $C_{w,\varphi}$ with symbol φ and weight w by

$$C_{w,\varphi} : f \longmapsto w(f \circ \varphi), \quad f \in H^2.$$

Kumar and Partington [14] proved that $C_{w,\varphi}$ is an isometry on H^2 if and only if φ is inner, $\|w\|_2 = 1$ and $\langle w, w\varphi^n \rangle_2 = 0$ for every $n \geq 1$.

Chalendar and Partington [8] obtained the following result: if φ is inner, then there exists a weight $w \in H^2$ such that $C_{w,\varphi}$ is an isometry on H^2.

The combination of these two results gives the following sufficient condition about the embedding of weighted composition operators. The main interest of this assertion

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is that, provided that we make an appropriate choice of the weight, it is not required that the symbol of the composition operator has a fixed point in $\mathbb{D}$.

Theorem 3.7 *Let φ be an inner function. Then there exists a weight $w \in H^2$ such that $C_{w,\varphi}$ is embeddable into a C_0-semigroup on H^2.*

Proof Since φ is inner, there exists a weight $w \in H^2$ such that $C_{w,\varphi}$ is an isometry on H^2. Moreover, w satisfied $\|w\|_2 = 1$ and $\langle w, w\varphi^n \rangle_2 = 0$ for every $n \geq 1$. It remains to show that $\operatorname{codim}(\operatorname{Im}(C_{w,\varphi})) = \infty$. Note that, for every $\lambda \in \mathbb{D}$ and $f \in H^2$, we have

$$\langle C_{w,\varphi} f, k_\lambda \rangle_2 = w(\lambda) C_\varphi f(\lambda) = w(\lambda)(f \circ \varphi(\lambda)).$$

Then, we deduce that $\langle C_{w,\varphi} f, k_\lambda \rangle_2 = 0$ if and only if $w(\lambda) = 0$ or $f \circ \varphi(\lambda) = 0$. Take $w = Bm$ where B is an infinite Blaschke product associated with a sequence $(\lambda_n)_{n \geq 1} \subset \mathbb{D}$ satisfying $\sum_{n \geq 1}(1 - |\lambda_n|) < \infty$ and where $m \in H^2$ satisfies $\|m\|_2 = 1$ and $\langle m, m\varphi^n \rangle_2 = 0$ for every $n \geq 1$. Notice that since B is inner, we get $\|w\|_2 = 1$ and $\langle w, w\varphi^n \rangle_2 = 0$ for every $n \geq 1$. Thus, $C_{w,\varphi}$ is an isometry on H^2 such that

$$\langle C_{w,\varphi} f, k_\lambda \rangle_2 = B(\lambda) m(\lambda) C_\varphi f(\lambda) = 0, \quad \lambda \in \{\lambda_n : n \geq 1\} \subset \mathbb{D}.$$

In other words, we deduce that $\operatorname{Span}_{H^2}(k_\lambda : \lambda \in \operatorname{Zero}(B)) \subset \operatorname{Im}(C_{w,\varphi})^\perp$ and

$$\operatorname{codim}(\operatorname{Im}(C_{w,\varphi})) = \dim(\operatorname{Im}(C_{w,\varphi})^\perp) = \infty.$$

Finally, for such a $w \in H^2$, $C_{w,\varphi}$ is embeddable into a C_0-semigroup on H^2 by Theorem 1.1. ∎

Remark 3.8 The form of the C_0-semigroup in which $C_{w,\varphi}$ is embeddable is less explicit than the one given in Corollary 3.5. Indeed, for φ an inner function and $w \in H^2$ the weight such that $C_{w,\varphi}$ is an isometry on H^2, by the Wold's decomposition, $H^2 = F \oplus^\perp G$ where $(C_{w,\varphi})_{|F}$ is unitary and $(C_{w,\varphi})_{|G}$ is unitarily equivalent to the right shift on $\ell^2(\mathbb{N}, \operatorname{Im}(C_{w,\varphi})^\perp)$. Then, by [10, Theorem V.1.14], $C_{w,\varphi}$ is embeddable into the C_0-semigroup

$$\left(Z^* \left(e^{t \log(m)}\right) Z \oplus U^* S_t U\right)_{t \geq 0},$$

where μ is a Borel measure, $m \in L^\infty(\sigma((C_{w,\varphi})_{|F}), \mu)$ is measurable and

$$Z : F \to L^2(\sigma((C_{w,\varphi})_{|F}), \mu), \;\; U : G \to L^2(\mathbb{R}_+, \operatorname{Im}(C_{w,\varphi})^\perp),$$

are unitary operators.

3.2 Analytic Toeplitz operators on H^2

Let $\varphi \in L^\infty(\mathbb{T})$. We define the *Toeplitz operator* T_φ with symbol φ by

$$T_\varphi : f \longmapsto P_+(\varphi f), \quad f \in H^2,$$

where P_+ denotes here the Riesz projection, i.e., the orthogonal projection of $L^2(\mathbb{T})$ onto H^2. It is a bounded operator on H^2 whose norm is equal to $\|\varphi\|_\infty$. See [12, Section 4] for the main properties about Toeplitz operators with symbols in $L^\infty(\mathbb{T})$. From now

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on, assume that $\varphi \in H^\infty$ and note that T_φ is then the multiplication operator by φ. We have $\ker(T_\varphi) = \{0\}$ and $\ker(T_\varphi^*) = \mathcal{K}_\theta$, thus

$$\operatorname{codim}(\operatorname{Im}(T_\varphi)) = \dim(\ker(T_\varphi^*)) = \dim(\mathcal{K}_\theta),$$

where $\mathcal{K}_\theta := (\theta H^2)^\perp$ is the *model space* associated with θ the inner part of φ. We refer the reader to [12] for a very nice introduction to model space theory. We also recall that T_φ is an isometry on H^2 if and only if φ is inner.

Theorem 3.9 *Let φ be a non constant inner function. Then T_φ is embeddable into a C_0-semigroup on H^2 if and only if φ is not a finite Blaschke product. Moreover, the operators of the semigroup are analytic Toeplitz operators if and only if φ does not have any zero in $\mathbb{D}$.*

Proof Let φ be a non constant inner function. Then T_φ is an isometry on H^2. By Theorem 1.1, T_φ is embeddable if and only if $\operatorname{codim}(\operatorname{Im}(T_\varphi)) = \dim(\mathcal{K}_\varphi) = \infty$. We deduce then easily that T_φ is embeddable into a C_0-semigroup on H^2 if and only if φ is not a finite Blaschke product (see [12, Proposition 5.5.19]). Denote by $(R_t)_{t\geq 0}$ this semigroup. Let us recall that the commutant $\{S\}'$ of S on H^2 is given by

$$\{S\}' = \left\{T_\psi : \psi \in H^\infty\right\}.$$

In that case, $(R_t)_{t\geq 0}$ is a semigroup of analytic Toeplitz operators if and only there exists $C \in \operatorname{Hol}(\mathbb{D})$ satisfying $\sup\{\operatorname{Re}(C(z)) : z \in \mathbb{D}\} < \infty$ and such that $R_t = T_{e^{tC}}$ for every $t \geq 0$ [16]. In particular, we get $R_1 = T_\varphi = T_{e^C}$ and $\varphi = e^C$, which does not vanish on $\mathbb{D}$. Reciprocally, if φ does not vanish on $\mathbb{D}$, then φ is an inner singular function of the form

$$\varphi(z) = \exp\left\{-\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right\}, \quad z \in \mathbb{D},$$

for μ a finite positive measure of Borel on $\mathbb{T}$ such that μ is singular with respect to the Lebesgue measure m. It is easy to see, by considering, for every $t \geq 0$, the bounded analytic functions

$$\varphi_t(z) = \exp\left\{-t\int_{\mathbb{T}} \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)\right\}, \quad z \in \mathbb{D},$$

that T_φ is embeddable into the C_0-semigroup $(M_{\varphi_t} = T_{\varphi_t})_{t\geq 0}$ of analytic Toeplitz operators on H^2. ∎

The case of the isometric Toeplitz operators is now complete. The aim of the rest of this section is to investigate other analytic Toeplitz operators.

Lemma 3.10 *Let φ be an outer function. Then T_φ is embeddable into a C_0-semigroup of analytic Toeplitz operators on H^2.*

Proof It is an immediate consequence of the canonical representation of an outer function. ∎

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Proposition 3.11 *Let $\varphi = (BS_\mu)\varphi_e \in H^\infty$ where B is a Blaschke product, S_μ is an inner singular function and φ_e is an outer function. Assume that φ is not an outer nor an inner function. Then the following assertions hold:*

(i) *if $B \equiv 1$, T_φ is embeddable into a C_0-semigroup of analytic Toeplitz operators on H^2.*

(ii) *if $S_\mu \equiv 1$ and if B is a non constant finite Blaschke product, T_φ is not embeddable into a C_0-semigroup on H^2.*

Proof We have:

(i) If $B \equiv 1$, then $\varphi = S_\mu \varphi_e$ which does not vanish on $\mathbb{D}$. Let us remark that $T_\varphi = T_{S_\mu \varphi_e} = M_{S_\mu} M_{\varphi_e}$. Each term is embeddable into a C_0-semigroup on H^2 which commutes, then T_φ is embeddable into the product of these two semigroups.

(ii) If $S_\mu \equiv 1$ and B is a non constant finite Blaschke product, then we get

$$\operatorname{codim}(\operatorname{Im}(T_\varphi)) = \dim(\mathcal{K}_B) \notin \{0, \infty\}.$$

We conclude with [10, Theorem V.1.7]. ∎

Let us remark that according to Proposition 3.11, the remaining open question is the following.

Question 3.12 *Do we have the embedding of T_φ when $\varphi = B\phi$, with B a non constant Blaschke product and ϕ a nonvanishing analytic function on $\mathbb{D}$?*

For that purpose, let us just note that on one side T_B is embeddable if and only if B is an infinite Blaschke product by Theorem 3.9. On the other side, T_ϕ is embeddable from Proposition 3.11(i). The following examples show the difficulty and the interest of this open question.

- Let B_1 be a finite Blaschke product and B_2 be an infinite Blaschke product. Then $T_{B_1 B_2}$ is embeddable by Theorem 3.9, whereas T_{B_1} is not embeddable.
- Let B be an infinite Blaschke product and S_μ be a singular inner function. Then T_B and T_{S_μ} are embeddable into a C_0-semigroup denoted, respectively, by $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$ by Theorem 3.9. Moreover, T_{BS_μ} is also embeddable into a C_0-semigroup, which is not the product of $(A_t)_{t\geq 0}$ and $(B_t)_{t\geq 0}$, even though T_B and T_{S_μ} commute.

We end this section with a result on the embedding of Toeplitz operators whose symbol are polynomials.

Corollary 3.13 *Let $n \geq 1$ and $P \in \mathcal{P}_n$, where $\mathcal{P}_n$ is the space of polynomials of degree at most n. Then T_P is embeddable into a C_0-semigroup on H^2 if and only if P does not have any zero in $\mathbb{D}$.*

Proof Let $n \geq 1$ and $P \in \mathcal{P}_n$ of the form

$$P(z) = a \prod_{k=1}^{n} (z - \alpha_k) \prod_{j=1}^{m} (z - \beta_j),$$

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where $a \in \mathbb{C}\backslash\{0\}$, $|\alpha_k| < 1$ for every $1 \leq k \leq n$ and $|\beta_j| \geq 1$ for every $1 \leq j \leq m$. It follows that $P(z) = B(z)F(z)$ where B is the finite Blaschke product associated with the sequence $(\alpha_k)_{1\leq k\leq n}$, and F is the outer function defined by $F(z) = a\prod_{k=1}^{n}(1 - \overline{\alpha_k}z)\prod_{j=1}^{m}(z-\beta_j)$. Then:

- if P does not have any zero in $\mathbb{D}$, i.e., $\alpha_k \notin \mathbb{D}$ for every $1 \leq k \leq n$, then $B \equiv 1$ and P is outer. Therefore, T_P is embeddable into a C_0-semigroup on H^2 according to Lemma 3.10.
- if P has at least one zero in $\mathbb{D}$, then $B \not\equiv 1$. By Proposition 3.11 (ii), T_P is not embeddable into a C_0-semigroup on H^2. ■

4 Isometric operators and properties of semigroups

In this last section, we state two quite obvious results concerning the properties of the semigroup in terms of isometry or compactness where the operator embedded is isometric.

Proposition 4.1 *Let $V \in \mathcal{L}(H)$ be isometric. If V is embeddable into a C_0-semigroup of contractions $(V_t)_{t\geq 0}$ on H, then V_t is isometric for every $t \geq 0$.*

Proof Let us remark that since V is an isometry, $V^n = V_n$ is also an isometry for every $n \in \mathbb{N}$. Assume that there exists $t_0 > 0$ such that V_{t_0} is not isometric. In that case, since V_{t_0} is a contraction, there exists $x_0 \in H$, $\|x_0\| = 1$ such that $\|V_{t_0}x_0\| < 1$. But, for every $N > t_0$, we have on one hand

$$\|V_{N-t_0}V_{t_0}x_0\| = \|V_N x_0\| = 1,$$

and on the other hand

$$\|V_{N-t_0}V_{t_0}x_0\| \leq \|V_{t_0}x_0\| < 1.$$

We obtain a contradiction, and so V_t is isometric for every $t > 0$. ■

Proposition 4.2 *Let $V \in \mathcal{L}(H)$ be isometric. If V is embeddable into a C_0-semigroup $(V_t)_{t\geq 0}$ on H, then V_t is not compact for every $t \geq 0$.*

Proof Assume that there exists $t_0 > 0$ such that V_{t_0} is compact. Since $\mathcal{K}(H)$ is a bilateral ideal, it comes that V_t is compact for every $t \geq t_0$ from the algebraic property of the semigroup. It comes also that, for every orthonormal sequence $(e_n)_{n\geq 0}$ of H, $\|V_t e_n\| \underset{n\to\infty}{\longrightarrow} 0$ for every $t \geq t_0$. But, since V is isometric, $V_N = V^N$ is also isometric for every $N \in \mathbb{N}$, and we get

$$\|V_N e_n\| = \|e_n\| = 1.$$

For $N \geq t_0$, we obtain a contradiction and so V_t is not compact for every $t > 0$. ■

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