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A generalised Milne-Thomson theorem for the case of an elliptical inclusion

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An $\mathbb{R}$-linear conjugation problem modelling the process of power fields forming in a heterogeneous infinite planar structure with an elliptical inclusion is considered. Exact analytical solutions are derived in the class of piece-wise meromorphic functions with their principal parts fixed. Cases with internal singularities and with singularities of the given principal parts at the interface are investigated.

Key words: Heterogeneous medium; Elliptic inclusion; $\mathbb{R}$-linear conjugation problem; Analytic functions

1 Introduction

The study of heterogeneous media is of great importance in several branches of mechanics of continua. This is important for manufactured composite materials as well as for investigations of natural ones (soils, aquifers, biological tissues etc.) and designed inhomogeneous objects. Mathematical models describing transport processes are, in general, 3D and transient, and therefore of utmost difficulty for analytical solution. Two-dimensional plane or axisymmetrically heterogeneous structures exposed to steady physical fields (temperature, concentration, fluid pressure, electrostatic stress etc.) are easier to tackle because the Fourier, Fick, Hook, Darcy, Ohm and other 'linear laws' reduce the field problem to the Laplace's equation in the homogeneous and isotropic components of the composite and, therefore, the complex-analysis theory is applicable. Even for a plane structure the possibility to get an explicit solution is problematic: success in analytical treatment depends on the geometry of the composite. Among all plane structures, the most intensively studied, since the seminal contribution of Maxwell [11] and Lord Rayleigh [21], is a medium consisting of an infinite isotropic matrix with an elliptical, in particular circular [12], inclusion. There are numerous papers devoted to this specific topic and only those reporting analytical or semi-analytical solutions are cited below.

Apparently, the works [2, 8] were the first where the problem of determination of the elastic field induced by an elliptical inhomogeneity was considered. Results of the latter paper were generalised in [4, 5, 14] for the 3D case of ellipsoidal inclusions. In particular, Hardiman [8] and Eshelby [5] proved that a uniform stress applied at infinity induces a constant stress state within an elliptical or ellipsoidal inhomogeneity. Eshelby [5] put forward the following conjecture: The field inside a bounded inclusion will be uniform for any uniform elastic loading at infinity if and only if this inclusion is of an elliptic or an ellipsoidal shape. It was shown in [4] and [1] that a polynomial stress applied at infinity induces the same degree polynomial stress inside the elliptic inclusion for an isotropic and an anisotropic matrix, respectively. The explicit form of the complex potential was derived in [6] for an elastic field in the matrix and with an elliptical inhomogeneity under a plane loading condition. The magnetoelastic fields induced by applied magnetic fields in an infinite matrix containing an elliptic elastic inclusion were investigated in [9]. Shmid and Podladchikov [22] have presented a set of closed-form solutions for an isolated and deformable elliptical inclusion subjected to general shear, far-field flows. They have shown how to apply these solutions to geological problems. Kang and Milton [9] have given a new strict proof of Eshelby's conjecture.

We do not refer here to papers on multiple elliptical inclusions, as they are out of the scope of our present research, but readers interested in this can find a relevant bibliography in [13].

In almost all above cited works the complex variable approach for plane isotropic and anisotropic problems of elasticity was useful. Solutions were derived by a conformal mapping of the physical z-plane onto an auxiliary concentric annular domain in the ζ-plane and through the Laurent series expansion of characteristic functions. This method allows one to get a closed-form solution for the case when the *a priori* given complex potential has the only singularity at infinity. Cases of arbitrarily distributed singularities of the given complex potential have been investigated essentially less. In [19] Pilatovskii considered one-phase seepage with arbitrarily distributed sinks or sources modelling abstraction or injection wells in a rock formation with a lens. He suggested an original method of reduction of this flow problem to a linear integral equation. This approach allows the derivation of an explicit solution for an elliptic contour ([19], p. 102) when a finite number of sinks are located either inside or outside the inclusion. Golubeva and Shpilevoy [6] considered the same problem for the case of only external singularities but these singularities were arbitrary, i.e. unconditioned multipoles. To the best of our knowledge, the problem of an elliptical inhomogeneity has not been investigated in the general case when the given complex potential has a finite number of arbitrary singularities distributed arbitrarily with respect to the inclusion, i.e. inside, outside or at the interface. The last case is special and has not been considered at all. The aim of the present paper, in particular, is to fill this 'analytical gap' using the same methods as in [15,16] and [17], p. 16.

The paper is organised as follows. The strict formulation of the problem is provided in Section 2; the closed-form solutions are derived in Sections 3 and 4 for singularities located inside or outside the ellipse and singularities just on the interface, respectively; and Section 5 discusses the results.

2 Formulation

A planar infinite homogeneous medium (matrix) is considered as a complex plane $\mathbb{C}$ of variable z, and $f(z)$ is a given complex potential. We restrict ourselves to a finite number of singularities of $f(z)$. Under such an assumption the derivative $f'(z)$ as a single valued meromorphic function is just a rational function and it can be represented as a sum of

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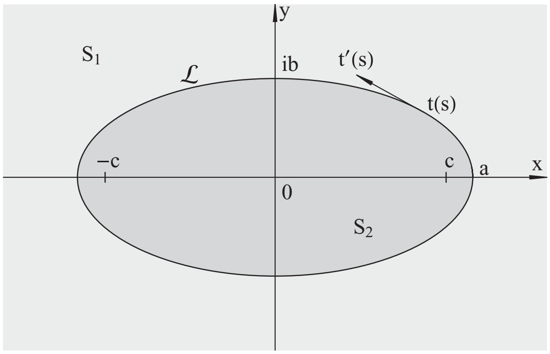

FIGURE 1. Elliptical inclusion.

common fractions and a polynomial:

$$f'(z) = \sum_{j=0}^{n} a_j z^j + \sum_{l=1}^{m} \sum_{j=1}^{m_j} \frac{a_{lj}}{(z - z_l)^j}. \tag{2.1}$$

The problem is to determine the corresponding disturbed complex potential $w(z)$ after insertion of an elliptical inclusion into $\mathbb{C}$ such that the medium becomes two-phase with arbitrary situated singularities of $f(z)$ about the inclusion, including the case with singularities at the interface. The usual boundary conditions are imposed on $w(z)$, that is the continuity of the stream function, $\psi(z) = \Im w(z)$, and product potential, $\varphi(z) = \Re w(z)$, and resistivity, $\rho(z)$, across the interface. Generally, a complex potential is a multi-valued function, while the derivative $v(z) = w'(z)$ is a single-valued one. Because of this reason $v(z)$ is more convenient for our forthcoming derivations.

Let S_1 and S_2 be the infinite matrix and elliptical inclusion, respectively (Figure 1). $\mathscr{L}$ is the boundary of inclusion S_2. A piece-wise meromorphic function $v(z) = w'(z) = v_x(x, y) - \mathrm{i}\, v_y(x, y)$ is complex conjugated with the complex velocity function $\mathbf{v}(z) = v_x(x, y) + \mathrm{i}\, v_y(x, y) = \overline{v(z)}$. Function $v(z) = v_k(z)$, $z \in S_k$ is meromorphic in S_k and continuous in $\overline{S}_k$ everywhere, except at singular points of its principal part coinciding with the sum of corresponding summands of function (2.1) ($k = 1, 2$). Two real boundary conditions, above imposed on $w(z)$, are equivalent ([3], p. 53) to the following $\mathbb{R}$-linear conjugation problem,

$$v_2(t) = Av_1(t) - B[t'(s)]^{-2}\overline{v_1(t)}, \qquad t \in \mathscr{L}, \tag{2.2}$$

where s is the arc length of the contour $\mathscr{L}$, the derivative $t'(s)$ is the unit tangential vector to the contour $\mathscr{L}$ at the point $t \in \mathscr{L}$ and

$$A = \frac{\rho_2 + \rho_1}{2\rho_2}, \quad B = \frac{\rho_2 - \rho_1}{2\rho_2}, \quad \lambda = B/A = \frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}. \tag{2.3}$$

Here ρ_k ($\sigma_k = 1/\rho_k$) is the resistivity (conductivity) of a medium S_k.

Thus, it is required to find a solution of the boundary-value problem (2.2) in the class of piece-wise meromorphic functions with principal part (2.1).

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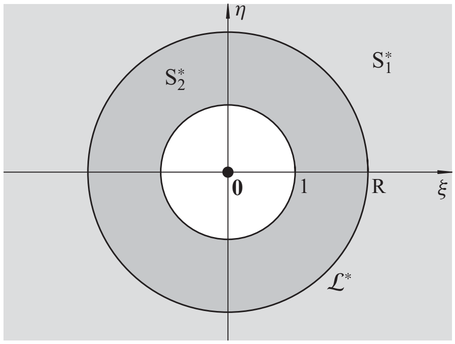

FIGURE 2. Image of $\mathbb{C}\backslash[-c, c]$ in ζ-plane.

We start with a solution of the problem (2.2) under the assumption that there are no singularities of $f'(z)$ at contour $\mathscr{L}$.

3 Solution of the problem (2.2) without singularities at the interface

For the sake of definiteness, let the coordinate axes of the z-plane coincide with the symmetry axes of the elliptical inclusion, i.e. $\mathscr{L} = \{z = x + iy \ : \ x^2/a^2 + y^2/b^2 = 1\}$, where a, b are positive parameters such that $a \geqslant b > 0$ and $\pm c$ are the foci of $\mathscr{L}$:

$$c = \sqrt{a^2 - b^2} \geqslant 0. \tag{3.1}$$

Let $a > b$. We consider the conformal mapping of the z-plane with the cut along the segment $[-c, c]$ onto the ζ-plane with the help of the branch of the function

$$\zeta(z) = \frac{1}{c}(z + \sqrt{z^2 - c^2}) \tag{3.2}$$

fixed by the condition $\zeta(\infty) = \infty$.

Function (3.2) maps the domains S_1 and $S_2\backslash[-c, c]$ onto the domains $S_1^* = \{\zeta \ : \ |\zeta| > R\}$ and $S_2^* = \{\zeta \ : \ 1 < |\zeta| < R\}$, respectively (Figure 2), where

$$R = \zeta(a) = \frac{a+b}{c} = \frac{c}{a-b} = \sqrt{\frac{a+b}{a-b}} > 1. \tag{3.3}$$

It has been proved in ([17], p. 36) that if $t \in \mathscr{L}$ and $\tau = \zeta(t) \in \mathscr{L}^*$ then

$$[t'(s)]^{-2} = -\frac{\overline{\tau - 1/\tau}}{\tau - 1/\tau}, \quad |\tau| = R. \tag{3.4}$$

We introduce a new unknown piece-wise meromorphic function

$$V(\zeta) = \tfrac{c}{2}\left(\zeta - \tfrac{1}{\zeta}\right) v_p[z(\zeta)] = V_p(\zeta), \quad \zeta \in S_p^*, \quad p = 1, 2, \tag{3.5}$$

where the multiplier $c/2$ is taken for the convenience in the forthcoming derivations and

$$z(\zeta) = \frac{c}{2}(\zeta + 1/\zeta) \tag{3.6}$$

is the inversion of function (3.2).

Due to (2.2) and (3.4), function (3.5) satisfies the boundary conditions

$$\begin{aligned} V_2(\tau) &= AV_1(\tau) + B\overline{V_1(\tau)}, \ |\tau| = R, \\ V_2(\tau) &= -V_2(1/\tau), \qquad\qquad |\tau| = 1. \end{aligned} \tag{3.7}$$

The latter condition (3.7) follows from definition (3.5) and the continuity of function $v_2(z)$ across segment $[-c, c]$.

It is clear that the function

$$\mathscr{V}(\zeta) = \begin{cases} V(\zeta), & |\zeta| \geqslant 1 \\ -V(1/\zeta), & |\zeta| < 1 \end{cases}, \tag{3.8}$$

is defined in the whole ζ-plane. This function gives, due to the second condition (3.7), an analytical continuation $\mathscr{V}_2(\zeta)$ of function $V_2(\zeta)$ from the annulus $S_2^* = \{\zeta\ :\ 1 < |\zeta| < R\}$ into the annulus $\Omega_2 = \{\zeta\ :\ 1/R < |\zeta| < R\}$. It defines the function $\mathscr{V}_1^+(\zeta)$ in the domain $\Omega_1^+ = \{\zeta\ :\ |\zeta| < 1/R\}$. For the sake of symmetry we designate $\mathscr{V}_1^-(\zeta) = V_1(\zeta)$ for $\zeta \in \Omega_1^- = S_1^*$. Then the following identities should be satisfied:

$$\mathscr{V}_1^+(1/\zeta) \equiv -\mathscr{V}_1^-(\zeta),\ \zeta \in \Omega_1^-; \quad \mathscr{V}_2(1/\zeta) \equiv -\mathscr{V}_2(\zeta),\ \zeta \in \Omega_2. \tag{3.9}$$

Thus, the problem (3.7) is reduced to the following one:

$$\begin{aligned} (1+\lambda)\mathscr{V}_2(\tau) &= \mathscr{V}_1^+(\tau) + \lambda\overline{\mathscr{V}_1^+(\tau)}, \ |\tau| = 1/R, \\ (1+\lambda)\mathscr{V}_2(\tau) &= \mathscr{V}_1^-(\tau) + \lambda\overline{\mathscr{V}_1^-(\tau)}, \quad\ |\tau| = R, \end{aligned} \tag{3.10}$$

where λ is defined in (2.3). A piece-wise meromorphic solution of the boundary-value problem (3.10) has to satisfy the symmetry condition (3.9) and its principal part has to coincide with

$$F(\zeta) = \frac{c}{2}\left(\zeta - \frac{1}{\zeta}\right) f'\left[\frac{c}{2}\left(\zeta + \frac{1}{\zeta}\right)\right]. \tag{3.11}$$

It is not difficult to show that the last rational function could be represented as a polynomial and a sum of common fractions combined in summands

$$F(\zeta) = F_1^+(\zeta) + F_1^-(\zeta) + F_2(\zeta),$$

which have poles in the domains $\Omega_1^\pm$ and Ω_2, respectively. Besides, the identities

$$F_1^+(1/\zeta) \equiv -F_1^-(\zeta), \qquad F_2(1/\zeta) \equiv -F_2(\zeta) \tag{3.12}$$

are true and hence $F_2(\pm 1) = 0$.

Indeed, in order to get the required representation we take $F_1^+(\zeta)$ as a sum of all common fractions, summands of $F(\zeta)$ having poles in Ω_1^+. We define $F_1^-(\zeta) = -F_1^+(1/\zeta)$, $F_2(\zeta) = F(\zeta) - (F_1^+(\zeta) - F_1^+(1/\zeta))$. In accordance with (3.11), function $F_2(\zeta)$ defined in this

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manner satisfies the second identity (3.12) and hence all its poles are in the domain Ω_2. Note that function $F_2(\zeta)$ differs, generally speaking, by a constant α from the sum of all summands of $F(\zeta)$ having poles in Ω_2. Hence, $F_2(\infty)=\alpha$, $F_1^+(\infty)=0$ and then $F_1^-(0)=0$ due to (3.12).

Now the required solution of the problem (3.10) could be taken in the form

$$\mathscr{V}_1^{\pm}(\zeta)=F_1^{\pm}(\zeta)+\mathscr{V}_{01}^{\pm}(\zeta), \qquad \mathscr{V}_2(\zeta)=F_2(\zeta)+\mathscr{V}_{02}(\zeta), \tag{3.13}$$

where $\mathscr{V}_{01}^{\pm}(\zeta)$ and $\mathscr{V}_{02}(\zeta)$ are new unknown functions holomorphic in the domains $\Omega_1^{\pm}$ and Ω_2 correspondingly. Both $\mathscr{V}(\zeta)$ and $F(\zeta)$ satisfy the identity (3.9) and therefore the function $\mathscr{V}_0(\zeta)=\{\mathscr{V}_{01}^{\pm}(\zeta),\zeta\in\Omega_1^{\pm};\ \mathscr{V}_{02},\zeta\in\Omega_2\}$ should satisfy the same identity. Wherefrom, in particular, it follows that $\mathscr{V}_{02}(\pm 1)=0$.

Due to the Laurent theorem,

$$\mathscr{V}_{02}(\zeta)=\mathscr{V}_{02}^{+}(\zeta)+\mathscr{V}_{02}^{-}(\zeta), \tag{3.14}$$

where functions $\mathscr{V}_{02}^{\pm}(\zeta)$ are holomorphic in the domains $\mathbb{C}\backslash\overline{\Omega_1^{\mp}}$, respectively. The summands of function (3.14) satisfy the identity

$$\mathscr{V}_{02}^{+}(1/\zeta)\equiv-\mathscr{V}_{02}^{-}(\zeta), \qquad \zeta\in\mathbb{C}\backslash\overline{\Omega_1^{+}}. \tag{3.15}$$

The uniqueness of the representation (3.14) is provided by the condition $\mathscr{V}_{02}^{+}(0)=0$ and hence $\mathscr{V}_{02}^{-}(\infty)=0$.

Using (3.13), we rewrite the second condition (3.10) in the form

$$\begin{aligned}(1+\lambda)(\mathscr{V}_{02}^{+}(\tau)+\mathscr{V}_{02}^{-}(\tau)+F_2(\tau))&=\mathscr{V}_{01}^{-}(\tau)\\ +F_1^{-}(\tau)+\lambda\big(\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\tau})}+\overline{F_1^{-}(R^2/\overline{\tau})}\big), &\qquad |\tau|=R.\end{aligned} \tag{3.16}$$

Let us now consider the function

$$\Phi(\zeta)=\begin{cases}(1+\lambda)(\mathscr{V}_{02}^{-}(\zeta)+F_2(\zeta))-\mathscr{V}_{01}^{-}(\zeta)-\lambda\overline{F_1^{-}(R^2/\overline{\zeta})}, & \zeta\in\Omega_1^{-}\\ F_1^{-}(\zeta)-(1+\lambda)\mathscr{V}_{02}^{+}(\zeta)+\lambda\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\zeta})}, & \zeta\in\mathbb{C}\backslash\overline{\Omega_1^{-}}\end{cases}. \tag{3.17}$$

It is clear that function (3.17) is holomorphic in the domains Ω_1^- and $\mathbb{C}\backslash\overline{\Omega_1^-}$, continues across the interface $|\zeta|=R$ due to (3.16) and is bounded at infinity. Hence, this function is holomorphic in $\overline{\mathbb{C}}$ and according to the Liouville theorem $\Phi(\zeta)\equiv C=\text{const}$. From (3.17) we get,

$$\begin{aligned}&(1+\lambda)(\mathscr{V}_{02}^{-}(\zeta)+F_2(\zeta))-\mathscr{V}_{01}^{-}(\zeta)-\lambda\overline{F_1^{-}(R^2/\overline{\zeta})}=C, && \zeta\in\Omega_1^{-},\\ &F_1^{-}(\zeta)-(1+\lambda)\mathscr{V}_{02}^{+}(\zeta)+\lambda\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\zeta})}=C, && \zeta\in\mathbb{C}\backslash\overline{\Omega_1^{-}}.\end{aligned} \tag{3.18}$$

Exclusion of function $\mathscr{V}_{01}^{-}(\zeta)$ from system (3.18) by the help of identities (3.12) and (3.15) leads to the following functional equation about $\mathscr{V}_{02}^{-}(\zeta)$:

$$\mathscr{V}_{02}^{-}(\zeta)=D-\lambda\overline{\mathscr{V}_{02}^{-}(\overline{\zeta}R^2)}+(1-\lambda)F_1^{+}(\zeta)-\lambda\overline{F_2(\overline{\zeta}R^2)}, \quad \zeta\in\mathbb{C}\backslash\overline{\Omega_1^{+}}, \tag{3.19}$$

where $D=(C+\lambda\overline{C})/(1+\lambda)$. The unique solution of equation (3.19) is a sum of series

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converging absolutely and uniformly in $\mathbb{C}\backslash\overline{\Omega_1^+}$:

$$\mathscr{V}_{02}^-(\zeta) = \frac{C}{1+\lambda} + (1-\lambda)\sum_{k=0}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k F_1^+(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k}) + \sum_{k=1}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k F_2(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k}), \quad \zeta \in \mathbb{C}\backslash\overline{\Omega_1^+}, \tag{3.20}$$

where $|\lambda| = |B/A| < 1$, $\mathrm{C}_{\mathbb{R}}$ is the operator of complex conjugation

$$\mathrm{C}_{\mathbb{R}} z = \overline{z}. \tag{3.21}$$

Now from (3.14), (3.15) and (3.20) it follows that

$$\mathscr{V}_{02}(\zeta) = (1-\lambda)\sum_{k=0}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k \left(F_1^-(\mathrm{C}_{\mathbb{R}}^k\zeta/R^{2k}) + F_1^+(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k})\right) + \sum_{k=1}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k \left(F_2(\mathrm{C}_{\mathbb{R}}^k\zeta/R^{2k}) + F_2(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k})\right), \quad \zeta \in \Omega_2. \tag{3.22}$$

From the first equality (3.18) we get

$$\mathscr{V}_{01}^-(\zeta) = \lambda\overline{F_1^+(\overline{\zeta}R^{-2})} + (1-\lambda^2)\sum_{k=0}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k F_1^+(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k}) + (1+\lambda)\sum_{k=0}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^k F_2(\mathrm{C}_{\mathbb{R}}^k\zeta R^{2k}), \quad \zeta \in \Omega_1^-. \tag{3.23}$$

At last the problem of finding w with singularities as in (2.1) and satisfying (2.2) can be determined via the formulae (3.2), (3.5) and (3.13). Thus, the following statement is proved.

Theorem 1 *If the given undisturbed complex potential $f(z)$ has no singular points at the elliptic interface $\mathscr{L} = \{z = x+iy : x^2/a^2 + y^2/b^2 = 1\}$, then the problem of finding a piecewise meromorphic function with principal part* (2.1) *subject to boundary condition* (2.2) *is unconditionally solvable. Its unique solution reads*

$$v_1(z) = \frac{F_1^-((z+\sqrt{z^2-c^2})/c) + \mathscr{V}_{01}^-((z+\sqrt{z^2-c^2})/c)}{\sqrt{z^2-c^2}}, \tag{3.24}$$

$$v_2(z) = \frac{F_2((z+\sqrt{z^2-c^2})/c) + \mathscr{V}_{02}((z+\sqrt{z^2-c^2})/c)}{\sqrt{z^2-c^2}}, \tag{3.25}$$

where $\mathscr{V}_{02}(\zeta)$ and $\mathscr{V}_{01}^-(\zeta)$ are defined in (3.22) *and* (3.23). *In tern $\lambda = (\rho_2-\rho_1)/(\rho_2+\rho_1)$, $R = \sqrt{(a+b)/(a-b)}$,*

$$F(\zeta) = c/2(\zeta - 1/\zeta)f'[c/2(\zeta + 1/\zeta)],$$

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$F_1^-(\zeta) = -F_1^+(1/\zeta)$, $F_2(\zeta) = F(\zeta) - \big(F_1^+(\zeta) - F_1^+(1/\zeta)\big)$, and $F_1^+(\zeta)$ is the sum of all common fractions, summands of $F(\zeta)$, having poles in the domain $\Omega_1^+ = \{\zeta\ :\ |\zeta| < 1/R\}$.

Remark 1 By integrating the complex velocities (3.24) and (3.25) the corresponding complex potentials can be found,

$$w_1(z) = \int_R^{\zeta(z)} (F_1^-(\zeta) + \mathscr{V}_{01}^-(\zeta))\frac{d\zeta}{\zeta}, \quad w_2(z) = \int_R^{\zeta(z)} (F_2(\zeta) + \mathscr{V}_{02}(\zeta))\frac{d\zeta}{\zeta}.$$

Remark 2 It is easy to check using identities (3.9) and (3.12) that the segment $[-c, c]$ is not a jump line and its end points $\pm c$ are not singular points for function $v_2(z)$ defined in (3.25).

Remark 3 If the potential $f(z)$ has only one dipole at infinity, i.e. $f(z) = V_0 z$, then $F(\zeta) = cV_0(\zeta - 1/\zeta)/2 = F_1^-(\zeta) + F_1^+(\zeta)$, $F_2(\zeta) \equiv 0$. Hence, formulae (3.22)–(3.25) give a well-known solution

$$v_1(z) = V_0 + \frac{\lambda(R^4-1)}{2}\left[\frac{V_x}{R^2+\lambda} + \mathrm{i}\frac{V_y}{R^2-\lambda}\right]\left(1 - \frac{z}{\sqrt{z^2-c^2}}\right),$$
$$v_2(z) = (1-\lambda)R^2\left[\frac{V_x}{R^2+\lambda} - \mathrm{i}\frac{V_y}{R^2-\lambda}\right].$$

Remark 4 Let $f(z) = P_n(z)$ be an arbitrary polynomial of the degree $n \geqslant 1$ with a zero-free term. Then $F_1^+(\zeta) = Q_n(1/\zeta)$ is a polynomial about $1/\zeta$ in the same degree n without free term, $F_1^-(\zeta) = -Q_n(\zeta)$, and $F_2(\zeta) \equiv 0$. The coefficients b_k of the polynomial

$$Q_n(\zeta) = \sum_{k=1}^{n} b_k \zeta^k$$

are defined as

$$b_k = \frac{1}{(n-k)!}\mathrm{D}^{n-k}\{\zeta^{n+1}\mathrm{D}[P_n(c/2(\zeta + 1/\zeta))]\}|_{\zeta=0}, \quad k = \overline{1, n},$$

where D is the differential operator $d/d\zeta$. Now the required solution could be expressed by formulae (3.22)–(3.25). It is not difficult to see that the complex velocity $v_2(z)$ found in this way is a polynomial of the degree $n - 1$. It gives a new proof of the generalised Eshelby's conjecture [5]. Indeed, by (3.22) we have

$$\mathscr{V}_{02}((z + \sqrt{z^2 - c^2})/c) = (1-\lambda)\sum_{k=0}^{\infty}(-1)^k\lambda^k \sum_{j=1}^{n} \frac{C_{\mathbb{R}}^k b_j}{R^{2kj}}(\zeta(z)^{-j} - \zeta(z)^j).$$

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From the last equality, (3.2) and (3.25) follows

$$v_2(z) = 2(1-\lambda)\sum_{j=1}^{n} \frac{\lambda R^{-2j}\bar{b}_j - b_j}{(1-\lambda^2 R^{-4j})c^j} \sum_{m=0}^{[(j-1)/2]} \binom{j}{2m+1} z^{j-2m-1}(z^2-c^2)^m,$$

where $[(j-1)/2]$ is the integer part of the number $(j-1)/2$. Thus, we have not only proved that the complex velocity within an elliptic inclusion is a polynomial of the degree $n-1$ but we have also found, as well as in [20], the exact expression for this polynomial.

Remark 5 Eshelby's conjecture was formulated and proved for a bounded inclusion. Within a parabolic inclusion, considered in [16], a uniform flow occurs inside this unbounded lens. It was also proved in [18] that the generalised Eshelby's conjecture is true for a parallel-layered medium and for a medium with a parabolic inclusion. Based on what was said above, we can put forward the following conjecture: *If a given complex potential is a polynomial then the field inside an inserted simply connected inclusion will be a polynomial of the same degree if and only if this inclusion is a strip, an ellipse or a parabola.*

Example 1 Let the given complex potential $f(z)$ have logarithmic singularities at two finite points z_1, z_2 outside and inside the inclusion, respectively, i.e.

$$f(z) = \gamma_1 \ln(z-z_1) + \gamma_2 \ln(z-z_2), \quad \gamma_k = \frac{\Gamma_k + \mathrm{i}\,Q_k}{2\pi \mathrm{i}}.$$

Note that infinity is not a singular point of $f(z)$ iff $\gamma_1+\gamma_2 = 0$.

Let $z_k = c(\zeta_k + 1/\zeta_k)/2$, where $\zeta_k = (z_k - \sqrt{z_k^2 - c^2})/c$, $0 < |\zeta_1| < 1/R$, $1/R < |\zeta_2| < 1$, then in accordance with designation (3.11) we get

$$F(\zeta) = \sum_{k=1}^{2} \gamma_k \left(\frac{\zeta_k}{\zeta - \zeta_k} + \frac{\zeta}{\zeta - 1/\zeta_k} \right).$$

Thus,

$$F_1^+(\zeta) = \frac{\gamma_1\zeta_1}{\zeta-\zeta_1}, \quad F_1^-(\zeta) = \frac{\gamma_1\zeta}{\zeta - 1/\zeta_1}, \quad F_2(\zeta) = \frac{\gamma_2(\zeta - 1/\zeta)}{\zeta + 1/\zeta - \zeta_2 - 1/\zeta_2}.$$

The corresponding disturbed complex velocity, found from (3.22)–(3.25), has the following form:

$$\begin{aligned} v_1(z) = {} & \frac{\gamma_1}{z-z_1} + \frac{c\lambda}{\sqrt{z^2-c^2}} \left(\frac{\bar{\gamma}_1\bar{\zeta}_1 R^2}{z+\sqrt{z^2-c^2}-c\bar{\zeta}_1 R^2} - \frac{\lambda\gamma_1\zeta_1}{z+\sqrt{z^2-c^2}-c\zeta_1} \right) \\ & + \frac{1+\lambda}{2\pi \mathrm{i}\sqrt{z^2-c^2}} \left[(1-\lambda)\sum_{k=1}^{\infty} \lambda^k \frac{(\Gamma_1 + (-1)^k\,\mathrm{i}\,Q_1)c\mathrm{C}_{\mathbb{R}}^k \zeta_1 R^{-2k}}{z+\sqrt{z^2-c^2} - c\mathrm{C}_{\mathbb{R}}^k\zeta_1 R^{-2k}} \right. \\ & \left. + \sum_{k=0}^{\infty} \lambda^k \frac{(\Gamma_2 + (-1)^k\,\mathrm{i}\,Q_2)[z(R^{2k}-R^{-2k}) + \sqrt{z^2-c^2}(R^{2k}+R^{-2k})]}{z(R^{2k}+R^{-2k}) + \sqrt{z^2-c^2}(R^{2k}-R^{-2k}) - 2\mathrm{C}_{\mathbb{R}}^k z_2} \right], \end{aligned} \tag{3.26}$$

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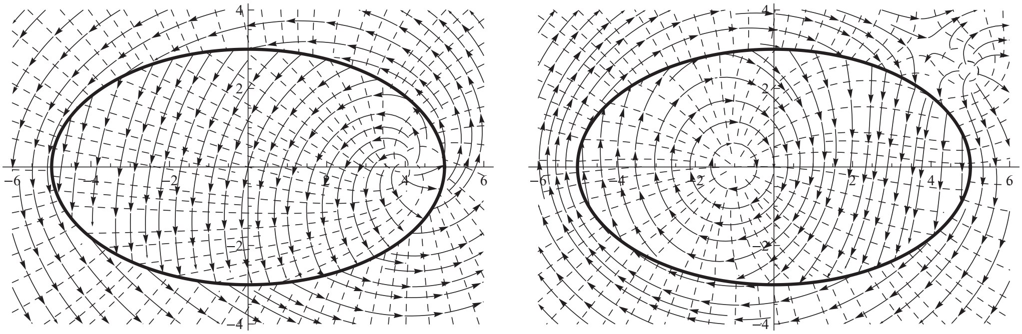

FIGURE 3. The case of a single source at the ellipse's focus and the case of vortex inside and source outside of the ellipse.

$$v_2(z) = \frac{\gamma_2}{z - z_2} + \frac{1-\lambda}{2\pi \mathrm{i}} \sum_{k=0}^{\infty} \frac{\lambda^k(\Gamma_1 + (-1)^k \mathrm{i}\, Q_1)}{z - c\mathrm{C}_{\mathbb{R}}^k(\zeta_1/R^{2k} + R^{2k}/\zeta_1)/2}$$
$$+ \frac{1}{\pi \mathrm{i}} \sum_{k=1}^{\infty} \frac{\lambda^k(\Gamma_2 + (-1)^k \mathrm{i}\, Q_2)[z - \mathrm{C}_{\mathbb{R}}^k z_2(R^{2k} + R^{-2k})/2)]}{[z - c\mathrm{C}_{\mathbb{R}}^k(R^{2k}\zeta_2 + R^{-2k}/\zeta_2)/2)][z - c\mathrm{C}_{\mathbb{R}}^k(\zeta_2/R^{2k} + R^{2k}/\zeta_2)/2)]}. \quad (3.27)$$

The last formulae can be significantly simplified if the given complex potential $f(z)$ has a single vortex-source of strength $\Gamma + \mathrm{i}\, Q$ at the ellipse's focus ($z_2 = c, \gamma_2 = \gamma = (\Gamma + \mathrm{i}\, Q)/(2\pi \mathrm{i})$, $\gamma_1 = 0$). Namely,

$$v_1(z) = \frac{1+\lambda}{2\pi \mathrm{i}\sqrt{z^2 - c^2}} \sum_{k=0}^{\infty} \lambda^k(\Gamma + (-1)^k \mathrm{i}\, Q) \frac{z + c(R^{2k} + R^{-2k})/2}{\sqrt{z^2 - c^2} + c(R^{2k} - R^{-2k})/2},$$
$$v_2(z) = \frac{\Gamma + \mathrm{i}\, Q}{2\pi \mathrm{i}(z - c)} + \frac{1}{\pi \mathrm{i}} \sum_{k=1}^{\infty} \lambda^k \frac{\Gamma + (-1)^k \mathrm{i}\, Q}{z - c(R^{2k} + R^{-2k})/2}.$$

Distributions of streamlines and equipotential (dashed) lines are shown in Figure 3 for the case $a = 5$, $b = 3$, $\lambda = 2/3$, $\Gamma_2 = \Gamma = 1$, $Q_2 = Q = 1$, $z_2 = c$, $\Gamma_1 = Q_1 = 0$ (left panel) and for the case $\lambda = 2/3$, $\Gamma_1 = 0$, $Q_1 = 1$, $\Gamma_2 = -1$, $Q_2 = 0$, $z_1 = a + 0.8\,\mathrm{i}\, b$, $z_2 = c - a$ (right panel).

4 Solution of the problem (2.2) with finite number of singularities at the interface

Here, as well as in [15], it is enough to investigate the case of the complex potential $f_0(z)$ with a single singularity at a point $t_0 \in \mathscr{L}$, i.e.

$$f_0'(z) = \sum_{k=0}^{n} a_{0k}(z - t_0)^{-k-1}. \quad (4.1)$$

We suppose that the principal parts, $f_{1,2}(z)$, of a disturbed complex potential $w_{1,2}(z)$ have the same structure as $f_0(z)$, i.e.

$$f_j'(z) = \sum_{k=0}^{n} a_{jk}(z - t_0)^{-k-1}, \quad j = 1, 2.$$

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Moreover, the condition

$$f_1(z) + f_2(z) = 2f_0(z) \tag{4.2}$$

is satisfied.

As in the previous case, using the representation (3.5), the problem (2.2) is reduced to problem (3.7) and then to (3.10) with an additional symmetry condition (3.9).

It could be shown that function

$$F_j(\zeta) = c/2(\zeta - 1/\zeta)f_j'(c/2(\zeta + 1/\zeta)), \quad j = 0, 1, 2, \tag{4.3}$$

satisfying the identity

$$F_j(\zeta) \equiv -F_j(1/\zeta) \tag{4.4}$$

can be represented in the following form, convenient for forthcoming evaluations,

$$F_j(\zeta) = \sum_{k=0}^{n} a_{jk}^{+}\left(\frac{\tau_0}{\zeta - \tau_0}\right)^{k+1} - \sum_{k=0}^{n} a_{jk}^{+}\left(\frac{\zeta}{1/\tau_0 - \zeta}\right)^{k+1} = F_j^{+}(\zeta) - F_j^{+}(1/\zeta), \tag{4.5}$$

where $c(\tau_0 + 1/\tau_0)/2 = t_0$, $|\tau_0| = 1/R$ and R is given by (3.3).

Indeed, it is clear that the principal part of $F_j(\zeta)$ at the point τ_0, $F_j^{+}(\zeta)$ can be taken as in (4.5). Then the difference $\Psi(\zeta) = F_j(\zeta) - (F_j^{+}(\zeta) - F_j^{+}(1/\zeta))$ satisfies the identity (4.4) and hence, as it does not have a singularity at the point τ_0, there is no singularity at the point $1/\tau_0$ either. Thus, the function $\Psi(\zeta)$ is holomorphic and bounded in $\overline{\mathbb{C}}$. Due to the Liouville theorem and according to (4.4) we have $\Psi(\zeta) \equiv \text{const} = 0$. This proves the representation (4.5).

Coefficients a_{0k}^{+} in (4.5) are uniquely defined through coefficients a_{0k} of the given complex potential (4.1) in accordance with definition (4.3). After simple algebra we get

$$a_{0k}^{+} = a_{0k}(\tau_0^2 - c^2)^{-k/2}. \tag{4.6}$$

Coefficients a_{jk}^{+}, $j = 1, 2$, have to be found through the boundary condition (3.10).

From the identity (4.2) and representations (4.3) and (4.5) follow the following equalities:

$$a_{1k}^{+} + a_{2k}^{+} = 2a_{0k}^{+}, \quad k = \overline{0, n}. \tag{4.7}$$

We take a solution of the problem (3.10) in the form (3.13) with $F_j^{-}(\zeta) = -F_j^{+}(1/\zeta)$. Then using (3.14) and (3.15), the second condition (3.10) could be written as

$$\begin{aligned}(1 + \lambda)(\mathscr{V}_{02}^{+}(\tau) + \mathscr{V}_{02}^{-}(\tau) + F_2^{+}(\tau) - F_2^{+}(1/\tau)) = \mathscr{V}_{01}^{-}(\tau)\\ - F_1^{+}(1/\tau) + \lambda\left(\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\tau})} - \overline{F_1^{+}(\overline{\tau}/R^2)}\right), \quad |\tau| = R.\end{aligned} \tag{4.8}$$

Let us now consider the function

$$\Phi(\zeta) = \begin{cases}(1+\lambda)(\mathscr{V}_{02}^{-}(\zeta) + F_2^{+}(\zeta)) - \mathscr{V}_{01}^{-}(\zeta) + \lambda\overline{F_1^{+}(\overline{\zeta}/R^2)}, & \zeta \in \Omega_1^{-}\\ -F_1^{+}(1/\zeta) - (1+\lambda)(\mathscr{V}_{02}^{+}(\zeta) - F_2^{+}(1/\zeta)) + \lambda\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\zeta})}, & \zeta \notin \overline{\Omega_1^{-}}\end{cases}. \tag{4.9}$$

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From the boundary equality (4.8) it is clear that function (4.9) is holomorphic everywhere in the closed ζ-plane except at point $1/\tau_0$, where it has a pole of the order $n+1$. In accordance with the generalised Liouville theorem we get

$$\Phi(\zeta) = C + P_{n+1}(1/(\zeta - 1/\tau_0)), \tag{4.10}$$

where C is a constant and $P_{n+1}(\zeta)$ is a polynomial of degree $n+1$ without free term. One has to choose this polynomial in a such way that functions $\mathscr{V}^-_{01}, \mathscr{V}^-_{02}$ defined by (4.9) would be holomorphic at point τ_0. The last condition is satisfied if the following identities are simultaneously true:

$$P_{n+1}(1/(\zeta - 1/\tau_0)) \equiv \lambda\overline{F_1^+(\overline{\zeta}/R^2)} \equiv (1+\lambda)F_2^+(1/\zeta) - F_1^+(1/\zeta) + C_1, \tag{4.11}$$

where

$$C_1 = \lambda\overline{F_1^+(0)} = F_1^+(0) - (1+\lambda)F_2^+(0),$$

as $F_{1,2}^+(\infty) = 0$. From (4.11) it follows that

$$\lambda\overline{F_1^+(1/(R^2\overline{\zeta}))} \equiv (1+\lambda)F_2^+(\zeta) - F_1^+(\zeta) + C_1. \tag{4.12}$$

Now in accordance with definition (4.5), we successively derive

$$\overline{F_1^+(1/(R^2\overline{\zeta}))} = \sum_{k=0}^{n}\overline{a_{1k}^+}\left(\frac{\overline{\tau}_0 R^2\zeta}{1-\overline{\tau}_0 R^2\zeta}\right)^{k+1} = \sum_{k=0}^{n}\overline{a_{1k}^+}\left(\frac{\zeta/\tau_0}{1-\zeta/\tau_0}\right)^{k+1}$$
$$= \sum_{k=0}^{n}\overline{a_{1k}^+}(-1)^{k+1}\left(1+\frac{\tau_0}{\zeta-\tau_0}\right)^{k+1} = \sum_{k=0}^{n}\overline{a_{1k}^+}(-1)^{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}\left(\frac{\tau_0}{\zeta-\tau_0}\right)^{j}.$$

Changing the order of summation we transform the last expression to

$$\overline{F_1^+(1/(R^2\overline{\zeta}))} = \overline{F_1^+(0)} + \sum_{k=0}^{n}\left(\frac{\tau_0}{\zeta-\tau_0}\right)^{k+1}\sum_{j=k}^{n}\overline{a_{1j}^+}(-1)^{j+1}\binom{j+1}{k+1}.$$

Using the derived relation and (4.5) and (4.12) we equal the coefficients at the same degrees of $\tau_0/(\zeta - \tau_0)$ and get the system

$$\lambda\sum_{j=k}^{n}\overline{a_{1j}^+}(-1)^{j+1}\binom{j+1}{k+1} = (1+\lambda)a_{2k}^+ - a_{1k}^+, \quad k = \overline{0,n}.$$

Then

$$\lambda\overline{a_{1k}^+}(-1)^{k+1} + a_{1k}^+ - (1+\lambda)a_{2k}^+ = c_k, \quad k = \overline{0,n}, \tag{4.13}$$

where

$$c_k = \lambda\sum_{j=k+1}^{n}\overline{a_{1j}^+}(-1)^{j}\binom{j+1}{k+1}, \quad k = \overline{0,n-1}, \quad c_n = 0. \tag{4.14}$$

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Systems (4.7) and (4.13) has a unique solution:

$$a_{1k}^{+} = \begin{cases} a_{0k}^{+} + \lambda\Re a_{0k}^{+} + \dfrac{c_k + \lambda\Re c_k}{2(1+\lambda)}, & k = 2m \\ a_{0k}^{+} + \mathrm{i}\lambda\Im a_{0k}^{+} + \dfrac{c_k + \mathrm{i}\lambda\Im c_k}{2(1+\lambda)}, & k = 2m+1 \end{cases}, \tag{4.15}$$

$$a_{2k}^{+} = \begin{cases} a_{0k}^{+} - \lambda\Re a_{0k}^{+} - \dfrac{c_k + \lambda\Re c_k}{2(1+\lambda)}, & k = 2m \\ a_{0k}^{+} - \mathrm{i}\lambda\Im a_{0k}^{+} - \dfrac{c_k + \mathrm{i}\lambda\Im c_k}{2(1+\lambda)}, & k = 2m+1 \end{cases}. \tag{4.16}$$

From (4.15), (4.16) and (4.14) we first define successively a_{1n}^{+}, a_{2n}^{+}, then c_{n-1}, $a_{1\,n-1}^{+}$, $a_{2\,n-1}^{+}$, and so on up to a_{10}^{+}, a_{20}^{+}.

From relations (4.9)–(4.11) we get

$$\begin{cases} (1+\lambda)(\mathscr{V}_{02}^{-}(\zeta) + F_2^{+}(\zeta)) - \mathscr{V}_{01}^{-}(\zeta) = C, & \zeta \in \Omega_1^{-} \\ -(1+\lambda)\mathscr{V}_{02}^{+}(\zeta) + \lambda\overline{\mathscr{V}_{01}^{-}(R^2/\overline{\zeta})} - C_1 = C, & \zeta \in \mathbb{C}\backslash\overline{\Omega_1^{-}} \end{cases}. \tag{4.17}$$

Exclusion of $\mathscr{V}_{01}^{-}(\zeta)$ from system (4.17) results in the following equation about $\mathscr{V}_{02}^{-}(\zeta)$:

$$\mathscr{V}_{02}^{-}(\zeta) = D - \lambda\overline{\mathscr{V}_{02}^{-}(\overline{\zeta}R^2)} - \lambda\overline{F_2^{+}(\overline{\zeta}R^2)}, \; \zeta \in \mathbb{C}\backslash\overline{\Omega_1^{+}}, \tag{4.18}$$

where $D = (C + \lambda\overline{C} + C_1)/(1+\lambda)$.

A unique solution of equation (4.18) can be written down as

$$\mathscr{V}_{02}^{-}(\zeta) = \frac{D - \lambda\overline{D}}{1-\lambda^2} + \sum_{k=1}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^{k} F_2^{+}(\mathrm{C}_{\mathbb{R}}^{k}\zeta R^{2k}),$$

where $\mathrm{C}_{\mathbb{R}}^{k}$ is the k-th power of the conjugation operator (3.21). Using (3.14) and (3.15) we derive

$$\mathscr{V}_{02}(\zeta) = \sum_{k=1}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^{k}\left(F_2^{+}(\mathrm{C}_{\mathbb{R}}^{k}\zeta R^{2k}) - F_2^{+}(R^{2k}/\mathrm{C}_{\mathbb{R}}^{k}\zeta)\right). \tag{4.19}$$

From the first equality (4.17) follows

$$\mathscr{V}_{01}^{-}(\zeta) = F_1^{+}(0) - \frac{F_2^{+}(0)}{1-\lambda} + (1+\lambda)\sum_{k=0}^{\infty}(-1)^k\lambda^k \mathrm{C}_{\mathbb{R}}^{k} F_2^{+}(\mathrm{C}_{\mathbb{R}}^{k}\zeta R^{2k}). \tag{4.20}$$

It completes the construction of the required solution of the problem (2.2) for the case of boundary singularities $f_0(z)$.

Theorem 2 *If the given undisturbed complex potential $f_0(z)$ has an only singular point t_0 at the elliptic interface $\mathscr{L}$ and corresponding complex velocity is a rational function* (4.1) *then the problem of finding a piece-wise meromorphic function with principal part $f_0'(z)$ subject to boundary condition* (2.2) *is unconditionally solvable. Its unique solution is given by the formulae* (2.3), (3.3), (3.24), (3.25), (4.19) *and* (4.20) *with functions $F_{1,2}^{+}$ defined in* (4.5), *$F_1^{-}(\zeta) = -F_1^{+}(1/\zeta)$, and coefficients a_{1k}^{+}, a_{2k}^{+} given by* (4.14)–(4.16).

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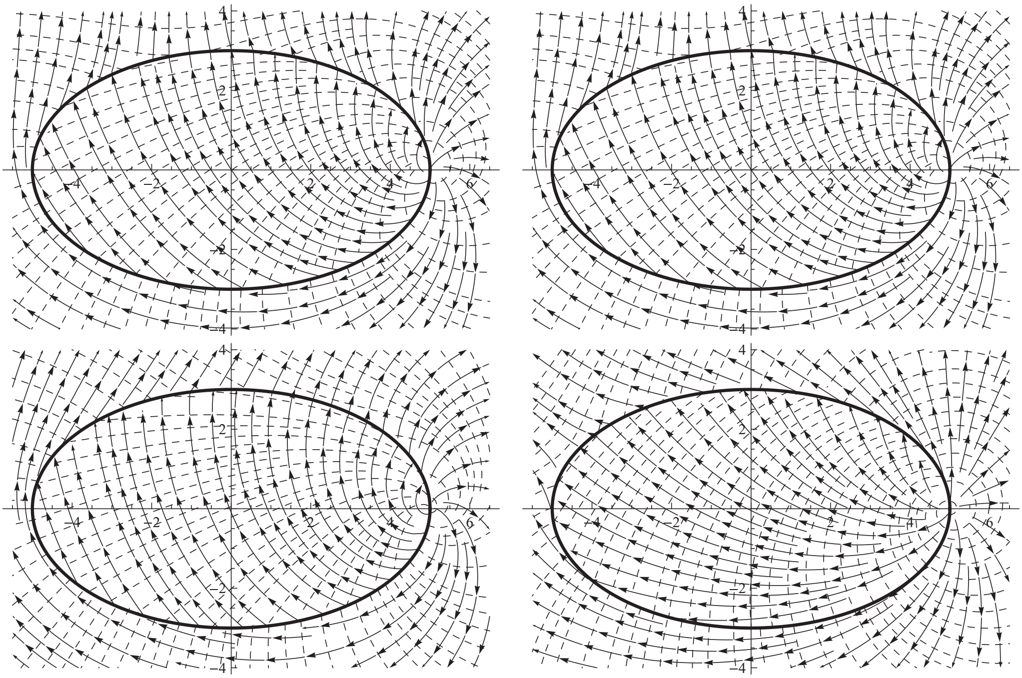

Figure 4. The source at the border point $z = a$.

Example 2 Let a given complex potential $f_0(z)$ has a single vortex-source of strength $\Gamma + \mathrm{i}\,Q$ at the point $z = a$, i.e. $f_0'(z) = (\Gamma + \mathrm{i}Q)/(2\pi\mathrm{i}(z-a))$, and consequently in accordance with (4.3) and (4.5)

$$F_0(\zeta) = \frac{(\zeta - 1/\zeta)(\Gamma + \mathrm{i}Q)}{2\pi\mathrm{i}((\zeta + 1/\zeta) - (R + 1/R))} = \frac{(\Gamma + \mathrm{i}Q)}{2\pi\mathrm{i}}\left(\frac{1/R}{\zeta - 1/R} - \frac{\zeta}{R - \zeta}\right),$$

wherefrom, as well as from (4.6), it follows that $a_{00}^+ = (\Gamma + \mathrm{i}Q)/(2\pi\mathrm{i})$. Using (4.14)–(4.16) we find

$$a_{10}^+ = \frac{\Gamma + \mathrm{i}(1+\lambda)Q}{2\pi\mathrm{i}}, \quad a_{20}^+ = \frac{\Gamma + \mathrm{i}(1-\lambda)Q}{2\pi\mathrm{i}}. \tag{4.21}$$

Via formulae (3.24), (3.25), (4.5) and (4.19)–(4.21) the disturbed complex velocity can be expressed in the following form:

$$\begin{aligned} v_1(z) &= \frac{\Gamma + \mathrm{i}(1+\lambda)Q}{2\pi\mathrm{i}(z-a)} - \frac{1}{\sqrt{z^2 - c^2}}\left[\frac{\lambda\overline{a_{20}^+}}{1-\lambda} + \frac{\lambda\overline{a_{10}^+}}{R\zeta(z) - 1}\right. \\ &\quad \left. + \frac{1+\lambda}{2\pi\mathrm{i}}\sum_{k=1}^{\infty}\lambda^k\frac{(\Gamma + \mathrm{i}\,(-1)^k(1-\lambda)Q)R^{-2k-1}}{\zeta(z) - R^{-2k-1}}\right], \\ v_2(z) &= \frac{\Gamma + \mathrm{i}\,(1-\lambda)Q}{2\pi\mathrm{i}(z-a)} + \frac{1}{2\pi\mathrm{i}}\sum_{k=1}^{\infty}\lambda^k\frac{\Gamma + (-1)^k\mathrm{i}\,(1-\lambda)Q}{z - c(R^{2k+1} + R^{-2k-1})/2}. \end{aligned}$$

In Figure 4, in the upper left panel, the streamlines and equipotential lines are plotted using the last formulae for the case $a = 5$, $b = 3$, $\Gamma = -1$, $Q = 3$, $\lambda = 2/3$, i.e. the

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conductivity of the inclusion is five times less than matrix's conductivity. The other three panels give the corresponding pictures when, instead of the last formulae, the formulae (3.26) and (3.27) were used. The upper right panel gives the flow net for the case of two vortex-sources of half intensity located close to a, namely $z_1 = a + 0.0001$, $z_2 = a - 0.0001$ and $\Gamma_1 = \Gamma_2 = -0.5$, $Q_1 = Q_2 = 1.5$ ($\lambda = 2/3$). The lower row represents the flow nets for the cases of an internal vortex-source (left panel, $z_2 = a - 0.0001$) and an external vortex-source (right panel, $z_1 = a + 0.0001$) of the same intensity as in the first case. The comparison of these four pictures shows that the flow nets are practically the same in the upper row, but the lower row pictures slightly differ from the upper ones. This difference is insignificant for the bottom left panel but more pronounced for the bottom right one. We surmise that the same effect of practical coincidence of flow nets for the case of separated (internal or external singularities) and coalesced singularity right on the interface will be observed for arbitrary singularities.

5 Conclusion

An explicit analytical solution of the problem of refraction of two harmonic 2D fields in a two-conductivity medium with an elliptical lens is obtained. The most general case of arbitrarily distributed singularities of the given complex potential is considered.

A special case of a sink-source pair, placed either inside or outside the lens, is elaborated with potential applications to subsurface mechanics (well hydraulics).

As an application of the results, presented in Theorem 1, a new proof of the generalised Eshelby's conjecture is provided.

A novel explicit solution is derived for a singular point of the given complex potential located exactly at the interface.

Flow nets are plotted for several possible loci of field-inducing singularities. The asymptotic behaviour of these nets is investigated, as the singularities placed in different media converge to the interface and eventually coalesce.

The results of the present paper can be generalised for the case of an infinite number of singularities of the given complex potentials as well as for the case of multiple elliptic inclusions (in particular, periodical and double periodical lattices of inclusions).

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References

[1] Asaro, R. J. & Barnett, D. M. (1975) The non-uniform transformation strain problem for an anisotropic ellipsoidal inclusion. *J. Mech. Phys. Solids* **23**, 77–83.

[2] Donnell, L. H. (1941) Stress concentration due to elliptical discontinuities in plates under edge stress. In: *Theodore von Karman Anniversary Volume*, California Institute of Technology, Pasadena, CA, pp. 293–309.

https://doi.org/10.1017/S0956792512000058 Published online by Cambridge University Press

[3] Emets, Y. P. (1987) *Boundary-Value Problems of Electrodynamics of Anisotropically Conducting Media*, Naukova Dumka, Kiev, Ukraine. (in Russian)
[4] Eshelby, J. D. (1957) The determination of the elastic field of an ellipsoidal inclusion, and related problems. *Proc. R. Soc. Lond.* A **241**, 376–396.
[5] Eshelby, J. D. (1961) Elastic inclusions and inhomogeneities. In: I. N. Sneddon II & R. Hill (editors), *Progress in Solid Mechanics*, North-Holland, Amsterdam, Netherlands, pp. 87–140.
[6] Golubeva, O. V. & Shpilevoy, A. Y. (1967) On plane seepage in media with continuously changing permeability along the curves of second order. *Izv. Akad. Nauk SSSR Mekh. Zhid. Gaza* **2**, 174–179. (in Russian)
[7] Gong, S. X. & Meguid, S. A. (1993) On the elastic fields of an elliptical inhomogeniety under plane deformation. *Proc. R. Soc. Lond.* A **443**, 457–471.
[8] Hardiman, N. J. (1954) Elliptical elastic inclusion in an infinite elastic plane. *Quart. J. Mech. Appl. Math.* **7**, 226–230.
[9] Kang, H. & Milton, G. W. (2008) Solutions to the conjectures of Pólya-Szegö and Eshelby. *Arch. Rat. Mech. Anal.* **188**(1), 93–116.
[10] Lin, C.-B. (2003) On a bounded elliptic inclusion in plane magneto elasticity. *Int. J. Solids Struct.* **40**, 1547–1565.
[11] Maxwell, J. C. (1904) *A Treaties on Electricity and Magnetism*, Clarendon Press, Oxford, UK.
[12] Milne-Thomson, L. M. (1968) *Theoretical Hydrodynamics*, 5th ed., Macmillan, London.
[13] Mityushev, V. (2009) Conductivity of a two-dimensional composite containing elliptical inclusions. *Proc. R. Soc. Lond.* A **465**, 2991–3010.
[14] Obdam, A. N. V. & Veiling, E. J. M. (1987) Elliptical inhomogeneities in groundwater flow – an analitical description. *J. Hydrology* **95**, 87–96.
[15] Obnosov, Y. V. (2006) A generalized Milne-Thomson theorem. *Appl. Math. Lett.* **19**, 581–586.
[16] Obnosov, Y. V. (2009a) A generalized Milne-Thomson theorem for the case of parabolic inclusion. *Appl. Math. Modelling* **33**, 1970–1981.
[17] Obnosov, Y. V. (2009b) *Boundary-Value Problems of Heterogeneous Medium Theory*, Kazan University, Kazan, Russia. (in Russian)
[18] Obnosov, Y. V. & Egorova, M. A. (2009) An R-linear conjugation problem for a confocal parabolic annulus. *Uchen. zap. Kazan Uniiversity. Ser. Phys.-Math. Nauk.* **151**(3), 170–178. (in Russian)
[19] Pilatovskii, V. P. (1966) *Basic Hydromechanics of a Thin Formation*, Nedra, Moscow, Russia. (in Russian)
[20] Rahman, M. (2002) The isotropic ellipsoidal inclusion with a polynomial distribution of eigenstrain. *J. Appl. Mech.* **69**(5), 593–601, doi:10.1115/1.1491270
[21] Rayleigh, Lord (1892) On the influence of obstacles arranged in rectangular order upon the properties of medium. *Phil. Mag.* **34**, 481–502.
[22] Schmid, D. W. & Podladchikov, Y. Y. (2003) Analytical solutions for deformable elliptical inclusions in general shear. *Geophys. J. Int.* **155**, 269–288.

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