Proceedings of the Edinburgh Mathematical Society (2017) **60**, 635–649
DOI:10.1017/S0013091516000146

ON SEMIPOSITONE NON-LOCAL BOUNDARY-VALUE PROBLEMS WITH NONLINEAR OR AFFINE BOUNDARY CONDITIONS

CHRISTOPHER S. GOODRICH

Department of Mathematics, Creighton Preparatory School, Omaha, NE 68114, USA (cgood@prep.creighton.edu)

(Received 22 April 2015)

Abstract We consider the boundary-value problem

$$\begin{aligned}
&-y'' = \lambda f(t, y(t)), \quad 0 < t < 1,\\
&y(0) = H(\varphi(y)),\\
&y(1) = 0,
\end{aligned}$$

where $H\colon [0, +\infty) \to \mathbb{R}$ and $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous and $\lambda > 0$ is a parameter. We show that if H satisfies a boundedness condition on a specified compact set, then this, together with an assumption that H is either affine or superlinear at $+\infty$, implies existence of at least one positive solution to the problem, even in the case where we impose no growth conditions on f. Finally, since it can hold that $f(t, y) < 0$ for all $(t, y) \in [0, 1] \times \mathbb{R}$, the semipositone problem is included as a special case of the existence result.

Keywords: positive solution; nonlinear boundary condition; non-local boundary condition; fixed-point index; boundary-value problem

2010 *Mathematics subject classification:* Primary 34B09; 34B10; 34B18
Secondary 47B40; 47H11

1. Introduction

In this paper we consider the boundary-value problem (BVP)

$$\left.\begin{aligned}
&-y'' = \lambda f(t, y(t)), \quad 0 < t < 1,\\
&y(0) = H(\varphi(y)),\\
&y(1) = 0,
\end{aligned}\right\} \tag{1.1}$$

where $H\colon [0, +\infty) \to \mathbb{R}$ and $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $\lambda > 0$ is a parameter. The functional $\varphi\colon \mathcal{C}([0, 1]) \to \mathbb{R}$ is assumed to be linear and realizable as a Stieltjes integral with (possibly) signed measure, which allows for the possibility that $\varphi(y) < 0$ even if y is a non-negative function. Since $f(t, 0) < 0$ is allowed, our results include as a special case the so-called semipositone problem. The principal existence theorem we provide for problem (1.1) essentially states (see Theorem 3.1 for the precise statement of the result) that if

- H is either affine or superlinear at $+\infty$ and satisfies a certain bound on a specified closed interval and

- φ decomposes in a particular way,

then for each $\lambda > 0$ sufficiently small, where λ can be explicitly and easily calculated, it follows that problem (1.1) has at least one non-trivial, positive solution; see conditions (A0), (A1) and (A4) in what follows for a precise description of these hypotheses. In addition to Theorem 3.1, we also provide a companion result, Theorem 3.2.

Importantly, we require no conditions on f other than that it is continuous on its domain and that we have some lower control over its magnitude in the sense that there is a non-negative function $u \in L^1([0,1])$ such that $f(t,y) \geqslant -u(t)$ for each $(t,y) \in [0,1] \times \mathbb{R}$, which is a standard condition for this type of problem. In particular, this means that there is no restriction on the type of growth that f has either at 0 or at $+\infty$. Moreover, it is possible that f is negative on the entirety of its domain. As suggested in the bullet points above, it is also possible that H has the form $H(z) = N_1 z + N_2$, with some restriction on the choice of the constants N_1 and N_2. In particular, then, we can treat affine boundary conditions in (1.1) of the form $y(0) = N_1\varphi(y) + N_2$ (see Remark 2.2 and Example 3.6 in what follows).

In addition to the fact that no growth conditions are imposed on f, we wish to draw attention to the fact that the function f may be strictly negative on its entire domain. Obviously, in the local boundary condition setting (i.e. Dirichlet boundary conditions) this would force any non-trivial solution to be non-positive. In this paper the non-local nonlinearity provides 'enough' positivity to ensure the existence of a positive solution even if f is always negative, and this works even if H is merely affine rather than in possession of superlinear growth at $+\infty$ (see Theorem 3.1). The recognition of this possibility is a consequence of the decomposition of φ, whose use we describe next.

Indeed, the effective transfer of all growth conditions to the function H is possible because of the decomposition hypothesis imposed on the functional φ. Introduced in [**6**], this hypothesis requires that φ be able to be written in the form $\varphi(y) = \varphi_1(y) + \varphi_2(y)$ for each $y \in \mathcal{C}([0,1])$, where φ_1 effectively 'isolates' the negativity of φ, whereas φ_2 satisfies a sort-of coercivity condition. For example, if, say,

$$\varphi(y) := \tfrac{1}{2}y(\tfrac{1}{2}) - \tfrac{1}{3}y(\tfrac{1}{3}), \tag{1.2}$$

then $\varphi(y)$ may be zero even if $\|y\|$ is very large. This presents a problem if we wish to use a condition that involves the behaviour of $H(z)/z$ as $z \to +\infty$. To circumvent this problem, we make the simple observation that (1.2) may be rewritten in the form, say,

$$\varphi(y) := \underbrace{\tfrac{1}{4}y(\tfrac{1}{2}) - \tfrac{1}{3}y(\tfrac{1}{3})}_{:=\varphi_1(y)} + \underbrace{\tfrac{1}{4}y(\tfrac{1}{2})}_{:=\varphi_2(y)}. \tag{1.3}$$

Due to the fact that we work in a particular cone, we can then establish that there is a constant $C_0 > 0$ such that $\varphi_2(y) \geqslant C_0\|y\|$ for each y in the cone, and, moreover, that $\varphi_1(y) \geqslant 0$ for each y in the cone. These two facts then provide sufficient control over $\varphi(y)$ so as to use, for example, a superlinear growth condition imposed on H at $+\infty$. Moreover, recognition of the decomposition makes the existence proof clean and elegant.

To conclude our introduction, we provide a brief synopsis of the current state of the literature and the way in which our results fit into this. In addition to the intrinsic mathematical interest, part of the interest in such problems is their potential use in modelling physical problems such as, for example, beam deformation, steady-state heat flow, and chemical reactor theory. For example, the boundary conditions in (1.1), in the context of a steady-state heat flow problem, can be considered as providing some nonlinear feedback to the temperature at the left endpoint, where this feedback depends in a possibly nonlinear way, via the map $H \circ \varphi$, on the temperature of the bar at various points on its lateral surface (see, for example, Infante *et al*. [**14**, **19**, **21**]).

In recent years there have been many papers attending to the theory of non-local BVPs. Some earlier results of Karakostas and Tsamatos [**25**, **26**] studied problems in cases in which the boundary conditions were linear, i.e. $H(z) := Cz$ for some constant $C > 0$. Later, Yang [**32**, **33**] studied non-local BVPs possessing nonlinear boundary conditions. Of note is that these authors studied the case in which the non-locality was realized in the form of a Riemann–Stieltjes integral. Webb and Infante [**28**] then produced a general approach for studying non-local BVPs with Stieltjes integral boundary conditions by considering a new cone, which allowed for the Stieltjes measures to be signed; for example, multipoint problems with signed coefficients were allowed. This sweeping approach has been further developed in subsequent work by Webb and Infante [**29**, **30**] in both the higher-order and semipositone settings. On the other hand, modifications of their general approach have been applied to BVPs with non-local and, in some cases, nonlinear boundary conditions in our recent papers [**3**–**10**, **12**] and those by Infante *et al*. [**14**–**21**] and Jankowski [**22**]. Also of interest is a recent paper by Anderson [**1**] in which the existence of at least three positive solutions was demonstrated for a first-order problem with non-local, nonlinear boundary conditions. Another recent paper by Karakostas [**24**] also presents an interesting approach for studying non-local BVPs, potentially with nonlinear boundary conditions. The classical papers by Picone [**27**] and Whyburn [**31**] are also to be recommended for their historic value in understanding the initial development of this area of mathematics.

Especially as concerns the intersection of the semipositone setting and non-local boundary conditions, the only works of which we are aware are [**11**, **13**, **23**, **30**]. However, Webb and Infante [**30**] and Jiang *et al*. [**23**] consider only the case of linear non-local boundary conditions. On the other hand, as concerns our recent works [**11**, **13**] in the semipositone setting, we also considered problems of the form given by (1.1). In [**11**] we considered the case in which $\lambda = 1$, whereas in [**13**] we considered the case in which $\lambda \in (0, \lambda_0)$ for some small computable number $\lambda_0 > 0$. However, in each of our papers, just as in each of the papers described in the previous paragraph, growth conditions were imposed on f; thus, here we improve that aspect of those papers by eliminating such conditions and showing that all conditions can be effectively transferred to the nonlinear non-local element. Moreover, the computation of λ in this work is rather more simple than in [**11**,**13**]. Thus, this paper shows that by taking a different approach and requiring the previously described condition on H, we can provide better and simpler results, and thus not only complement the works in the previous paragraph but also complement and/or improve [**11**,**13**,**23**,**30**], too.

With this in mind, the outline of this paper is as follows. In § 2 we describe briefly the preliminary results we require in order to prove the existence results. In § 3 we state and prove our existence theorems. We then conclude the paper by providing two different numerical examples in order to demonstrate the use and scope of our results. Finally, we note that although we elect to study this type of problem in the specific incarnation of (1.1), it is obvious that our methods can be adapted trivially to treat other boundary conditions, perturbed Hammerstein integral equations, or solutions of elliptic partial differential equations with radial solution structure; we simply prefer the concreteness that (1.1) provides.

2. Preliminaries

In this section we provide some background on the techniques and notation that are used in § 3. Furthermore, we introduce the conditions that are assumed in the existence results. To this end, we begin by noting that, as is standard in the semipositone setting (see, for example, [**2**]), letting $u \in L^1([0,1];[0,+\infty))$, we will make use of both the auxiliary problem

$$\left.\begin{aligned} -w'' &= \lambda u(t), \quad 0 < t < 1,\\ w(0) &= 0 = w(1), \end{aligned}\right\} \tag{2.1}$$

and the modified problem

$$\left.\begin{aligned} &-y'' = \lambda[f(t,(y-w)^*(t)) + u(t)], \quad 0 < t < 1,\\ &y(0) = H^*(\varphi(y-w)),\\ &y(1) = 0, \end{aligned}\right\} \tag{2.2}$$

where we put $(y-w)^*(t) := \max\{0,(y-w)(t)\}$ and $H^*(z) := H(\max\{0,z\})$. The purpose of the function u is to serve as a lower bound on f (see condition (A3) in what follows). Henceforth, we shall denote by w the unique solution of problem (2.1); observe that w may be written in the form

$$w(t) = \lambda \int_0^1 G(t,s)u(s)\,\mathrm{d}s. \tag{2.3}$$

Note that in both (2.3) and what follows the map $G\colon [0,1]\times[0,1] \to [0,1]$ denotes the Green function defined by

$$G(t,s) := \begin{cases} t(1-s), & 0 \leqslant t \leqslant s \leqslant 1,\\ s(1-t), & 0 \leqslant s \leqslant t \leqslant 1. \end{cases}$$

It is well known that for a given $(a,b) \Subset (0,1)$ the constant $\gamma^* := \min_{t\in[a,b]}\{t, 1-t\} = \min\{a, 1-b\} \in (0,1)$ satisfies the Harnack-like inequality

$$\min_{t\in[a,b]} G(t,s) \geqslant \gamma^* \max_{t\in[0,1]} G(t,s) = \gamma^* G(s,s).$$

Now, define the completely continuous operator $T\colon \mathcal{C}([0,1]) \to \mathcal{C}([0,1])$ by

$$(Ty)(t) := (1-t)H^*(\varphi(y-w)) + \lambda \int_0^1 G(t,s)[f(s,(y-w)^*(s)) + u(s)]\,\mathrm{d}s. \tag{2.4}$$

Then a fixed point of T is a solution of problem (2.2). This is important due to the following standard result, whose proof we omit.

Lemma 2.1. *Suppose that y is a solution of problem (2.2). In addition, suppose both that $(y-w)(t) \geqslant 0$ for each $t \in [0,1]$ and that $\varphi(y-w) \geqslant 0$. Then the function $\varUpsilon\colon [0,1] \to \mathbb{R}$ defined by $\varUpsilon(t) := (y-w)(t)$ is a positive solution of problem (1.1).*

We next state the hypotheses we make in § 3; note that conditions (A0)–(A2) ensure that the non-locality φ has the correct structure, whereas conditions (A4) and (A5) impose the correct growth and local boundedness conditions on the nonlinearity H. Observe that in what follows, for any finite $r_1, r_2, r > 0$, with $r_2 > r_1$, we use the notation

$$\tilde{H}_{r_1,r_2} := \max_{z\in[r_1,r_2]} H(z),$$
$$\tilde{f}_r := \max_{(t,z)\in[0,1]\times[0,r]} f(t,z)$$

for the functions H and f appearing in (1.1).

(A0) Assume that there are two linear functionals $\varphi_1, \varphi_2\colon \mathcal{C}([0,1]) \to \mathbb{R}$ such that

$$\varphi(y) = \varphi_1(y) + \varphi_2(y).$$

Moreover, assume that there exists a constant $C_0 > 0$ such that $\varphi_2(y) \geqslant C_0\|y\|$ for each $y \in \mathcal{K}$, where $\mathcal{K} \subseteq \mathcal{C}([0,1])$ is a cone defined below. Furthermore, since φ is linear, we let C_1 denote a constant such that $|\varphi(y)| \leqslant C_1\|y\|$ for each $y \in \mathcal{C}([0,1])$.

(A1) The functionals described in condition (A0) have the form

$$\varphi(y) := \int_{[0,1]} y(t)\,\mathrm{d}\alpha(t), \quad \varphi_1(y) := \int_{[0,1]} y(t)\,\mathrm{d}\alpha_1(t), \quad \varphi_2(y) := \int_{[0,1]} y(t)\,\mathrm{d}\alpha_2(t),$$

where $\alpha, \alpha_1, \alpha_2\colon [0,1] \to \mathbb{R}$ satisfy $\alpha_1, \alpha_2 \in \mathrm{BV}([0,1])$ so that $\alpha \in \mathrm{BV}([0,1])$.

(A2) It holds that

$$\int_{[0,1]} G(t,s)\,\mathrm{d}\alpha_1(t), \qquad \int_{[0,1]} (1-t)\,\mathrm{d}\alpha_1(t) > 0,$$

where the first inequality holds for every $s \in [0,1]$.

(A3) The functions $f\colon [0,1]\times[0,+\infty)\to\mathbb{R}$ and $H\colon\mathbb{R}\to\mathbb{R}$, satisfying $H([0,+\infty))\subseteq[0,+\infty)$, are continuous, and there exists a map $u\in L^1([0,1];(0,+\infty))$ such that $f(t,y)>-u(t)$ for each $(t,y)\in[0,1]\times\mathbb{R}$.

(A4) There exists a number $N\in(0,+\infty)$ such that $\tilde{H}_{C_0N,C_1N}<N$.

(A5) It holds that

$$\lim_{z\to+\infty}\frac{H(z)}{z}>M,$$

where $M>0$ is a real number satisfying $M>1/C_0$. It is also allowed that $\lim_{z\to+\infty}H(z)/z=+\infty$; in this case M can be any number such that $MC_0>1$.

Remark 2.2. Condition (A4) is the precise statement of the 'bound' that H must satisfy on a 'specified closed interval', to which we referred in § 1. Observe that this is a condition on the graph of H, and essentially requires that there exists at least one rectangle, with vertices $(C_0N,0)$, $(C_1N,0)$, (C_0N,N) and (C_1N,N), such that the graph of $H|_{[C_0N,C_1N]}$ is contained in this rectangle.

Remark 2.3. It is worth noting that condition (A5) does allow the map $z\mapsto H(z)$ to be affine at $+\infty$. In particular, this means that the boundary condition in (1.1) at $t=0$ may have the form $y(0)=N_1\varphi(y)+N_2$ for some constants N_1 and N_2 (see Example 3.6 for an explicit example of this case). In addition, $z\mapsto H(z)$ may be linear away from $+\infty$.

When studying the operator T, which was defined in (2.4), we will work within the cone $\mathcal{K}\subseteq\mathcal{C}([0,1])$ defined by

$$\mathcal{K}:=\{y\in\mathcal{C}([0,1])\colon y(t)\geqslant q(t)\|y\| \text{ for all } t\in[0,1],\ \varphi_1(y)\geqslant 0\},$$

where $q\colon[0,1]\to[0,\frac{1}{4}]$ is defined by $q(t):=t(1-t)$. It is standard to show both that $T(\mathcal{K})\subseteq\mathcal{K}$ and that T is a completely continuous operator on $\mathcal{K}$. It is clear that $\mathcal{K}$ is neither empty nor trivial since $1-t\in\mathcal{K}$. In addition, it can be shown that $w\in\mathcal{K}$, a fact that is important in § 3 and whose proof is left to the reader.

Next we recall the following lemma. Its proof can be found, for example, in [**11**, **13**].

Lemma 2.4. *For each* $(t,s)\in[0,1]\times[0,1]$ *it holds that* $G(t,s)\leqslant q(t)$.

Remark 2.5. Observe that we do not need to include the condition $\varphi_2(y)\geqslant 0$ in the cone $\mathcal{K}$ defined above since, due to the coercivity condition of (A0), it follows at once that φ_2 is non-negative on the cone $\mathcal{K}$.

For use in the statement and proof of Theorems 3.1 and 3.2, we give the following notation. That η_0 is positive will be shown in the proof of Theorem 3.1.

Notation 2.6. Define the number $\xi_0>0$ by

$$\xi_0:=\max_{t\in[0,1]}\int_0^1 G(t,s)u(s)\,\mathrm{d}s.$$

In addition, define the number $\eta_0 > 0$ by

$$\eta_0 := \sup\{\tilde{\eta}_0 \in (0, NC_0)\colon 0 < \tilde{H}_{C_0N-\tilde{\eta}_0, C_1N+\tilde{\eta}_0} + \tfrac{1}{2}(N - \tilde{H}_{C_0N, C_1N}) < N\},$$

where the number N is from condition (A4).

Finally, we conclude this section by stating formally the fixed-point theorems we use (see, for example, either [**21**] or [**34**] for additional information).

Lemma 2.7. *Let $\mathcal{B}$ be a Banach space and let $\mathcal{K} \subseteq \mathcal{B}$ be a cone. Assume that Ω_1 and Ω_2 are bounded open sets contained in $\mathcal{B}$ such that $0 \in \Omega_1$ and $\bar{\Omega}_1 \subseteq \Omega_2$. Furthermore, suppose that $T\colon \mathcal{K} \cap (\bar{\Omega}_2 \setminus \Omega_1) \to \mathcal{K}$ is a completely continuous operator. If either*

(1) $\|Ty\| \leqslant \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \geqslant \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$ *or*

(2) $\|Ty\| \geqslant \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_1$ *and* $\|Ty\| \leqslant \|y\|$ *for* $y \in \mathcal{K} \cap \partial\Omega_2$,

then T has at least one fixed point in $\mathcal{K} \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

Lemma 2.8. *Let $D \supset \{0\}$ be a bounded open set and, with $\mathcal{K}$ a cone in a Banach space $\mathscr{X}$, suppose both that $D \cap \mathcal{K} \neq \emptyset$ and that $\bar{D} \cap \mathcal{K} \neq \mathcal{K}$. Let D_1 be open in $\mathscr{X}$ with $\bar{D}_1 \subseteq D \cap \mathcal{K}$. Assume that $T\colon \bar{D} \cap \mathcal{K} \to \mathcal{K}$ is a compact map such that $Tx \neq x$ for $x \in \mathcal{K} \cap \partial D$. If $i_{\mathcal{K}}(T, D \cap \mathcal{K}) = 1$ and $i_{\mathcal{K}}(T, D_1 \cap \mathcal{K}) = 0$, then T has a fixed point in $(D \cap \mathcal{K}) \setminus (\overline{D_1 \cap \mathcal{K}})$. Moreover, the same result holds if $i_{\mathcal{K}}(T, D \cap \mathcal{K}) = 0$ and $i_{\mathcal{K}}(T, D_1 \cap \mathcal{K}) = 1$.*

3. Main results and discussion

In this section we first state and prove the existence results for problem (1.1). Then we give a couple of examples to illustrate the results and explicate their use. We remark that in this section we use the notation

$$\Omega_\rho := \{y \in \mathcal{C}([0, 1])\colon \|y\| < \rho\}.$$

Theorem 3.1. *Suppose that conditions (A0)–(A5) hold. Define the real number $\lambda_0 > 0$ by*

$$\lambda_0 := \min\left\{\frac{\eta_0}{C_1\xi_0}, N\left[\int_0^1 u(s)\,\mathrm{d}s\right]^{-1}, \tfrac{1}{2}(N - \tilde{H}_{C_0N, C_1N})\left[\int_0^1 G(s, s)[\tilde{f}_N + u(s)]\,\mathrm{d}s\right]^{-1}\right\}. \tag{3.1}$$

Then for each $\lambda \in (0, \lambda_0)$ problem (1.1) has at least one positive solution.

Proof. To begin the argument we provide some preliminary estimates, which will be useful throughout the proof. Furthermore, we henceforth assume that λ is fixed and selected so that $\lambda \in (0, \lambda_0)$, where λ_0 is as in (3.1) above; we show in the next paragraph that $\lambda_0 > 0$. So, using Lemma 2.4 we first of all notice that whenever $y \in \mathcal{K} \setminus \Omega_N$ it holds that

$$\begin{aligned}(y-w)(t) \geqslant q(t)\|y\| - w(t) &= q(t)\|y\| - \lambda \int_0^1 G(t,s)u(s)\,\mathrm{d}s \\ &\geqslant q(t)\left[\|y\| - \lambda \int_0^1 u(s)\,\mathrm{d}s\right] \\ &\geqslant 0, \end{aligned} \tag{3.2}$$

where the final inequality in (3.2) follows from the fact that

$$\lambda < N\left[\int_0^1 u(s)\,\mathrm{d}s\right]^{-1}$$

and $\|y\| \geqslant N$.

Next, we notice that

$$0 < \tilde{H}_{C_0N,C_1N} + \tfrac{1}{2}(N - \tilde{H}_{C_0N,C_1N}) < N, \tag{3.3}$$

where the inequalities in (3.3) follow from assumption (A4). Furthermore, since the map $z \mapsto \tilde{H}_{C_0N-z,C_1N+z}$ is continuous, it follows that $\lim_{z\to 0^+} \tilde{H}_{C_0N-z,C_1N+z} = \tilde{H}_{C_0N,C_1N}$. This implies that there exists $z_0 > 0$ such that whenever $\tilde{\eta}_0 \in [0, z_0)$ it follows that

$$0 < \tilde{H}_{C_0N-\tilde{\eta}_0,C_1N+\tilde{\eta}_0} + \tfrac{1}{2}(N - \tilde{H}_{C_0N,C_1N}) < N. \tag{3.4}$$

Consequently, defining η_0 as in § 2, we see that (3.4) implies that $\eta_0 > 0$, which, in particular, implies that $\lambda_0 > 0$ in (3.1).

Finally, for any $y \in \mathcal{K} \cap \partial\Omega_N$ we notice that each of

$$\varphi(y-w) = \varphi(y) - \varphi(w) \leqslant C_1\|y\| - C_0\|w\| = C_1N - C_0\lambda\xi_0 \tag{3.5}$$

and

$$\varphi(y-w) \geqslant C_0\|y\| - \varphi(w) \geqslant C_0N - C_1\|w\| = C_0N - C_1\lambda\xi_0 \tag{3.6}$$

holds, using in (3.5) and (3.6) both the fact that $\|y\| = N$ plus assumption (A0) and the fact that $\|w\| = \lambda\xi_0$. Combining estimates (3.5) and (3.6), we deduce that

$$C_0N - C_1\lambda\xi_0 \leqslant \varphi(y-w) \leqslant C_1N - C_0\lambda\xi_0 \tag{3.7}$$

for each $y \in \mathcal{K} \cap \partial\Omega_N$; notice that the bound in (3.7) is well defined since we compute

$$(C_1N - C_0\lambda\xi_0) - (C_0N - C_1\lambda\xi_0) = \underbrace{(C_1 - C_0)}_{>0} N + \underbrace{(C_1 - C_0)}_{>0} \lambda\xi_0 > 0.$$

In any case, (3.7) implies that

$$\varphi(y-w) \in (C_0N - \eta_0, C_1N + \eta_0) \tag{3.8}$$

https://doi.org/10.1017/S0013091516000146 Published online by Cambridge University Press

provided that

$$\lambda < \frac{\eta_0}{C_1\xi_0}. \tag{3.9}$$

Since (3.9) holds from the choice of λ_0 in (3.1), it follows that (3.8) holds for each $y \in \mathcal{K} \cap \partial\Omega_N$. At the same time, let us also note that for any $y \in \mathcal{K} \setminus \Omega_N$ it holds that

$$\varphi(y-w) \geqslant C_0N - \eta_0 \geqslant 0, \tag{3.10}$$

where (3.8) and the fact that $\eta_0 \in (0, NC_0]$ jointly imply the final inequality in (3.10). The significance of estimates (3.8) and (3.10) is that we can then estimate

$$H^*(\varphi(y-w)) = H(\varphi(y-w)) \leqslant \tilde{H}_{C_0N-\eta_0,C_1N+\eta_0} \tag{3.11}$$

for each $y \in \mathcal{K} \cap \partial\Omega_N$, which will be used in the next part of the proof; in fact, the first equality in (3.11) actually holds for each $y \in \mathcal{K} \setminus \Omega_N$.

With the preceding preliminary estimates dispatched, we now show that

$$\|Ty\| \leqslant \|y\| \quad \text{for all } y \in \mathcal{K} \cap \partial\Omega_N. \tag{3.12}$$

To this end, let $y \in \mathcal{K} \cap \partial\Omega_N$ be fixed but otherwise arbitrary. Now, note that

$$0 \leqslant (y-w)^*(t) \leqslant y(t) \leqslant N. \tag{3.13}$$

Then, in light of estimates (3.2), (3.4), (3.11) and (3.13) together with the definition of λ_0 in (3.1), we observe that

$$\begin{aligned}
(Ty)(t) &\leqslant H^*(\varphi(y-w)) + \lambda\int_0^1 G(t,s)[f(s,(y-w)^*(s)) + u(s)]\,\mathrm{d}s \\
&\leqslant H(\varphi(y-w)) + \lambda\int_0^1 G(s,s)[f(s,(y-w)(s)) + u(s)]\,\mathrm{d}s \\
&\leqslant H(\varphi(y-w)) + \lambda\int_0^1 G(s,s)[\tilde{f}_N + u(s)]\,\mathrm{d}s \\
&< \tilde{H}_{C_0N-\eta_0,C_1N+\eta_0} + \tfrac{1}{2}(N - \tilde{H}_{C_0N,C_1N}) \\
&\leqslant N \\
&= \|y\|
\end{aligned} \tag{3.14}$$

for each $t \in [0,1]$. Thus, (3.14) implies that (3.12) holds.

On the other hand, we next argue, putting $\tau_w := (\rho_2 + \varphi(w))/C_0$, that

$$\|Ty\| \geqslant \|y\| \quad \text{for all } y \in \mathcal{K} \cap \partial\Omega_{\tau_w}, \tag{3.15}$$

for a suitable number $\rho_2 > 0$ to be fixed momentarily. To this end, by assumption (A5) we deduce that there exists a number $\varepsilon_0 > 0$ such that whenever $\tilde{\varepsilon} \in [0, \varepsilon_0)$ it holds that $M > (1/C_0)(1+\tilde{\varepsilon})$. Furthermore, also by assumption (A5) there exists a number $\rho_2 > 0$ such that $H(z) > Mz$ whenever $z \in [\rho_2, +\infty)$.

Now, observe that λ has been previously fixed and, in particular, does not depend on ρ_2. Thus, we may assume without loss of generality that ρ_2 is selected so large that

$$0 \leqslant \frac{\lambda \int_0^1 \int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s}{\rho_2} < \varepsilon_0. \tag{3.16}$$

And from both (3.16) and the choice of M provided in the previous paragraph it follows that

$$M > \frac{1}{C_0}\left[1 + \frac{\lambda \int_0^1 \int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s}{\rho_2}\right]. \tag{3.17}$$

Moreover, we can furthermore assume by selecting ρ_2 even larger if necessary that $\tau_w > N$ holds. Also, observe that whenever $y \in \mathcal{K} \cap \partial\Omega_{\tau_w}$ we may estimate

$$\varphi(y-w) \geqslant C_0\|y\| - \varphi(w) = \rho_2,$$

from which it follows that $H^*(\varphi(y-w)) \geqslant M\varphi(y-w)$. Then, having fixed $\rho_2 > 0$ so large that (3.17) is true, we estimate for each $y \in \mathcal{K} \cap \partial\Omega_{\tau_w}$ that

$$\begin{aligned}(Ty)(0) \geqslant M\varphi(y-w) \geqslant M\rho_2 > \rho_2\left[\frac{1}{C_0}\left(1 + \frac{\lambda \int_0^1 \int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s}{\rho_2}\right)\right] \\ = \frac{\rho_2 + \lambda \int_0^1 \int_0^1 G(t,s)u(s)\,\mathrm{d}\alpha(t)\,\mathrm{d}s}{C_0} \\ = \tau_w \\ = \|y\|,\end{aligned} \tag{3.18}$$

whence $(Ty)(0) > \|y\|$. Consequently, (3.18) implies estimate (3.15), as desired.

Now, an application of Lemma 2.7 implies the existence of

$$y_0 \in \mathcal{K} \cap (\bar{\Omega}_{\tau_w} \setminus \Omega_N) \tag{3.19}$$

such that $Ty_0 = y_0$. The choice of ρ_2 in the preceding part of the argument ensures that the intersection in (3.19) is non-empty. Define $\Upsilon\colon [0,1] \to \mathbb{R}$ by $\Upsilon(t) := (y_0 - w)(t)$. Since $\|y_0\| \geqslant N$, estimates (3.2) and (3.10) yield both that $\Upsilon(t) \geqslant 0$ and that $\varphi(y_0 - w) \geqslant 0$. Consequently, applying Lemma 2.1 implies that Υ is a positive solution to the original problem (1.1). This completes the proof. □

We now provide an alternative existence proof. To prove this theorem we shall appeal to Lemma 2.8 instead of Lemma 2.7.

Theorem 3.2. *Suppose that conditions (A0)–(A3) and (A5) hold. In addition, putting*

$$\sigma := \tilde{H}_{NC_0,NC_1} \int_0^1 (1-t)\,\mathrm{d}\alpha(t),$$

suppose that there exists $N \in (0, +\infty)$ such that each of the following inequalities holds:

$$\left.\begin{aligned} 0 \leqslant \sigma < NC_0,& \\ \int_0^1 \mathrm{d}\alpha(t) \geqslant 0,& \\ \tilde{H}_{\sigma,NC_1} \int_0^1 (1-t)\,\mathrm{d}\alpha(t) < NC_0.& \end{aligned}\right\} \tag{3.20}$$

Then problem (1.1) has at least one positive solution for each $\lambda \in (0, \lambda_0^*)$, where

$$\lambda_0^* := \min\left\{\frac{NC_0 - \tilde{H}_{\sigma,NC_1}\int_0^1 (1-t)\,\mathrm{d}\alpha(t)}{C_1\int_0^1 G(s,s)[\tilde{f}_N + u(s)]\,\mathrm{d}s}, \frac{NC_0 - \sigma}{C_1\int_0^1 G(s,s)u(s)\,\mathrm{d}s}, N\left[\int_0^1 u(s)\,\mathrm{d}s\right]^{-1}\right\}. \tag{3.21}$$

Proof. We first argue that

$$i_{\mathcal{K}}(T, \Omega_N) = 0. \tag{3.22}$$

To prove (3.22), we shall show that for each $\mu \geqslant 1$ it holds that $\mu y \neq Ty$ for each $y \in \mathcal{K} \cap \partial\Omega_N$. So, for a contradiction suppose not. Then we find $\mu \geqslant 1$ and $y \in \mathcal{K} \cap \partial\Omega_N$ such that $\mu y(t) = (Ty)(t)$ for each $t \in [0, 1]$. By applying φ to each side of this equality we thus obtain the estimate

$$\mu\varphi(y) = H(\varphi(y-w))\int_0^1 (1-t)\,\mathrm{d}\alpha(t) + \lambda\int_0^1\int_0^1 G(t,s)[f(s,(y-w)(s)) + u(s)]\,\mathrm{d}\alpha(t)\,\mathrm{d}s. \tag{3.23}$$

Note that in (3.23) we use the fact that $(y - w)(t) \geqslant 0$ due to the choice of λ_0^* and the fact that $\|y\| \geqslant N$. We have also used the fact that $\varphi(y - w) \geqslant 0$, which will be proved in the next paragraph.

Next we recall some of the preliminary estimates from the proof of Theorem 3.1. Indeed, in light of estimate (3.7) an important consideration in the proof is to ensure that the quantity $C_0N - C_1\lambda\xi_0$ remains non-negative, since from (3.6) we know that $\varphi(y - w) \geqslant C_0\|y\| - C_1\lambda\xi_0$. However, by the choice of λ_0^* given in (3.21) we see that for each $\lambda \in (0, \lambda_0^*)$ we may estimate

$$\begin{aligned} C_0N - C_1\lambda\xi_0 &> C_0N - \frac{C_1(NC_0 - \sigma)}{C_1\int_0^1 G(s,s)u(s)\,\mathrm{d}s}\left(\max_{t\in[0,1]}\int_0^1 G(t,s)u(s)\,\mathrm{d}s\right) \\ &\geqslant C_0N - \frac{C_1(NC_0 - \sigma)}{C_1\int_0^1 G(s,s)u(s)\,\mathrm{d}s}\left(\int_0^1 G(s,s)u(s)\,\mathrm{d}s\right) \\ &= \sigma. \end{aligned} \tag{3.24}$$

Observe that due to condition (3.20) we know that $NC_0 - \sigma > 0$, from which it follows that the direction of the second inequality in (3.24) is valid. Thus, from (3.7) and (3.24) we conclude that for each $y \in \mathcal{K} \cap \partial\Omega_N$ it holds that

$$H(\varphi(y-w)) \leqslant \tilde{H}_{\sigma,NC_1} := \max_{\sigma\leqslant z\leqslant NC_1} H(z). \tag{3.25}$$

Note that (3.25) is well defined since from condition (3.20) together with the fact that $C_1 > C_0 > 0$ we obtain that $\sigma/(NC_1) < \sigma/(NC_0) < 1$, whence $\sigma < NC_1$. In fact, the preceding calculations show that $\varphi(y-w) \geqslant 0$ whenever $\|y\| \geqslant N$.

So, putting (3.25) into (3.23) and observing both that $\mu\varphi(y) \geqslant \mu C_0\|y\| = \mu C_0 N$ and that $0 < \lambda < \lambda_0^*$, we obtain the estimate

$$\begin{aligned}\mu &\leqslant \frac{1}{NC_0}\left(\tilde{H}_{\sigma,NC_1}\int_0^1 (1-t)\,\mathrm{d}\alpha(t) + \lambda\int_0^1\int_0^1 G(t,s)\underbrace{[\tilde{f}_N + u(s)]}_{\geqslant 0}\,\mathrm{d}\alpha(t)\,\mathrm{d}s\right)\\ &\leqslant \frac{1}{NC_0}\left(\tilde{H}_{\sigma,NC_1}\int_0^1 (1-t)\,\mathrm{d}\alpha(t) + \lambda C_1\int_0^1 G(s,s)[\tilde{f}_N + u(s)]\,\mathrm{d}s\right)\\ &< \frac{1}{NC_0}\left(\tilde{H}_{\sigma,NC_1}\int_0^1 (1-t)\,\mathrm{d}\alpha(t) + \left[NC_0 - \tilde{H}_{\sigma,NC_1}\int_0^1 (1-t)\,\mathrm{d}\alpha(t)\right]\right)\\ &= 1. \end{aligned} \tag{3.26}$$

All in all, since (3.26) implies that $\mu < 1$, we obtain a contradiction, and so (3.22) holds, as desired.

Letting τ_w denote the same quantity as in the proof of Theorem 3.1, we next argue that

$$i_{\mathcal{K}}(T, \Omega_{\tau_w}) = 1 \tag{3.27}$$

for $\tau_w > N$ sufficiently large, just as in the proof of Theorem 3.1. To argue that (3.27) holds, we show that $y \neq Ty + \mu e$ for each $\mu \geqslant 0$ with $e(t) \equiv 1$; observe that $e \in \mathcal{K}$ due to condition (3.20). Select $\rho_2 > 0$ sufficiently large in exactly the same way as in the second part of the proof of Theorem 3.1, and for contradiction assume the existence of $\mu \geqslant 0$ and $y \in \mathcal{K} \cap \partial\Omega_{\tau_w}$ such that $y(t) = (Ty)(t) + \mu e(t)$ for each $t \in [0,1]$. Then one can repeat (3.18) and the entire second part of the proof of Theorem 3.1 verbatim to obtain that $y(0) \geqslant (Ty)(0) > \|y\|$, which is a contradiction. Thus, we obtain that (3.27) holds.

Finally, putting (3.22) and (3.27) together, we obtain by Lemma 2.8 the existence of

$$y_0 \in \mathcal{K} \cap (\Omega_{\tau_w} \setminus \bar{\Omega}_N) \tag{3.28}$$

such that $Ty_0 = y_0$. Note that since $\|y_0\| > N$ and due to the choice of λ_0^*, we may repeat verbatim inequality (3.2). But then concluding as in the proof of Theorem 3.1 we obtain that the function y_0 identified in (3.28) can be used to construct a map $t \mapsto \Upsilon(t) := (y_0 - w)(t)$ such that this map is a positive solution of (1.1), which completes the proof. □

Remark 3.3. Note that due to condition (A3) and (3.20) it follows that λ_0^* in (3.21) is well defined and, in particular, satisfies $\lambda_0^* > 0$.

Remark 3.4. In comparison to Theorem 3.1, one possible advantage of Theorem 3.2 is that the calculation of λ_0^* is simpler since one need not calculate the number η_0.

We conclude this paper with a couple of examples to illustrate the application of the preceding existence theorems.

Example 3.5. Suppose that $H(z) := z^2$ and that φ, φ_1 and φ_2 are defined by

$$\varphi(y) := \tfrac{7}{10}y(\tfrac{1}{2}) - \tfrac{1}{10}y(\tfrac{1}{3}) = \underbrace{(\tfrac{3}{5}y(\tfrac{1}{2}) - \tfrac{1}{10}y(\tfrac{1}{3}))}_{:=\varphi_1(y)} + \underbrace{\tfrac{1}{10}y(\tfrac{1}{2})}_{:=\varphi_2(y)}.$$

Routine calculations reveal that $C_0 = 1/40$ and $C_1 = 4/5$ in this case. Moreover, condition (A5) is trivially satisfied. Furthermore, condition (A4) holds, for example, with $N = 1$ since we notice that $\tilde{H}_{C_0,C_1} = \tilde{H}_{1/40,4/5} = 16/25 < 1 = N$. Now, let us suppose for the sake of simplicity that $u(t) \equiv u_0$. Then additional calculations demonstrate that in light of (3.1) we may set (to three decimal places of accuracy)

$$\lambda_0 := \min\{1.055u_0^{-1}, u_0^{-1}, 1.080[\tilde{f}_1 + u_0]^{-1}\} = \min\{u_0^{-1}, 1.080[\tilde{f}_1 + u_0]^{-1}\} > 0. \quad (3.29)$$

Consequently, Theorem 3.1 implies that for any continuous function f satisfying (A3) and such that $\tilde{f}_1 + u(t)$ does not vanish on any non-degenerate subinterval of $[0, 1]$, problem (1.1) has at least one positive solution for each $\lambda \in (0, \lambda_0)$ with λ_0 defined as in (3.29).

Example 3.6. Suppose that φ is defined by

$$\varphi(y) := \tfrac{27}{140}y(\tfrac{1}{2}) - \tfrac{1}{7}y(\tfrac{1}{3}) = \underbrace{(\tfrac{1}{7}y(\tfrac{1}{2}) - \tfrac{1}{7}y(\tfrac{1}{3}))}_{:=\varphi_1(y)} + \underbrace{\tfrac{1}{20}y(\tfrac{1}{2})}_{:=\varphi_2(y)}. \quad (3.30)$$

Moreover, suppose, in contrast to Example 3.5, that

$$H(z) := \begin{cases} z, & 0 \leqslant z < 2, \\ 2 + 84(z - 2), & 2 \leqslant z < +\infty. \end{cases}$$

From (3.30) we calculate $C_0 = \frac{1}{80}$ and $C_1 = \frac{47}{140}$. In addition, for each $N \in (0, \frac{1660}{177})$ it can be shown that condition (3.20) is satisfied. Moreover, we note that condition (A5) remains satisfied since

$$\lim_{z\to+\infty} \frac{H(z)}{z} = 84 > \frac{1}{C_0} = 80. \quad (3.31)$$

In light of (3.31), we could, for instance, pick $M := 81$. Finally, a simple calculation affirms that conditions (A1) and (A2) also hold.

Consequently, Theorem 3.2 implies that problem (1.1) has at least one positive solution in this setting for any function f satisfying condition (A3) and each $\lambda > 0$ sufficiently small. Note that (3.30) implies that the boundary condition in (1.1) at $t = 0$ is affine since

$$y(0) = \begin{cases} \tfrac{27}{140}y(\tfrac{1}{2}) - \tfrac{1}{7}y(\tfrac{1}{3}), & \varphi(y) \in [0, 2], \\ \tfrac{81}{5}y(\tfrac{1}{2}) - 12y(\tfrac{1}{3}) - 166, & \varphi(y) \in [2, +\infty). \end{cases} \quad (3.32)$$

Finally, to estimate the admissible range of the parameter λ, as in Example 3.5 let us suppose that $u(t) \equiv u_0$. Then in light of (3.21) it can be shown that 'sufficiently small' means that $\lambda \in (0, \lambda_0^*)$, where

$$\lambda_0^* := \tfrac{2846}{2209} \min\{u_0^{-1}, (\tilde{f}_{280/47} + u_0)^{-1}\} > 0,$$

provided that we use $N = \frac{280}{47}$ to generate the estimation of the number λ_0 in (3.21). In fact, as one can easily demonstrate, the coefficient of the minimum in the above estimate for λ_0^* is maximized precisely when $N = \frac{280}{47}$ is chosen, as above.

Remark 3.7. Observe that the function f can be strictly negative on the entirety of its domain in each of Examples 3.5 and 3.6. This is due to the fact that growth requirements are imposed only on H. Consequently, so long as $f(t, y) \geqslant -u(t)$ on $[0, 1] \times \mathbb{R}$, as required by condition (A3), then $f(t, y) < 0$ may hold for all (t, y).

Remark 3.8. It can be shown that conditions (A4) and (A5) jointly imply that H cannot be linear on its entire domain. However, as Example 3.6 demonstrates, an affine map is admissible. Moreover, H can be linear away from $+\infty$.

Acknowledgements. The author gratefully thanks the anonymous referee, whose careful reading of the original manuscript considerably improved both the accuracy and exposition of the results.

References

1. D. R. Anderson, Existence of three solutions for a first-order problem with nonlinear nonlocal boundary conditions, *J. Math. Analysis Applic.* **408** (2013), 318–323.
2. V. Anuradha, D. D. Hai and R. Shivaji, Existence results for superlinear semipositone BVPs, *Proc. Am. Math. Soc.* **124** (1996), 757–763.
3. C. S. Goodrich, Positive solutions to boundary value problems with nonlinear boundary conditions, *Nonlin. Analysis* **75** (2012), 417–432.
4. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically superlinear boundary conditions, *Commentat. Math. Univ. Carolinae* **53** (2012), 79–97.
5. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically sublinear boundary conditions, *Appl. Analysis Discr. Math.* **6** (2012), 174–193.
6. C. S. Goodrich, On nonlocal BVPs with boundary conditions with asymptotically sublinear or superlinear growth, *Math. Nachr.* **285** (2012), 1404–1421.
7. C. S. Goodrich, On nonlinear boundary conditions satisfying certain asymptotic behavior, *Nonlin. Analysis* **76** (2013), 58–67.
8. C. S. Goodrich, On a nonlocal BVP with nonlinear boundary conditions, *Results Math.* **63** (2013), 1351–1364.
9. C. S. Goodrich, Positive solutions to differential inclusions with nonlocal, nonlinear boundary conditions, *Appl. Math. Computat.* **219** (2013), 11071–11081.
10. C. S. Goodrich, An existence result for systems of second-order boundary value problems with nonlinear boundary conditions, *Dynam. Syst. Applic.* **23** (2014), 601–618.
11. C. S. Goodrich, A note on semipositone boundary value problems with nonlocal, nonlinear boundary conditions, *Arch. Math.* **103** (2014), 177–187.
12. C. S. Goodrich, On nonlinear boundary conditions involving decomposable linear functionals, *Proc. Edinb. Math. Soc.* **58** (2015), 421–439.
13. C. S. Goodrich, Semipositone boundary value problems with nonlocal, nonlinear boundary conditions, *Adv. Diff. Eqns* **20** (2015), 117–142.
14. G. Infante, Nonlocal boundary value problems with two nonlinear boundary conditions, *Commun. Appl. Analysis* **12** (2008), 279–288.
15. G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlin. Analysis* **71** (2009), 1301–1310.

16. G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlin. Studies* **16** (2009), 187–196.
17. G. Infante and P. Pietramala, Perturbed Hammerstein integral inclusions with solutions that change sign, *Commentat. Math. Univ. Carolinae* **50** (2009), 591–605.
18. G. Infante and P. Pietramala, A third order boundary value problem subject to nonlinear boundary conditions, *Math. Bohem.* **135** (2010), 113–121.
19. G. Infante and P. Pietramala, Multiple non-negative solutions of systems with coupled nonlinear BCs, *Math. Meth. Appl. Sci.* **37** (2014), 2080–2090.
20. G. Infante, F. Minhós and P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions, *Commun. Nonlin. Sci. Numer. Simulation* **17** (2012), 4952–4960.
21. G. Infante, P. Pietramala and M. Tenuta, Existence and localization of positive solutions for a nonlocal BVP arising in chemical reactor theory, *Commun. Nonlin. Sci. Numer. Simulation* **19** (2014), 2245–2251.
22. T. Jankowski, Positive solutions to fractional differential equations involving Stieltjes integral conditions, *Appl. Math. Computat.* **241** (2014), 200–213.
23. J. Jiang, L. Liu and Y. Wu, Positive solutions for second-order singular semipositone differential equations involving Stieltjes integral conditions, *Abstr. Appl. Analysis* **2012** (2012), 696283.
24. G. L. Karakostas, Existence of solutions for an n-dimensional operator equation and applications to BVPs, *Electron. J. Diff. Eqns* **2014** (2014), No. 71.
25. G. L. Karakostas and P. Ch. Tsamatos, Existence of multiple positive solutions for a nonlocal boundary value problem, *Topolog. Meth. Nonlin. Analysis* **19** (2002), 109–121.
26. G. L. Karakostas and P. Ch. Tsamatos, Multiple positive solutions of some Fredholm integral equations arisen from nonlocal boundary-value problems, *Electron. J. Diff. Eqns* **2002** (2002), No. 30.
27. M. Picone, Su un problema al contorno nelle equazioni differenziali lineari ordinarie del secondo ordine, *Annali Scuola Norm. Sup. Pisa* **10** (1908), 1–95.
28. J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. Lond. Math. Soc.* **74** (2006), 673–693.
29. J. R. L. Webb and G. Infante, Non-local boundary value problems of arbitrary order, *J. Lond. Math. Soc.* **79** (2009), 238–258.
30. J. R. L. Webb and G. Infante, Semi-positone nonlocal boundary value problems of arbitrary order, *Commun. Pure Appl. Analysis* **9** (2010), 563–581.
31. W. M. Whyburn, Differential equations with general boundary conditions, *Bull. Am. Math. Soc.* **48** (1942), 692–704.
32. Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlin. Analysis* **62** (2005), 1251–1265.
33. Z. Yang, Positive solutions of a second-order integral boundary value problem, *J. Math. Analysis Applic.* **321** (2006), 751–765.
34. E. Zeidler, *Nonlinear functional analysis and its applications, I: fixed-point theorems* (Springer, 1986).