FUNDAMENTAL DEFINITION OF THE SOLVENCY CAPITAL REQUIREMENT IN SOLVENCY II

BY

MARCUS C. CHRISTIANSEN AND ANDREAS NIEMEYER

ABSTRACT

It is essential for insurance regulation to have a clear picture of the risk measures that are used. We compare different mathematical interpretations of the Solvency Capital Requirement (SCR) definition from Solvency II that can be found in the literature. We introduce a mathematical modeling framework that enables us to make a mathematically rigorous comparison. The paper shows similarities, differences, and properties such as convergence of the different SCR interpretations. Moreover, we generalize the SCR definition to future points in time based on a generalization of the value at risk. This allows for a sound definition of the Risk Margin. Our study helps to make the Solvency II insurance regulation more consistent.

KEYWORDS

Solvency II, Solvency Capital Requirement, Risk Margin, dynamic value at risk, minimal SCR.

1. INTRODUCTION

Solvency II is the new regulatory framework of the European Union for insurance and reinsurance companies. It will replace the Solvency I regime and is set to become effective probably in 2016. One main aspect of Solvency II is the calculation of the Solvency Capital Requirement (SCR), which is the amount of its own funds (i.e. capital) that an insurance company is required to hold. For the calculation of the market-consistent values of liabilities, Solvency II suggests using a cost-of-capital method and defines the Risk Margin (RM). In order to calculate the SCR, each company can choose between setting up its own internal model and using a provided standard formula. The calculation standards were defined in the documents of the Committee of European Insurance and Occupational Pensions Supervisors (CEIOPS), but they are mainly described only in words. To our knowledge, the only truly

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mathematical definitions for the SCR and the RM currently exist for the standard formula.

Since Solvency II will have a significant impact on the European insurance industry, a large number of papers have already been published on that topic. For example, Devolder (2011) studies the capital requirement under different risk measurements; Eling *et al.* (2007) outline the characteristics of Solvency II; Doff (2008) makes a critical analysis of the Solvency II proposal; Steffen (2008) gives an overview of the project; Filipović (2009) analyzes the aggregation in the standard formula; Holzmüller (2009) focuses on the relationship between the United States' risk-based capital standards, Solvency II, and the Swiss Solvency Test; Sandström (2006) shows the path from historical regulation to Solvency II; and Sandström (2011) gives a detailed overview over the Solvency II project. Wüthrich and Merz (2013) discuss general solvency aspects mathematically rigorously, but do not focus on Solvency II problems. Only a few papers give a mathematically substantiated definition of the Solvency II SCR (e.g. Barrieu *et al.*, 2012; Bauer *et al.*, 2012; Devineau and Loisel, 2009; Kochanski, 2010). All of these papers define the SCR only at time 0. Möhr (2011) gives a rigorous definition for any point in time, but does not specify the definition of a conditional value at risk. Ohlsson and Lauzeningks (2009) also define the SCR for any point in time, but only within a chain ladder framework. Kriele and Wolf (2007) present a fairly general definition for the future value at risk, assuming that the underlying probability space has some specific structure. Another problem is that different mathematical definitions are used. The reason is that the directive of the European Parliament and the Council (2009) describes the SCR only in words, and from a mathematical point of view, there is room for interpretation. This paper yields the first mathematical analysis of similarities and differences of the various interpretations of the SCR.

The RM is supposed to enable the calculation of the liabilities' market-consistent values; however, it is discussed less in the literature. For example, Floreani (2011) studies conceptual issues relating to the RM in a one-period model, Kriele and Wolf (2007) consider different approaches for a RM, Möhr (2011) proposes a framework for the market-consistent valuation of insurance liabilities using cost of capital and shows that the resulting value is sometimes smaller than the sum of best estimate and RM, Salzmann and Wüthrich (2010) analyze the RM in a chain ladder framework, and Wüthrich *et al.* (2011) use the probability distortion to define a RM. Generally, the RM is defined by a cost-of-capital approach and is based on minimal future SCRs. However, no broad definitions for minimal future SCRs currently exist in the literature, which subsequently lacks a mathematically correct definition of the RM. This paper contributes to this problem by introducing a dynamic and minimizing SCR definition.

The paper is structured as follows. In Section 2 we present different interpretations of the fundamental SCR definition. In order to end up with mathematically well-defined SCR definitions, in Section 3 we discuss necessary and

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sufficient assumptions and restrictions. All following sections base on these specifications. Section 4 transfers the SCR definitions to any point in time. Section 5 compares the different definitions. In Sections 6 and 7, we study convergence and invariance properties of the SCR definitions. With the help of the generalized SCR definitions of Section 4, we present a sound definition of the RM in Section 8. Section 9 gives an overview of the main findings.

2. The regulatory framework

In this section we discuss the fundamental definition of the SCR, taking into account regulatory requirements. In the directive of the European Parliament and the Council (2009), which is the binding framework for Solvency II, we find the following two definitions of the SCR:

- Article 101 of the directive requires that the SCR "shall correspond to the value at risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99.5% over a one-year period."
- At the beginning of the directive, an enumeration of recitals is given that has been attached to the directive. Recital 64 of the directive (European Parliament and the Council, 2009, page 24) says that "the Solvency Capital Requirement should be determined as the economic capital to be held by insurance... undertakings in order to ensure... that those undertakings will still be in a position with a probability of at least 99.5%, to meet their obligations to policy holders and beneficiaries over the following 12 months."

However, from a mathematical point of view, there is room for interpretation, and we have to clarify the fundamental definition of the SCR. First we introduce some notation.

Definition 2.1 (assets and liabilities). Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$. We assume that $(K_t^i)_{t\in\mathbb{N}_0}, (H_t^i)_{t\in\mathbb{N}_0}, (L_t)_{t\in\mathbb{N}_0}, (Z_t)_{t\in\mathbb{N}}$ are adapted real-valued stochastic processes $(1 \leq i \leq m)$.

1. Let $K_t^i > 0$ be the capital accumulation function that gives the market value of investment i at time t $(1 \leq i \leq m)$ and let $K_t = (K_t^1, ..., K_t^m)$.
2. Let H_t^i be the units of asset K_t^i that the insurer holds at time $t \in \mathbb{N}_0$ $(1 \leq i \leq m)$ and let $H_t = (H_t^1, ..., H_t^m)$.
3. We define $A_t := H_t \cdot K_t := \sum_{i=1}^m H_t^i K_t^i$ as the market value of the assets that an insurer holds at time t. For technical reasons, we generally assume that $A_t = 0$ implies $H_t = (0, ..., 0)$ which is needed in (2.4).
4. If $A_t \in \mathbb{R} \setminus \{0\}$, we define $\theta_t^i := \frac{H_t^i K_t^i}{H_t \cdot K_t}$ as the proportion that the market value of investment i has on the total market value of the asset portfolio. We call $(\theta_t)_t := (\theta_t^1, ..., \theta_t^m)_t$ the asset strategy of the insurer.
5. Let L_t be the time t market-consistent value of liabilities of the insurer, which is also called technical provisions in Solvency II.
6. Let $N_t = A_t - L_t$ be the net value of assets minus liabilities at time t. In the literature, it is also denoted as available capital. For simplification, we

assume that the basic own funds and the eligible own funds are equal and correspond to the net value.

7. Let Z_t be the sum of all payments that the insurer makes or receives on the interval $(t-1, t]$ in time t money, including any premiums, costs, and benefits from new and old business as well as payments to and from the shareholder. Payments from the insurance company have a negative and payments to the company a positive sign.

The framework focuses on the assets, while we do not specify here how the market-consistent value of liabilities are calculated. For the following investigation, it does not matter if the market-consistent value is derived from a RM approach, a risk-neutral martingale measure, some real-world risk measure, or any other approach. We discuss the RM approach which is in line with the Solvency II framework in Section 8. For a general discussion of the valuation of insurance liabilities, we refer to Wüthrich *et al.* (2010).

The discount factor $\overline{v}(t, t+1)$ for the time period $(t, t+1]$ of any investment $\overline{H}_t$ with corresponding strategy $\overline{\theta}_t$ is defined as

$$\overline{v}(t, t+1) := \frac{\overline{H}_t \cdot K_t}{\overline{H}_t \cdot K_{t+1}} = \left(\sum_{i=1}^{m} \overline{\theta}_t^i \, \frac{K_{t+1}^i}{K_t^i} \right)^{-1} . \tag{2.1}$$

We define $\frac{0}{0} = 1$ and $\frac{1}{0} = \infty$. Furthermore, we define $\mathrm{VaR}_\alpha(Y) := \inf\{y \in \mathbb{R} : \mathrm{P}(Y \le y) \ge \alpha\}$ for $\alpha \in (0, 1)$. The following definitions are possible interpretations of the SCR definition in the Solvency II framework.

(a) One possible interpretation of Article 101 is

$$SCR_0 := \mathrm{VaR}_{0.995}(N_0 - v(0, 1) N_1) \tag{2.2}$$

for a positive random variable $v(0, 1)$, not necessarily of the form (2.1). The proper choice of the discount factor $v(0, 1)$ is unclear. Article 101 does not give a definite answer.

(a1) Let θ_t^{num} be an asset strategy that correspond to a *numeraire* such that the appendant discount factor $v^{num}(t, t+1)$ has a representation of the form (2.1). Thus, a possible specification of definition (2.2) is

$$SCR_0^{num} := \mathrm{VaR}_{0.995}(N_0 - v^{num}(0, 1) N_1) . \tag{2.3}$$

Note that the discount factor $v^{num}(t, t+1)$ is usually random and that Artzner *et al.* (2009) prefer the use of the term *eligible asset* instead of numeraire. They also discuss a framework with several eligible assets. Such a SCR definition can be found, for example, in Ohlsson and Lauzeningks (2009), Floreani (2011), Möhr (2011), and Bauer *et al.* (2012). They all specify the numeraire to be the discount factor corresponding to a riskless interest rate. Such a rate can be theoretically derived from a model, or it can simply be defined as the returns on government bonds or real bank accounts.

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Only Devineau and Loisel (2009) use such a definition without further restricting the choice of the numeraire.

(a2) Let $v^{real}(t, t+1)$ be a discount factor that relates to the *real* capital gains that the insurance company from Definition 2.1 actually earns on its assets in the time period $(t, t+1]$, which is defined with the strategy θ_t and (2.1). The following representations are equivalent

$$v^{real}(t, t+1) = \frac{H_t \cdot K_t}{H_t \cdot K_{t+1}} = \frac{A_t}{A_{t+1} - Z_{t+1}}. \tag{2.4}$$

For $A_t = 0$ we have by Definition 2.1 $H_t = 0$ and consequently $H_t \cdot K_{t+1} = 0$, recall that $\frac{0}{0} = 1$ and $\frac{1}{0} = \infty$. Thus, another possible specification of definition (2.2) is

$$SCR_0^{real} := \mathrm{VaR}_{0.995}(N_0 - v^{real}(0,1)N_1). \tag{2.5}$$

We are not aware of such a definition in the literature, although it has advantageous properties, as we will see later on.

(b) Assuming the existence of a martingale measure Q that allows for a risk-neutral valuation of assets and liabilities, some authors (e.g. Kochanski, 2010; Barrieu *et al.*, 2012) define the SCR according to Article 101 as

$$SCR_0^Q := \mathrm{VaR}_{0.995}\left(\mathbb{E}_Q\left(v^{num}(0,1)N_1\right) - v^{num}(0,1)N_1\right). \tag{2.6}$$

If $v^{num}(0,t)N_t$ is a Q-martingale, we obtain $N_0 = \mathbb{E}_Q(v^{num}(0,1)N_1)$, and SCR_0^{num} and SCR_0^Q are equal.

For the calculation of the subsequent SCR definitions, we have to minimize the value of the asset portfolio. Because upsizing and downsizing of the asset portfolio can be disproportional to the existing portfolio, we have to extend our modeling framework.

Definition 2.2 (upsizing and downsizing of assets).

1. The new net value at time t is defined by $\widetilde{N}_t = A_t + \widetilde{A}_t - L_t$ where $\widetilde{A}_t$ is one potential shift of the assets at time t. We assume that A_t and L_t are invariant with respect to the hypothetical capital inflow/outflow $\widetilde{A}_t$, i.e. when $\widetilde{A}_t$ changes, A_t and L_t do not change.
2. We assume that an upsizing or downsizing of the asset portfolio at time $t \in \mathbb{N}_0$ follows an adapted management strategy function h_t that gives for every shift $\widetilde{A}_t$ the corresponding units of the investments, i.e. $h_t(\widetilde{A}_t)$ is $\mathcal{F}_t$-measurable and $h_t(\widetilde{A}_t) \cdot K_t = \widetilde{A}_t$ for all possible market value shifts $\widetilde{A}_t$.
3. We also use the notation $\widetilde{H}_t = h_t(\widetilde{A}_t)$ to describe possible shift of the units of assets at time $t \in \mathbb{N}_0$ with corresponding investment strategy $\widetilde{\theta}_t$. Note that $\widetilde{H}_t$ and $\widetilde{\theta}_t$ change when $\widetilde{A}_t$ changes. Again, we assume that $\widetilde{A}_t = 0$ implies $\widetilde{H}_t = (0, ..., 0)$, i.e. $h_t(0) = 0$.

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The intuition behind the assumption that A_t and L_t are invariant with respect to $\widetilde{A}_t$ is that the asset shift is only meant to adapt the solvency level of the insurance company but shall not interact with its business plan.

(c) A mathematical interpretation of Recital 64 of the directive leads to

$$SCR_0 := \inf_{\widetilde{A}_0} \left\{ N_0 + \widetilde{A}_0 : \mathrm{P}\left(N_1^{\widetilde{A}_0} \geq 0\right) \geq 0.995 \right\},$$

where $N_1^{\widetilde{A}_0}$ is the net value at time one under the hypothetical assumption that the asset value at time zero was shifted by $\widetilde{A}_0$. Actually, the net value should be positive in the whole time interval [0, 1]. Since no insurance company is able to provide this information, we reduce our investigation to a yearly time grid. Bauer *et al.* (2012) state that this is the intuitive definition of the SCR, while SCR_0^{num} is an approximation of it.

For the minimization of $N_0 + \widetilde{A}_0$, there are two possibilities: The assets can be reduced or increased by a given strategy, or we even minimize with respect to all reducing strategies. Consequently, we split up definition (c) into (c1) and (c2).

(c1) Let h_0 be a given management strategy as in Definition 2.2. Then one interpretation of Recital 64 is

$$SCR_0^h := \inf_{\widetilde{A}_0} \left\{ N_0 + \widetilde{A}_0 : \mathrm{P}\left(N_1 + h_0(\widetilde{A}_0) \cdot K_1 \geq 0\right) \geq 0.995 \right\}, \quad (2.7)$$

where we assume that the infimum exists.

(c2) By minimizing over all possible strategies h_0 in the first interpretation, we get another interpretation of Recital 64

$$\begin{aligned} SCR_0^{inf} &:= \inf_{h_0} SCR_0^h \\ &= \inf_{\widetilde{H}_0} \left\{ N_0 + \widetilde{H}_0 \cdot K_0 : \mathrm{P}\left(N_1 + \widetilde{H}_0 \cdot K_1 \geq 0\right) \geq 0.995 \right\}, \end{aligned} \quad (2.8)$$

where we assume that the infimum exists. The second equality holds, since for every $\widetilde{H}_0$ there is a h_0 with $h_0(\widetilde{H}_0 \cdot K_0) = \widetilde{H}_0$ and since for every h_0 and $\widetilde{A}_0$ there is a $\widetilde{H}_0$ with $\widetilde{H}_0 = h_0(\widetilde{A}_0)$. Note that by Definition 2.2 we have that $\widetilde{A}_0 = 0$ implies $\widetilde{H}_0 = 0$ which corresponds to $h_0(0) = 0$. The existence problem of the infimum is discussed in Artzner *et al.* (2009) for finite Ω. Their condition for existence is that no acceptability arbitrage exists. Here we also allow for infinite Ω, and some sufficient conditions for the existence of the infimum will be given in the next section.

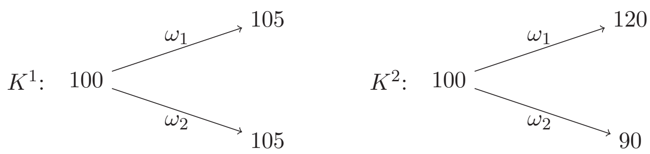

FIGURE 1: Development of the assets.

The different interpretations of the SCR lead us to the following questions:

(1) Are the aforementioned interpretations of Article 101 and Recital 64 consistent? If not, which additional assumptions do we need to make them consistent?
(2) Are (some of) the different definitions equivalent? If not, can we find additional conditions that make (some of) them equivalent?
(3) If the different definitions cannot be harmonized, are there other arguments that support or disqualify some versions?

So far, we have only discussed the definition of a present SCR that gives the solvency requirement for today. However, for the calculation of the RM, which will be discussed in more detail in Section 8, we also have to define future SCRs that describe solvency requirements at future points in time.

(4) How can we mathematically define an SCR_s that describes the solvency requirement at a future time $s > 0$?

In the Solvency II standard formula, the one-year perspective is replaced by shocks that happen instantaneously. Consequently, there is no discount factor, and so the standard formula does not answer the questions.

The following example illustrates the SCR definitions. The example is kept very simple in order to make the differences between the definitions more clear.

Example 2.3 (SCR of a riskless insurer). We consider a time horizon of one year and a financial market with two assets, a riskless bond K^1 and a stock K^2, which both have a price of $K_0^1 = K_0^2 = 100$ at time 0. Two scenarios $\Omega = \{\omega_1, \omega_2\}$ may occur; see Figure 1. Both scenarios shall have the same probability, $\mathrm{P}(\{\omega_1\}) = \mathrm{P}(\{\omega_2\}) = 0.5$. We consider a simplified insurance company that is closed to new business and which has an asset portfolio with no bonds and two stocks, $H_0 = (H_0^1, H_0^2) = (0, 2)$. The insurance portfolio consists of just one unit-linked life-insurance with a sum insured of K_1^2 at time 1. As no assets are traded during the year, we obtain $N_0 = 100$ and $N_1 = K_1^2$. In the following, we calculate the SCR according to the different definitions.

(a1) We choose the riskless bond as the numeraire, which leads to a riskless discount factor of $v^{num}(0, 1) = 1.05^{-1}$. By definition (2.3), we obtain

$$SCR_0^{num} = \mathrm{VaR}_{0.995}\left(100 - 1.05^{-1} K_1^2\right) = \frac{100}{7}.$$

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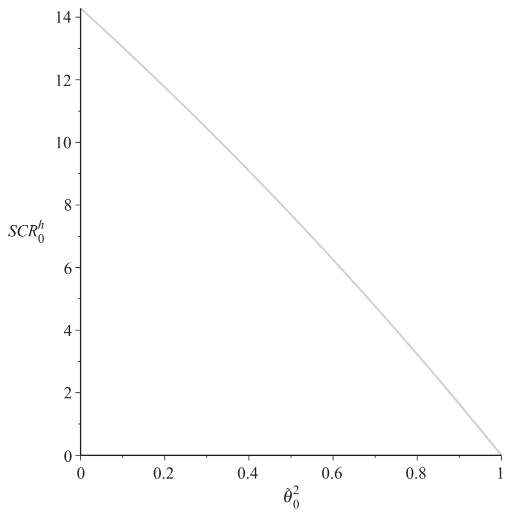

FIGURE 2: SCR_0^h calculated for different choices of $\widetilde{\theta}_0^2$.

(a2) Since $v^{real}(0,1) = \frac{100}{K_1^2}$, the SCR according to definition (2.5) is

$$SCR_0^{real} = \mathrm{VaR}_{0.995}\Big(100 - \frac{100}{K_1^2} K_1^2\Big) = 0\,.$$

(b) Since we can show that $Q(\{\omega_i\}) = P(\{\omega_i\}) = 0.5$ for $i = 1, 2$, we have $\mathbb{E}_Q(v^{num}(0,1)\,N_1) = 100 = N_0$, and consequently SCR_0^Q is equal to SCR_0^{num}.

(c1) Definition (2.7) requires minimization of the additional assets $\widetilde{A}_0$ by a strategy h_0. We assume that h_0 is a nonnegative linear function, i.e. there are $\widetilde{\theta}_0^1, \widetilde{\theta}_0^2 \in [0,1]$ with $\widetilde{\theta}_0^1 + \widetilde{\theta}_0^2 = 1$ and $h_0(\widetilde{A}_0) = (\widetilde{A}_0\,\widetilde{\theta}_0^1/K_0^1, \widetilde{A}_0\,\widetilde{\theta}_0^2/K_0^2)^T$. Thus, we get

$$N_1 + h_0(\widetilde{A}_0)\cdot K_1 = N_1 + \widetilde{A}_0\left(\widetilde{\theta}_0^1\frac{K_1^1}{K_0^1} + \widetilde{\theta}_0^2\frac{K_1^2}{K_0^2}\right).$$

By minimizing $\widetilde{A}_0$ under the condition $P(N_1 + h_0(\widetilde{A}_0)\cdot K_1 \geq 0) \geq 0.995$, we get

$$SCR_0^h = 100 + \max\left\{-120(1.05 + 0.15\,\widetilde{\theta}_0^2)^{-1}; -90(1.05 - 0.15\,\widetilde{\theta}_0^2)^{-1}\right\},$$

which is visualized in Figure 2. We observe that the SCR highly depends on the choice of the reduction strategy. Since stocks are the only assets of the company, it seems to be a proper choice to reduce the assets by selling stocks. This would lead to an SCR of 0.

(c2) We use the right-hand side of (2.8) to calculate SCR_0^{inf}. Straightforward calculations show that the condition $\mathrm{P}(N_1 + \widetilde{H}_0 \cdot K_1 \geq 0) \geq 0.995$ is fulfilled if and only if

$$\widetilde{H}_0^1 \geq \begin{cases} -\frac{90}{105}\left(1 + \widetilde{H}_0^2\right), & \widetilde{H}_0^2 \geq -1 \\ -\frac{120}{105}\left(1 + \widetilde{H}_0^2\right), & \widetilde{H}_0^2 \leq -1 \end{cases}.$$

The infimum of $N_0 + \widetilde{H}_0 \cdot K_0$ is in both cases 0 with the strategy $\widetilde{H}_0 = (0, -1)$. Consequently, we obtain $SCR_0^{inf} = 0$.

The numerical example shows that the different SCR definitions are not generally equivalent. Then which SCR is adequate here, zero or greater than zero? Let us recall the fundamental intention of the Solvency II project. According to Recital 16 (European Parliament and the Council, 2009, page 8) "the main objective of insurance and reinsurance regulation and supervision is the adequate protection of policy holders and beneficiaries." Consequently, if the company holds one stock, it has a perfect hedge for the liabilities, and the policy holder is sufficiently protected. Hence, it seems reasonable to set SCR_0 equal to zero. However, from a shareholder's perspective it is not necessarily optimal to perfectly hedge the liabilities (see Wüthrich and Merz, 2013). The SCR definitions that are most frequently used in the literature, namely SCR_0^{num} and SCR_0^{Q}, both lead to an SCR larger than zero.

To keep the example as simple as possible, we used a binomial model for the development of the stock. However, we can construct similar examples for many kinds of stock models. For the calculation of SCR_0^{real} the stock price at time 1 is irrelevant, and for reasonable stock developments SCR_0^h is still equal to zero for $\widetilde{\theta}_0^2 = 1$ and positive for $\widetilde{\theta}_0^2 \neq 1$.

3. Discussion of Recital 64

Analogously to (2.1) and (2.4), we define the discount factor for one possible additional asset portfolio $\widetilde{A}_t$ as

$$v^h(t, t+1) := \frac{\widetilde{H}_t \cdot K_t}{\widetilde{H}_t \cdot K_{t+1}} = \frac{h_t(\widetilde{A}_t) \cdot K_t}{h_t(\widetilde{A}_t) \cdot K_{t+1}} = \frac{\widetilde{A}_t}{\widetilde{A}_{t+1}}, \tag{3.1}$$

where $\widetilde{A}_{t+1}$ is the time $t+1$ value of $\widetilde{A}_t$. In general, the return of the additional assets can depend on the amount of additional assets. In order to simplify the mathematical structure in (2.7), we assume that the discount factor $v^h(t, t+1)$

is invariant with respect to $\widetilde{A}_t$, i.e.

$$v^h(t, t+1) = \frac{\widetilde{A}_t}{h_t(\widetilde{A}_t) \cdot K_{t+1}} = \frac{a}{h_t(a) \cdot K_{t+1}}, \quad \text{for all } a \in \mathbb{R}.$$

This assumption does not fully agree with insurance practice, but it will often be an acceptable approximation.

Proposition 3.1. *Suppose that all investment portfolios with value zero (almost surely) at time* 1 *necessarily have value zero at time* 0*. Then the discount factor* $v^h(0, 1)$ *is invariant with respect to* $\widetilde{A}_0$ *for all possible capital markets* $(K_t)_t$ *if and only if* h_0 *is a linear function, i.e.* $h_0(\widetilde{A}_0) = \widetilde{A}_0 B_0$ *for a constant vector* B_0.

Proof. Let $v^h(0, 1)$ be invariant with respect to $\widetilde{A}_0$. For any $\widetilde{A}_0$ we get that

$$\frac{\widetilde{A}_0}{h_0(\widetilde{A}_0) \cdot K_1} = v^h(0, 1) = \frac{1}{h_0(1) \cdot K_1}, \text{ almost surely,}$$

which is equivalent to $(\widetilde{A}_0 h_0(1) - h_0(\widetilde{A}_0)) \cdot K_1 = 0$ almost surely. By assumption this implies that $(\widetilde{A}_0 h_0(1) - h_0(\widetilde{A}_0)) \cdot K_0 = 0$. Note that the denominator $h_0(1) \cdot K_1$ is never zero: $h_0(1) \cdot K_1 = 0$ implies $h_0(1) \cdot K_0 = 0$ which leads to the contradiction $1 = h_0(1) \cdot K_0 = 0$ because of Definition 2.2. As the set of all possible capital markets K_0 at time 0 includes a basis of $\mathbb{R}^d$, we necessarily conclude that $\widetilde{A}_0 h_0(1) - h_0(\widetilde{A}_0) = 0$. Hence, h_0 is linear and we can define $B_0 := h_0(1)$. The other way round it is obvious that h_0 linear implies that $v^h(0, 1)$ is invariant with respect to $\widetilde{A}_0$. ■

If there existed an investment portfolio with value zero at time 1 but with a value unequal to zero at time 0, there exists arbitrage in the market.

Because of this proposition, in the following we always let h_0 be a linear function. Analogously, we say that also the h_t, $t \in \mathbb{N}$, shall be linear, which implies that $v^h(t, t+1)$ is invariant with respect to $\widetilde{A}_t$ for each $t \in \mathbb{N}$. As v^{num} corresponds to a linear management strategy, all Solvency II SCR definitions that we found in the literature implicitly assume linear management strategies. In practice, management strategies for the excess assets may be non-linear, and linearity can be seen as a first-order approximation of non-linear management decisions.

Proposition 3.2. *Suppose that* $h_0(\widetilde{A}_0) = \widetilde{A}_0 B_0$*, where we find both positive and negative signs in the components of* B_0*. Furthermore, we assume that there exists a measurable set* $M \subset \Omega$ *with* $0.005 < \mathrm{P}(M) < 0.995$*. Then there are* H_0*,* $(L_t)_{t \in \mathbb{N}_0}$*,* $(Z_t)_{t \in \mathbb{N}}$*, and* $(K_t)_{t \in \mathbb{N}_0}$ *such that*

$$\inf_{\widetilde{A}_0} \left\{ N_0 + \widetilde{A}_0 : \mathrm{P}\big(N_1 + h_0(\widetilde{A}_0) \cdot K_1 \geq 0\big) \geq 0.995 \right\} = \inf \emptyset$$

while $\mathrm{VaR}_{0.995}(N_0 - v(0, 1) N_1)$ *is a finite real number for any real random variable* $v(0, 1)$.

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Proof. We choose $(K_t)_{t\in\mathbb{N}_0}$ in such a way that we have $M = \{\omega \in \Omega : v^h(0,1)(\omega) < 0\}$. Let $\overline{M} := \Omega \setminus M$. We define H_0, $(L_t)_{t\in\mathbb{N}_0}$, $(Z_t)_{t\in\mathbb{N}}$, and $(K_t)_{t\in\mathbb{N}_0}$ such that $N_0 = 0$ and

$$N_1 = \begin{cases} \dfrac{1}{v^h(0,1)} & \text{on } M, \\ -\dfrac{1}{v^h(0,1)} & \text{on } \overline{M}. \end{cases}$$

Then the inequality $N_1 + h(\widetilde{A}_0) \cdot K_1 \geq 0$ holds on M and $\overline{M}$ if and only if $\widetilde{A}_0 \leq -1$ and $\widetilde{A}_0 \geq 1$, respectively. Consequently, $\{N_0 + \widetilde{A}_0 : \mathrm{P}(N_1 + h_0(\widetilde{A}_0) \cdot K_1 \geq 0) \geq 0.995\} = \emptyset$. On the other hand, for any real random variable $v(0,1)$, all α-quantiles, $\alpha \in (0,1)$, of the random variable $N_0 - v(0,1)N_1$ exist and are finite. ■

The proposition shows that — under the assumptions that we made so far — it can happen that the SCR according to Article 101 of the Solvency II directive exists while the SCR according to Recital 64 of the Solvency II directive does not. In the proof we construct such inconsistencies by letting $v^h(0,1)$ be negative on the set M. In order to avoid such inconsistencies we make the following assumption.

Assumption 3.3 (linearity and nonnegativity of h_t). We assume that the management strategy function h_t is linear, i.e. $h_t(\widetilde{A}_t) = \widetilde{A}_t B_t$, and that the $\mathcal{F}_t$-measurable random vector B_t takes only values in $[0,\infty)^m \setminus \{0\}$.

Assumption 3.3 excludes acceptability arbitrage in the sense of Artzner *et al.* (2009), but only for the shifting portfolio $\widetilde{H}_t$. The total portfolio $H_t + \widetilde{H}_t$ may have short-long positions. From Definition 2.2 and Assumption 3.3 we get that $1 = h_t(1) \cdot K_t = B_t \cdot K_t$.

Remark 3.4 (nonnegativity of the asset strategy vector B_t). The assumption that B_t takes only values in $[0,\infty)^m \setminus \{0\}$ is sufficient to ensure that $v^h(t,t+1)$ is strictly positive. In some sense this assumption is also necessary: If B_t equals the zero vector with positive probability, then the assumption $h_t(\widetilde{A}_t) \cdot K_t = \widetilde{A}_t$ is violated. If B_t has negative entries with positive probability, then we can construct a financial market $(K_t)_t$ for which $B_t \cdot K_{t+1}$ is negative with positive probability. Thus, in order to ensure $v^h(t,t+1) = (B_t \cdot K_{t+1})^{-1} > 0$ for all kinds of financial market models $(K_t)_t$, the random vector B_t must almost surely take values in $[0,\infty)^m \setminus \{0\}$. This positivity property will play a crucial role later on. It implies that a decrease (an increase) of A_t by $\widetilde{A}_t < 0$ ($\widetilde{A}_t > 0$) at time t cannot result in an increase (a decrease) of the asset value one year later, i.e. we always have

$$A_t + \widetilde{A}_t < A_t \quad \Longrightarrow \quad A_{t+1} + \widetilde{A}_{t+1} < A_{t+1},$$
$$A_t + \widetilde{A}_t > A_t \quad \Longrightarrow \quad A_{t+1} + \widetilde{A}_{t+1} > A_{t+1}.$$

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Capital that is added at time t in order to increase the solvency level lets the insurer always be financially better off after one year and never the other way round.

The denominator $h(\widetilde{A}_t) \cdot K_{t+1}$ of the discount factor $v^h(t, t+1)$ defined in (3.1) can get zero, but $B_t \cdot K_{t+1}$ is always positive since all entries of K_{t+1} are positive, all entries of B_t are nonnegative, and B_t is not the zero vector. Therefore in the following we always use the definition

$$v^h(t, t+1) := \frac{B_t \cdot K_t}{B_t \cdot K_{t+1}} = \frac{1}{B_t \cdot K_{t+1}}. \tag{3.2}$$

Under the condition of Assumption 3.3 definitions (2.7) and (2.8) change. Unless stated otherwise, from now on we assume that the management strategy in SCR_0^h (cf. (2.7)) fulfills Assumption 3.3 and that SCR_0^{inf} is defined as follows

$$SCR_0^{inf} = \inf_{B_0 \in [0,\infty)^m \setminus \{0\},\, B_0 \cdot K_0 = 1} \inf_{\widetilde{A}_0} \left\{ N_0 + \widetilde{A}_0 : \mathrm{P}(N_1 + \widetilde{A}_0\, B_0 \cdot\ K_1 \geq 0) \geq 0.995 \right\}. \tag{3.3}$$

These changes in the definitions of SCR_0^h and SCR_0^{inf} cannot be found in Recital 64, but we make them for two reasons. First, we showed in Proposition 3.2 that without Assumption 3.3 it is possible to construct examples where the SCR is equal to the infimum of the empty set, which is not a sound definition for the SCR. Second, one key aspect of this paper is the question whether Article 101 and Recital 64 are consistent. When Assumption 3.3 does not hold we know from Proposition 3.2 that Article 101 and Recital 64 cannot be consistent in general. Consequently, we restrict the further investigation to Assumption 3.3.

Note that Assumption 3.3 also implies that (3.3) is finite. Suppose we had a sequence of $\widetilde{A}_0$ that goes to $-\infty$ and satisfies the infimum condition in (3.3). Applying limit theorems we obtain that $\mathrm{P}(-B_0 \cdot\ K_1 \geq 0) \geq 0.995$ for some $B_0 \in [0,\infty)^m \setminus \{0\}$ with $B_0 \cdot K_0 = 1$, which is a contradiction to the fact that $B_0 \cdot\ K_1$ is always positive.

Theorem 3.5. *Under Assumption 3.3 we have*

$$SCR_0^h = \mathrm{VaR}_{0.995}\big(N_0 - v^h(0,1) N_1\big). \tag{3.4}$$

Furthermore, there exists a linear management strategy h_0^{inf} and a corresponding discount factor $v^{inf}(0,1)$ such that SCR_0^{inf} defined according to (3.3) *has the representation*

$$\begin{aligned} SCR_0^{inf} &= \inf_{\widetilde{A}_0} \left\{ N_0 + \widetilde{A}_0 : \mathrm{P}(N_1 + h_0^{inf}(\widetilde{A}_0) \cdot K_1 \geq 0) \geq 0.995 \right\} \\ &= \mathrm{VaR}_{0.995}\big(N_0 - v^{inf}(0,1) N_1\big). \end{aligned} \tag{3.5}$$

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Proof. From (3.1) we get $\widetilde{A}_{s+1} = v^h(s, s+1)^{-1}\widetilde{A}_s$, which leads to

$$\widetilde{N}_{s+1} = N_{s+1} + v^h(s, s+1)^{-1}\widetilde{A}_s = N_{s+1} + v^h(s, s+1)^{-1}(\widetilde{N}_s - N_s),$$

and since $v^h(s, s+1)$ is always positive, we obtain

$$\{\widetilde{N}_{s+1} \geq 0\} = \{v^h(s, s+1)\,\widetilde{N}_{s+1} \geq 0\} = \{N_s - v^h(s, s+1)\,N_{s+1} \leq \widetilde{N}_s\} \quad (3.6)$$

for all $s \in \mathbb{N}_0$. As N_0 is deterministic, the left-hand side of (3.4) is well defined and equals the right-hand side of (3.4) because

$$\mathrm{P}\big(\widetilde{N}_1 \geq 0\big) = \mathrm{P}\big(N_0 - v^h(0, 1)\,N_1 \leq \widetilde{N}_0\big).$$

By the definition of SCR_0^{inf} given in (3.3), there exists a series $(a^k, b^k)_{k\in\mathbb{N}}$, $a^k \in \mathbb{R}$, $b^k \in [0, \infty)^m\backslash\{0\}$, with $b^k\cdot K_0 = 1$, $\mathrm{P}(N_1 + a^k\,b^k\cdot K_1 \geq 0) \geq 0.995$, and $N_0 + a^k \to SCR_0^{inf}$. As $\{b \in [0, \infty)^m : b \cdot K_0 = 1\}$ is a compact subset of $\mathbb{R}^m$, we may assume that b^k converges to some $b^\infty \in [0, \infty)^m$ with $b^\infty \cdot K_0 = 1$. Furthermore, $N_1 + a^k\,b^k\cdot K_1$ converges stochastically to the random variable $N_1 + (SCR_0^{inf} - N_0)b^\infty\cdot K_1$. For any sequence $(X_k)_k$ of random variables that stochastically converges to a limit X, the distribution functions satisfy $F_X(t-) \leq \liminf_k F_{X_k}(t-)$ for all $t \in \mathbb{R}$ (compare Milbrodt, 2010, pages 270–271). Therefore we can conclude that

$$\mathrm{P}(N_1 + (SCR_0^{inf} - N_0)b^\infty\cdot\, K_1 \geq 0) \geq \limsup_k \mathrm{P}(N_1 + a^k\,b^k\cdot\, K_1 \geq 0) \geq 0.995.$$

Hence, by defining $h_0^{\inf}(x) = xb^\infty$, in the first line in (3.5) the right hand side is not greater than the left hand side. On the other hand, the right hand side cannot be truly smaller than the left hand side since b^∞ is one of the management strategies in the first infimum in (3.3). The second equality in (3.5) follows from (3.4). ■

The theorem allows us to substitute definition (2.7) with (3.4). This is especially useful when Monte-Carlo simulations are used. For calculating (2.7) a starting level $N_0 + \widetilde{A}_0$ is needed before $N_1 + h_0(\widetilde{A}_0)\cdot K_1$ can be simulated, and the simulation only approximates the ruin probability for this starting level. Consequently, we need methods such as nested intervals, and the simulation has to be run over and over again until the desired ruin probability is reached. In contrast, (3.4) can be calculated with one run of simulations. One might think that similarly the value at risk representation on the right hand side of (3.5) can serve as a substitute for definition (2.8). However, Theorem 3.5 yields only an existence result but not a construction principle for v^{inf}.

Corollary 3.6. *Under Assumption 3.3* SCR_0^{inf} *has the representation*

$$SCR_0^{inf} = \inf_{b_0\in[0,\infty)^m\backslash\{0\}} \mathrm{VaR}_{0.995}(N_0 - v^b(0, 1)\,N_1)\,, \quad (3.7)$$

where $v^b(0, 1) := \frac{b_0\cdot K_0}{b_0\cdot K_1}$.

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Proof. Using (3.3) and (3.4), SCR_0^{inf} has the representation

$$SCR_0^{inf} = \inf_{B_0 \in [0,\infty)^m \setminus \{0\},\, B_0 \cdot K_0 = 1} \mathrm{VaR}_{0.995}(N_0 - v^h(0,1)\, N_1)$$
$$= \inf_{b_0 \in [0,\infty)^m \setminus \{0\}} \mathrm{VaR}_{0.995}(N_0 - v^b(0,1)\, N_1)\,,$$

where the second equality follows from the fact that for all $b_0 \in [0,\infty)^m \setminus \{0\}$ we can define a corresponding $B_0 = \frac{b_0}{b_0 \cdot K_0}$ that satisfies Assumption 3.3 and for which $v^h(0,1) = \frac{B_0 \cdot K_0}{B_0 \cdot K_1} = \frac{b_0 \cdot K_0}{b_0 \cdot K_1}$. ■

4. The SCR for an arbitrary point in time

We now generalize all our SCR definitions to future points in time with the help of a dynamic value at risk definition. The following proposition helps to define such a dynamic value at risk. The proof can be found in the appendix. A dynamic value at risk definition is also given in Kriele and Wolf (2012) that is based on a sophisticated construction of the probability space. Our construction includes the model of Kriele and Wolf (2012), but requires less effort.

Proposition 4.1. *Let* $(\Omega, \mathcal{F}, \mathrm{P})$ *be a probability space with random variables* $X_{[0,s]} : (\Omega, \mathcal{F}) \to (\Omega', \mathcal{F}'_s)$ *and* $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathfrak{B}(\mathbb{R}))$, $Y \in \mathcal{Y}$, *where* $(\Omega', \mathcal{F}'_s)$ *is some measurable space and* $\mathcal{Y}$ *is some countable set of real-valued random variables. Then the function* $q_{Y,\alpha} : \Omega' \to \mathbb{R}$ *defined by*

$$q_{Y,\alpha}(x) := \inf\{y \in \mathbb{R} : \mathrm{P}(Y \leq y | X_{[0,s]} = x) \geq \alpha\}$$

and the function $q^{inf}_{\mathcal{Y},\alpha}(x) : \Omega' \to \mathbb{R}$ *defined by*

$$q^{inf}_{\mathcal{Y},\alpha}(x) := \inf\{Y \in \mathcal{Y} : q_{Y,\alpha}(x)\}$$

are $\mathcal{F}'_s$-$\mathfrak{B}(\mathbb{R})$*-measurable.*

Definition 4.2 (dynamic value at risk). Suppose that the assumptions of Proposition 4.1 hold, and let $X_{[0,s]}$ be a generator of the filtration from Definition 2.1; i.e. $\mathcal{F}_s = \sigma(X_{[0,s]})$. For $\alpha \in (0,1)$ and a random variable Y we define

$$\mathrm{VaR}_\alpha\big(Y\big|\mathcal{F}_s\big) := q_{Y,\alpha}(X_{[0,s]})\,,$$

and for a countable set $\mathcal{Y}$ of random variables we define

$$\inf{}_{Y\in\mathcal{Y}} \mathrm{VaR}_\alpha\big(Y\big|\mathcal{F}_s\big) := q^{inf}_{\mathcal{Y},\alpha}(X_{[0,s]})\,.$$

With the help of Definition 4.2, we can generalize SCR definitions (2.2), (2.3), (2.5), (2.6), (3.4), and (3.7) to future points in time by replacing the values at risk by dynamic values at risk.

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Definition 4.3 (present and future SCR). Given that Assumption 3.3 holds, the SCR at time $s \in \mathbb{N}_0$ is defined as

(a) $SCR_s^x := \mathrm{VaR}_{0.995}(N_s - v^x(s, s+1)\, N_{s+1}|\mathcal{F}_s),\ x \in \{num, real, h\}$,
(b) $SCR_s^Q := \mathrm{VaR}_{0.995}(\mathbb{E}_Q(v^{num}(s, s+1)\, N_{s+1}|\mathcal{F}_s) - v^{num}(s, s+1)\, N_{s+1})|\mathcal{F}_s)$,
(c) $SCR_s^{inf} := \inf_{b_s \in [0,\infty)^m \setminus \{0\}} \mathrm{VaR}_{0.995}(N_s - v^b(s, s+1)\, N_{s+1}|\mathcal{F}_s)$,

where Q is a martingale measure and $v^b(s, s+1) := \frac{b_s \cdot K_s}{b_s \cdot K_{s+1}}$.

The definition of SCR_s^{inf} is motivated by (3.7). The following proposition shows that the uncountable set $[0, \infty)^m \setminus \{0\}$ may be reduced to the countable set $[0, \infty)^m \cap \mathbb{Q}^m \setminus \{0\}$ such that SCR_s^{inf} is a well-defined $\mathcal{F}_s$-measurable random variable according to Definition 4.2. Measurability ensures that SCR_s^{inf} has a probability distribution and will be crucial in Definition 8.4.

Proposition 4.4. *Under Assumption 3.3, we have*

$$SCR_s^{inf} = \inf_{b_s \in [0,\infty)^m \cap \mathbb{Q}^m \setminus \{0\}} \mathrm{VaR}_{0.995}(N_s - v^b(s, s+1)\, N_{s+1}|\mathcal{F}_s). \tag{4.1}$$

Proof. Given that $X_{[0,s]} = x$, K_s almost surely equals a deterministic vector $K_s^x \in (0, \infty)^m$ and N_s almost surely equals a real variable $N_s^x \in \mathbb{R}$. Using (3.6) and the fact that $v^b(s, s+1)$ is invariant with respect to any positive scaling of b_s, we get

$$\begin{aligned}
SCR_s^{inf}(x) &= \inf_{b_s \in [0,\infty)^m \setminus \{0\}} \mathrm{VaR}_{0.995}(N_s - v^b(s, s+1)\, N_{s+1}| X_{[0,s]} = x) \\
&= \inf_{b_s \in [0,\infty)^m \setminus \{0\}} \inf\{N_s^x + a \in \mathbb{R} : \\
&\quad \mathrm{P}(N_s^x - v^b(s, s+1)\, N_{s+1} \leq N_s^x + a | X_{[0,s]} = x) \geq 0.995\} \\
&= \inf_{b_s \in [0,\infty)^m \setminus \{0\}} \inf \Bigg\{ N_s^x + a \in \mathbb{R} : \\
&\quad \mathrm{P}\Bigg(N_{s+1} + a\, \frac{b_s \cdot K_{s+1}}{b_s \cdot K_s^x} \geq 0 \Bigg| X_{[0,s]} = x \Bigg) \geq 0.995 \Bigg\} \\
&= \inf\{N_s^x + b_s' \cdot K_s^x : b_s' \in [0, \infty)^m \cup (-\infty, 0]^m, \\
&\quad \mathrm{P}(N_{s+1} + b_s' \cdot K_{s+1} \geq 0 | X_{[0,s]} = x) \geq 0.995\}.
\end{aligned}$$

The last inequality is true since the vector

$$b_s' := \frac{a}{b_s \cdot K_s^x} b_s \tag{4.2}$$

meets the quantile condition in the last line and satisfies $b_s' \cdot K_s^x = a$ whenever the pair (a, b_s) satisfies the quantile condition in the second to last line, and since the pair

$$a := b'_s \cdot K^x_s,$$
$$b_s := \begin{cases} b'_s & : a > 0 \\ -b'_s & : a < 0 \\ (1, \ldots, 1) & : a = 0 \end{cases} \tag{4.3}$$

meets the quantile condition in the second to last line whenever b'_s satisfies the quantile condition in the last line. Since $c \cdot K_{s+1} \geq 0$ for all $c \in [0, \infty)^m$, the inequality

$$\mathrm{P}(N_{s+1} + (b'_s + c) \cdot K_{s+1} \geq 0 | X_{[0,s]} = x) \geq 0.995 \tag{4.4}$$

is true for all nonnegative vectors c. Thus, the infimum set of the latter infimum contains a sequence of rationale vectors that converges to b'_s. Hence, the above infimum does not change if we replace the infimum set $b'_s \in [0, \infty)^m \cup (-\infty, 0]^m$ by the dense subset $b'_s \in ([0, \infty)^m \cup (-\infty, 0]^m) \cap \mathbb{Q}^m$. We now use (4.2) and (4.3) again in order to get back to the double infimum representation. The transformations (4.2) and (4.3) do not necessarily lead to rationale quantities, but, analogously to the arguments in (4.4), we always find rationale sequences that meet the corresponding quantile conditions and converge to the transformations (4.2) and (4.3). Hence, we obtain

$$\begin{aligned} &SCR^{inf}_s(x) \\ &= \inf\{N^x_s + b'_s \cdot K^x_s : b'_s \in ([0, \infty)^m \cup (-\infty, 0]^m) \cap \mathbb{Q}^m, \\ &\quad \mathrm{P}(N_{s+1} + b'_s \cdot K_{s+1} \geq 0 | X_{[0,s]} = x) \geq 0.995\} \\ &= \inf_{b'_s \in [0,\infty)^m \cap \mathbb{Q}^m \setminus \{0\}} \inf\{N^x_s + a : a \in \mathbb{Q}, \\ &\quad \mathrm{P}(N^x_s - v^{b'}(s, s+1)\, N_{s+1} \leq N^x_s + a | X_{[0,s]} = x) \geq 0.995\} \\ &= \inf_{b'_s \in [0,\infty)^m \cap \mathbb{Q}^m \setminus \{0\}} \mathrm{VaR}_{0.995}(N_s - v^{b'}(s, s+1)\, N_{s+1} | X_{[0,s]} = x). \end{aligned}$$

■

Interestingly, in Example 2.3 SCR^{num}_t and SCR^Q_t are equal. The following remark clarifies their relation.

Remark 4.5 (relation between SCR^{num}_t and SCR^Q_t). Suppose that $v^{num}(0, t)\, K_t$ is a Q-martingale. We have $N_t = \mathbb{E}_Q(v^{num}(t, t+1)\, N_{t+1} | \mathcal{F}_t)$ for all times t if and only if

$$L_t = \mathbb{E}_Q\Big(\sum_{s>t} -v^{num}(t, s) Z_s \Big| \mathcal{F}_t \Big) \tag{4.5}$$

for all times t. This can be proven by using the martingale property of $v^{num}(0,t)K_t$ to show that $\mathbb{E}_Q\big(v^{num}(t,t+1)(v^{real}(t,t+1))^{-1}|\mathcal{F}_t\big) = 1$ for all $t \in \mathbb{N}_0$. Because of equation (2.4), $N_t = \mathbb{E}_Q(v^{num}(t,t+1)\,N_{t+1}|\mathcal{F}_t)$ is equivalent to $L_t = \mathbb{E}_Q(v^{num}(t,t+1)\,(L_{t+1} - Z_{t+1})|\mathcal{F}_t)$. Using the latter equation iteratively, we arrive at (4.5) and vice versa.
So if Q is a martingale measure that allows for a risk-neutral valuation of assets and liabilities, then (4.5) is satisfied and, in turn, SCR_s^{num} and SCR_s^{Q} are equal. Therefore, in the following we do not consider SCR_s^{Q} anymore.

5. Comparison of the different SCR definitions

In a next step we want to learn if and when the different SCR definitions are equivalent.

Proposition 5.1. *Under Assumption 3.3 we almost surely have*

$$SCR_s^{inf} \leq SCR_s^{h}, \quad s \in \mathbb{N}_0.$$

Let H_s^{num} be a portfolio that replicates the numeraire between times s and $s+1$ and H_s the actual portfolio of the company according to Definition 2.1. If the $\mathcal{F}_s$-measurable random vectors H_s and H_s^{num} almost surely assume values in $[0,\infty)^m \cup (-\infty,0]^m$ *and are non-zero, then we also have*

$$SCR_s^{inf} \leq SCR_s^{real}, \quad SCR_s^{inf} \leq SCR_s^{num}, \qquad s \in \mathbb{N}_0,$$

almost surely.

Proof. By setting $h_s(x) := \frac{x}{H_s \cdot K_s} H_s$ and $h_s(x) := \frac{x}{H_s^{num} \cdot K_s} H_s^{num}$, we get $SCR_s^{h} = SCR_s^{real}$ and $SCR_s^{h} = SCR_s^{num}$, respectively. The condition that H_s and H_s^{num} almost surely assume values in $[0,\infty)^m \cup (-\infty,0]^m$ ensures that $h_s(x)$ fulfills Assumption 3.3. Hence, we just have to prove that $SCR_s^{inf} \leq SCR_s^{h}$ almost surely. The latter inequality follows directly from the definition of SCR_s^{inf}, because under the condition $X_{[0,s]} = x$ the management strategy vector B_s equals almost surely a deterministic vector $b_s \in [0,\infty)^m \setminus \{0\}$. ∎

The proposition shows that under the proposed conditions SCR_s^{inf} is a lower bound of the other definitions. This explains the superscript in the notation SCR_s^{inf}, which stands for *infimum*.

Theorem 5.2. *Let $\overline{H}_s$ be an arbitrary but fixed non-zero asset portfolio with values in* $[0,\infty)^m \cup (-\infty,0]^m$ *and with corresponding discount factor* $\overline{v}(s,s+1)$ *according to definition* (2.1).

(i) *Under Assumption 3.3 we have* $\mathrm{VaR}_{0.995}(N_s - \overline{v}(s,s+1)\,N_{s+1}\big|\mathcal{F}_s) = SCR_s^{h}$ *for all insurance companies $(H_t)_t$, $(L_t)_t$, $(Z_t)_t$, and all probability measures equivalent to* P *if and only if* $\overline{v}(s,s+1) = v^{h}(s,s+1)$ *almost surely.*

(ii) *Given that the asset strategy of the insurer θ_s from Definition 2.1 and the asset strategy of a possible shift $\widetilde{\theta}_s$ from Definition 2.2 exist, we have $v^{real}(s, s+1) = v^h(s, s+1)$ for all financial markets $(K_t)_t$ if and only if $\theta_s = \widetilde{\theta}_s$ almost surely.*

(iii) *Given that the asset strategy that correspond to a numeraire θ_s^{num} defined before* (2.3) *and the asset strategy of a possible shift $\widetilde{\theta}_s$ exist, we have $v^{num}(s, s+1) = v^h(s, s+1)$ for all financial markets $(K_t)_t$ if and only if $\theta_s^{num} = \widetilde{\theta}_s$ almost surely.*

Proof. (i) If $\overline{v}(s, s+1) = v^h(s, s+1)$ almost surely, then $\mathrm{VaR}_{0.995}(N_s - \overline{v}(s, s+1)\, N_{s+1} \big| \mathcal{F}_s)$ is almost surely equal to the definition of SCR_s^h. Suppose now that $\overline{v}(s, s+1) \neq v^h(s, s+1)$. Without loss of generality let $\{\overline{v}(s, s+1) < v^h(s, s+1)\}$ be a non-zero set (a set of positive measure). Then there must be an $\varepsilon > 0$ such that $\{\overline{v}(s, s+1)/v^h(s, s+1) < 1-\varepsilon\}$ is also a non-zero set, otherwise we could write $\{\overline{v}(s, s+1) < v^h(s, s+1)\}$ as a countable union of zero sets. Thus we can find a probability measure P^* equivalent to P for which $\mathrm{P}^*(\overline{v}(s, s+1)/v^h(s, s+1) < 1-\varepsilon | \mathcal{F}_s) > 0.995$ almost surely for some $\varepsilon > 0$. We further define $(H_t)_t$, $(L_t)_t$, and $(Z_t)_t$ in such a way that $N_s = 0$ and

$$N_{s+1} = -\frac{1}{v^h(s, s+1)}\, \mathbf{1}_{\overline{v}(s,s+1)/v^h(s,s+1)<1-\varepsilon}.$$

Then we obtain $SCR_s^h \geq 1$ (calculated on the basis of P^*), since $\mathrm{P}^*(N_s - v^h(s, s+1)\, N_{s+1} \geq 1 | \mathcal{F}_s) > 0.995 > 0.005$, and $SCR_s < 1-\varepsilon$ (calculated on the basis of P^*) since $\mathrm{P}^*(N_s - \overline{v}(s, s+1)\, N_{s+1} < 1-\epsilon | \mathcal{F}_s) > 0.995$. Hence, $\mathrm{VaR}_{0.995}(N_s - \overline{v}(s, s+1)\, N_{s+1} | \mathcal{F}_s) \neq SCR_s^h$ (calculated on the basis of P^*).

(ii) If $\theta_s = \widetilde{\theta}_s$ almost surely, then we also have by (2.1) $v^{real}(s, s+1) = v^h(s, s+1)$ almost surely. On the other hand, suppose now that there exists an $i_0 \in \{1, ..., m\}$ for which $\mathrm{P}(\theta_s^{i_0} \neq \widetilde{\theta}_s^{i_0}) > 0$. By defining $K_t^{i_0}(\omega) := 1 + \mathbf{1}_{[s+0.5,\infty)}(t)$ and $K_t^j(\omega) := 1$ for all $j \neq i_0$ and $t \in \mathbb{R}^+$, from (2.1) we get

$$v^{real}(s, s+1) = \frac{1}{1+\theta_s^{i_0}}, \quad v^h(s, s+1) = \frac{1}{1+\widetilde{\theta}_s^{i_0}},$$

since the sum of the θ_t^i equals 1 by definition. Thus, we obtain $\mathrm{P}(v^{real}(s, s+1) = v^h(s, s+1)) = \mathrm{P}(\theta_s^{i_0} = \widetilde{\theta}_s^{i_0}) < 1$.

(iii) The proof is analogous to the proof of part (ii). From $\theta_s^{num} = \widetilde{\theta}_s$ almost surely we can conclude that $v^{num}(s, s+1) = v^h(s, s+1)$ almost surely. On the other hand, if there exists a component $i_0 \in \{1, ..., m\}$ where the vectors θ_s^{num} and $\widetilde{\theta}_s$ differ with positive probability, then we can construct a financial market $(K_t)_t$ with $\mathrm{P}(v^{num}(s, s+1) = v^h(s, s+1)) < 1$.

■

In particular, Theorem 5.2 shows that the SCR definitions SCR_s^{real}, SCR_s^{num}, and SCR_s^h are equivalent for all insurance companies and market situations if

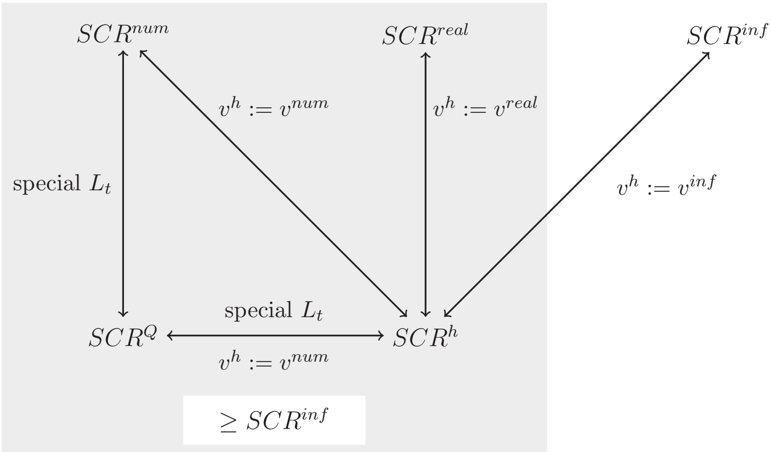

FIGURE 3: Relationship between the SCR definitions under Assumption 3.3.

and only if the corresponding investment strategies for the excess capital are equal.

Remark 5.3. SCR^h_s can change considerably depending on the choice of $v^h(s, s+1)$. While in all other definitions the discount factor is largely determined by pre-existing circumstances, the discount factor $v^h(s, s+1)$ is mainly a management decision that the insurer has to make by appointing a management strategy function h_s. A more detailed discussion of management strategies can be found in Wüthrich and Merz (2013). In case $\widetilde{H}_s$ is nearly proportional to H_s then $v^h(s, s+1)$ is well approximated by $v^{real}(s, s+1)$.

The relationship among the different SCR definitions is summarized in Figure 3.

6. CONVERGENCE OF SCR DEFINITIONS

For the next result we need the following setting: An insurer has a net value of $y^{(0)} := N_s$ and calculates the SCR

$$y^{(1)} := SCR_s = \mathrm{VaR}_{0.995}(N_s - v(s, s+1)\, N_{s+1}|\mathcal{F}_s)\,,$$

where $v(s, s+1)$ is any arbitrary discount factor. In contrast to the previous sections, the discount factor $v(s, s+1)$ is not necessarily of the form (2.1) but just some positive random variable. In a next step, the company reduces the asset portfolio with the linear management function h_s as in Assumption 3.3 by $\widetilde{A}_s := y^{(1)} - N_s$ such that the new net value is $\widetilde{N}_s = N_s + \widetilde{A}_s = y^{(1)}$. According to (3.1), we have $\widetilde{A}_{s+1} = v^h(s, s+1)^{-1}\widetilde{A}_s$, and thus we get $\widetilde{N}_{s+1} = N_{s+1} + (y^{(1)} - N_s)v^h(s, s+1)^{-1}$. With $y^{(2)}$ we denote the SCR that corresponds to the altered net value. As $y^{(2)}$ is not necessarily equal to $y^{(1)}$, the asset portfolio is again

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re-organized such that $\widetilde{N}_s = N_s + \widetilde{A}_s = y^{(2)}$. By repeating this procedure n-times, we obtain

$$y^{(n)} = \mathrm{VaR}_{0.995}(y^{(n-1)} - v(s, s+1)\,(N_{s+1} + (y^{(n-1)} - N_s)v^h(s, s+1)^{-1})|\mathcal{F}_s). \tag{6.1}$$

Theorem 6.1. *Let h_s be as in Assumption 3.3.*

(i) *If there exists an $\epsilon > 0$ such that*

$$\epsilon < \frac{v(s, s+1)}{v^h(s, s+1)} < \frac{1}{\epsilon} \text{ almost surely,} \tag{6.2}$$

then the random variable SCR_s^h is the (almost surely) unique fix-point of iteration (6.1).

(ii) *If there exists an $\epsilon \in (0, 1)$ such that*

$$\epsilon < \frac{v(s, s+1)}{v^h(s, s+1)} < 2 - \epsilon \text{ almost surely,} \tag{6.3}$$

then $\lim_{n\to\infty} y^{(n)} = SCR_s^h$ *almost surely, where $y^{(n)}$ is defined as in* (6.1).

Proof. (i) If y is a fix-point of (6.1), then almost surely

$$\begin{aligned}
& y = \mathrm{VaR}_{0.995}(y - v(s, s+1)(N_{s+1} + (y - N_s)v^h(s, s+1)^{-1})|\mathcal{F}_s) \\
\Leftrightarrow\ & 0 = \mathrm{VaR}_{0.995}\left(\frac{v(s, s+1)}{v^h(s, s+1)}\left(N_s - v^h(s, s+1)N_{s+1} - y\right)\middle|\mathcal{F}_s\right) \\
\Leftrightarrow\ & 0 = \mathrm{VaR}_{0.995}(N_s - v^h(s, s+1)N_{s+1} - y|\mathcal{F}_s) \\
\Leftrightarrow\ & y = \mathrm{VaR}_{0.995}(N_s - v^h(s, s+1)N_{s+1}|\mathcal{F}_s) = SCR_s^h\,.
\end{aligned}$$

In the third line, we use that $\frac{v(s,s+1)}{v^h(s,s+1)}$ can be omitted by Proposition A.1 in the appendix. The requirements for the proposition are fulfilled because of (6.2). The equivalences yield that SCR_s^h is always a fix-point and that all fix-points equal SCR_s^h.

(ii) The following calculations are all true almost surely. From (6.1) we get

$$\begin{aligned}
y^{(n+1)} = \mathrm{VaR}_{0.995}\Bigg(& SCR_s^h + \left(1 - \frac{v(s, s+1)}{v^h(s, s+1)}\right)(y^{(n)} - SCR_s^h) \\
& + \frac{v(s, s+1)}{v^h(s, s+1)}(N_s - v^h(s, s+1)N_{s+1} - SCR_s^h)\Bigg|\mathcal{F}_s\Bigg).
\end{aligned} \tag{6.4}$$

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By multiplying equation (6.3) with -1, adding 1, and multiplying the result with $y^{(n)} - SCR_s^h$ separately for $y^{(n)} - SCR_s^h \geq 0$ or $y^{(n)} - SCR_s^h < 0$, we get

$$-(1-\epsilon)\left|y^{(n)} - SCR_s^h\right| \leq (y^{(n)} - SCR_s^h)\left(1 - \frac{v(s,s+1)}{v^h(s,s+1)}\right) \leq (1-\epsilon)\left|y^{(n)} - SCR_s^h\right|.$$

Since the dynamic values at risk satisfy the monotonicity and the translation invariance property as the standard values at risk do, and by applying the inequalities, we obtain from (6.4) that

$$y^{(n+1)} \leq SCR_s^h + (1-\epsilon)\left|y^{(n)} - SCR_s^h\right| + \mathrm{VaR}_{0.995}\left(\frac{v(s,s+1)}{v^h(s,s+1)}(N_s - v^h(s,s+1)N_{s+1} - SCR_s^h)\middle|\mathcal{F}_s\right).$$

Since h_s is a nonnegative linear function, we get with Theorem 3.5 $\mathrm{VaR}_{0.995}(N_s - v^h(s,s+1)N_{s+1} - SCR_s^h) = 0$ and by Proposition A.1 we can multiply the inner of the value at risk with $\frac{v(s,s+1)}{v^h(s,s+1)}$, since the fraction satisfies (6.3). Consequently, the last summand is equal to zero. In total we obtain that $y^{(n+1)}$ has the upper bound $SCR_s^h + (1-\epsilon)\left|y^{(n)} - SCR_s^h\right|$ and, analogously, the lower bound $SCR_s^h - (1-\epsilon)\left|y^{(n)} - SCR_s^h\right|$. By induction we can show that

$$\left|y^{(n+1)} - SCR_s^h\right| \leq (1-\epsilon)^{n+1}\left|y^{(0)} - SCR_s^h\right| \to 0 \quad (n \to \infty)$$

pointwise. Hence, $\lim_{n\to\infty} y^{(n)} = SCR_s^h$.

■

Setting $v(s,s+1) = v^{num}(s,s+1)$ and $v(s,s+1) = v^{real}(s,s+1)$ and assuming that they fulfill (6.3) accordingly, we get that iterative calculations of SCR_s^{num} and SCR_s^{real} converge to SCR_s^h. If the probability space Ω is finite, condition (6.3) can be relaxed to $0 < \frac{v(s,s+1)}{v^h(s,s+1)} < 2$. We have $\frac{v(s,s+1)}{v^h(s,s+1)} < 2$ if and only if the return ϕ^h of the additional assets is smaller than $1+2\phi$, or $\phi^h < 1+2\phi$, where ϕ is the return corresponding to the discount factor $v(s,s+1)$. This restriction is usually met in practice. The proof also shows that, for a discount ratio $\frac{v(s,s+1)}{v^h(s,s+1)} \geq 2$, the sequence $y^{(n)}$ never converges. This fact is illustrated in Example 6.2. If the discount ratio is random and takes values both less and greater than 2, a general convergence result is out of reach.

Example 6.2 (speed of convergence). We modify the payoff of the stock from Example 2.3, such that its payoff is deterministic. We use this variable to analyze different discount ratios $\rho := \frac{v(s,s+1)}{v^h(s,s+1)}$. With a starting point of $N_0 = 100$, we calculate the iteration (6.1) with $h(x) = (x\widetilde{\theta}_0^1/K_0^1, x(1-\widetilde{\theta}_0^1)/K_0^2)^T$ and

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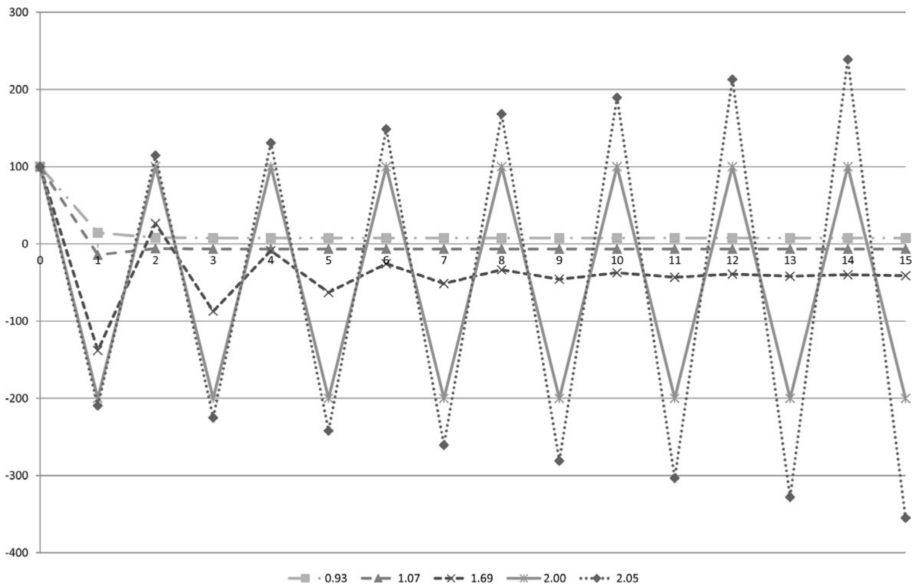

FIGURE 4: Iteration (6.1) for different discount ratios.

$\widetilde{\theta}_t^i = \theta_t^i$, such that $v^h(s, s+1) = v^{real}(s, s+1)$ and $SCR_s^h = SCR_s^{real}$. The results for the first 15 steps are shown in Figure 4. For $\rho = 0.93$ the convergence is rapid. This is still the case for $\rho = 1.07$, even though the iteration is not monotonous anymore. If the discount ratio is equal to $\rho = 2.00$, the iteration has two accumulation points, and the iteration jumps between them. For values larger than 2, we see divergent behavior. It should be mentioned that these examples are quite extreme. For example, in the case that $\rho = 2.00$, the stock performance has to be at least 215%, given that the riskless interest rate is 5% and the stock ratio is 50%. In practice, condition (6.3) is hardly a restriction.

7. INVARIANCE PROPERTY WITH APPLICATION TO INSURANCE GROUPS

In this section, we consider the question of how additional assets (excess capital) change the SCR. It sounds reasonable to require an invariance property so that the SCR is independent of the amount of excess capital that the insurer holds. However, Example 7.5 will show that such a property is not always advantageous.

Let SCR_s^h be the SCR calculated with market-consistent values of liabilities $(L_t)_t$, payments $(Z_t)_t$, and present asset portfolio H_s. By altering the present asset portfolio to $\overline{H}_s$, we get a different SCR, which we denote by $\overline{SCR}_s^h$.

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Proposition 7.1 (invariance of SCR^h). *For each management strategy function h_s as in Assumption 3.3, we have*

$$SCR_s^h = \overline{SCR}_s^h \text{ a.s.}$$

for all $\overline{H}_s \in \{H_s + h_s(\widetilde{A}_s) | \widetilde{A}_s$ *is a* $\mathcal{F}_s$*-measurable random variable*$\}$.

Proof. The difference of H_s and $\overline{H}_s$ has a representation of the form $\overline{H}_s - H_s = h_s(\widetilde{A}_s)$ for some $\mathcal{F}_s$-measurable random variable $\widetilde{A}_s$. The difference of the corresponding asset values is $\overline{A}_s - A_s = \widetilde{A}_s$. Since (3.2) implies that $v^h(s, s+1)\,\widetilde{A}_{s+1} = \widetilde{A}_s$, we have

$$\begin{aligned}\overline{N}_s - v^h(s, s+1)\overline{N}_{s+1} &= N_s + \widetilde{A}_s - v^h(s, s+1)(N_{s+1} + \widetilde{A}_{s+1}) \\ &= N_s - v^h(s, s+1)\, N_{s+1}\,.\end{aligned}$$

Consequently,

$$\begin{aligned}SCR_s^h &= \mathrm{VaR}_{0.995}(N_s - v^h(s, s+1)\, N_{s+1} | \mathcal{F}_s) \\ &= \mathrm{VaR}_{0.995}(\overline{N}_s - v^h(s, s+1)\overline{N}_{s+1} | \mathcal{F}_s) = \overline{SCR}_s^h\,. \qquad (7.1)\end{aligned}$$

■

The set $\{H_s + h_s(\widetilde{A}_s) | \widetilde{A}_s$ is a $\mathcal{F}_s$-measurable random variable$\}$ is a subset in the set of all (theoretically possible) asset portfolios at time s. The invariance of SCR_s^h is only true on this subset.

Remark 7.2 (invariance of SCR_s^{num} and SCR_s^{real}). Proposition 7.1 gives us an invariance property for SCR_s^h. Analogously to the proof of Proposition 7.1 we get that SCR_s^{num} and SCR_s^{real} are invariant with respect to the initial net value if additional capital is invested in the numeraire and proportionally to the existing asset portfolio, respectively. In other words, the SCR is invariant with respect to excess capital if the discount factor v^x in Definition 4.3(a) describes the return that the insurer really earns on its excess capital. Such an invariance property for SCR_s^{num} is also mentioned in Artzner and Eisele (2010). With Theorem 5.2 we get that, for having invariance with respect to all financial markets and market-consistent values of liabilities, it is not only sufficient but also necessary that excess capital is invested according to the discount factor in Definition 4.3(a). For SCR_s^{num} and SCR_s^{real} this means that we necessarily have to invest the excess capital in the numeraire and proportional to the existing portfolio, respectively.

Let SCR_0^{inf} be the minimal SCR of an insurer according to (2.8) with market-consistent values of liabilities $(L_t)_t$, payments $(Z_t)_t$, and present asset portfolio H_0, and let $\overline{SCR}_0^{inf}$ be the minimal SCR of the same insurer but with present asset portfolio $\overline{H}_0$. The following corollary is a direct consequence from definition (2.8), which is only valid at time 0.

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Corollary 7.3 (invariance of SCR^{inf} at time 0). *We have*

$$SCR_0^{inf} = \overline{SCR}_0^{inf} .$$

The corollary is not true anymore if we restrict possible asset shifts as in Assumption 3.3, so we cannot generalize this result to Definition 4.3(c).

Since insurance groups have the possibility to shift money between their subsidiary undertakings up to a certain extent, an invariance property is particularly useful for insurance groups, because in that situation, a shifting of money between subsidiary undertakings does not change the SCR.

Suppose we have an insurance group that consists of n insurance companies with asset portfolios $({}^iH_t)_t$ and market-consistent values of liabilities $({}^iL_t)_t$ $(1 \leq i \leq n)$. Let $({}^i\widetilde{\theta}_t)_t$ be the asset strategy of company i for additional assets within a linear management strategy ${}^ih_s$. We assume that the total assets of the n insurance companies are reallocated at time s and that ${}^i\widehat{H}_s$ are the units of assets that insurer i delivers or receives (depending on the sign) at time s. Let ${}^i\widehat{\theta}_s^j = \frac{{}^i\widehat{H}_s^j K_s^j}{{}^i\widehat{H}_s \cdot K_s}$ be the corresponding proportions of transferred assets, and let ${}^iSCR_s$ and ${}^i\widehat{SCR}_s$ be the SCRs for company i before and after the asset transfer.

Corollary 7.4. *Under the above framework, we have the following properties:*

(i) ${}^iSCR_0^{inf} = {}^i\widehat{SCR}_0^{inf}$*, where the SCR is defined according to* (2.8).

(ii) *Given that Assumption 3.3 holds,* ${}^iSCR_s^h = {}^i\widehat{SCR}_s^h$ *for all* $s \in \mathbb{N}_0$, $1 \leq i \leq n$ *and all* $(K_t)_t$, $(L_t)_t$, $(Z_t)_t$, *and probability measures equivalent to* P *if and only if* ${}^i\widehat{\theta}_s = {}^i\widetilde{\theta}_s$ *for all* $1 \leq i \leq n$.

The corollary is a direct consequence of the previous results and Theorem 5.2. As stated in the directive (European Parliament and the Council, 2009, Chapter II, Section 1), the SCR of a group should be calculated on the basis of the consolidated accounts (default method) or by aggregating the stand-alone SCRs (alternative method). The default method is similar to the calculation of a single SCR. In the technical specifications to QIS 5 (CEIOPS, 2010, Section 6), the alternative method is basically described as

$$SCR_{\text{group}} = \sum {}^iSCR_{\text{solo-adjusted}} ,$$

where ${}^iSCR_{\text{solo-adjusted}}$ is the SCR of company i adjusted according to some group effects. Consequently, in case the assumptions of Corollary 7.4 hold, shifting money between subsidiary undertakings does not change the group SCR.

Example 7.5 (invariance property for insurance group). We consider Example 2.3 but add one more asset, which has price $K_0^3 := 100$ at time zero and payoff $K_1^3 := 210 - K_1^2$ after one year. We consider two insurance companies that belong to an insurance group, and both have liabilities of $L_0 = 100$ and $L_1 = 105$. The asset structure of the companies assumed for this example, their

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TABLE 1

SCRs BEFORE AND AFTER THE TRANSFER TOOK PLACE.

		H_0	N_0	N_1	SCR_0^{num}	SCR_0^{real}	SCR_0^h	SCR_0^{inf}
Before Transfer	Company 1	$(1, 1, 0)^T$	100	K_1^2	$\frac{100}{7}$	$\frac{100}{13}$	25	0
	Company 2	$(1, 0, 1)^T$	100	K_1^3	$\frac{100}{7}$	$\frac{100}{13}$	0	0
After Transfer	Company 1	$(1, 1, 1)^T$	200	210	0	0	25	0
	Company 2	$(1, 0, 0)^T$	0	0	0	0	0	0

net values and the resulting SCRs are shown in the first two rows of Table 1. Suppose that company 2 transfers one unit of investment 3 at time zero to company 1, ${}^1\widehat{H}_0 = {}^1\widetilde{H}_0 = (0, 0, 1)$ and ${}^2\widehat{H}_0 = {}^2\widetilde{H}_0 = (0, 0, -1)$. These strategies are used for the calculation of SCR^h. The resulting SCRs are shown in last two rows of Table 1.

Since the requirements of Corollary 7.4 are fulfilled, SCR^h and SCR^{inf} are invariant with respect to the exchange of assets, while SCR^{num} and SCR^{real} are not invariant, not only individually for each company but also in total. The transfer of investment K^3 from company 2 to company 1 is reasonable, since K^3 is a perfect hedge for investment K^2. After the asset transfer, both companies have no longer any risk, and SCRs of zero seem to be appropriate. Then why is ${}^1\widehat{SCR}_0^h > 0$? As ${}^1\widetilde{H}_0 = (0, 0, 1)$, and the definition of SCR^h implicitly assumes that redundant assets are paid out (cf. definition (2.7) and Remark 7.2), shares of K^3 are paid out, and the perfect hedge is disrupted.

The example illustrates that an invariance property is not always desirable.

8. RISK MARGIN

This section deals with the RM according to Solvency II and its interaction with the SCR. Since there is no universally valid definition of the RM in the academic literature, we resolve some of the unsolved problems to find a RM that is consistent with the directive. The key is that we defined the SCR in Definition 4.3 also for future points in time and that we have with SCR^{inf} a minimizing SCR definition, which is a key property for the definition of the SCR of a reference undertaking.

The purpose of the RM is to decompose the calculation of the market-consistent value of liabilities into $L_s = BE_s + RM_s$, where BE_s denotes the best estimate of the liabilities. Article 77 paragraph 5 of the Solvency II directive requires that the "risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof"; i.e. the directive suggests a cost-of-capital approach with respect to the SCR. A possible implementation of these requirements is given

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in the technical specifications to the fifth Quantitative Impact Study (CEIOPS, 2010). We interpret the RM given there as

$$RM = c \sum_{k \geq 0} \frac{SCR_{0,k}^{RU}}{(1 + r_{k+1})^{k+1}}, \tag{8.1}$$

where $SCR_{t,k}^{RU}$ is the SCR of a reference undertaking for the time period $[k, k+1)$ conditional on the information available at time t, c is the cost-of-capital rate, and r_t is the risk-free rate for maturity t. In CEIOPS (2010) the numerator in (8.1) is given just as SCR_k^{RU}, but from our point of view the notation $SCR_{0,k}^{RU}$ is more appropriate. The calculation of the RM is based on a transfer scenario, where the liabilities are taken over by an artificial insurance company that is capitalized exactly to SCR^{RU}. Furthermore, it is assumed that "the assets should be considered to be selected in such a way that they minimize the SCR for market risk that the reference undertaking is exposed to" (compare CEIOPS, 2010). CEIOPS (2009) requires that the RM contains the non-hedgeable risks. If there are two reference undertakings with the same liabilities, but one company has a different asset portfolio such that it also has a lower SCR, then an investor prefers the one with the smaller SCR. Hence, we conclude that the SCR of a reference undertaking should have an asset portfolio such that the resulting SCR is minimal. Formula (8.1) has the following three deficiencies, which cannot be clarified from the official documents, since a mathematically rigorous definition is missing.

- The risk margin RM and the quantities $SCR_{0,k}^{RU}$ shall be $\mathcal{F}_0$-measurable, but it is not specified how the $\mathcal{F}_k$-measurable random variable SCR_k that gives the SCR for time period $[k, k+1)$ is transferred to a $\mathcal{F}_0$-measurable random variable.
- Formula (8.1) defines the RM only for time $s = 0$.
- A precise mathematical definition of the SCR of a reference undertaking is missing.

In the following we want to deduce a mathematically sound definition for the SCR of a reference undertaking which chooses an optimal asset portfolio such that the SCR is minimized. However, in SCR^{inf} according to Definition 4.3 the SCR is minimized with respect to all asset portfolios that can be built from the originally portfolio with a positive or a negative shift. More precisely, let H_s be again the current asset portfolio of the insurer, then SCR^{inf} considers all portfolios in the set $\{H_s + \widetilde{H}_s : \widetilde{H}_s \in [0, \infty)^m \cup (-\infty, 0]^m\}$. Since this excludes a lot a reasonable asset portfolios we introduce a minimum asset portfolio $\underline{H}_s$ which has in all components very negative values such that all reasonable asset portfolios can be reached with a positive shift. This is specified in the following assumption.

Assumption 8.1. We assume that for all $s \in \mathbb{N}_0$ there is a deterministic minimal portfolio $\underline{H}_s \in \mathbb{R}^m$ such that $\underline{H}_s \leq H_s + \widetilde{H}_s$ a.s., where H_s is the asset portfolio of the insurer (compare Definition 2.1), $\widetilde{H}_s$ is a possible shift of the portfolio (compare Definition 2.2) that fulfills Assumption 3.3 and can be arbitrary within these restrictions, and for vectors $w, \widetilde{w} \in \mathbb{R}^m$ we specify that $w \leq \widetilde{w}$ if and only if $w_i \leq \widetilde{w}_i$ for all $1 \leq i \leq m$.

This assumption means that there is a minimal asset portfolio such that all possible asset portfolios that we consider are in all components larger. Such a minimal portfolio exists if we restrict possible short sales of each asset. For example, if we completely rule out short sales, we set $\underline{H}_s = 0 \in \mathbb{R}^m$. Assumption 8.1 can be seen as a restriction from two different point of views: First, given that we know all relevant portfolios $H_s + \widetilde{H}_s$ we assume that there is a minimal portfolio $\underline{H}_s$. Second, given that we have a minimal portfolio it restricts the admissible portfolios $H_s + \widetilde{H}_s$. Consequently, Assumption 8.1 restricts the possible asset portfolios with respect to which SCR_s^h and SCR_s^{inf} are minimizing. Let SCR_s^h be the SCR of an insurance company with market-consistent values of liabilities $(L_t)_t$, payments $(Z_t)_t$, present asset portfolio H_s, and management strategy vector B_s according to Definition 2.2, Assumption 3.3, and Assumption 8.1. Under these requirements SCR_s^h can be specified with Proposition 4.1 and Definition 4.3, given that $X_{[0,s]} = x$, as

$$\begin{aligned} SCR_s^h = \inf\{&N_s^x + \widetilde{A}_s^x \in \mathbb{R} : H_s^x + \widetilde{A}_s^x\, B_s^x \geq \underline{H}_s, \\ &\mathrm{P}(N_{s+1} + \widetilde{A}_s^x\, B_s^x \cdot K_{s+1} \geq 0 | X_{[0,s]} = x) \geq 0.995\}, \end{aligned} \tag{8.2}$$

since $\widetilde{H}_s^x = \widetilde{A}_s^x\, B_s^x$. For the remainder of this section we use (8.2) as the definition of SCR_s^h. Let $\underline{SCR}_s^{inf}$ be the SCR of the same insurance company, but with asset portfolio $\underline{H}_s$ from Assumption 8.1, such that $\underline{N}_s = \underline{H}_s \cdot K_s - L_s$. Under Assumption 8.1 we define $\underline{SCR}_s^{inf}$, given that $X_{[0,s]} = x$, analogously to the notation used in the proof of Proposition 4.4 as

$$\begin{aligned} \underline{SCR}_s^{inf} = \inf_{b_s \in [0,\infty)^m \setminus \{0\}} \inf\{&\underline{N}_s^x + \underline{y} b_s \cdot K_s^x \in \mathbb{R} : \underline{H}_s + \underline{y} b_s \geq \underline{H}_s, \\ &\mathrm{P}(\underline{N}_{s+1} + \underline{y} b_s \cdot K_{s+1} \geq 0 | X_{[0,s]} = x) \geq 0.995\}. \end{aligned} \tag{8.3}$$

Proposition 8.2. *Under Assumptions 3.3 and 8.1, we have for all portfolios H_s that are admissible in the sense of Assumption 8.1*

$$\underline{SCR}_s^{inf} \leq SCR_s^h \text{ almost surely}, \tag{8.4}$$

where SCR_s^h and $\underline{SCR}_s^{inf}$ are defined as in (8.2) *and* (8.3)*, respectively.*

Proof. Given that $X_{[0,s]} = x$ and under the restrictions of Assumption 8.1 SCR_s^h and $\underline{SCR}_s^{inf}$ are given by (8.2) and (8.3), respectively. For all $\widetilde{A}_s^x \in \mathbb{R}$ that satisfy the inequalities in (8.2), we set

$$\underline{y} = \widetilde{A}_s^x + N_s^x - \underline{N}_s^x \quad \text{and} \quad b_s = \begin{cases} \underline{y}^{-1}(H_s^x + \widetilde{A}_s^x\, B_s^x - \underline{H}_s) & \text{for } \underline{y} \neq 0 \\ (1, 0, ..., 0)^T & \text{for } \underline{y} = 0 \end{cases}.$$

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With Assumption 8.1 , $\widetilde{H}_s^x = \widetilde{A}_s^x B_s^x$, and $\underline{N}_{s+1} = \underline{H}_s \cdot K_{s+1} + Z_{s+1} - L_{s+1}$, we get that

- $\underline{y} = \widetilde{H}_s^x \cdot K_s^x + H_s^x \cdot K_s^x - \underline{H}_s \cdot K_s^x \geq 0$,
- $b_s \in [0, \infty)^m \setminus \{0\}$, since $H_s^x + \widetilde{A}_s^x B_s^x - \underline{H}_s = H_s^x + \widetilde{H}_s^x - \underline{H}_s \geq 0$,
- for $\underline{y} \neq 0$: $\underline{N}_s^x + \underline{y} b_s \cdot K_s^x = N_s^x + (\underline{H}_s - H_s^x + \underline{y} b_s) \cdot K_s^x = N_s^x + \widetilde{A}_s^x$, since $B_s^x \cdot K_s^x = 1$, and for $\underline{y} = 0$ this a direct consequence from the definition of $\underline{y}$,
- $\underline{H}_s + \underline{y} b_s \geq \underline{H}_s$, since $\underline{y} b_s = H_s^x + \widetilde{H}_s^x - \underline{H}_s \geq 0$ for $\underline{y} \neq 0$,
- $\underline{N}_{s+1} + \underline{y} b_s \cdot K_{s+1} = N_{s+1} + \underline{H}_s \cdot K_{s+1} - H_s^x \cdot K_{s+1} + \underline{y} b_s \cdot K_{s+1} = N_{s+1} + \widetilde{A}_s^x B_s^x \cdot K_{s+1}$, where the last step follows from $\underline{H}_s - H_s^x + \underline{y} b_s = \widetilde{A}_s^x B_s^x$ for $\underline{y} \neq 0$ and for $\underline{y} = 0$ we get $0 = \widetilde{A}_s^x + N_s^x - \underline{N}_s^x = (\widetilde{H}_s^x + H_s^x - \underline{H}_s^x) \cdot K_s^x$ which implies that $0 = \widetilde{H}_s^x + H_s^x - \underline{H}_s^x = \widetilde{A}_s^x B_s^x + H_s^x - \underline{H}_s^x$, since K_s^x is always strictly positive in all components by definition and $\widetilde{H}_s^x + H_s^x - \underline{H}_s^x \geq 0$ by Assumption 8.1.

Consequently, the infimum set of SCR_s^h is a subset of the infimum set of $\underline{SCR}_s^{inf}$ and so we get $\underline{SCR}_s^{inf} \leq SCR_s^h$ almost surely. ■

With Proposition 8.2 we see that $\underline{SCR}^{inf}$ chooses implicitly the optimal portfolio to minimize the SCR and is smaller than SCR^h independent of the choice of h_s. For this reason, we define the SCR of a reference undertaking on the basis of $\underline{SCR}^{inf}$.

Definition 8.3 (SCR of a reference undertaking). Under the assumptions of Proposition 8.2, we define the SCR for time period $[s, s+1)$ and conditional on $\mathcal{F}_s$ of a reference undertaking with market-consistent values of liabilities $(L_t)_t$ and payments $(Z_t)_t$ by

$$SCR_{s,s}^{RU} := \underline{SCR}_s^{inf} .$$

As the RM is intended for the calculation of the market-consistent value of current liabilities, we exclude the new business in L_t. This is required in Assumption 5 in CEIOPS (2009). The choice of $\underline{H}_s$ according to Assumption 8.1 can have a significant impact on $SCR_{s,s}^{RU}$, especially when short selling is a crucial part of the optimal strategy. However, if $\underline{H}_s$ is way below optimal short selling strategies, then $SCR_{s,s}^{RU}$ is rather insensitive with respect to $\underline{H}_s$.

CEIOPS (2009) mention in the context of the SCR of a reference undertaking that the projected SCR should be considered. This allows us to conclude that we need some measure that makes a transfer from time k ($k \geq s$) to time s. One possible choice is to take the expected value of the future SCRs which we will do in the following. A natural choice is the expected value under a martingale measure Q, given such a measure exists. However, one could also argue for other measures, e.g. the real-world measure. For a general valid definition, we allow for a random and time-dependent cost of capital rate. Since the SCR, the cost-of-capital rate, and the discount factor might be dependent, we have to take the

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expected value of all three factors together. In conclusion, we generalize (8.1) by the following definition.

Definition 8.4 (RM cost-of-capital version). The RM at time s is defined as

$$RM_s := \sum_{k\geq s} \mathbb{E}_{\mathrm{Q}}\big(c(k)\, v^{num}(s, k+1)\, SCR^{RU}_{k,k} \big| \mathcal{F}_s\big),$$

where $c(k)$ is the cost-of-capital rate at time k, $SCR^{RU}_{k,k}$ only includes new business up to time s, and v^{num} is the discount factor corresponding to a bank-account, such that $\mathbb{E}_{\mathrm{Q}}(v^{num}(s, t)|\mathcal{F}_s)$ is the price of a zero-coupon bond at time s with maturity t.

In practice, usually there does not exist a unique martingale measure Q that comprises all risk modules of Solvency II. For those risks where market prices are available (e.g. interest rate risk, equity risk), Q should be defined as the corresponding market measure. For those risks that are not traded on a deep and liquid market (e.g. longevity risk, lapse risk), we suggest defining Q on a first-order valuation basis. For example, for the valuation of longevity risk, let Q correspond to a first-order mortality table (that means that we interpret the difference between first-order and second-order mortality table as the market price for longevity risk). A discussion of calculating the risk-adjusted expected value for the risk margin can also be found in Wüthrich *et al.* (2011).
For the practical purpose of calculating the RM, this definition is usually too complex, so that simplifications are needed. Suppose that the cost-of-capital rate is constant, that is, $c(t) \equiv c$, and that $v^{num}(s, k+1)$ and SCR^{inf}_k are conditionally independent given $\mathcal{F}_s$. Note that these two requirements are often not fulfilled. Then we obtain the following, simplified version of the RM,

$$RM_s = c \sum_{k\geq s} \mathbb{E}_{\mathrm{Q}}\big(v^{num}(s, k+1)\big|\mathcal{F}_s\big)\, \mathbb{E}_{\mathrm{Q}}\big(SCR^{RU}_{k,k}\big|\mathcal{F}_s\big),$$

which is inter alia similar to a definition of the RM in Möhr (2011). Implicitly we have here

$$SCR^{RU}_{s,k} = \mathbb{E}_{\mathrm{Q}}\big(SCR^{RU}_{k,k}\big|\mathcal{F}_s\big).$$

In practice, further simplifications are used, which cannot be theoretically established.

9. Conclusion

We started the paper with a comparison of Article 101 and Recital 64 of the Solvency II directive and discussed in particular three interpretations of Article 101 and two interpretations of Recital 64 which are based on an asset minimization concept. Our main findings are as follows:

- For nonnegative linear management strategies, Recital 64 can be represented as a value at risk and, thus, the mathematical structure is similar to Article 101.
- Article 101 and Recital 64 are consistent (in the sense that they are equal for all financial markets and liability market-consistent values) if and only if the discount factor in Article 101 corresponds to the investment strategy for excess capital in Recital 64.
- Interpreting Recital 64 as the infimum with respect to all linear, nonnegative management strategies, the resulting SCR is indeed smaller or equal than all other definitions.
- When assets are minimized iteratively by applying a linear management strategy, the SCR from Recital 64 is the unique fix point of this iteration, and under weak requirements the resulting SCRs even converge towards the SCR from Recital 64.
- Invariance of the SCR with respect to additional capital corresponds to investing this capital according to the discount factor v in (2.2). The minimal SCR interpretation of Recital 64 is at time 0 always invariant. However, invariance is not necessarily a desirable property.

In practice, management strategies for the excess assets may be non-linear. Our focus on linear management strategies can be seen as a first-order approximation, allowing us to write all SCR definitions in form of a value at risk and making the different definitions more comparable. The SCR definitions that we found in the literature all implicitly imply linear management strategies. Future research should also examine non-linear management strategies.

For the calculation of the market-consistent value of liabilities, Solvency II suggests using a cost of capital method and calculating a RM. However, the definition of the RM depends not only on a present SCR but also on future SCRs, which are in fact random.

- We showed how to define future SCRs based on a generalization of the value at risk to a dynamic value at risk.
- We give a general RM definition that takes into account the randomness of the future SCRs.
- We defined the SCR of a reference undertaking with the help of SCR^{inf}.

The definition of SCR^{inf} calculated with a minimal asset portfolio implicitly assumes that the market risk is minimized by buying corresponding securities. However, insurance companies might not invest their money using this optimal strategy, so that their market risk is much higher than the definition implicitly assumes. Article 101 of the directive of the European Parliament and the Council (2009) is often interpreted in such a way that the real market risk of the insurer shall be considered, and thus the minimal SCR interpretation of Recital 64 is not necessarily in line with Article 101. A clarification of that question by the regulator would be helpful.

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MARCUS C. CHRISTIANSEN
Institut für Versicherungswissenschaften
Universität Ulm, D-89069 Ulm, Germany
E-Mail: marcus.christiansen@uni-ulm.de

ANDREAS NIEMEYER (Corresponding author)
Institut für Versicherungswissenschaften
Universität Ulm, D-89069 Ulm, Germany
E-Mail: andreas.niemeyer@uni-ulm.de

APPENDIX

Proof of Proposition 4.1. As $\mathbb{R}$ is a Polish space (see Kechris, 1995), according to Bauer (1981, Chapter 10) there exists a Markov kernel Q such that $x \mapsto Q(x, A)$ is a version of

$$\mathrm{P}(Y \in A | X_{[0,s]} = x), \quad A \in \mathfrak{B}(\mathbb{R}).$$

Setting $A = (-\infty, y]$, we get that the function $x \mapsto F_x(y) := Q(x, (-\infty, y]) = \mathrm{P}(Y \leq y | X_{[0,s]} = x)$ is $\mathcal{F}'_s$-$\mathfrak{B}(\mathbb{R})$-measurable for each $y \in \mathbb{R}$. For any fixed x, $F_x(y)$ is a cumulative distribution function, and we define the quantile function or generalized inverse as

$$F_x^{-1}(\alpha) = \inf\{y \in \mathbb{R} | F_x(y) \geq \alpha\}.$$

Since $x \mapsto F_x(y)$ is $\mathcal{F}'_s$-$\mathfrak{B}(\mathbb{R})$-measurable, it holds that

$$\{x \in \Omega : F_x(r) \geq \alpha\} \in \mathcal{F}'_s, \quad \forall r \in \mathbb{R}.$$

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From Milbrodt (2010, page 229) we know that $F(r) \geq \alpha \Leftrightarrow F^{-1}(\alpha) \leq r$ for all $r \in \mathbb{R}$ and $\alpha \in (0, 1]$ such that we obtain

$$\{x \in \Omega : F_x^{-1}(\alpha) \leq r\} = \{x \in \Omega : F_x(r) \geq \alpha\} \in \mathcal{F}, \quad \forall r \in \mathbb{R}.$$

Hence, the function

$$x \mapsto F_x^{-1}(\alpha) = \inf\{y \in \mathbb{R} : \mathrm{P}(Y \leq y | X_{[0,s]} = x) \geq \alpha\} = q_{Y,\alpha}(x)$$

is $\mathcal{F}_s'$-$\mathfrak{B}(\mathbb{R})$-measurable. The pointwise infimum of countably many real-valued measurable functions is measurable as well. ■

Proposition A.1. *Let* $(\Omega, \mathcal{F}, \mathrm{P})$ *be a probability space with random variables* W *and* Z*, where* Z *is bounded and satisfies* $Z > \varepsilon > 0$ *almost surely. For all* $\alpha \in (0, 1)$ $\mathrm{Var}_\alpha(W|\mathcal{F}_s) = 0$ *if and only if* $\mathrm{Var}_\alpha(WZ|\mathcal{F}_s) = 0$.

Proof. Let $\mathcal{F}_s = \sigma(X_{[0,s]}) \subset \mathcal{F}$. Since $\mathrm{P}(W \leq 0 | X_{[0,s]} = x) = \mathrm{P}(WZ \leq 0 | X_{[0,s]} = x)$ almost surely, we have

$$\begin{aligned} &\inf\{w \in \mathbb{R} : \mathrm{P}(W \leq w | X_{[0,s]} = x) \geq \alpha\} = 0 \\ &\quad \Longrightarrow \inf\{w \in \mathbb{R} : \mathrm{P}(WZ \leq w | X_{[0,s]} = x) \geq \alpha\} \leq 0 \end{aligned}$$

and

$$\begin{aligned} &\inf\{w \in \mathbb{R} : \mathrm{P}(WZ \leq w | X_{[0,s]} = x) \geq \alpha\} = 0 \\ &\quad \Longrightarrow \inf\{w \in \mathbb{R} : \mathrm{P}(W \leq w | X_{[0,s]} = x) \geq \alpha\} \leq 0. \end{aligned}$$

Now we show that the infima on the right hand side cannot be smaller than 0. We start with the first line. If $\mathrm{Var}_\alpha(W|\mathcal{F}_s) = 0$, we necessarily have $\mathrm{P}(-\delta < W \leq 0 | X_{[0,s]} = x) > 0$ for all $\delta > 0$. Since by assumption $Z < c$ almost surely for some $c > 0$, we obtain that also

$$\mathrm{P}(-\delta c < WZ \leq 0 | X_{[0,s]} = x) \geq \mathrm{P}(-\delta < W \leq 0 \leq w | X_{[0,s]} = x) > 0, \quad \delta > 0.$$

This implies that $\mathrm{Var}_\alpha(WZ|\mathcal{F}_s)$ is not smaller than zero. Analogously, using $Z > \varepsilon$, we can show that $\mathrm{Var}_\alpha(W|\mathcal{F}_s)$ cannot be negative if $\mathrm{Var}_\alpha(WZ|\mathcal{F}_s) = 0$. ■

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