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AN ANALOGUE OF AN IDENTITY OF JACOBI

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Abstract

H. H. Chan, K. S. Chua and P. Solé ['Quadratic iterations to π associated to elliptic functions to the cubic and septic base', *Trans. Amer. Math. Soc.* **355**(4) (2002), 1505–1520] found that, for each positive integer d, there are theta series A_d, B_d and C_d of weight one that satisfy the Pythagoras-like relationship $A_d^2 = B_d^2 + C_d^2$. In this article, we show that there are two collections of theta series $A_{b,d}, B_{b,d}$ and $C_{b,d}$ of weight one that satisfy $A_{b,d}^2 = B_{b,d}^2 + C_{b,d}^2$, where b and d are certain integers.

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1. Introduction

One of the most famous identities of Jacobi states that

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2}\Big)^2 + \Big(\sum_{m,n=-\infty}^{\infty} q^{(m+1/2)^2+(n+1/2)^2}\Big)^2. \tag{1.1}$$

One can view (1.1) as a solution to

$$A^2 = B^2 + C^2, \tag{1.2}$$

where A, B and C are theta series of weight one. This identity is instrumental in the parametrisation of Gauss' arithmetic–geometric mean by modular forms [2, 8].

In [5], Chan *et al.*, motivated by the study of codes and lattices, found that, for any positive integer d,

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{2(m^2+mn+dn^2)}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+mn+dn^2}\Big)^2 + \Big(\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+(m+1/2)n+dn^2)}\Big)^2. \tag{1.3}$$

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Identity (1.3) provides an infinite number of solutions in theta functions of weight one to (1.2). For more information on this generalised Jacobi identity, see [6, 7].

Recently, while studying theta series associated with binary quadratic forms of discriminant −15, we discovered the identity

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{2m^2+mn+2n^2}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{2m^2+mn+2n^2}\Big)^2 + \Big(2\sum_{m,n=-\infty}^{\infty} q^{2(2(m+1/2)^2+(m+1/2)n+2n^2)}\Big)^2. \tag{1.4}$$

We establish the following analogue of (1.3) for which (1.4) is a special case.

THEOREM 1.1. *Let d be any positive integer and let $1 \le b \le d-1$. Then*

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{dm^2+bmn+dn^2}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{dm^2+bmn+dn^2}\Big)^2 + \Big(2\sum_{m,n=-\infty}^{\infty} q^{2(d(m+1/2)^2+b(m+1/2)n+dn^2)}\Big)^2. \tag{1.5}$$

When $d = 2$ and $b = 1$, we recover (1.4) from (1.5). The proof of (1.5) is given in Section 2.

Our discovery of (1.5) provides a motivation for deriving the following two-variable version of (1.3): that is,

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{2(bm^2+bmn+dn^2)}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{bm^2+bmn+dn^2}\Big)^2 + \Big(\sum_{m,n=-\infty}^{\infty} q^{2(b(m+1/2)^2+b(m+1/2)n+dn^2)}\Big)^2. \tag{1.6}$$

Observe that, when $b = 1$, (1.6) implies (1.3). We give a proof of (1.6) in Section 3.

2. Proof of (1.5)

The Jacobi one-variable theta functions are defined by

$$\vartheta_2(q) = \sum_{j=-\infty}^{\infty} q^{(j+1/2)^2},$$

$$\vartheta_3(q) = \sum_{j=-\infty}^{\infty} q^{j^2}$$

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and

$$\vartheta_4(q) = \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2}.$$

We first express the theta functions in (1.5) in terms of $\vartheta_j(q), j = 2, 3, 4$.

LEMMA 2.1. *For* $|q| < 1$,

$$\mathcal{A}_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{dm^2+bmn+dn^2} = \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) + \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b}), \tag{2.1}$$

$$\mathcal{B}_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{dm^2+bmn+dn^2} = \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) - \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b}) \tag{2.2}$$

and

$$C_{b,d} = 2\sum_{m,n=-\infty}^{\infty} q^{2(d(m+1/2)^2+b(m+1/2)n+dn^2)} = \vartheta_2(q^{d+b/2})\vartheta_2(q^{d-b/2}). \tag{2.3}$$

PROOF. We observe that

$$dm^2 + bmn + dn^2 = \begin{pmatrix} m & n \end{pmatrix} \begin{pmatrix} d & b/2 \\ b/2 & d \end{pmatrix} \begin{pmatrix} m \\ n \end{pmatrix}.$$

Next, since

$$\begin{pmatrix} d & b/2 \\ b/2 & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} d+b/2 & 0 \\ 0 & d-b/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

we find that

$$dm^2 + bmn + dn^2 = \frac{2d+b}{4}(m+n)^2 + \frac{2d-b}{4}(m-n)^2.$$

Therefore,

$$\begin{aligned}\sum_{m,n=-\infty}^{\infty} q^{dm^2+bmn+dn^2} &= \sum_{m,n=-\infty}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} \\ &= \sum_{\substack{m,n=-\infty \\ m+n \text{ even}}}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} + \sum_{\substack{m,n=-\infty \\ m+n \text{ odd}}}^{\infty} q^{(2d+b)(m+n)^2/4+(2d-b)(m-n)^2/4} \\ &= \vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}) + \vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b}),\end{aligned}$$

which completes the proof of (2.1). The proof of (2.2) is similar to the proof of (2.1).

To prove (2.3), we need the identity

$$\sum_{m=-\infty}^{\infty} (-1)^m q^{(m+1/2)^2} = 0. \tag{2.4}$$

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Identity (2.4) is true because

$$\sum_{m=-\infty}^{\infty}(-1)^m q^{(m+1/2)^2}=\sum_{s=-\infty}^{\infty}(-1)^s q^{(s-1/2)^2}=\sum_{t=-\infty}^{\infty}(-1)^{t+1} q^{(t+1/2)^2}.$$

From (2.4), we deduce that, for any integer ℓ,

$$\sum_{m=-\infty}^{\infty}(-1)^m q^{(m+\ell+1/2)^2}=0. \tag{2.5}$$

A consequence of (2.5) is that

$$\sum_{n=-\infty}^{\infty} q^{(2n+\ell+1/2)^2}=\sum_{n=-\infty}^{\infty} q^{(2n+1+\ell+1/2)^2}. \tag{2.6}$$

We are now ready to prove (2.3). Write

$$C_{b,d}=2\sum_{m,n=-\infty}^{\infty} q^{(2d+b)(m+1/2+n)^2/2+(2d-b)(m+1/2-n)^2/2}.$$

Let $k=m-n$. Then

$$\begin{aligned}C_{b,d}&=2\sum_{k=-\infty}^{\infty} q^{(2d-b)(k+1/2)^2/2}\sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2}\\&=\sum_{k=-\infty}^{\infty} q^{(2d-b)(k+1/2)^2/2}\sum_{s=-\infty}^{\infty} q^{(2d+b)(s+1/2)^2/2}=\vartheta_2(q^{(2d-b)/2})\vartheta_2(q^{(2d+b)/2}),\end{aligned}$$

which is (2.3). The last equality follows by writing

$$\begin{aligned}2\sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2}&=\sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1/2)^2/2}+\sum_{n=-\infty}^{\infty} q^{(2d+b)(2n+k+1+1/2)^2/2}\\&=\sum_{s=-\infty}^{\infty} q^{(2d+b)(s+1/2)^2/2},\end{aligned}$$

where we have used (2.6) in the first equality. □

Using (2.1) and (2.2), we deduce that

$$\mathcal{A}_{b,d}^2-\mathcal{B}_{b,d}^2=4\vartheta_2(q^{2d+b})\vartheta_2(q^{2d-b})\vartheta_3(q^{2d+b})\vartheta_3(q^{2d-b}).$$

Next, it is known from Jacobi's triple product identity that

$$\vartheta_2(q)=2q^{1/4}\prod_{j=1}^{\infty}(1-q^{2j})(1+q^{2j})^2$$

and

$$\vartheta_3(q)=\prod_{j=1}^{\infty}(1-q^{2j})(1+q^{2j-1})^2.$$

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Therefore,

$$2\vartheta_2(q^2)\vartheta_3(q^2) = \vartheta_2^2(q). \tag{2.7}$$

Replacing q^2 by q and using (2.3), we deduce that

$$\mathcal{A}_{b,d}^2 - \mathcal{B}_{b,d}^2 = C_{b,d}^2$$

and the proof of (1.5) is complete.

It is possible to derive (2.7) without using Jacobi's triple product identity. For more details, see [4, page 58].

When $d = 1$ and $b = 0$, (1.5) becomes

$$\Big(\sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}\Big)^2 = \Big(\sum_{m,n=-\infty}^{\infty} (-1)^{m+n} q^{m^2+n^2}\Big)^2 + \Big(2\sum_{m,n=-\infty}^{\infty} q^{2((m+1/2)^2+n^2)}\Big)^2,$$

which reduces to

$$\vartheta_3^4(q) = \vartheta_4^4(q) + 4\vartheta_2^2(q^2)\vartheta_3^2(q^2). \tag{2.8}$$

By (2.7), we arrive at (1.1). Next, (2.8) can then be written as

$$\vartheta_3^4(q) + \vartheta_2^4(q) = \vartheta_3^4(q) - \vartheta_2^4(q) + 8\vartheta_2^2(q^2)\vartheta_3^2(q^2). \tag{2.9}$$

Identity (2.9) appeared in [1, page 140] and the functions

$$\vartheta_3^4(q) + \vartheta_2^4(q), \quad \vartheta_3^4(q) - \vartheta_2^4(q) = \vartheta_4^4(q) \quad \text{and} \quad 2\vartheta_2^2(q)\vartheta_3^2(q)$$

play important roles in Ramanujan's theory of elliptic functions to the quartic base (see [3, Theorem 2.6(b)] and [1, (1.10) and (1.11)]).

3. Proof of (1.6)

The proof of (1.6) is similar to the proof of (1.3). First, we need a lemma.

LEMMA 3.1. *Let* $0 < b < 4d$. *Then*

$$A_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2(bm^2+bmn+dn^2)} = \vartheta_3(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_2(q^{2b})\vartheta_2(q^{2(4d-b)}), \tag{3.1}$$

$$B_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{bm^2+bmn+dn^2} = \vartheta_4(q^b)\vartheta_4(q^{4d-b}) \tag{3.2}$$

and

$$C_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2(b(m+1/2)^2+b(m+1/2)n+dn^2)} = \vartheta_2(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_3(q^{2b})\vartheta_2(q^{2(4d-b)}). \tag{3.3}$$

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PROOF. The proof of (3.1) follows by writing $A_{b,d}$ as

$$A_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2b(m+n/2)^2+n^2(4d-b)/2}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$, we find that

$$\begin{aligned} A_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell)^2+2\ell^2(4d-b)} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(\ell+1/2)^2(4d-b)} \\ &= \vartheta_3(q^{2b})\vartheta_3(q^{2(4d-b)}) + \vartheta_2(q^{2b})\vartheta_2(q^{2(4d-b)}), \end{aligned}$$

and this completes the proof of (3.1). Next, write $B_{b,d}$ as

$$B_{b,d} = \sum_{m,n=-\infty}^{\infty} (-1)^{m-n} q^{b(m+n/2)^2+n^2(4d-b)/4}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$ and using (2.5), we find that

$$\begin{aligned} B_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} (-1)^m q^{2b(m+\ell)^2+2\ell^2(4d-b)} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(\ell+1/2)^2(4d-b)} \\ &= \sum_{m,\ell=-\infty}^{\infty} (-1)^\ell q^{(4d-b)\ell^2} \sum_{m=-\infty}^{\infty} (-1)^{m+\ell} q^{b(m+\ell)^2} \\ &= \vartheta_4(q^{4d-b})\vartheta_4(q^b), \end{aligned}$$

and (3.2) follows. Finally, to prove (3.3), write

$$C_{b,d} = \sum_{m,n=-\infty}^{\infty} q^{2b(m+1/2+n/2)^2+2n^2(4d-b)/4}.$$

Splitting the sum into two sums with one summing over even integers $n = 2\ell$ and the other summing over odd integers $n = 2\ell + 1$, we deduce that

$$\begin{aligned} C_{b,d} &= \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1/2)^2+2(2\ell)^2(4d-b)/4} + \sum_{m,\ell=-\infty}^{\infty} q^{2b(m+\ell+1)^2+2(2\ell+1)^2(4d-b)/4} \\ &= \vartheta_2(q^{2b})\vartheta_3(q^{8d-2b}) + \vartheta_3(q^{2b})\vartheta_2(q^{8d-2b}), \end{aligned}$$

and the proof of (3.3) is complete. □

To complete the proof of (1.6), we note that

$$A_{b,d} - C_{b,d} = (\vartheta_3(q^{2b}) - \vartheta_2(q^{2b}))(\vartheta_3(q^{8d-2b}) - \vartheta_2(q^{8d-2b}))$$

and

$$A_{b,d} + C_{b,d} = (\vartheta_3(q^{2b}) + \vartheta_2(q^{2b}))(\vartheta_3(q^{8d-2b}) + \vartheta_2(q^{8d-2b})).$$

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But it is immediate that

$$\vartheta_3(q^4) - \vartheta_2(q^4) = \vartheta_4(q)$$

and

$$\vartheta_3(q^4) + \vartheta_2(q^4) = \vartheta_3(q).$$

Therefore,

$$(\vartheta_3(q^4) - \vartheta_2(q^4))(\vartheta_3(q^4) + \vartheta_2(q^4)) = \vartheta_4(q)\vartheta_3(q) = \vartheta_4^2(q^2),$$

where the last equality follows from [2, page 34]. Therefore,

$$\begin{aligned} A_{b,d}^2 - C_{b,d}^2 &= (\vartheta_3(q^{2b}) - \vartheta_2(q^{2b}))(\vartheta_3(q^{8d-2b}) - \vartheta_2(q^{8d-2b})) \\ &\quad \times (\vartheta_3(q^{2b}) + \vartheta_2(q^{2b}))(\vartheta_3(q^{8d-2b}) + \vartheta_2(q^{8d-2b})) \\ &= \vartheta_4^2(q^b)\vartheta_4^2(q^{4d-b}) = B_{b,d}^2, \end{aligned}$$

and the proof of (1.6) is complete.

4. Concluding remarks

We have found infinitely many solutions to $X^2 + Y^2 = Z^2$, where X, Y and Z are theta series of weight one. The Borweins' identity states that

$$\begin{aligned} \Big(\sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}\Big)^3 &= \Big(\sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}\Big)^3 \\ &\quad + \Big(\sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}\Big)^3, \end{aligned} \tag{4.1}$$

where $\omega = e^{2\pi i/3}$. This is the only example of a solution to $X^3 + Y^3 = Z^3$ with X, Y and Z being theta series of weight one. Are there infinitely many solutions to $X^3 + Y^3 = Z^3$, where X, Y and Z are theta series of weight one, apart from (4.1)? This appears to be an interesting question.

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