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PURE INJECTIVE AND ABSOLUTELY PURE SHEAVES

EDGAR ENOCHS[1], SERGIO ESTRADA[2] AND SINEM ODABAŞI[2]

[1] *Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA* (enochs@ms.uky.edu)
[2] *Departamento de Matemáticas, Universidad de Murcia, 30100 Murcia, Spain* (sestrada@um.es; sinem.odabasi1@um.es)

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Abstract We study two notions of purity in categories of sheaves: the categorical and the geometric. It is shown that pure injective envelopes exist in both cases under very general assumptions on the scheme. Finally, we introduce the class of locally absolutely pure (quasi-coherent) sheaves with respect to the geometrical purity, and characterize locally Noetherian closed subschemes of a projective scheme in terms of the new class.

Keywords: local purity; categorical purity; concentrated scheme; locally finitely presented category; pure injective sheaf; absolutely pure sheaf

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1. Introduction

The history of purity goes back to the work of [**23**] for abelian groups. Later, the notion was introduced into module categories by [**1**]. The notion, which is a problem of solving equations with one variable in abelian groups, turned out to be that of solving equations in several variables in module categories. The notion was developed further in [**9**,**24**,**25**,**27**]. More recently it was shown by Crawley-Boevey [**3**] that locally finitely presented additive categories are the most general additive setup in which to define a good purity theory. We recall that a short exact sequence in a locally finitely presented category is said to be pure whenever it is projectively generated by the class of finitely presented objects. Several problems in algebra and in relative homological algebra can be solved by purity arguments. For example, to show that a class of objects of a class $\mathcal{F}$ in a locally finitely presented category $\mathcal{A}$ allows us to define unique up to homotopy minimal resolutions, it suffices to check that $\mathcal{F}$ is closed under direct limits and under pure subobjects or under pure quotients (see [**4**,**5**]). In the case of the category of R-modules (R any commutative ring with identity), it is well known that purity can also be defined in terms of the tensor product, that is, a short exact sequence $\mathbb{E} = 0 \to L \to M \to N \to 0$ of R-modules is pure provided that the functor $\mathbb{E} \otimes T$ leaves the sequence exact for each R-module T. But in general, for an arbitrary monoidal locally finitely presented category, these two notions need not be equivalent. For example, when X is a concentrated (i.e. quasi-compact and quasi-separated) scheme, the category $\mathfrak{Qcoh}(X)$ of quasi-coherent sheaves on X is locally finitely presented (see [**12**, Chapitre 1, § 6, Corollaire 6.9.12] or [**11**, Proposition 7] for a precise formulation) and it comes equipped with a canonical tensor product, so one might wonder about the relationship (if any) between the two possible definitions of purity: the categorical one arising from the general fact that we are working with a locally finitely presented category, or the second one, arising from the usual tensor product in $\mathfrak{Qcoh}(X)$. We shall denote by fp-pure the notion of purity in the first sense and just by pure in the second. Thus, the first part of this paper is devoted to exploring the relationship between the two notions on $\mathfrak{Qcoh}(X)$. Actually, we will consider a slightly different notion of purity in $\mathfrak{Qcoh}(X)$. Namely, we say that a short exact sequence $\mathbb{E}$ in $\mathfrak{Qcoh}(X)$ is pure exact provided that $\mathbb{E} \otimes \mathscr{M}$ is exact for each sheaf of $\mathcal{O}_X$-modules $\mathscr{M}$ (so not just the quasi-coherent ones). As we point out in Remark 3.5, this is equivalent to $\mathbb{E} \otimes \cdot$ being exact in $\mathfrak{Qcoh}(X)$ provided that X is quasi-separated. The reason for considering this more general definition is that it is always equivalent to purity on the stalks (Proposition 3.2). So this justifies its geometrical nature.

From this point of view, the present paper can be seen as a continuation of the ongoing program initiated in [**8**] where a wide class of projective schemes was exhibited that do not have non-trivial *categorical flat* quasi-coherent sheaves (that is, quasi-coherent sheaves such that each short exact sequence ending on them is fp-pure). If we denote by $\mathrm{Pure}_{\mathrm{fp}}$ and by Pure the classes of fp-pure and pure short exact sequences in $\mathfrak{Qcoh}(X)$, we prove the following result (Proposition 3.9).

Proposition. *If X is a concentrated scheme,* $\mathrm{Pure}_{\mathrm{fp}} \subseteq \mathrm{Pure}$.

In particular, this allows us to clarify the general relationship between categorical and geometrical flatness for concentrated schemes (Corollary 3.12).

Proposition. *Assume that X is quasi-compact and semi-separated. Then each categorical flat sheaf in $\mathfrak{Qcoh}(X)$ is geometrical flat.*

The converse is not true, in general, for non-affine schemes. This is one of the main results of Estrada and Saorín [**8**, Theorem 4.4].

Section 4 of the paper is devoted to showing that pure injective envelopes do exist with respect to both notions of purity. The first proof is a particular instance of a theorem due to Herzog [**15**] (see also [**10**]) on the existence of pure injective envelopes in locally finitely presented additive categories.

Theorem. *Let X be a concentrated scheme. Then every quasi-coherent sheaf in $\mathfrak{Qcoh}(X)$ admits an fp-pure injective envelope, which is an fp-pure monomorphism.*

However, we can show that pure injective envelopes with respect to the geometrical purity always exist, without assuming any condition on the scheme (Theorem 4.10).

Theorem. *Let X be any scheme. Each quasi-coherent sheaf in $\mathfrak{Qcoh}(X)$ has a pure injective envelope that is a pure monomorphism.*

In § 5 we will focus on the geometrical pure notion in $\mathcal{O}_X$-Mod and $\mathfrak{Qcoh}(X)$ and we introduce the classes of (locally) absolutely pure sheaves of modules and quasi-coherent

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sheaves. Given an associative ring R with unit, a left R-module is absolutely pure if every finite system of linear equations whose independent terms lie in M possesses a solution in M. This is equivalent to saying that M is a pure submodule of any R-module that contains it. In some aspects these behave like injective R-modules (see [**9**, **17**, **18**, **21**, **26**] for a general treatment of absolutely pure modules and [**20**] for a revisited study). In fact, Noetherian rings can be characterized in terms of properties of absolutely pure modules. Namely, R is Noetherian if and only if the class of absolutely pure R-modules coincides with the class of injective R-modules (see [**18**]). We will exhibit the main properties of (locally) absolutely pure sheaves of modules, both in $\mathfrak{Qcoh}(X)$ and in $\mathcal{O}_X$-Mod, for the case in which X is a locally coherent scheme. For example, we show in Proposition 5.7 that local absolutely purity in $\mathfrak{Qcoh}(X)$ can be checked on a particular affine covering of X. And we also see that locally absolutely quasi-coherent sheaves are precisely the absolutely pure $\mathcal{O}_X$-modules that are quasi-coherent. This is analogous to the question posted in [**14**, Chapter II, § 7, p. 135] for locally Noetherian schemes (see [**2**, Lemma 2.1.3]). We then characterize locally Noetherian closed subschemes of the projective space $\mathbb{P}^n(A)$ in terms of its class of absolutely pure quasi-coherent sheaves.

Theorem. *A locally coherent closed subscheme $X \subseteq \mathbb{P}^n(A)$ is locally Noetherian if and only if every locally absolutely pure quasi-coherent sheaf is locally injective.*

If X is a Noetherian scheme, it is known that the class of locally injective quasi-coherent sheaves is a covering class in $\mathfrak{Qcoh}(X)$. We finish this section by extending this result to the class of locally absolutely pure quasi-coherent sheaves on a locally coherent scheme X.

Theorem. *Let X be a locally coherent scheme. Then every quasi-coherent sheaf in $\mathfrak{Qcoh}(X)$ admits a locally absolutely pure cover.*

2. Preliminaries

In this paper, all rings used will be commutative with identity.

Following [**3**], an additive category $\mathcal{A}$ with direct limits is said to be *locally finitely presented* provided that the skeleton of the subcategory of finitely presented objects in $\mathcal{A}$ is small, and each object of $\mathcal{A}$ is a direct limit of finitely presented objects. Here, an object A in $\mathcal{A}$ is called *finitely presented* if the functor $\mathrm{Hom}_{\mathcal{A}}(A, \cdot)$ preserves direct limits.

For example, for any ring S (not necessarily commutative and with unit) the category S-Mod of left S-modules, is locally finitely presented [**16**]. If X has a basis of compact open sets, then the category $\mathcal{O}_X$-Mod of all sheaves of $\mathcal{O}_X$-modules is locally finitely presented (see, for example, [**22**, Theorem 5.6] or [**21**, Theorem 16.3.17]). If X is a concentrated scheme (i.e. quasi-compact and quasi-separated), then the category $\mathfrak{Qcoh}(X)$ of quasi-coherent sheaves of $\mathcal{O}_X$-modules is also locally finitely presented (see [**11**, Proposition 7] for a proof based on [**12**, Chapitre 1, § 6, Corollaire 6.9.12]).

We recall that a short exact sequence of R-modules $0 \to L \to M \to N \to 0$ is called *pure* if $T \otimes L \to T \otimes M$ is a monomorphism for every R-module T. This is equivalent to $0 \to \mathrm{Hom}_R(F, L) \to \mathrm{Hom}_R(F, M) \to \mathrm{Hom}_R(F, N) \to 0$ being exact for each finitely presented R-module M. The last condition can be adapted to give the usual definition of a sequence $0 \to L \to M \to N \to 0$ of morphisms to be pure exact in an arbitrary locally finitely presented category $\mathcal{A}$.

Definition 2.1. Let $\mathcal{C}$ be a Grothendieck category. A direct system of objects of $\mathcal{C}$, $(M_\alpha \mid \alpha \leqslant \lambda)$, is said to be a *continuous system of monomorphisms* if $M_0 = 0$, $M_\beta = \varinjlim_{\alpha<\beta} M_\alpha$ for each limit ordinal $\beta \leqslant \lambda$ and all the morphisms in the system are monomorphisms.

Let $\mathcal{S}$ be a class of objects that is closed under isomorphisms. An object M of $\mathcal{C}$ is said to be $\mathcal{S}$*-filtered* if there is a continuous system $(M_\alpha \mid \alpha \leqslant \lambda)$ of subobjects of M such that $M = M_\lambda$ and $M_{\alpha+1}/M_\alpha$ is isomorphic to an object of $\mathcal{S}$ for each $\alpha < \lambda$.

The class of $\mathcal{S}$-filtered objects in $\mathcal{C}$ is denoted by $\mathrm{Filt}(\mathcal{S})$. The relation $\mathcal{S} \subseteq \mathrm{Filt}(\mathcal{S})$ always holds. In the case in which $\mathrm{Filt}(\mathcal{S}) \subseteq \mathcal{S}$, the class $\mathcal{S}$ is said to be *closed under* $\mathcal{S}$*-filtrations.*

Definition 2.2. Let $\mathcal{F}$ be a class of objects of a Grothendieck category $\mathcal{A}$. A morphism $\phi\colon F \to M$ of $\mathcal{C}$ is said to be an $\mathcal{F}$*-precover* of M if $F \in \mathcal{F}$ and if $\mathrm{Hom}(F', F) \to \mathrm{Hom}(F', M) \to 0$ is exact for every $F' \in \mathcal{F}$. If any morphism $f\colon F \to F$ is such that $\phi \circ f = \phi$ is an isomorphism, then it is called an $\mathcal{F}$*-cover* of M. If the class $\mathcal{F}$ is such that every object has an $\mathcal{F}$-cover, then $\mathcal{F}$ is called a *precovering class*. The dual notions are those of $\mathcal{F}$-envelope and enveloping class.

3. Purity in $\mathfrak{Qcoh}(X)$

Let X be a scheme and let $\mathscr{F}, \mathscr{G}$ be $\mathcal{O}_X$-modules. The tensor product $\mathscr{F} \otimes \mathscr{G}$ is defined as the sheafification of the presheaf $U \to \mathscr{F}(U) \otimes_{\mathcal{O}_X(U)} \mathscr{G}(U)$ for each open subset $U \subseteq X$. There is also an internal Hom functor in $\mathcal{O}_X$-Mod, $\mathscr{H}\mathrm{om}(\cdot,\cdot)$. The image $\mathscr{H}\mathrm{om}(\mathscr{F},\mathscr{G})(U)$ on an open subset $U \subseteq X$ is $\mathrm{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$. It is known that the pair $(\cdot \otimes \cdot, \mathscr{H}\mathrm{om}(\cdot,\cdot))$ makes $\mathcal{O}_X$-Mod a closed symmetric monoidal category.

Definition 3.1. Let $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. This is called *pure exact* if, for each $\mathscr{M} \in \mathcal{O}_X$-Mod, the induced sequence

$$0 \to \mathscr{M} \otimes \mathscr{F} \xrightarrow{\mathrm{id} \otimes \tau} \mathscr{M} \otimes \mathscr{G}$$

is exact.

Proposition 3.2. *Let $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. The following conditions are equivalent:*

(1) *the sequence is pure exact,*

(2) *for each $x \in X$ the monomorphism $0 \to \mathscr{F}_x \xrightarrow{\tau} \mathscr{G}_x$ in $\mathcal{O}_{X,x}$-Mod is pure.*

Proof. (1) $\Longrightarrow$ (2) Let $M \in \mathcal{O}_{X,x}$-Mod. Then $i_{x,*}M$ (the skyscraper sheaf with respect to M) is an $\mathcal{O}_X$-module such that $(i_{x,*}M)_x = M$. Since $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ is pure,

$$0 \to i_{x,*}M \otimes \mathscr{F} \to i_{x,*}M \otimes \mathscr{G}$$

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is exact, that is, for each $x \in X$,

$$0 \to (i_{x,*}M \otimes \mathscr{F})_x \to (i_{x,*}M \otimes \mathscr{G})_x$$

is exact in $\mathcal{O}_{X,x}$-Mod. But for each $\mathscr{A} \in \mathcal{O}_{X,x}$-Mod, $(i_{x,*}M \otimes \mathscr{A})_x \cong M \otimes \mathscr{A}_x$. Hence, from the previous map, it follows that

$$0 \to M \otimes \mathscr{F}_x \to M \otimes \mathscr{G}_x$$

is exact in $\mathcal{O}_{X,x}$-Mod. So $0 \to \mathscr{F}_x \xrightarrow{\tau} \mathscr{G}_x$ is pure.

(2) $\Longrightarrow$ (1) Let $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ be an exact sequence in $\mathcal{O}_X$-Mod (so, for each $x \in X$, $0 \to \mathscr{F}_x \xrightarrow{\tau_x} \mathscr{G}_x$ is exact in $\mathcal{O}_{X,x}$-Mod). Given $\mathscr{M} \in \mathcal{O}_X$-Mod, the induced $\mathscr{M} \otimes \mathscr{F} \xrightarrow{\mathrm{id} \otimes \tau} \mathscr{M} \otimes \mathscr{G}$ will be a monomorphism if and only if for each $x \in X$ the morphism of $\mathcal{O}_{X,x}$-modules $(\mathscr{M} \otimes \mathscr{F})_x \xrightarrow{(\mathrm{id} \otimes \tau)_x} (\mathscr{M} \otimes \mathscr{G})_x$ is a monomorphism. But, for each $x \in X$ and $\mathscr{A} \in \mathcal{O}_X$-Mod, $(\mathscr{M} \otimes \mathscr{A})_x \cong \mathscr{M}_x \otimes \mathscr{A}_x$. So by (2) it follows that $\mathscr{M} \otimes \mathscr{F} \xrightarrow{\mathrm{id} \otimes \tau} \mathscr{M} \otimes \mathscr{G}$ is a monomorphism. Therefore, $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ is pure. □

Proposition 3.3. *Let X be a scheme and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Qcoh}(X)$. The following conditions are equivalent:*

(1) $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ *is pure exact,*

(2) $0 \to \mathscr{F}(U) \xrightarrow{\tau_U} \mathscr{G}(U)$ *is pure in $\mathcal{O}_X(U)$-Mod for each open affine $U \subseteq X$.*

Proof. (1) $\Longrightarrow$ (2) Let U be an affine open subset of X and let $\imath \colon U \hookrightarrow X$ be the inclusion. Let $M \in \mathcal{O}_X(U)$-Mod. Then $\imath_*(\tilde{M})$ is an $\mathcal{O}_X$-module. Therefore,

$$0 \to \imath_*(\tilde{M}) \otimes \mathscr{F} \to \imath_*(\tilde{M}) \otimes \mathscr{G}$$

is exact. But then

$$0 \to (\imath_*(\tilde{M}) \otimes \mathscr{F})(U) \to (\imath_*(\tilde{M}) \otimes \mathscr{G})(U)$$

is exact in $\mathcal{O}_X(U)$-Mod, that is,

$$0 \to \imath_*(\tilde{M})(U) \otimes \mathscr{F}(U) \to \imath_*(\tilde{M})(U) \otimes \mathscr{G}(U)$$

is exact. Since, for each $\mathcal{O}_X(U)$-module A, $\imath_*(\tilde{A})(U) = A$, we obtain that $0 \to M \otimes \mathscr{F}(U) \to M \otimes \mathscr{G}(U)$ is exact. Thus, $0 \to \mathscr{F}(U) \to \mathscr{G}(U)$ is pure.

(2) $\Longrightarrow$ (1) This is immediate just by observing that, for each affine open set $U \subseteq X$, $(\mathscr{F} \otimes \mathscr{G})(U) \cong \mathscr{F}(U) \otimes \mathscr{G}(U)$, and that a morphism τ in $\mathcal{O}_X$-Mod is a monomorphism if and only if τ_U is a monomorphism in $\mathcal{O}_X(U)$-Mod. □

Proposition 3.4. *Let X be a scheme and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Qcoh}(X)$. The following statements are equivalent:*

(1) $0 \to \mathscr{F} \xrightarrow{\tau} \mathscr{G}$ *is pure exact,*

(2) *there exists an open covering of X by affine open sets $\mathscr{U} = \{U_i\}$ such that* $0 \to \mathscr{F}(U_i) \xrightarrow{\tau_{U_i}} \mathscr{G}(U_i)$ *is pure in $\mathcal{O}_X(U_i)$-Mod,*

(3) $0 \to \mathscr{F}_x \xrightarrow{\tau_x} \mathscr{G}_x$ *is pure in $\mathcal{O}_{X,x}$-Mod for each $x \in X$.*

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Proof. (1) $\Longrightarrow$ (2) It follows from Proposition 3.3.

(2) $\Longrightarrow$ (3) Let $x \in X$. There then exists $U_i \in \mathscr{U}$ such that $x \in U_i = \operatorname{Spec}(A_i)$ for some ring A_i. But then the claim follows by observing that

$$\mathscr{F}_x = (\widetilde{\mathscr{F}(U_i)})_x \cong \widetilde{\mathscr{F}(U_i)}_x$$

and noticing that if $0 \to M \to N$ is pure exact in A_i-Mod, then $0 \to M_x \to N_x$ is pure exact in $(A_i)_x$-Mod.

(3) $\Longrightarrow$ (1) By Proposition 3.2, we know that τ is pure in $\mathcal{O}_X$-Mod. $\square$

Remark 3.5. Note that $\mathfrak{Qcoh}(X)$ is a monoidal category with the tensor product induced from $\mathcal{O}_X$-Mod. So we could also define a notion of purity in $\mathfrak{Qcoh}(X)$ by using this monoidal structure, that is, $0 \to \mathscr{F} \to \mathscr{G}$ is pure exact provided that it is $\mathscr{M} \otimes \cdot$ exact for each $\mathscr{M} \in \mathfrak{Qcoh}(X)$. For the case in which X is quasi-separated, this notion agrees with the one we have considered (that is, $0 \to \mathscr{F} \to \mathscr{G}$ is pure in $\mathfrak{Qcoh}(X)$ if it is pure in $\mathcal{O}_X$-Mod). This is because the direct image functor $\imath_*(\tilde{M})$ preserves quasi-coherence when X is quasi-separated in the proof of Proposition 3.3.

Over an affine scheme X, the category of quasi-coherent sheaves on X is equivalent to the category $\mathcal{O}_X(X)$-Mod. So the following lemma can be easily obtained.

Lemma 3.6. *Let $\mathscr{F} \in \mathfrak{Qcoh}(X)$ and let U be an affine open subset of X. Then $\mathscr{F}|_U$ is finitely presented in $\mathfrak{Qcoh}(U)$ if and only if $\mathscr{F}(U)$ is finitely presented.*

Proposition 3.7. *Assume that X is semi-separated or concentrated. Let $\mathscr{F} \in \mathfrak{Qcoh}(X)$ and consider the following assertions.*

(1) *$\mathscr{F}$ is a finitely presented object in $\mathfrak{Qcoh}(X)$.*

(2) *$\mathscr{F}|_U$ is finitely presented in $\mathfrak{Qcoh}(U)$ for all affine open subsets $U \subseteq X$.*

(3) *$\mathscr{F}_x$ is finitely presented for each $x \in X$.*

*Then the implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) hold. If X is concentrated, then (1) $\Longleftrightarrow$ (2) (see [**19**, Proposition 75]).*

Proof. (1) $\Longrightarrow$ (2) We have to show that the canonical morphism

$$\psi\colon \varinjlim \operatorname{Hom}(\mathscr{F}|_U, \widetilde{B_i}) \to \operatorname{Hom}(\mathscr{F}|_U, \varinjlim \widetilde{B_i})$$

is an isomorphism for any direct system $\{\widetilde{B_i}, \varphi_{ij}\}_I$ of quasi-coherent $\mathcal{O}_X|_U$-modules. We have the following commutative diagram:

$$\begin{array}{ccccc}
\operatorname{Hom}(\mathscr{F}|_U, \widetilde{B_i}) & \longrightarrow & \varinjlim \operatorname{Hom}(\mathscr{F}|_U, \widetilde{B_i}) & \xrightarrow{\psi} & \operatorname{Hom}(\mathscr{F}|_U, \varinjlim \widetilde{B_i}) \\
\downarrow & & \downarrow & & \downarrow \\
\operatorname{Hom}(\mathscr{F}, \imath_*(\widetilde{B_i})) & \longrightarrow & \varinjlim \operatorname{Hom}(\mathscr{F}, \imath_*(\widetilde{B_i})) & \xrightarrow{\psi'} & \operatorname{Hom}(\mathscr{F}, \varinjlim \imath_*(\widetilde{B_i}))
\end{array}$$

Columns are isomorphisms because of the adjoint pair $(\mathrm{res}_U, \imath_*)$. For the third column we also need to observe that, under the hypothesis on X, the direct image functor $\imath_*$ preserves direct limits. Since $\mathscr{F}$ is finitely presented, the canonical morphism ψ' is an isomorphism. So ψ is an isomorphism.

(3) $\Longrightarrow$ (4) $\mathscr{F}_x \cong M_p$ for some finitely presented R-module M and prime ideal p. Then (4) follows because the localization of a finitely presented R-module is a finitely presented R_p-module. □

Definition 3.8 (Crawley-Boevey [3, § 3]). An exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{T} \to 0$ in $\mathfrak{Qcoh}(X)$ is called *categorical pure* if the functor $\mathrm{Hom}(\mathscr{H}, \cdot)$ leaves the sequence exact for every finitely presented quasi-coherent $\mathcal{O}_X$-module $\mathscr{H}$.

We shall denote by $\mathrm{Pure}_{\mathrm{fp}}$ the class of categorical pure short exact sequences in $\mathfrak{Qcoh}(X)$ and by Pure the class of pure short exact sequences in $\mathfrak{Qcoh}(X)$, as in Proposition 3.4.

Proposition 3.9. *If $\mathfrak{Qcoh}(X)$ is a locally finitely presented category, then categorical pure short exact sequences are pure exact, that is,* $\mathrm{Pure}_{\mathrm{fp}} \subseteq \mathrm{Pure}$.

Proof. Let $\mathbb{E} \equiv 0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ be an exact sequence in $\mathrm{Pure}_{\mathrm{fp}}$. By assumption, $\mathscr{H} = \varinjlim \mathscr{H}_i$, where $\mathscr{H}_i$ is a finitely presented object in $\mathfrak{Qcoh}(X)$ for each i. Now, for each i, the top row of the pullback diagram

$$\begin{array}{ccccccccc} \mathbb{E}^i = & 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G}_i & \longrightarrow & \mathscr{H}_i & \longrightarrow & 0 \\ & & & \downarrow & & \downarrow & & \downarrow & & \\ & 0 & \longrightarrow & \mathscr{F} & \longrightarrow & \mathscr{G} & \longrightarrow & \mathscr{H} & \longrightarrow & 0 \end{array}$$

is a categorical pure exact sequence ending with a finitely presented object $\mathscr{H}_i$. Therefore, each $\mathbb{E}^i$ splits for every i. That is, $\mathbb{E} = \varinjlim \mathbb{E}^i$, where $\mathbb{E}^i$ is a splitting exact sequence for every i. Now, taking the stalk at $x \in X$, we obtain $\mathbb{E}_x = \varinjlim \mathbb{E}^i_x$. Then $\mathbb{E}^i_x$ is pure exact in $\mathcal{O}_{X,x}$-Mod for each $x \in X$, and so is $\mathbb{E}_x$. Hence, by Proposition 3.4, $\mathbb{E}$ is a pure exact sequence in $\mathfrak{Qcoh}(X)$. □

We recall that a quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ is *flat* if $\mathscr{F} \otimes \cdot$ is exact in $\mathcal{O}_X$-Mod. Equivalently, $\mathscr{F}(U)$ is flat as an $\mathcal{O}_X(U)$-module for each affine open subset $U \subseteq X$, or $\mathscr{F}_x$ is flat as an $\mathcal{O}_{X,x}$-module for each $x \in X$. We will denote by $\mathcal{F}\mathrm{lat}$ the class of all flat quasi-coherent sheaves.

Definition 3.10. A quasi-coherent $\mathcal{O}_X$-module $\mathscr{F}$ is called *tensor flat* (respectively, *fp-flat*) if every short exact sequence in $\mathfrak{Qcoh}(X)$ ending in $\mathscr{F}$ is pure exact (respectively, is categorical pure). We shall denote by $\mathcal{F}\mathrm{lat}_{\otimes}$ (respectively, by $\mathcal{F}\mathrm{lat}_{fp}$) the class of all tensor flat quasi-coherent sheaves (respectively, the class of all fp-flat quasi-coherent sheaves).

Proposition 3.11. *Let $\mathscr{F} \in \mathfrak{Qcoh}(X)$. If $\mathscr{F}$ is flat, then it is also tensor flat. In the case in which X is semi-separated, the converse also holds.*

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Proof. Let $0 \to \mathscr{T} \to \mathscr{G} \to \mathscr{F} \to 0$ be an exact sequence in $\mathfrak{Qcoh}(X)$. Given an affine open $U \subseteq X$, $0 \to \mathscr{T}(U) \to \mathscr{G}(U) \to \mathscr{F}(U) \to 0$ is also exact. Since $\mathscr{F}(U)$ is a flat $\mathcal{O}_X(U)$-module, we deduce from Proposition 3.3 that $\mathscr{F}$ is tensor flat.

If X is a semi-separated scheme, then the direct image functor $\imath_*$ for the inclusion map $\imath: U \hookrightarrow X$, where U is affine, is exact.

Let $\mathscr{F} \in \mathfrak{Qcoh}(X)$ be tensor flat. We need to show that $\mathscr{F}(U)$ is a flat $\mathcal{O}_X(U)$-module for each affine open subset $U \subseteq X$. Let

$$0 \longrightarrow A \longrightarrow B \longrightarrow \mathscr{F}(U) \longrightarrow 0$$

be an exact sequence of $\mathcal{O}(U)$-modules. By the previous observation, we have an exact sequence

$$0 \longrightarrow \imath_*(\tilde{A}) \longrightarrow \imath_*(\tilde{B}) \longrightarrow \imath_*(\widetilde{\mathscr{F}|_U}) \longrightarrow 0$$

If we take the pullback of the morphism $\imath_*(\tilde{B}) \to \imath_*(\widetilde{\mathscr{F}|_U})$ and the canonical morphism $\mathscr{F} \to \imath_*(\widetilde{\mathscr{F}|_U})$, we obtain the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \imath_*(\tilde{A}) & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{F} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \imath_*(\tilde{A}) & \longrightarrow & \imath_*(\tilde{B}) & \longrightarrow & \imath_*(\widetilde{\mathscr{F}|_U}) & \longrightarrow & 0 \end{array}$$

Since $\imath_*(\tilde{A})$ and $\mathscr{F}$ are quasi-coherent, $\mathscr{H}$ is quasi-coherent. By assumption, the first row is pure exact, so by Proposition 3.3 each image under affine open subset is pure exact. From this diagram, it can be deduced that $\mathscr{H}(U) \cong \imath_*(\tilde{B})(U) = B$. So the short exact sequence $0 \to A \to B \to \mathscr{F}(U) \to 0$ is pure and then $\mathscr{F}(U)$ is a flat $\mathcal{O}_X(U)$-module. □

Corollary 3.12. *Assume that $\mathfrak{Qcoh}(X)$ is locally finitely presented (for instance if X is concentrated). Then $\mathcal{F}\mathrm{lat}_{fp} \subseteq \mathcal{F}\mathrm{lat}_{\otimes}$. If X is semi-separated, then $\mathcal{F}\mathrm{lat}_{fp} \subseteq \mathcal{F}\mathrm{lat}_{\otimes} = \mathcal{F}\mathrm{lat}$.*

Proof. This follows from Proposition 3.9 and Proposition 3.11. □

Remark 3.13. The inclusions in Corollary 3.12 are strict. Namely, in [**8**, Corollary 4.6] it was shown that $\mathcal{F}\mathrm{lat}_{fp} = 0$ for the case in which $X = \boldsymbol{P}^n(R)$. In general there is a large class of projective schemes X such that $\mathcal{F}\mathrm{lat}_{fp} = 0$ in $\mathfrak{Qcoh}(X)$ (see [**8**, Theorem 4.4]).

4. Pure injective envelopes

Definition 4.1. A quasi-coherent $\mathcal{O}_X$-module $\mathscr{M}$ is said to be *fp-pure injective* (respectively, *pure injective*) if for every short exact sequence $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ in $\mathrm{Pure}_{\mathrm{fp}}$ (respectively, in Pure) the sequence $0 \to \mathrm{Hom}(\mathscr{H}, \mathscr{M}) \to \mathrm{Hom}(\mathscr{G}, \mathscr{M}) \to \mathrm{Hom}(\mathscr{F}, \mathscr{M}) \to 0$ is exact. We shall denote by $\mathcal{P}\mathrm{inj}_{fp}$ (respectively, by $\mathcal{P}\mathrm{inj}$) the class of all fp-pure injective quasi-coherent sheaves (respectively, the class of all pure injective quasi-coherent sheaves). In general, when we say that an $\mathcal{O}_X$-module is pure injective, we mean that it is 'injective' with respect to all pure exact sequences in $\mathcal{O}_X$-Mod.

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Remark 4.2.

- If X is concentrated, then, by Proposition 3.9, $\mathcal{P}\mathrm{inj} \subseteq \mathcal{P}\mathrm{inj}_{fp}$.
- Clearly, every injective quasi-coherent $\mathcal{O}_X$-module is both fp-pure injective and pure injective.

Theorem 4.3. *Let X be a concentrated scheme. Then every $\mathscr{M} \in \mathfrak{Qcoh}(X)$ admits an fp-pure injective envelope $\eta\colon \mathscr{M} \to \mathrm{PE}_{fp}(\mathscr{M})$. That is, $\mathcal{P}\mathrm{inj}_{fp}$ is enveloping.*

Moreover, the induced short exact sequence

$$0 \to \mathscr{M} \xrightarrow{\eta} \mathrm{PE}_{fp}(\mathscr{M}) \to \frac{\mathrm{PE}_{fp}(\mathscr{M})}{\mathscr{M}} \to 0$$

is in $\mathrm{Pure}_{\mathrm{fp}}$.

Proof. Since X is concentrated, $\mathfrak{Qcoh}(X)$ is a locally finitely presented Grothendieck category. So the result follows from [**15**, Theorem 6] (see also [**10**]). □

Now we will recall the definition of an internal Hom functor in $\mathfrak{Qcoh}(X)$ (X is an arbitrary scheme). The category $\mathfrak{Qcoh}(X)$ is Grothendieck abelian (see [**6**, Corollary 3.5] for the existence of a generator for $\mathfrak{Qcoh}(X)$) and the inclusion functor $\mathfrak{Qcoh}(X) \to \mathcal{O}_X$-Mod has a right adjoint functor C by the special adjoint functor theorem. This right adjoint functor is known in the literature as the coherator. The internal Hom functor is thus defined as $\mathscr{H}\mathrm{om}_{\mathrm{qc}}(\mathscr{F}, \mathscr{G}) = C\mathscr{H}\mathrm{om}(\mathscr{F}, \mathscr{G})$, where $\mathscr{H}\mathrm{om}(\cdot,\cdot)$ is the usual sheafhom functor. Therefore, $\mathfrak{Qcoh}(X)$ is a closed symmetric monoidal category with the usual tensor product and the $\mathscr{H}\mathrm{om}_{\mathrm{qc}}(\cdot,\cdot)$ a bifunctor, and there is a natural isomorphism

$$\mathrm{Hom}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}(\mathscr{F}, \mathscr{H}\mathrm{om}_{\mathrm{qc}}(\mathscr{G}, \mathscr{H})).$$

The unit object of the monoidal structure is given by $\mathcal{O}_X$. Thus, one gets a natural equivalence $\mathrm{Hom}(\mathcal{O}_X, \mathscr{H}\mathrm{om}_{\mathrm{qc}}(\cdot,\cdot)) \simeq \mathrm{Hom}(\cdot,\cdot)$, and so for each $\mathscr{F}, \mathscr{G} \in \mathfrak{Qcoh}(X)$ there is a bijection $\mathrm{Hom}(F, G) \cong \mathscr{H}\mathrm{om}_{\mathrm{qc}}(\mathscr{F}, \mathscr{G})(X)$.

Now, since $\mathcal{O}_X$-Mod is a Grothendieck category, it has injective envelopes. Let $\Lambda = \{\mathscr{S}_i \colon i \in I\}$ be a set of generators for $\mathcal{O}_X$-Mod (see, for example, [**28**, Corollary 6.8]). We pick an injective embedding

$$\bigoplus_{\Lambda, \mathscr{T}} \mathscr{S}_i/\mathscr{T} \to \mathscr{E},$$

where $\mathscr{E} \in \mathcal{O}_X$-Mod is injective and the sum also runs over all $\mathcal{O}_X$-submodules of each $\mathscr{S}_i$. Then it is clear that such an $\mathscr{E}$ is an *injective cogenerator* for $\mathcal{O}_X$-Mod. This is an injective $\mathcal{O}_X$-module with the property that for every non-zero $\mathscr{G} \in \mathcal{O}_X$-Mod there exists a non-zero morphism $\mathscr{G} \to \mathscr{E}$. Note that $C(\mathscr{E})$ is an injective cogenerator in $\mathfrak{Qcoh}(X)$. Indeed, the inclusion functor $\mathfrak{Qcoh}(X) \to \mathcal{O}_X$-Mod is an exact functor with right adjoint C.

We shall denote by $\mathscr{M}^\vee$ the *character $\mathcal{O}_X$-module* given by $\mathscr{M}^\vee = \mathscr{H}\mathrm{om}(\mathscr{M}, \mathscr{E})$. There is a canonical map $\mathrm{ev}\colon \mathscr{M} \to \mathscr{M}^{\vee\vee}$.

Proposition 4.4. *Given $\mathscr{M} \in \mathcal{O}_X$-Mod, the character $\mathcal{O}_X$-module $\mathscr{M}^\vee$ is pure injective in $\mathcal{O}_X$-Mod.*

Proof. Let $0 \to \mathscr{T} \to \mathscr{N} \to \mathscr{H} \to 0$ be a pure exact sequence in $\mathcal{O}_X$-Mod. Then

$$\operatorname{Hom}(\mathscr{N}, \mathscr{M}^\vee) \to \operatorname{Hom}(\mathscr{T}, \mathscr{M}^\vee) \to 0$$

is exact if and only if

$$\operatorname{Hom}(\mathscr{N} \otimes \mathscr{M}, \mathscr{E}) \to \operatorname{Hom}(\mathscr{T} \otimes \mathscr{M}, \mathscr{E}) \to 0$$

is exact. But the latter follows since $0 \to \mathscr{T} \otimes \mathscr{M} \to \mathscr{N} \otimes \mathscr{M}$ is exact and $\mathscr{E}$ is an injective cogenerator. □

Proposition 4.5. *A short exact sequence in $\mathcal{O}_X$-Mod,*

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{T} \to 0,$$

is pure exact if and only if

$$0 \to \mathscr{T}^\vee \to \mathscr{G}^\vee \to \mathscr{F}^\vee \to 0$$

splits.

Proof. The proof is the same as that in categories of modules (see, for example, [**7**, Proposition 5.3.8]). It is necessary to point out that in any Grothendieck category $\mathcal{C}$, with an injective cogenerator E, a sequence $0 \to M \to L \to N \to 0$ is exact if and only if $0 \to \operatorname{Hom}(N, E) \to \operatorname{Hom}(L, E) \to \operatorname{Hom}(M, E) \to 0$ is exact. □

Corollary 4.6. *For any $\mathscr{M} \in \mathcal{O}_X$-Mod the evaluation map $\mathrm{ev}\colon \mathscr{M} \to \mathscr{M}^{\vee\vee}$ is a pure monomorphism.*

Proof. First we will see that ev is injective. Let $0 \neq x \in \mathscr{M}(U)$ for some affine open U. Then there exists a non-zero $\mathcal{O}_X$-module $\mathscr{S}/\mathscr{T} \subseteq \mathscr{M}$, where $\mathscr{S} \in \Lambda$, with $x \in \mathscr{S}/\mathscr{T}$. By the definition of $\mathscr{E}$, there is a monomorphism $\alpha\colon \mathscr{S}/\mathscr{T} \to \mathscr{E}$ with $\alpha(x) \neq 0$. Then α extends to $\alpha'\colon \mathscr{M} \to \mathscr{E}$. And $\mathrm{ev}(x)(\alpha') = \alpha'(x) \neq 0$. So we are done. To show that $\mathrm{ev}\colon \mathscr{M} \to \mathscr{M}^{\vee\vee}$ is pure exact we need to show, by Proposition 4.5, that $\mathscr{M}^{\vee\vee\vee} \to \mathscr{M}^\vee$ admits a section, but $\mathrm{ev}^\vee\colon \mathscr{M}^\vee \to \mathscr{M}^{\vee\vee\vee}$ is such a section. □

Lemma 4.7. *Let $\mathscr{M}$ be a pure-injective $\mathcal{O}_X$-module. Then its coherator $C(\mathscr{M})$ is pure injective in $\mathfrak{Qcoh}(X)$ as well.*

Proof. Let $0 \to \mathscr{F} \to \mathscr{G}$ be a pure exact sequence in $\mathfrak{Qcoh}(X)$. This means that it is pure exact in $\mathcal{O}_X$-Mod. So we have an exact sequence

$$\operatorname{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathscr{G}, \mathscr{M}) \to \operatorname{Hom}_{\mathcal{O}_X\text{-Mod}}(\mathscr{F}, \mathscr{M}) \to 0.$$

Since (ι, C) is an adjoint pair, where $\iota\colon \mathfrak{Qcoh}(X) \hookrightarrow \mathcal{O}_X$-Mod, this implies that

$$\operatorname{Hom}_{\mathfrak{Qcoh}(X)}(\mathscr{G}, C(\mathscr{M})) \to \operatorname{Hom}_{\mathfrak{Qcoh}(X)}(\mathscr{F}, C(\mathscr{M})) \to 0$$

is exact. □

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Corollary 4.8. *Every quasi-coherent sheaf $\mathscr{M}$ can be purely embedded into a pure injective quasi-coherent sheaf. In particular, the class of pure injective quasi-coherent sheaves is pre-enveloping.*

Proof. Let $\mathscr{M}$ be a quasi-coherent sheaf. By Corollary 4.6, there is a pure monomorphism $\mathrm{ev}\colon \mathscr{M} \to \mathscr{M}^{\vee\vee}$, where $\mathscr{M}^{\vee\vee}$ is a pure injective $\mathcal{O}_X$-module. So we apply the coherator functor on $\mathscr{M}^{\vee\vee}$, $C(\mathscr{M}^{\vee\vee})$. By Lemma 4.7, it is a pure injective quasi-coherent sheaf. The adjoint pair (ι, C) allows us to factorize ev over $C(\mathscr{M}^{\vee\vee})$. Indeed, $\mathfrak{Qcoh}(X)$ is a coreflective subcategory of $\mathcal{O}_X$-Mod, and $\mathscr{M}$ is quasi-coherent. So there is a unique morphism $\varphi\colon \mathscr{M} \to C(\mathscr{M}^{\vee\vee})$ over which ev is factorized. Then φ is a pure monomorphism as well. □

In order to show that the class Pure in $\mathfrak{Qcoh}(X)$ is enveloping, we will apply [**29**, Theorem 2.3.8] (this, in turn, uses [**29**, Theorem 2.2.6]). The arguments in these proofs are categorical and can be easily extended to our setup in $\mathfrak{Qcoh}(X)$ by taking into account the following lemma.

Lemma 4.9. *For a given $\mathscr{M} \in \mathfrak{Qcoh}(X)$, the class of sequences in* Pure *of the form*

$$0 \to \mathscr{M} \to \mathscr{L} \to \mathscr{T} \to 0$$

varying $\mathscr{L}, \mathscr{T} \in \mathfrak{Qcoh}(X)$ is closed under direct limits.

Proof. The argument is local and so it can be deduced from the corresponding result on module categories (see, for example, [**29**, Proposition 2.3.7]). □

Combining Lemma 4.9 and Corollary 4.8, and applying the analogue to [**29**, Theorem 2.3.8] for the category $\mathfrak{Qcoh}(X)$, we obtain the following theorem.

Theorem 4.10. *Every $\mathscr{M} \in \mathfrak{Qcoh}(X)$ admits a pure injective envelope $\eta\colon \mathscr{M} \to \mathrm{PE}(\mathscr{M})$. That is, $\mathcal{P}$inj is enveloping.*

Moreover, the induced short exact sequence

$$0 \to \mathscr{M} \xrightarrow{\eta} \mathrm{PE}(\mathscr{M}) \to \frac{\mathrm{PE}(\mathscr{M})}{\mathscr{M}} \to 0$$

is in Pure.

5. Locally absolutely pure quasi-coherent sheaves and absolutely pure sheaves

An R-module A is *absolutely pure* (see [**17**]) if it is pure in every module containing it as a submodule. Absolutely pure modules are also studied with the terminology of FP-injectives (see [**26**]). It follows immediately from the definition that A is absolutely pure if and only if it is a pure submodule of some injective module. And therefore A is absolutely pure if and only if $\mathrm{Ext}^1_R(M, A) = 0$ for each finitely presented R-module M.

In this section we will study (locally) absolutely pure sheaves in both $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$. Since we have pure exact sequences in categories of sheaves rather than categorical ones, we deal with tensor purity to define absolutely pure sheaves in $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$.

Definition 5.1. Let $(X, \mathcal{O}_X)$ be a scheme.

(1) Let $\mathscr{F}$ be in $\mathcal{O}_X$-Mod. $\mathscr{F}$ is *absolutely pure* in $\mathcal{O}_X$-Mod if every exact sequence $0 \to \mathscr{F} \to \mathscr{G}$ in $\mathcal{O}_X$-Mod is pure exact in $\mathcal{O}_X$-Mod.

(2) Let $\mathscr{F}$ be a quasi-coherent sheaf on X. $\mathscr{F}$ is called *absolutely pure* in $\mathfrak{Qcoh}(X)$ if every exact sequence $0 \to \mathscr{F} \to \mathscr{G}$ in $\mathfrak{Qcoh}(X)$ is pure exact.

(3) Let $\mathscr{F}$ be a quasi-coherent sheaf on X. $\mathscr{F}$ is called *locally absolutely pure* if $\mathscr{F}(U)$ is absolutely pure over $\mathcal{O}_X(U)$ for every affine open $U \subseteq X$.

Lemma 5.2. *All these notions of locally absolute purity of quasi-coherent sheaves and absolute purity in $\mathcal{O}_X$-Mod and in $\mathfrak{Qcoh}(X)$ are closed under taking pure subobjects.*

Proof. It follows from the fact that if $f \circ g$ is a pure monomorphism with monomorphisms f and g, then g is a pure monomorphism. □

Lemma 5.3. *Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. The following are equivalent:*

(1) *$\mathscr{F}$ is absolutely pure in $\mathcal{O}_X$-Mod,*

(2) *$\mathscr{F}|_{U_i}$ is absolutely pure in $\mathcal{O}_X|_{U_i}$-Mod for a cover $\{U_i\}$ of X.*

Proof. (1) $\Longrightarrow$ (2) Let $U \subseteq X$ be open. Then the extension of $\mathscr{F}|_U$ to zero outside U, $j_!(\mathscr{F}|_U)$, is contained in $\mathscr{F}$. Since the stalk of $j_!(\mathscr{F}|_U)$ is $\mathscr{F}_x$ if $x \in U$ and 0 otherwise, $j_!(\mathscr{F}|_U)$ is a pure subsheaf of $\mathscr{F}$ in $\mathcal{O}_X$-Mod. So $j_!(\mathscr{F}|_U)$ is absolutely pure in $\mathcal{O}_X$-Mod, too.

Now let $\mathscr{G}$ be any $\mathcal{O}_X|_U$-module with an exact sequence $0 \to \mathscr{F}|_U \to \mathscr{G}$. Then $0 \to j_!(\mathscr{F}|_U) \to j_!(\mathscr{G})$ is still exact in $\mathcal{O}_X$-Mod. So it is pure in $\mathcal{O}_X$. But this means that $0 \to [j_!(\mathscr{F}|_U)]_x \to [j_!(\mathscr{G})]_x$ is pure for all $x \in X$. For $x \in U$ that exact sequence is equal to the exact sequence $0 \to (\mathscr{F}|_U)_x \to (\mathscr{G})_x$ and $j_!(\mathscr{F}|_U)|_U = \mathscr{F}|_U$ and $(j_!(\mathscr{G}))|_U = \mathscr{G}$. This proves the desired implication.

(2) $\Longrightarrow$ (1) Let $0 \to \mathscr{F} \to \mathscr{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. In order to show that it is pure exact, we need to show that the morphism induced on the stalk is pure exact for every $x \in X$. But the restriction functor to open subsets is left exact and $(\mathscr{F}|_U)_x = \mathscr{F}_x$. So the claim follows. □

Lemma 5.4. *Let $\mathscr{F}$ be an $\mathcal{O}_X$-module. If $\mathscr{F}_x$ is absolutely pure for all $x \in X$, then $\mathscr{F}$ is absolutely pure in $\mathcal{O}_X$-Mod.*

Proof. Let $0 \to \mathscr{F} \to \mathscr{G}$ be an exact sequence in $\mathcal{O}_X$-Mod. To be pure in $\mathcal{O}_X$-Mod is equivalent to being pure at the induced morphism on the stalk for every $x \in X$. So that proves our implication. □

Let $X = \operatorname{Spec}(R)$ be an affine scheme. The next proposition shows that in order to check that a quasi-coherent $\mathcal{O}_X$-module $\tilde{A}$ is absolutely pure, it suffices that its restrictions $\tilde{A}|_{D(s_i)}$, $i = 1, \ldots, n$, be absolutely pure, where $\bigcup_{i=1}^{n} D(s_i) = X$, and $s_1, \ldots, s_n \in R$.

Proposition 5.5. *Let R be a ring and let $s_1, s_2, \dots, s_n$ be a finite number of elements of R that generate the unit ideal. Let A be an R-module. If A_{s_i} is absolutely pure over R_{s_i} for every $i = 1, \dots, n$, then A is absolutely pure over R.*

Proof. Given $A \subseteq B$, we want to prove that the canonical morphism $M \otimes A \to M \otimes B$ is injective for every module M. Let $K = \mathrm{Ker}(M \otimes A \to M \otimes B)$. Then, by our hypothesis, we obtain $K_{s_i} = 0$ for each $i = 1, \dots, n$. So if $x \in K$, then ${s_i}^{h_i} x = 0$ for some $h_i \geqslant 0$. But the set $\{s_1, s_2, \dots, s_n\}$ generates R. So we have $s_1 t_1 + \dots + s_n t_n = 1$ for some $t_1, \dots, t_n \in R$. And also $(s_1 t_1 + \dots + s_n t_n)^h x = 0$ if $h > h_1 + \dots + h_n - 1$, i.e. $x = 1 \times x = 1^h x = 0$. □

Let $X = \mathrm{Spec}(R)$ be an affine scheme. Now we will see that in order to check that a quasi-coherent $\mathcal{O}_X$-module $\tilde{A}$ is absolutely pure, it suffices to check that for each $P \in X$ each stalk $\tilde{M}_P$ is an absolutely pure $\mathcal{O}_{X,P}$-module.

Proposition 5.6. *If A_P is absolutely pure over R_P for every prime ideal P, then A is absolutely pure over R.*

Proof. Let M be a finitely presented R-module. We want to prove that $\mathrm{Ext}^1_R(M, A) = 0$. Since M is finitely presented,

$$(\mathrm{Ext}^1_R(M, A))_P \cong \mathrm{Ext}^1_{R_P}(M_P, A_P) = 0.$$

Since this is true for each prime ideal P, $\mathrm{Ext}^1_R(M, A) = 0$. So A is absolutely pure. □

Neither Proposition 5.5 nor 5.6 assume any condition on the ring R. Their converses are not true in general. However, they are if R is coherent (see [**20**, Theorem 3.21]). So it makes sense to define a notion of locally absolutely pure quasi-coherent sheaves over a locally coherent scheme. A scheme $(X, \mathcal{O}_X)$ is *locally coherent* provided that $\mathcal{O}_X(U)$ is a coherent ring for each affine open subset $U \subseteq X$. Since coherence descends along faithfully flat morphisms of rings (see [**13**, Corollary 2.1]), it follows that X is locally coherent if and only if $\mathcal{O}_X(U_i)$ is coherent for each $i \in I$ of some affine open covering $\{U_I\}_{i \in I}$ of X. So, over a locally coherent scheme, the next proposition states that in order to prove whether a quasi-coherent sheaf is locally absolutely pure, it is sufficient to look at some cover by affine subsets of X. And these show that locally absolute purity is a stalkwise property.

Proposition 5.7. *Let $(X, \mathcal{O}_X)$ be a locally coherent scheme. Then the following conditions are equivalent for a quasi-coherent sheaf $\mathscr{F}$:*

(1) *$\mathscr{F}(U)$ is absolutely pure for every affine U,*

(2) *$\mathscr{F}(U_i)$ is absolutely pure for all $i \in I$ for some cover $\{U_i\}_{i \in I}$ of affine open subsets,*

(3) *$\mathscr{F}_x$ is absolutely pure for all $x \in X$.*

Proof. We just need to prove the implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1). By [**20**, Theorem 3.21], the localization of an absolutely pure module over a coherent ring is again absolutely pure, so the first implication follows. For the second, let $\mathscr{F}(U) \cong M$ for an $\mathcal{O}_X(U)$-module M. By assumption, $\mathscr{F}(U)_P \cong M_P$ is absolutely pure for every prime ideal P of $\mathcal{O}_X(U)$. Hence, $\mathscr{F}(U) = M$ is also absolutely pure by Proposition 5.6. □

The next lemma shows that the locally absolutely pure objects in $\mathfrak{Qcoh}(X)$ on a locally coherent scheme X are exactly the absolutely pure $\mathcal{O}_X$-modules that are quasi-coherent.

Lemma 5.8. *Let X be a locally coherent scheme and let $\mathscr{F}$ be a quasi-coherent sheaf. Then $\mathscr{F}$ is locally absolutely pure if and only if $\mathscr{F}$ is absolutely pure in $\mathcal{O}_X$-Mod.*

Proof. It follows by Lemma 5.3 and Proposition 5.7. □

At this point, we may consider the relation between absolutely pure quasi-coherent sheaves and locally absolutely pure quasi-coherent sheaves.

Lemma 5.9. *Let X be a locally coherent scheme. Every locally absolutely pure quasi-coherent sheaf is absolutely pure in $\mathfrak{Qcoh}(X)$.*

Proof. This follows from Proposition 5.7 and Proposition 3.4. □

The converse of Lemma 5.9 is not clear in general. But it is true if $X = \mathrm{Spec}(R)$ is affine and R is coherent, or if X is locally Noetherian. The first case is clear since $\mathfrak{Qcoh}(X) \cong \mathcal{O}_X(X)$-Mod. For the second, let $\mathscr{F}$ be absolutely pure in $\mathfrak{Qcoh}(X)$ and let $E(\mathscr{F})$ be its injective envelope in $\mathfrak{Qcoh}(X)$. Then $0 \to \mathscr{F} \to E(\mathscr{F})$ is pure exact. So, for each affine open subset $U \subseteq X$, $0 \to \mathscr{F}(U) \to E(\mathscr{F})(U)$ is pure exact in $\mathcal{O}_X(U)$-Mod. But $E(\mathscr{F})(U)$ is an injective $\mathcal{O}_X(U)$-module and $\mathscr{F}(U)$ is a pure submodule of it. Hence, $\mathscr{F}(U)$ is absolutely pure for each affine $U \subseteq X$. So, $\mathscr{F}$ is a locally absolutely pure quasi-coherent sheaf.

Proposition 5.10. *Let X be a locally coherent scheme. If the class of injective sheaves in $\mathcal{O}_X$-Mod is equal to the class of absolutely pure sheaves in $\mathcal{O}_X$-Mod, then X is a locally Noetherian scheme.*

Proof. Suppose that these classes are equal. Let M be an absolutely pure $\mathcal{O}_X(U)$-module, where U is an affine open subset. Then the sheaf $j_!(\tilde{M})$ obtained by extending $\tilde{M}$ to zero outside U is an absolutely pure $\mathcal{O}_X$-module by Lemma 5.4. By assumption, it is injective in $\mathcal{O}_X$-Mod. So, its restriction $(j_!(\tilde{M}))|_U = \tilde{M}$ is injective in $\mathcal{O}_X|_U$-Mod. Since $\tilde{M}$ is quasi-coherent, it is injective in $\mathfrak{Qcoh}(U)$, which implies that M is an injective $\mathcal{O}_X(U)$-module. So $\mathcal{O}_X(U)$ is a Noetherian ring and X is a locally Noetherian scheme. □

Now we will extend the known fact that a ring R is Noetherian if and only if each absolutely pure R-module is injective (see [**18**, Theorem 3]) for closed subschemes of $\mathbb{P}^n(R)$ that are locally coherent. Let R be a commutative ring and let $X = \mathbb{P}^n(R)$ be a projective scheme over R, where $n \in \mathbb{N}$. Then take a cover of X consisting of affine open

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subsets $D_+(x_i)$ for all $i = 0, \dots, n$ and all possible intersections. In this case, our cover contains basic open subsets of the form

$$D_+\biggl(\prod_{i\in v} x_i\biggr),$$

where $v \subseteq \{0, 1, \dots, n\}$. It is known that the category of quasi-coherent sheaves over a scheme is equivalent to the class of certain module representations over some quiver satisfying the cocycle condition (see [**6**]). In our case, the vertices of our quiver are all subsets of $\{0, 1, \dots, n\}$ and we have only one edge $v \to w$ for each $v \subseteq w \subseteq \{0, 1, \dots, n\}$ since $D_+(\prod_{i\in w} x_i) \subseteq D_+(\prod_{i\in v} x_i)$. Its ring representation has

$$\mathcal{O}_{\mathbb{P}^n(R)}\biggl(D_+\biggl(\prod_{i\in v} x_i\biggr)\biggr) = R[x_0, \dots, x_n]_{(\prod_{i\in v} x_i)}$$

on each vertex v, which is the subring of the localization $R[x_0, \dots, x_n]_{\prod_{i\in v} x_i}$ containing its degree zero elements. It is isomorphic to the polynomial ring on the ring R with the variables x_j/x_i, where $j = 0, \dots, n$ and $i \in v$. We denote this polynomial ring by $R[v]$. Then the representation $\mathscr{R}$ with respect to this quiver with relations is defined as $\mathscr{R}(v) = R[v]$ for each vertex v, and there is an edge $\mathscr{R}(v) \hookrightarrow \mathscr{R}(w)$ provided that $v \subseteq w$. Finally, a quasi-coherent sheaf $\mathscr{M}$ on $\mathfrak{Qcoh}(X)$ is uniquely determined by a compatible family of $\mathscr{R}(v)$-modules $\mathscr{M}(v)$ satisfying that

$$S_{vw}^{-1} f_{vw} \colon S_{vw}^{-1}\mathscr{M}(v) \to S_{vw}^{-1}\mathscr{M}(w) = \mathscr{M}(w)$$

is an isomorphism as $R[w]$-modules for each $f_{vw} \colon \mathscr{M}(v) \to \mathscr{M}(w)$, where S_{vw} is the multiplicative set generated by $\{x_j/x_i \mid j \in w \setminus v,\ i \in v\} \cup \{1\}$ and $v \subset w$.

Recall that a closed subscheme X of $\mathbb{P}^n(R)$ is given by a quasi-coherent sheaf of ideals, i.e. we have an ideal $I_v \subseteq R[v]$ for each v with $R[w] \otimes_{R[v]} I_v \cong I_w$ when $v \subseteq w$. This means that $I_v \to I_w$ is the localization by the same multiplicative set as above. But then $R[v]/I_v \to R[w]/I_w$ is also a localization. So, by abusing the notation, we shall also denote by $\mathscr{R}$ the structural sheaf of rings attached to X.

Proposition 5.11. *A closed subscheme $X \subseteq \mathbb{P}^n(R)$ that is locally coherent (for example, if $X = \mathbb{P}^n(R)$ and R is stably coherent) is locally Noetherian if and only if locally absolutely pure quasi-coherent sheaves are locally injective.*

Proof. The 'if' part is clear. Indeed, if a scheme is locally Noetherian, then all classes of locally absolutely pure, absolutely pure, locally injective and injective quasi-coherent sheaves are equal by [**14**, Chapter II, Proposition 7.17, Theorem 7.18].

For the 'only if' part, suppose that the class of locally injective and locally absolutely pure quasi-coherent sheaves are equal. As explained above, we deal with a cover $\{D_+(\prod_{i\in v} x_i)\}_{v\subseteq\{1,\dots,n\}}$ of basic affine open subsets of X since locally absolute purity is independent of choice of the base by Proposition 5.7. Let M be an absolutely pure $R[v]$-module for some $v \subseteq \{1, \dots, n\}$. By taking its direct image $\iota_*(\tilde{M})$, we obtain a locally

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absolutely pure quasi-coherent sheaf on X. Indeed, $\iota_*(\tilde{M})(D_+(\prod_{i\in w} x_i)) = S_{vw}^{-1}M(v)$ for $v \subseteq w$ is an absolutely pure $R[w]$-module by [**20**, Theorem 3.21], and

$$\iota_*(\tilde{M})\left(D_+\left(\prod_{i\in w} x_i\right)\right) = \tilde{M}\left(D_+\left(\prod_{i\in w} x_i\right) \cap D_+\left(\prod_{i\in v} x_i\right)\right)$$

as an $R[w]$-module for $v \nsubseteq w$. But

$$\tilde{M}\left(D_+\left(\prod_{i\in w} x_i\right) \cap D_+\left(\prod_{i\in v} x_i\right)\right) = S_{v(v\cup w)}^{-1}M(v)$$

is absolutely pure as an $R[(v\cup w)]$-module and, since $R[(v\cup w)] = S_{v(v\cup w)}^{-1}R[w]$, it is also absolutely pure as an $R[w]$-module by [**20**, Theorem 3.20]. By assumption, $\iota_*(\tilde{M})$ is locally injective, that is, $(\iota_*(\tilde{M}))(D_+(\prod_{i\in v} x_i)) = M$ is injective. So, $R[v]$ is Noetherian by [**18**, Theorem 3]. This implies that X is locally Noetherian. □

Note that the class of locally absolutely pure quasi-coherent sheaves over a locally coherent scheme is closed under direct limits and coproducts since absolutely pure modules over coherent rings are closed under direct limits [**20**, Proposition 2.4].

Theorem 5.12. *Let X be a locally coherent scheme. The class of locally absolutely pure quasi-coherent sheaves is a covering class.*

Proof. First, note that over a coherent ring a quotient of an absolutely pure module by a pure submodule is again absolutely pure [**20**, Proposition 4.2]. So, using that, we can say that a quotient of a locally absolutely pure by a pure quasi-coherent subsheaf is again locally absolutely pure.

Let λ be the cardinality of the scheme X, that is, the supremum of all cardinalities of $\mathcal{O}_X(U)$ for all affine open subsets $U \subseteq X$. By [**6**, Corollary 3.5], there is an infinite cardinal κ such that every quasi-coherent sheaf can be written as a sum of quasi-coherent subsheaves of type κ. In fact, every subsheaf with type κ of a quasi-coherent sheaf $\mathscr{F}$ can be embedded in a quasi-coherent subsheaf of type κ that is pure in $\mathscr{F}$. Let $\mathcal{S}$ be the set of locally absolutely pure quasi-coherent sheaves of type κ. By combining this with the fact that the class of locally absolutely pure quasi-coherent sheaves is closed under taking a quotient by a pure quasi-coherent sheaf, it follows that each locally absolutely pure quasi-coherent sheaf admits an $\mathcal{S}$-filtration. So, every locally absolutely pure sheaf is filtered by those of type κ.

On the other hand, since absolutely pure modules are closed under extensions and direct limits over a coherent ring, every quasi-coherent sheaf on a locally coherent scheme possessing an $\mathcal{S}$-filtration is also locally absolutely pure quasi-coherent. So, the class of locally absolutely pure quasi-coherent sheaves is equal to the class $\mathrm{Filt}(\mathcal{S})$ of all $\mathcal{S}$-filtered quasi-coherent sheaves. So, that class is precovering. Being closed under direct limits also implies that the class of locally absolutely pure quasi-coherent sheaves is covering. □

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