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Density fluctuations and polarization features of magnetohydrodynamic waves

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Abstract. In the interstellar medium (ISM), a broad range of turbulent plasmas can be observed by using radio waves. However, only density fluctuations are directly measurable with these radio astronomic tools. If one wants to deduce the underlying features of the turbulence, it is necessary to calculate the relation between density and magnetic fluctuations. Here a magnetohydrodynamic (MHD)-ansatz is taken into account for the ISM turbulence to calculate the fluctuations for the three low-frequency MHD waves.

1. Introduction

In the field of cosmic ray astrophysics it is necessary to know the power spectra of magnetic fluctuations in the interstellar medium (ISM) to determine transport parameters as there are parallel and perpendicular diffusion coefficients, momentum diffusion coefficients and the rate of adiabatic deceleration of cosmic rays. Those fluctuations in the ISM are non-relativistic and have frequencies below the proton gyrofrequency Ω_{i}.

Usually the fluctuations are interpreted as a superposition of magnetohydrodynamic (MHD) waves, which model the turbulent electromagnetic field (i.e. the plasma wave viewpoint). The different MHD waves (fast and slow magnetosonic, Alfvén) have unique relations between electric and magnetic field expressed by their dispersion relation and Maxwell's equations.

The main issue we will deal with in this publication is the basic problem that arises from the observations. The main tools for the investigation of the ISM are radio scintillations and dispersion measures, as described by Armstrong et al. (1995). Actually, these observations are only capable of determining the fluctuations in density and the mean parallel (to the line-of-sight) magnetic field. So we have to figure out the relation between magnetic field and density fluctuations for the MHD waves.

Schlickeiser and Lerche (2002) have calculated this relation in a full kinetic way in order to find a wavenumber-dependent compressibility for Alfvén waves, which cannot be found within the MHD framework. In contrast to that work, we will stick to the plasma wave viewpoint and develop a theory for small perturbations of an MHD system with a background magnetic field B_0.

We will apply our calculations to the warm intercloud medium, which is a low-β plasma (β is the ratio of thermal and magnetic pressure) with a neutral density $n_{\mathrm{H}} = 0.2\ \mathrm{cm}^{-3}$ and an ion density of $n_{\mathrm{i}} = 0.08\ \mathrm{cm}^{-3}$. Temperatures are ranging between 6000 and 10^4 K. The warm intercloud medium is the prominent phase of the ISM with its enormous filling factor and very high degree of ionization.

Observations provide some knowledge about the power spectrum of the density fluctuations, it seems as if we can assume a Kolmogorov-like power law behaviour, with an spectral index $s = 5/3$ for the density fluctuations. However, as mentioned before, this gives no insight into the nature of the magnetic field fluctuations. In particular, it cannot solve the problem if the magnetic spectra are anisotropic (as proposed by Spangler 1991 and Goldreich and Sridhar 1995).

2. Basic equations

We start with the MHD equations

$$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{m}} \cdot \mathbf{v}) = 0, \tag{2.1}$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}}{\partial t} + \rho_{\mathrm{m}} (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p + \mathbf{j} \times \mathbf{B}, \tag{2.2}$$

$$\nabla p = v_{\mathrm{s}}^2 \nabla \rho_{\mathrm{m}}, \tag{2.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \tag{2.4}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{2.5}$$

Here ρ_{m} is the mass density, $\mathbf{v}$ is the mass fluid velocity, v_{s} is the ion sound speed, $\mathbf{E}$ and $\mathbf{B}$ are electric and magnetic field respectively.

We neglect the extra Hall term in (2.5)

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} + \frac{m_{\mathrm{i}}}{\rho e c} \mathbf{j} \times \mathbf{B} = 0, \tag{2.6}$$

as the scales involved in ISM turbulence are mostly above the Hall scale $c v_{\mathrm{A}} / (v_0 \omega_{pi}) = 10^8$ cm.

We investigate the relation between the fluctuations of magnetic field and density. The outline of the calculation is given in Koskinen (2001).

The above equations are rewritten as

$$\frac{\partial \rho_{\mathrm{m}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{m}} \mathbf{v}) = 0, \tag{2.7}$$

$$\rho_{\mathrm{m}} \frac{\partial \mathbf{v}}{\partial t} + \rho_{\mathrm{m}} (\mathbf{v} \cdot \nabla) \mathbf{v} + v_{\mathrm{s}}^2 \nabla \rho_{\mathrm{m}} - \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{\mu_0} = 0, \tag{2.8}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) = \frac{\partial \mathbf{B}}{\partial t}. \tag{2.9}$$

In the next step we linearize (2.7)–(2.9) by assuming that the magnetic field and the density exhibit small perturbations of the mean field. In addition, we transform the plasma into the rest system, so that there is no bulk velocity,

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1(\mathbf{r}, t), \tag{2.10}$$

$$\rho_{\mathrm{m}} = \rho_0 + \rho_1(\mathbf{r}, t), \tag{2.11}$$

$$\mathbf{v} = 0 + \mathbf{v}_1(\mathbf{r}, t). \tag{2.12}$$

Hence from (2.7)–(2.9), we derive

$$\frac{\partial \rho_1}{\partial t} + \rho_0(\nabla \cdot \mathbf{v}_1) = 0 \tag{2.13}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + v_{\mathrm{s}}^2 \nabla \rho_1 + \frac{\mathbf{B}_0 \times (\nabla \times \mathbf{B}_1)}{\mu_0} = 0 \tag{2.14}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} - \nabla \times (\mathbf{v}_1 \times \mathbf{B}_0) = 0. \tag{2.15}$$

We assume the time and space variations of the perturbed quantities to be of the form

$$f_1 = \hat{f}_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}. \tag{2.16}$$

Inserting this form into the MHD equations (2.13)–(2.15), the time and space derivatives can now be written as functions of the wave frequency ω and the wavevector $\mathbf{k}$, and we obtain

$$\omega \rho_1 - (\mathbf{k} \cdot \mathbf{v}_1)\rho_0 = 0, \tag{2.17}$$

$$\omega \mathbf{v}_1 \rho_0 - v_{\mathrm{s}}^2 \rho_1 \mathbf{k} - \frac{1}{\mu_0} \mathbf{B}_0 \times (\mathbf{k} \times \mathbf{B}_1) = 0, \tag{2.18}$$

$$\omega \mathbf{B}_1 + \mathbf{k} \times (\mathbf{v}_1 \times \mathbf{B}_0) = 0. \tag{2.19}$$

Combining (2.17)–(2.19) with each other, we can derive a vector equation linear in the density fluctuations

$$\left[\omega^2 \mathbf{B}_0 - (\mathbf{k} \cdot \mathbf{B}_0) v_{\mathrm{s}}^2 \mathbf{k}\right] \frac{\rho_1}{\rho_0} = \left[\omega^2 - \frac{(\mathbf{k} \cdot \mathbf{B}_0)^2}{\mu_0 \rho_0}\right] \mathbf{B}_1 + \frac{(\mathbf{k} \cdot \mathbf{B}_0)(\mathbf{B_0} \cdot \mathbf{B}_1)\mathbf{k}}{\mu_0 \rho_0}. \tag{2.20}$$

By introducing the Alfvén velocity $\mathbf{v}_{\mathrm{A}} = \mathbf{B}_0/\sqrt{\mu_0 \rho_0}$ we can simplify the above equation. To solve it for the magnetic field fluctuations we will separate the equations into their different components

$$(-k_z B_0 v_{\mathrm{s}}^2 k_x) \frac{\rho_1}{\rho_0} = (\omega^2 - k_z^2 v_{\mathrm{A}}^2) B_{1x} + v_{\mathrm{A}}^2 k_z k_x B_{1z} \tag{2.21}$$

$$(-k_z B_0 v_{\mathrm{s}}^2 k_y) \frac{\rho_1}{\rho_0} = (\omega^2 - k_z^2 v_{\mathrm{A}}^2) B_{1y} + v_{\mathrm{A}}^2 k_z k_y B_{1z} \tag{2.22}$$

$$(\omega^2 B_0 - k_z^2 B_0 v_{\mathrm{s}}^2) \frac{\rho_1}{\rho_0} = (\omega^2 - k_z^2 v_{\mathrm{A}}^2) B_{1z} + v_{\mathrm{A}}^2 k_z^2 B_{1z}. \tag{2.23}$$

Without further loss of generality, we restrict the k-vector to be in the x-z-plane. So our calculation results in

$$\frac{1}{B_0} \begin{pmatrix} B_{1x} \\ B_{1y} \\ B_{1z} \end{pmatrix} = \frac{\rho_1}{\rho_0} \begin{pmatrix} k_x k_z \dfrac{(v_{\mathrm{A}}^2 k_z^2 v_{\mathrm{s}}^2/\omega^2 - v_{\mathrm{A}}^2 - v_{\mathrm{s}}^2)}{\omega^2 - k_z^2 v_{\mathrm{A}}^2} \\ k_y k_z \dfrac{(v_{\mathrm{A}}^2 k_z^2 v_{\mathrm{s}}^2/\omega^2 - v_{\mathrm{A}}^2 - v_{\mathrm{s}}^2)}{\omega^2 - k_z^2 v_{\mathrm{A}}^2} \\ \dfrac{\omega^2 - k_z^2 v_{\mathrm{s}}^2}{\omega^2} \end{pmatrix}. \tag{2.24}$$

3. Dispersion relation

From Swanson (1989) we can take the three branches of low frequency MHD waves, which are the fast (FM) and slow (SM) magnetosonic waves and Alfvén waves (AW).

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The latter is an incompressible wave, while the other two are compressible. Their dispersion relations are

$$\text{SM}\ \omega^2 = \frac{k^2 v_{\rm A}^2}{2}\left((1+\beta) - \sqrt{(1+\beta)^2 - 4\beta\cos^2\theta}\right), \tag{3.1}$$

$$\text{FM}\ \omega^2 = \frac{k^2 v_{\rm A}^2}{2}\left((1+\beta) + \sqrt{(1+\beta)^2 - 4\beta\cos^2\theta}\right), \tag{3.2}$$

$$\text{AW}\ \omega^2 = k_z^2 v_{\rm A}^2, \tag{3.3}$$

where β is the ratio of the gas pressure and the magnetic pressure. The dispersion relations can be simplified for the case of $\beta \ll 1$, as found in the ISM. We then expand the square root in a series around $\beta = 0$. We find for the FM waves

$$\omega^2 \approx k^2 v_{\rm A}^2 \left(1 + \beta - \frac{\beta}{1+\beta}\cos^2\theta\right). \tag{3.4}$$

For the case of SM waves we will have to do a second-order expansion in β because the first-order expansion would cancel out the denominator of (2.24) to derive

$$\omega^2 \approx v_{\rm A}^2 k^2 \left(\frac{\beta\cos^2\theta}{1+\beta} + \frac{\beta^2\cos^4\theta}{(1+\beta)^3}\right). \tag{3.5}$$

4. Magnetic field

For each type of wave we will substitute the respective dispersion relation into (2.24) to derive the relation of the magnetic field and density fluctuations in each case. We consider each wave mode in turn.

Alfvén waves. These waves exhibit no density fluctuations, as can be seen from the x-component of (2.24). The denominator $\omega^2 - k_z^2 v_{\rm A}^2$ will vanish if one inserts the dispersion relation for Alfvén waves. To fulfil this equation, ρ_1 has to be zero and also B_z will be zero. Therefore, Alfvén waves are strictly incompressible in the MHD framework.

Slow magnetosonic waves. For the x-component of the magnetic field we find that

$$\frac{\rho_1}{\rho_0} = \frac{B_{1x}}{B_0}\frac{(1+\beta(1+\cos^2\theta))\sin\theta}{\beta(\beta+\sin^2\theta)\cos\theta}, \tag{4.1}$$

and analogically for the z-component

$$\frac{\rho_1}{\rho_0} = -\frac{B_{1z}}{B_0}\frac{1+\beta(1+\cos^2\theta)}{\beta(\beta+\sin^2\theta)}. \tag{4.2}$$

From this follows that there is a non-vanishing longitudinal magnetic field component, while the transversal magnetic field vanishes for waves propagating parallel to the magnetic field. If one calculates the polarization feature of the wave from both aforementioned equations, they yield

$$\frac{B_{1x}}{B_{1z}} = -\frac{\cos\theta}{\sin\theta}. \tag{4.3}$$

Fast magnetosonic waves. Using the same technique as before, we derive from Eq. (2.24)

$$\frac{\rho_1}{\rho_0} = \left[1 + \frac{\beta\cos^2\theta}{1+2\beta\sin^2\theta}\right]\frac{B_{1z}}{B_0}, \tag{4.4}$$

$$\frac{\rho_1}{\rho_0} = -\frac{\sin\theta}{\cos\theta}(1-\beta\cos^2\theta)\frac{1+2\beta}{1-2\beta\sin^2\theta}\frac{B_{1x}}{B_0}. \tag{4.5}$$

The polarization can now be calculated from these density fluctuation relations, yielding

$$\frac{B_{1x}}{B_{1z}} = -\frac{\cos\theta}{\sin\theta}\frac{1-2\beta\sin^2\theta}{1+2\beta}\frac{1+\beta(1+\sin^2\theta)}{1+2\beta\sin^2\theta}. \tag{4.6}$$

5. Electric field

From the known polarization of the magnetic field it is straightforward to derive the properties of the electric field starting from (2.18)

$$\omega\mathbf{v}_1\rho_0 = v_\mathrm{s}^2\rho_1\mathbf{k} + \frac{1}{\mu_0}\mathbf{B}_0\times(\mathbf{k}\times\mathbf{B}_1). \tag{5.1}$$

To calculate the velocity $\mathbf{v}_1$ we use the relation

$$\rho_1 = \mathbf{k}\cdot\mathbf{v}_1\frac{\rho_0}{\omega}, \tag{5.2}$$

which was derived from the continuity equation. Inserting this relation into (5.1) results in

$$\mathbf{v}_1 = \frac{v_\mathrm{s}^2}{\omega\rho_0}\mathbf{k}(\mathbf{k}\cdot\mathbf{v}_1)\frac{\rho_0}{\omega} + \frac{(\mathbf{B}_0\cdot\mathbf{B}_1)\mathbf{k}}{\omega\rho_0} - \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\omega\rho_0} \tag{5.3}$$

$$= \frac{v_\mathrm{s}^2}{\omega^2}\mathbf{k}(\mathbf{k}\cdot\mathbf{v}_1) + \frac{(\mathbf{B}_0\cdot\mathbf{B}_1)\mathbf{k}}{\omega\rho_0} - \frac{(\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\omega\rho_0}. \tag{5.4}$$

These equations can be solved componentwise, here for example, performed for the x-component, we have

$$v_{1x} - \frac{v_\mathrm{s}^2}{\omega^2}k_x(\mathbf{k}\cdot\mathbf{v}_1) = \frac{(\mathbf{B}_0\cdot\mathbf{B}_1)\mathbf{k} - (\mathbf{B}_0\cdot\mathbf{k})\mathbf{B}_1}{\mu_0\omega\rho_0}, \tag{5.5}$$

with the other two components similar to the x-component. By considering a special geometry ($\mathbf{B}_0 = B_0\mathbf{e}_z$ and $\mathbf{k} = k_x\mathbf{e}_x + k_z\mathbf{e}_z$), we obtain

$$v_{1x} = \frac{v_\mathrm{s}^2}{\omega^2}k_x(k_xv_{1x} + k_zv_{1z}) + \frac{B_{0z}B_{1z}k_x - B_{0z}k_zB_{1x}}{\mu_0\omega\rho_0}, \tag{5.6}$$

$$v_{1y} = -\frac{B_{0z}B_{1y}k_z}{\mu_0\omega\rho_0}, \tag{5.7}$$

$$v_{1z} = \frac{v_\mathrm{s}^2}{\omega^2}k_z^2v_{1z} + \frac{v_\mathrm{s}^2}{\omega^2}k_xk_zv_{1x}. \tag{5.8}$$

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Again solving the above system of equations for the components, we have

$$v_{1x} = \frac{1}{\mu_0\rho_0\omega}\frac{\omega^2 - v_s^2k_z^2}{\omega^2 - v_s^2k_z^2 - v_s^2k_x^2}(B_{0z}B_{1z}k_x - B_{0z}B_{1x}k_z), \tag{5.9}$$

$$v_{1y} = -\frac{B_{0z}B_{1y}k_z}{\mu_0\omega\rho_0}, \tag{5.10}$$

$$v_{1z} = \frac{1}{\mu_0\rho_0\omega}\frac{v_s^2k_xk_z}{\omega^2 - v_s^2k_z^2 - v_s^2k_x^2}(B_{0z}B_{1z}k_x - B_{0z}B_{1x}k_z). \tag{5.11}$$

Using our results in (2.5) we obtain for the electrical field

$$\mathbf{E}_1 = \begin{pmatrix} v_{1y}B_{0z} \\ -v_{1x}B_{0z} \\ 0 \end{pmatrix} \tag{5.12}$$

$$= \begin{pmatrix} -\dfrac{B_{0z}^2B_{1y}k_z}{\omega\rho_0} \\ \dfrac{1}{\mu_0\rho_0\omega}\dfrac{\omega^2 - v_s^2k_z^2}{\omega^2 - v_s^2k_z^2 - v_s^2k_x^2}(B_{0z}B_{1z}k_x - B_{0z}B_{1x}k_z)B_0z \\ 0 \end{pmatrix}. \tag{5.13}$$

According to (2.24) $B_{1y} \propto k_y = 0$ so that in this geometry $v_{1y} = 0$ implying $E_{1x} = E_{1z} = 0$, independent of the wave type.

We will now take a detailed look at the velocity and the electric field for the different wave modes.

Slow magnetosonic. The detailed calculations show

$$v_{1x} = -v_A\sqrt{\beta(1+\beta)}\frac{\rho_1}{\rho_0}\frac{\cos\theta}{\sin\theta}\frac{\beta}{1+\beta(1+\cos^2\theta)}, \tag{5.14}$$

$$v_{1y} = 0, \tag{5.15}$$

$$v_{1z} = v_A.\sqrt{\beta(1+\beta)}\frac{\rho_1}{\rho_0}\frac{1}{1+\beta(1+\cos^2\theta)}. \tag{5.16}$$

We can immediately deduce from (5.12)

$$\mathbf{E} = -\mathbf{e}_yB_0v_A\sqrt{\beta(1+\beta)}\frac{\rho_1}{\rho_0}\frac{1}{1+\beta(1+\cos^2\theta)}. \tag{5.17}$$

Fast magnetosonic. with a similar calculation as above, we obtain

$$v_{1x} \simeq v_A\sqrt{\frac{1+\beta}{\beta(1+\sin^2\theta)}\frac{1+2\beta\sin^2\theta}{1+\beta\sin^2\theta}} \times \left(\sin\theta\frac{1+2\beta\sin^2\theta}{1+\beta(1+\sin^2\theta)} + \frac{\cos^2\theta}{\sin\theta}\frac{1+2\beta(1+\sin^2\theta)}{1+\beta\cos^2\theta}\right), \tag{5.18}$$

$$v_{1z} \simeq \sqrt{\frac{1+\beta}{\beta(1+\sin^2\theta)}\frac{\beta\sin\theta\cos\theta}{1+\beta\sin^2\theta}} \times \left(\sin\theta\frac{1+2\beta\sin^2\theta}{1+\beta(1+\sin^2\theta)} + \frac{\cos^2\theta}{\sin\theta}\frac{1+2\beta(1+\sin^2\theta)}{1+\beta\cos^2\theta}\right). \tag{5.19}$$

So for the electrical field we have

$$\mathbf{E} = \mathbf{e}_y B_0 v_{\mathrm{A}} \sqrt{\frac{1+\beta}{\beta(1+\sin^2\theta)} \frac{1+2\beta\sin^2\theta}{1+\beta\sin^2\theta}}$$
$$\times \left(\sin\theta \frac{1+\beta\sin^2\theta}{1+\beta} - \frac{\cos^2\theta}{\sin\theta} \frac{1-2\beta\sin^2\theta}{1+\beta(2-\cos^2\theta)} \right). \tag{5.20}$$

Alfvén waves. As we have already worked out, there are no density fluctuations in Alfvén waves and the only non-vanishing component of the magnetic field is the x-component. It is, therefore, straightforward to work out the electric field and the velocities from the induction equation and Ohm's law

$$\mathbf{k} \times \mathbf{E} = -\frac{\omega}{c}\mathbf{B}, \tag{5.21}$$

$$\Rightarrow E_y = \frac{v_{\mathrm{A}}}{c} B_{1x}, \tag{5.22}$$

$$\mathbf{E} = -\frac{v}{c} \times \mathbf{B}, \tag{5.23}$$

$$\Rightarrow v_{1x} = v_{\mathrm{A}} \frac{B_{1x}}{B_{0z}}. \tag{5.24}$$

6. Comparison with previous results

Sitenko (1967) has calculated the polarization of the **E**-vector from the dielectric tensor. We will summarize his results as follows.

6.1. *Alfvén*

$$\frac{\mathbf{E}}{E_x} = \begin{pmatrix} 1 \\ -ik_z \dfrac{\omega}{\Omega_{\mathrm{i}}} \tan^{-2}\theta \\ -\dfrac{v_{\mathrm{s}}^2}{v_{\mathrm{A}}^2} \dfrac{\omega^2}{\Omega_{\mathrm{i}}^2} \dfrac{1}{\sin\theta\cos\theta} \end{pmatrix} \tag{6.1}$$

$$\frac{\mathbf{B}}{E_x} = \frac{c}{\omega} \begin{pmatrix} ik_z \dfrac{\omega}{\Omega_{\mathrm{i}}} \tan^{-2}\theta \\ k_z + k_x \dfrac{v_{\mathrm{s}}^2}{v_{\mathrm{A}}^2} \dfrac{\omega}{\Omega_{\mathrm{i}}^2} \dfrac{1}{\sin\theta\cos\theta} \\ ik_x \dfrac{\omega}{\Omega_{\mathrm{i}}} \tan^{-2}\theta \end{pmatrix}. \tag{6.2}$$

Considering the physical boundary conditions we encounter in the diffuse ISM ($\omega \ll \Omega_{\mathrm{i}}$ and $\beta \ll 1$), we may limit the calculations for the magnetic field to

$$B_y = \frac{ck_z}{\omega} E_x = \frac{c}{v_{\mathrm{A}}} E_x \tag{6.3}$$

for $\theta < \pi/2$. Keeping the fact in mind that the x- and y-component are interchangeable, this resembles the result of (5.23).

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6.2. Fast magnetosonic waves

$$\frac{\mathbf{E}}{E_y} = \begin{pmatrix} -i\dfrac{\omega}{\Omega_{\rm i}}\sin^{-2}\theta \\ 1 \\ -i\dfrac{v_{\rm s}^2}{v_{\rm A}^2}\dfrac{\omega}{\Omega_{\rm i}}\sin\theta\cos\theta \end{pmatrix}, \tag{6.4}$$

$$\frac{\mathbf{B}}{E_y} = \frac{c}{\omega}\begin{pmatrix} -k_z \\ ik\dfrac{\omega}{\Omega_{\rm i}}(\cos\theta\sin^{-2}\theta + v_{\rm s}^2 v_{\rm A}^{-2}\sin^2\theta\cos\theta) \\ k_x \end{pmatrix}. \tag{6.5}$$

If we consider the ratio between the x- and z-component as in (4.6), we see that

$$\frac{B_x}{B_z} = -\frac{\cos\theta}{\sin\theta} \tag{6.6}$$

gives us the same result as the limit $\beta \to 0$ for (4.6). The B_y-component can only be calculated in a kinetic framework, as a finite Larmor-radius is included.

6.3. Slow magnetosonic waves

$$\frac{\mathbf{E}}{E_z} = \begin{pmatrix} \sin\theta \\ -i\dfrac{v_{\rm s}^2}{v_{\rm A}^2}\sin\theta\cos\theta \\ \cos\theta \end{pmatrix} \tag{6.7}$$

$$\frac{\mathbf{B}}{E_z} = \frac{c}{\omega}\begin{pmatrix} i\dfrac{v_{\rm s}^2}{v_{\rm A}^2}\dfrac{\Omega_{\rm i}}{\omega}\sin^2\theta k_z \\ 0 \\ i\dfrac{v_{\rm s}^2}{v_{\rm A}^2}\dfrac{\Omega_{\rm i}}{\omega}\cos^2\theta k_x \end{pmatrix}. \tag{6.8}$$

The Sitenko result shows the same B_x/B_z behaviour as our result, which is requested by the divergence-free condition for the magnetic field.

Schlickeiser and Lerche (2002) have derived the density fluctuation spectra for the Alfvén and FM waves, but as they used a kinetic theory to show compressibility for Alfvén waves we can compare only the results for the FM. Equation (6.9) in their paper reads

$$P_{\rm zz}(\mathbf{k}) = \tan^2\theta P_{\rm xx}(\mathbf{k}) \tag{6.9}$$

which is similar to (4.6) in the $\beta \to 0$ limit. Their (6.8) gives the fluctuation spectrum

$$\frac{P_{\rm nn}(\mathbf{k})}{n_e^2} = \frac{P_{\rm zz}(\mathbf{k})}{B_0^2}\frac{9}{4}(1+\beta\sin^2\theta)^2. \tag{6.10}$$

Contrary to this we have calculated

$$\frac{P_{\rm nn}(\mathbf{k})}{n_e^2} = \frac{P_{\rm zz}(\mathbf{k})}{B_0^2}\left(1 + \frac{\beta\cos^2\theta}{1+2\beta\sin^2\theta}\right)^2. \tag{6.11}$$

Hall (1980) has also calculated the plasma density fluctuations induced by waves, but he considered a different physical regime (high-β), so that his results are not comparable.

7. Discussion and conclusions

We were able to derive the relations between the density fluctuations and the resulting magnetic and electric field fluctuations. These are important results for the interpretation of astrophysical data in the context of analytical calculations.

The results are applicable to plasma waves with $\beta < 1$ and propagation angles $\theta < \pi/2$. Another limitation is that ω should be below $\Omega_{\rm i}$ so that the effects of a finite Larmor radius may be neglected. In order to neglect the Hall effect, the scales involved should be above the Hall scale.

Our theory proved the incompressibility of Alfvén waves in the MHD framework, which is important as a test case for the theory. Comparing our results with those of Sitenko (1967), who works with a theory based on the dielectric tensor, we can see that we are able to recover most properties of the magnetic field, except for those accounted for kinetic effects (i.e. B_y and E_x). We have a different result for the magnetic field by SM waves, that is explained by our use of a plasma with $\beta \neq 0$. Also, the electric field shows different properties that are a result of the deviations in the magnetic field and the chosen system of coordinates.

The major advantage of our calculation is the possibility to handle plasmas with $\beta \neq 0$.

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