DYNAMIC HEDGING OF LONGEVITY RISK: THE EFFECT OF TRADING FREQUENCY

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ABSTRACT

This paper investigates dynamic hedging strategies for pension and annuity liabilities that are exposed to longevity risk. In particular, we consider a hedger who wishes to minimize the variance of her hedging error using index-based longevity-linked derivatives. To cope with the fact that liquidity of longevity-linked derivatives is still limited, we consider a liquidity constrained case where the hedger can only trade longevity-linked derivatives at a frequency lower than other assets. Time-consistent, closed-form solutions of optimal hedging strategies are obtained under a forward mortality framework. In the numerical illustration, we show that lowering the trading of the longevity-linked derivatives to a 2-year frequency only leads to a slight loss of the hedging performance. Moreover, even when the longevity-linked derivatives are traded at a very low (5-year) frequency, dynamic hedging strategies still significantly outperform the static one.

KEYWORDS

Dynamic hedging, longevity risk, minimum variance, forward mortality model.

1. INTRODUCTION

This paper considers the question of how to hedge a portfolio of liabilities that are exposed to longevity risk, i.e., the risk due to unanticipated changes in the life expectancy of populations. Longevity risk is becoming a global challenge to the pension and annuity industry. For example, estimates of the global amount of annuity and pension-related longevity risk exposure range from \$15 to \$25 trillion (CRO Forum, 2010; Biffs and Blake, 2014). In particular, pension plans and annuity providers face substantial risk of making payments longer than anticipated due to the ongoing increase of post-retirement life expectancy. As reported by the Basel Committee's Joint Forum in 2013,[1] if life expectancy of a typical defined benefit pension fund's members increases by 1 year, the present value of its liabilities would increase by 3% to 4%.

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A pension plan or an annuity provider (hereafter the hedger) can reduce her longevity risk exposure via capital markets, for example, by trading index-based longevity-linked derivatives. By taking a long position of an index-based contract, the hedger receives payments that increase with the realized survival rates in her portfolio, and can thus mitigate unexpected increases in her liabilities. The payments of a index-based derivative are linked to a mortality index, which can be the weighted average of the actual survival rates of one or more national populations. Compared to indemnity derivatives, with which the hedger receives payments linked to the actual survival rates in her own liabilities, index-based derivatives may be attractive to a broader range of hedge suppliers, have lower data requirements, and provide potentially quicker executions (Li and Luo, 2012).[2] Popular longevity-linked derivatives include longevity bonds, q-forwards, and survivor swaps (Menoncin, 2008; Dawson *et al.*, 2010). For a detailed description of the longevity-linked capital market, see, for example, Biffs and Blake (2014).

Despite the importance of longevity risk management, the existing studies focus mainly on static hedging strategies (Blake *et al.*, 2006; Li and Hardy, 2011; Li and Luo, 2012; Cairns *et al.*, 2014; Li *et al.*, 2017). Although static hedging seems a reasonable choice when indemnity derivatives are used, it may be sub-optimal for index-based derivatives. In the latter case, the hedging decisions may be affected by population basis risk, i.e., the mismatch of the mortality index and the actual survival rates in the hedger's liabilities. Therefore, the optimal hedging strategy depends on the estimated correlations between the payments from the derivatives and the actual pension or annuity payments from the hedger, and shall be adjusted when this estimated correlation is updated (when new observations of survival rates are available). Moreover, when implementing a dynamic hedging strategy, special attention shall be paid to the trading frequency of the longevity-linked derivatives. Due to the fact that the longevity-linked capital market is in its early stage of development, the liquidity of longevity-linked derivatives is so far rather limited (Biffs and Blake, 2014). Therefore, a practical dynamic strategy should take into account the fact that trading frequency of longevity-linked derivatives are lower than other traditional financial assets, such as stocks and bonds.

In this paper, we consider dynamic hedging of longevity risk with index-based longevity-linked derivatives under limited trading frequency. Following most existing studies on the dynamic hedging of longevity risk, we use a continuous-time stochastic mortality framework. Continuous-time mortality models are gaining greater attention during the past two decades. For example, Biffis (2005), Dahl *et al.* (2008), and Biffis *et al.* (2010) propose different specifications of jump-diffusion mortality process; Bauer *et al.* (2008), Bauer *et al.* (2012), Blackburn and Sherris (2013), and Blackburn and Sherris (2014) propose different variations under the forward mortality framework; Wong *et al.* (2014) and Wong *et al.* (2015) consider diffusion mortality processes that are co-integrated. Continuous-time frameworks serve as a convenient tool for dynamic analysis. For discrete-time mortality models, it is difficult to model the evolvement of the values of the longevity-linked derivatives and their correlations with the hedger's liabilities without resorting to nested simulation (Cairns, 2011).[3] Many existing studies incorporate population basis risk in dynamic hedging. However, only Dahl *et al.* (2011) take into account the limitation of trading frequency in their hedging strategies, where a quadratic loss function is used.

As the hedging objective, we consider the minimal variance criterion. Specifically, the hedger aims to minimize the variance of her hedging error, which is defined as the deviation of the market value of her investments (in longevity-linked instruments and other financial assets) from the market value of her liabilities, at a specific future valuation date.[4] The variance criterion is widely used by researchers and practitioners in static settings. In the dynamic setting, the variance criterion is also a useful objective, as it is more interpretable than utility functions. For example, we can measure the hedging quality by looking at the optimal variance of the hedging error instead of resorting to the functional form of the utility functions and the choice of risk aversion parameters. The latter is not always easy to specify for a pension plan or annuity provider.

This paper contributes to the literature by analyzing the effect of trading frequency in dynamic minimum-variance longevity hedging. In particular, we extend the algorithm considered in Basak and Chabakauri (2010, 2012) and Wong *et al.* (2014) to derive time-consistent optimal hedging strategies in a liquidity constraint case, where part of the assets can only be traded at a limited, deterministic frequency. Closed-form solutions are derived in both cases under a forward interest rate and mortality rate framework with reasonable parametric assumptions. Up to date, most of the continuous-time mortality models have not been fitted to actual mortality data, possibly due to their complicated structures. There are only few exceptions, for example, Bauer *et al.* (2008) and Blackburn and Sherris (2013) for single-population, and Blackburn and Sherris (2014) for a two-population setting. In this paper, we follow the parametric specification proposed by Blackburn and Sherris (2014), and evaluate the hedging effectiveness in a numerical illustration. In particular, a one-factor and three-factor Hull–White specification is used for the interest rate and the mortality process, respectively. We use parameter estimates in Driessen *et al.* (2003) for the interest rate model, and parameter estimates in Blackburn and Sherris (2014) for the mortality model. From the numerical study, we find that mild trading frequency constraints, such as a 2-year frequency, leads to only a slight loss (about 3.7%) of the hedging quality compared to continuous trading. Moreover, even under a very low frequency, such as a 5-year frequency, a dynamic hedge still leads to 23% lower hedging error than the static hedge.

The rest of the paper is organized as follows. Section 2 introduces the hedger's optimization problem. Section 3 describes the hedger's assets and liabilities. Section 4 gives the optimal hedging strategy to the liquidity constrained problem. Section 5 gives a numerical illustration of the hedging strategy and hedge effectiveness. Section 6 provides sensitivity analysis of the numerical study. Section 7 concludes.

2. The hedging problem

We consider a setting where the hedger implements a dynamic hedging strategy at time 0, and assesses the financial outcome of the hedging strategy at a specific future valuation date, T_0. For ease of exposition, assume that the hedger's portfolio consists of N male participants in a single cohort aged x_0 at year 0.[5] Starting from year T_0, each member receives a continuous payment of \$1 per year until he dies or a terminal date, T_1, is reached. The hedger can invest in a money market account, a set of zero-coupon bonds, and a set of zero coupon longevity bonds (hereafter longevity bond) contingent on cohort x_0.[6] A longevity bond is a zero-coupon bond with a random principle repayment depending on the actual cumulative survival probability of a certain cohort in the reference population at maturity.[7] Denote by $k \in \{rp, pp\}$ the set of populations, with rp the reference population and pp the hedger's portfolio population. The relevant notations are introduced in the following:

- ${}_sp(x, t, k)$: the (future) probability that an individual aged x at time t in population k survives up to time $t+s$, given that he is alive at time t. The survival probability, ${}_sp(x, t, k)$, can be estimated at time $t + s$, but is random beforehand.
- $B(t)$: the time t value of the money market account with $B(0) = 1$.
- $B(t, T)$: the time t price of the zero-coupon bond which pays \$1 at time T with $B(t) = B(t, t)$.
- $L(t, T, x)$: the time t price of the longevity bond contingent on cohort x, which pays ${}_Tp(x, 0, rp)$ dollars at time T.

In this paper, we assume that N is large enough, so the number of survivors in year t is closely approximated by $N \times {}_tp(x_0, 0, pp)$. In other words, we focus only on macro longevity risk. For the hedging objective, we minimize the variance of the hedging error, which is defined as the deviation of the market value of hedger's investments from the market value of her liabilities evaluated at T_0. This setting applies to providers of deferred annuities, and defined benefit pension plans which are close to the start of the decumulation phase. Cairns (2013) and Cairns *et al.* (2014) consider a similar hedging problem in a static setting. We assume that there are M zero-coupon bonds with different maturities $\mathcal{T} = \{\tau_1, \tau_2, \ldots, \tau_M\}$ available at any time t. However, due to the fact there are not yet many tradable longevity-linked derivatives in the market, we assume that there exists only $\tilde{M} \leq M$ longevity bonds available in the market, with $\tilde{\mathcal{T}} \subset \mathcal{T}$.

Consider a finite horizon T^* and a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.[8] $\mathcal{F}_0$ is augmented by all the P-null subsets of $\mathcal{F}$, and $\mathcal{F}_t$ is the σ-field generated by an $N = (n_r + n_\mu)$-dimensional standard Brownian motion under P, $W^P(t) = (W_r^P(t)', W_\mu^P(t)')'$. All stochastic processes are assumed to be well-defined and adapted to $\{\mathcal{F}_t, t \in [0, T^*]\}$. The evolvement of the observed interest rate and mortality rates is driven by the n_r-dimensional process $W_r^P(t)$ and the n_μ-dimensional process $W_\mu^P(t)$, respectively. Furthermore, we assume

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independence of $W_r^P(t)$ and $W_\mu^P(t)$ for all t, i.e., the evolvement of mortality rates is assumed to be independent of the financial markets. The money market account is given by

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1, \tag{2.1}$$

where $r(t)$ is the instantaneous spot rate at time t. We assume the existence of a risk neutral measure, Q^λ, with the money market account as the numéraire, by which prices are determined. The existence of Q^λ guarantees the exclusion of arbitrage opportunities. In particular, Q^λ is characterized by the Radon–Nikodym density

$$\left(\frac{dQ^\lambda}{dP}\right)_t = \exp\left(-\int_0^t \lambda(s)' dW(t) - \frac{1}{2}\int_0^t \|\lambda(s)\|^2 ds\right), \tag{2.2}$$

where $\lambda(t) = (\lambda_r(t)', \lambda_\mu(t)')'$ is the market price of risk vector with respect to Q^λ, satisfying appropriate regularity conditions (see, for example, Proposition 1.7.31 in Jeanblanc *et al.* (2009)). In this paper, we assume that the zero-coupon bond market is complete, so $M = n_r$. However, we allow for the situation where $\tilde{M} < n_\mu$, in which case the longevity bond market is incomplete.[9] For simplicity of notation, we write Q^λ as Q in the sequel. Let $W^Q(t) = (W_r^Q(t)', W_\mu^Q(t)')'$ be an N-dimensional process satisfying

$$dW^Q(t) = dW^P(t) + \lambda(t)dt. \tag{2.3}$$

By Girsanov's Theorem (Karatzas, 1991), $W^Q(t)$ is an N-dimensional Brownian motion under Q. We assume that $\lambda_r(t)$ only depends on information regarding the financial markets, and $\lambda_\mu(t)$ only depends on information regarding the mortality processes. As a result, $W_r^Q(t)$ and $W_\mu^Q(t)$ are independent for all t.

For simplicity of notation, define $Y(t)$ as the normalized time t discounted market value of the hedger's liabilities, i.e., the hedger's time t market value of liabilities is $N \times Y(t)$ dollars. For $t \in [0, T_0]$, $Y(t)$ can be written as

$$\begin{aligned} Y(t) &= {}_tp(x_0, 0, pp) E_t^Q \left[B(t, T_0) \int_{T_0}^{T_1} B(T_0, s)\, {}_{s-t}p(x_0 + t, t, pp) ds \right] \\ &= E_t^Q \left[\int_{T_0}^{T_1} B(t, s)\, {}_sp(x_0, 0, pp) ds \right]. \end{aligned} \tag{2.4}$$

The hedger's initial assets are assumed to be equal to her initial market value of liabilities, i.e., $w(0) = Y(0)$.[10] At time t, her wealth is

$$w(t) = u_0(t)B(t) + \boldsymbol{u}_1(t)'\boldsymbol{L}(t) + \boldsymbol{u}_2(t)'\boldsymbol{B}(t) \tag{2.5}$$

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with the self-financing budget constraint

$$dw(t) = u_0(t)dB(t) + \boldsymbol{u}_1(t)'d\boldsymbol{L}(t) + \boldsymbol{u}_2(t)'d\boldsymbol{B}(t). \tag{2.6}$$

In (2.5)–(2.6), $u_0(t)$, $\boldsymbol{u}_1(t)$, and $\boldsymbol{u}_2(t)$ are the hedger's holdings of money market account, longevity bond, and zero-coupon bond at time t, respectively. Moreover, $\boldsymbol{L}(t)$ and $\boldsymbol{B}(t)$ are the $\tilde{M}$- and M-dimensional vector containing the time t price of the longevity bonds and the zero-coupon bonds with maturities in $\tilde{\mathcal{T}}$ and $\mathcal{T}$, respectively. Given the above definition, the time T_0 hedging error of the hedger is $w(T_0) - Y(T_0)$, and the objective of the hedger can be formulated as

$$\min_{\boldsymbol{u}\in\mathcal{U}} \ \mathrm{Var}_0\left[e^{-\int_0^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))\right] \tag{2.7}$$

subject to the budget constraint (2.6), where $\boldsymbol{u}(t) = (\boldsymbol{u}_1(t)', \boldsymbol{u}_2(t)')'$. $\mathcal{U}$ is the set of admissible strategies, i.e., each $\boldsymbol{u} \in \mathcal{U}$ is $\mathcal{F}$-predictable, and satisfies standard integrability conditions. We minimize the variance of the discounted hedging error since, as mentioned in Basak and Chabakauri (2010), optimizing with the discounted value would significantly facilitate the derivation of the dynamic hedging strategies. We consider the hedging problem in two cases: In the benchmark case, both the zero-coupon bonds and the longevity bonds can be traded continuously; while in the liquidity constrained case, the longevity bonds can only be traded at a predetermined lower frequency.

3. Assets and liabilities

In this section, we describe the dynamics of the forward interest rate and mortality rates, and the hedger's assets and liabilities.

3.1. Forward interest rates and mortality rates

3.1.1. *Interest rates.* The instantaneous forward interest rate, $f(t, T)$, is defined as

$$f(t, T) = -\frac{\partial}{\partial T}\log\{B(t, T)\}, \tag{3.1}$$

and the short rate, $r(t)$, is given by $r(t) = f(t, t)$. Under the Q measure, $f(t, T)$ is assumed to follow

$$f(t, T) = f(0, T) + \int_0^t a_f(s, T)ds + \int_0^t \sigma_f(s, T)dW_r^Q(s), \quad t \leq T, \tag{3.2}$$

with a given initial continuous forward curve $f(0, T)$. The dynamics of $r(t)$ under Q is given by

$$r(t) = f(0, t) + \int_0^t a_f(s, t)ds + \int_0^t \sigma_f(s, t)dW_r^Q(s). \tag{3.3}$$

3.1.2. *Mortality rates.* Following Bauer *et al.* (2008) and Blackburn and Sherris (2013), denote the forward force of mortality as

$$\mu^Q(t, T, x_0, k) = -\frac{\partial}{\partial T}\log\left\{E_t^Q[{}_{T-t}p(x_0 + t, t, k)]\right\}, \tag{3.4}$$

where $E_t^Q[.] = E^Q[.|\mathcal{F}_t]$ is the time t conditional expectation under Q. Given (3.4), we have

$$E_t^Q[{}_{T-t}p(x_0 + t, t, k)] = \exp\left(-\int_t^T \mu^Q(t, s, x_0, k)ds\right). \tag{3.5}$$

Let $\mu^Q(t, T, x_0) = (\mu^Q(t, T, x_0, rp), \mu^Q(t, T, x_0, pp))'$. $\mu^Q(t, T, x_0)$ is assumed to follow

$$\begin{aligned}\mu^Q(t, T, x_0) =&\mu(0, T, x_0) + \int_0^t a_\mu(s, T, x_0)ds + \int_0^t \sigma_\mu(s, T, x_0)dW_\mu^Q(s),\\ \mu(0, T, x_0) >&0,\end{aligned} \tag{3.6}$$

where $t \to a_\mu(t, T, x_0)$ and $t \to \sigma_\mu(t, T, x_0)$ are by assumption a continuous and deterministic vector-valued and matrix-valued function, respectively. The vector of the spot force of mortality, $\hat{\mu}^Q(t, x_0)$, is given by

$$\hat{\mu}^Q(t, x_0) \equiv \mu^Q(t, t, x_0). \tag{3.7}$$

Heath *et al.* (1992) gives a parsimonious dynamic for $a_f(t, T)$ under the equivalent martingale measure

$$a_f(t, T) = \sigma_f(t, T)\int_t^T \sigma_f(t, s)'ds. \tag{3.8}$$

Following the same argument, we have (see Bauer and Ruß 2006)

$$a_\mu(t, T, x_0) = \sigma_\mu(t, T, x_0)\int_t^T \sigma_\mu(t, s, x_0)'ds. \tag{3.9}$$

Preferably, we should calibrate $f(t, T)$ and $\mu^Q(t, T, x_0)$ from existing data of market prices. However, at the current stage, there is no enough price data of longevity-linked derivatives to derive meaningful estimation. As an alternative, following Bauer *et al.* (2008), we model the *best estimated* force of mortality, $\mu(t, T, x_0)$, which are calibrated from historical mortality data. The best estimated forward force of mortality for each k is given by

$$\mu(t, T, x_0, k) = -\frac{\partial}{\partial T}\log\{E_t^P[{}_{T-t}p(x_0 + t, t, k)]\}, \quad k = rp, pp, \tag{3.10}$$

where $E_t^P[.] = E^P[.|\mathcal{F}_t]$ is the time t conditional expectation under the physical probability measure P. As shown in Bauer *et al.* (2008), the relation between $\mu(t, T, x_0)$ and $\mu^Q(t, T, x_0)$ is

$$\mu(t, T, x_0) = \mu^Q(t, T, x_0) + \int_t^T \sigma_\mu(s, T, x_0)\lambda_\mu(s)ds. \tag{3.11}$$

From (3.6) and (3.11), the dynamic of $\mu(t, T, x_0)$ is given by

$$\begin{aligned}\mu(t, T, x_0) &= \mu(0, T, x_0) + \int_t^T a_\mu(s, T, x_0)dt + \int_t^T \sigma_\mu(s, T, x_0)dW_\mu^P(t),\\ \mu(0, T, x_0) &> 0.\end{aligned} \tag{3.12}$$

Similarly, define the best estimated spot force of mortality, $\hat{\mu}(t, x_0)$, as

$$\hat{\mu}(t, x_0) \equiv \mu(t, t, x_0). \tag{3.13}$$

In this paper, we assume that the market price of mortality risk process, $(\lambda_\mu(t))_{t\geq 0}$, is deterministic for mathematical convenience. Consequently, the risk neutral expected survival probabilities can be written as deterministic functions of the best estimated survival probabilities (Bauer *et al.*, 2008)

$$E_t^Q[{}_{T-t}p(x_0 + t, t, k)] = e^{\int_t^T \int_t^s \sigma_\mu(u,s,x_0)\lambda(u)duds} E_t^P[{}_{T-t}p(x_0 + t, t, k)]. \tag{3.14}$$

As a result, if the values of $(\lambda_\mu(t))_{t\geq 0}$ are given, then we can price the longevity bond and the longevity contingent liabilities using the best estimated forward force of mortality, $\mu(t, T, x_0)$.

3.2. Assets and liabilities

3.2.1. *Zero coupon bond.* For any $T \in \mathcal{T}$, the time t price of the zero-coupon bond can be written as

$$B(t,T) = \exp\left(-\int_t^T f(t,s)ds\right), \tag{3.15}$$

with the dynamics under Q given by

$$dB(t,T) = B(t,T)r(t)dt + B(t,T)\sigma_B(t,T)dW_r^Q(t), \quad B(T,T) = 1, \tag{3.16}$$

where $\sigma_B(u,T) = -\int_u^T \sigma_f(u,s)ds$. The dynamics of $B(t,T)$ under P is

$$dB(t,T) = B(t,T)(r(t) + b_B(t,T))dt + B(t,T)\sigma_B(t,T)dW_r^P(t), \quad B(T,T) = 1, \tag{3.17}$$

where $b_B(u,T) = -\int_u^T \sigma_f(u,s)ds\lambda_r(u)$.

3.2.2. *Longevity bond.* Following Blake *et al.* (2006) and Menoncin (2008), we consider a zero-coupon longevity bond which pays ${}_Tp(x_0, 0, rp)$ at maturity T. Since we only consider longevity bonds contingent on cohort x_0, we simplify the notation by writing $L(t,T) = L(t,T,x_0)$ for all $T \in \tilde{\mathcal{T}}$. The price of the longevity bond at time t is given by

$$\begin{aligned} L(t,T) &= B(t,T)E_t^Q[{}_Tp(x_0,0,rp)] \\ &= B(t,T){}_tp(x_0,0,rp)E_t^Q[{}_{T-t}p(x_0+t,t,rp)]. \end{aligned} \tag{3.18}$$

The first equality holds due to independence of $W_r^Q(t)$ and $W_\mu^Q(t)$. The dynamics of the longevity bond is given by the following Proposition.

Proposition 1. *The price of the zero coupon longevity bond with maturity $T \in \tilde{\mathcal{T}}$ satisfies*

$$\frac{dL(t,T)}{L(t,T)} = \{r(t) + b_B(t,T) + b_L(t,T)\}dt + \sigma_B(t,T)dW_r^P(t) + \sigma_L(t,T)dW_\mu^P(t), \tag{3.19}$$

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with $L(T, T) = e^{-\int_0^T \hat{\mu}(s, x_0, rp)ds}$, *and*

$$\sigma_L(u, T) = -\int_u^T \sigma_{\mu,1}(u, s, x_0)ds$$

$$b_B(u, T) = -\left(\int_u^T \sigma_f(u, s)ds\right)\lambda_r(u)$$

$$b_L(u, T) = -\left(\int_u^T \sigma_\mu(u, s, x_0, 1)ds\right)\lambda_\mu(u), \tag{3.20}$$

where $\sigma_\mu(u, s, x_0, 1)$ *is the first row of* $\sigma_\mu(u, s, x_0)$.

Proof. See Appendix A. ■

From (3.19), we see that, compared to the dynamics of the zero-coupon bond, the longevity bond is in addition affected by the longevity risk premium, $b_L(t, T)$, and the shocks to the mortality processes, $W_\mu^P(t)$. As suggested by Blake *et al.* (2006), zero coupon longevity bonds with structures given in (3.18) could be attractive in practice, as they provide building blocks for tailor-made positions for hedgers.

3.2.3. *Hedger's assets and liabilities.* As described in Section 2, the hedger's time t wealth is given by

$$w(t) = u_0(t)B(t) + \boldsymbol{u}_1(t)'\boldsymbol{L}(t) + \boldsymbol{u}_2(t)'\boldsymbol{B}(t). \tag{3.21}$$

For any $\boldsymbol{u}(t) \in \mathcal{U}$, $w(t)$ satisfies the stochastic differential equation

$$dw(t) = \{w(t)r(t) + \boldsymbol{u}(t)'a_w(t)\}dt + \boldsymbol{u}(t)'\sigma_w(t)dW^P(t), \tag{3.22}$$

where $a_w(t)$ is a $\tilde{M} + M$-dimensional vector which contains $L(t, T)(b_B(t, T) + b_L(t, T))$ for $T \in \tilde{\mathcal{T}}$ in the first $\tilde{M}$ components, and $B(t, T)b_B(t, T)$ for $T \in \mathcal{T}$ in the last M component. $\sigma_w(t)$ is the $(\tilde{M} + M) \times (n_r + n_\mu)$ instantaneous covariance matrix containing the following four blocks:

- $\sigma_{LB}(t)$: the upper-left $\tilde{M} \times n_r$ block, which contains $1 \times n_r$ vectors, $L(t, T)\sigma_B(t, T)$, for $T \in \tilde{\mathcal{T}}$.
- $\sigma_B(t)$: the lower-left $M \times n_r$ block, which contains $1 \times n_r$ vectors, $B(t, T)\sigma_B(t, T)$, for $T \in \mathcal{T}$.
- $\sigma_L(t)$: the upper-right $\tilde{M} \times n_\mu$ block, which contains $1 \times n_\mu$ vectors, $L(t, T)\sigma_L(t, T)$, for $T \in \tilde{\mathcal{T}}$.
- the lower-right $\tilde{M} \times n_\mu$ zero sub-matrix.

We assume that $\sigma_B(t)$ is invertible for any t. Given the assumption that $M = n_r$, this simply means that there is no redundant zero-coupon bond in the market.

As discussed in Section 2, the time t discounted market value of pension liabilities, $Y(t)$, can be formulated as

$$
\begin{aligned}
Y(t) &= \int_{T_0}^{T_1} B(t,s) E_t^Q[{}_s p(x_0, 0, pp)]ds, \\
&= {}_t p(x_0, 0, pp) \int_{T_0}^{T_1} B(t,s) E_t^Q[{}_{s-t} p(x_0 + t, t, pp)]ds, \\
&= e^{-\int_0^t \hat{\mu}(\tau, x_0, pp)d\tau} \int_{T_0}^{T_1} B(t,s) E_t^Q[{}_{s-t} p(x_0 + t, t, pp)]ds. \qquad (3.23)
\end{aligned}
$$

4. The optimal strategy

In this section, we derive the optimal strategy in the liquidity constraint case, where the hedger can only trade the longevity bonds at fixed and deterministic times. As a benchmark, we first state the optimal hedging strategy when all bonds can be traded continuously. Then we show the optimal strategy in the liquidity constraint case, and compare two sets of optimal strategies.

4.1. The benchmark strategy

First, we state the optimal strategy in the benchmark case, where all bonds can be traded continuously. Denote by $X(t, x_0)$ the vector of state variables, which includes the spot interest rate and the spot mortality rate. In a multi-factor setting, i.e., when $n_r > 1$ or $n_\mu > 1$ holds, extra state variables are needed to make the spot interest rate and mortality rates Markovian (see, e.g., Inui and Kijima 1998). In our setup, $X(t, x_0)$ is an $n_r + 2n_\mu$-dimensional vector, containing, as components, the state variables

$$
\begin{aligned}
\eta_{i,r}(t) &= \int_0^t \sigma_{i,f}(s,t)\lambda_{i,r}(s)ds + \int_0^t \sigma_{i,f}(s,t)dW_{i,r}^P(s), \quad i = 1, 2, 3, \ldots, n_r, \\
\eta_{j1,\mu}(t) &= \int_0^t \sigma_{j,\mu}(s, t, x_0, rp)dW_{j,\mu}^P(s), \quad j = 1, 2, 3, \ldots, n_\mu, \\
\eta_{j2,\mu}(t) &= \int_0^t \sigma_{j,\mu}(s, t, x_0, pp)dW_{j,\mu}^P(s), \quad j = 1, 2, 3, \ldots, n_\mu, \qquad (4.1)
\end{aligned}
$$

where $\sigma_{i,f}(s,t)$, $\lambda_{i,r}(s)$ and $\sigma_{j,\mu}(s,t,x_0,k)$ are the ith entry of $\sigma_f(s,t)$ and $\lambda_r(s)$, and the jth entry of $\sigma_\mu(s,t,x_0,k)$, $k=rp,pp$, respectively. In this section, we simply write the dynamics of $X(t,x_0)$ under the physical measure P as

$$dX(t,x_0)=a_X(t,X_t,x_0)dt+\sigma_X(t,X_t,x_0)dW^P(t). \tag{4.2}$$

The concrete representation of the dynamics can be derived when specific parametric choices of $\sigma_f(s,t)$ and $\sigma_\mu(s,t,x_0)$ are made. An illustration is provided in the next section. For simplicity of notation, we omit the term x_0 in the drift and volatility terms of the state variables in the sequel.

In the benchmark case, the hedger's optimization problem is

$$\min_{\boldsymbol{u}\in\mathcal{U}}\mathrm{Var}_0\left[e^{-\int_0^{T_0}r(\tau)d\tau}(w(T_0)-Y(T_0))\right] \tag{4.3}$$

subject to the budget constraint (2.6). We follow the recursive approach proposed by Strotz (1956), Caplin and Leahy (2006), Basak and Chabakauri (2010, 2012), and Wong *et al.* (2014) to obtain time-consistent optimal solutions to the minimum-variance problem. In particular, the recursive formulation at time t is expressed as the expected future value of the variance plus an adjustment term, which is the time t variance of the expected terminal net asset value. For each $t\in[0,T_0]$, define

$$U_t\equiv\mathrm{Var}_t\left[-e^{-\int_t^{T_0}r(\tau)d\tau}(Y(T_0)-w(T_0))\right]. \tag{4.4}$$

Applying the law of total variance to (4.4) yields

$$U_t=E_t[U_{t+\epsilon}]+\mathrm{Var}_t[E_{t+\epsilon}[-e^{-\int_t^{T_0}r(\tau)d\tau}(Y(T_0)-w(T_0))]]. \tag{4.5}$$

The hedger minimizes (4.5) subject to the budget constraints (2.6) and (4.2) by backward induction. The time-consistent optimal hedging strategy can be obtained by applying Propositions 3.1 and 3.2 of Wong *et al.* (2014). The results are stated below.

Result 1. *Under the budget constraints* (2.6) *and* (4.2)*, the minimum-variance optimal strategy is given by*

$$\boldsymbol{u}^*(t)=-(\sigma_w(t)\sigma_w(t)')^{-1}\sigma_w(t)\sigma_X(t)'\frac{\partial G(t)}{\partial X(t)}, \tag{4.6}$$

where $G(t)$ has the representation[11]

$$G(t)=-E_t^{\tilde{Q}}[e^{-\int_t^{T_0}r(\tau)d\tau}Y(T_0)]. \tag{4.7}$$

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$\tilde{Q}$ is a probability measure with the Radon–Nikodym density w.r.t. P

$$\left(\frac{d\tilde{Q}}{dP}\right)_t = \exp\left(-\frac{1}{2}\int_0^t ||\lambda^{\tilde{Q}}(s)||^2 ds - \int_0^t \lambda^{\tilde{Q}}(s) dW^P(s)\right), \tag{4.8}$$

with

$$\lambda^{\tilde{Q}}(t) = -a_w(t)'(\sigma_w(t)\sigma_w(t)')^{-1}\sigma_w(t). \tag{4.9}$$

4.2. Optimal strategy under trading constraint

Now we look at the hedging strategies under a liquidity constraint, where the hedger can only trade the longevity bond at fixed and deterministic times $t \in \{t_0, t_1, \ldots, t_n\}$. In this case, following Dahl *et al.* (2011) and Ang *et al.* (2014), we assume that $\boldsymbol{B}(t)$ and $\boldsymbol{L}(t)$ are $\mathcal{F}$-adapted, i.e., the price of both the zero-coupon bonds and the longevity bonds are still observable at any time t. Moreover, the holdings of all assets are allowed to jump. Assume that the trading opportunity arrives at time t, then the hedger is able to rebalance u_0 and u_2 according to the self-financing constraint

$$0 = (u_0(t)-u_0(t-))B(t)+(\boldsymbol{u}_1(t)-\boldsymbol{u}_1(t-))'\boldsymbol{L}(t)+(\boldsymbol{u}_2(t)-\boldsymbol{u}_2(t-))'\boldsymbol{B}(t). \tag{4.10}$$

4.2.1. *The constrained optimal strategy.* Denote by $\hat{\boldsymbol{u}}(.) = (\hat{\boldsymbol{u}}_1(.)', \hat{\boldsymbol{u}}_2(.)')'$ the hedging strategy under the liquidity constraint, and $\hat{\mathcal{U}}$ the corresponding admissible set. In this case, the hedger solves the optimization problem

$$\min_{\hat{\boldsymbol{u}}\in\hat{\mathcal{U}}} \mathrm{Var}_0[e^{-\int_0^{T_0} r(\tau)d\tau}[w(T_0) - Y(T_0)]] \tag{4.11}$$

under constraints (2.6) and (4.2). Compared with other objective functions, such as the quadratic loss function studied in Dahl *et al.* (2011), special attention should be paid to obtain time-consistent solutions to the minimum-variance problem. As shown in Theorem 1, we develop a recursive approach to obtain time-consistent optimal strategy in the constrained case.

Denote by $\sigma_w(s,1)$ and $\sigma_w(s,2)$ the $\tilde{M}\times(n_r+n_\mu)$ and $M\times(n_r+n_\mu)$ matrix containing the upper $\tilde{M}$ and lower M rows of $\sigma_w(s)$, respectively. Moreover, denote by $a_w(s,1)$ and $a_w(s,2)$ the $\tilde{M}$ and M vector containing the first $\tilde{M}$ and last M components of $a_w(s)$. The constrained optimal hedging strategy is given in the following theorem.

Theorem 1. *Denote $G(t)$ by*

$$G(t) = \sum_{j=i+1}^{n} E_t^{\bar{Q}}\left[\int_{t_j}^{t_{j+1}} e^{-\int_t^s r(\tau)d\tau}\hat{\boldsymbol{u}}_1^*(t_j)a_w(s,1)ds\right] - E_t^{\bar{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau}Y_{T_0}]$$
$$+\hat{\boldsymbol{u}}_1^*(t_i)'E_t^{\bar{Q}}\left[\int_t^{t_{i+1}} e^{-\int_t^s r(\tau)d\tau}a_w(s,1)ds\right]$$
$$\equiv G_i(t) + \hat{\boldsymbol{u}}_1(t_i)'H_i(t), \tag{4.12}$$

for $t \in [t_i, t_{i+1})$ with $t_{n+1} \equiv T_0$, and $\bar{Q}$ the probability measure with Radon–Nikodym density w.r.t. P given by

$$\left(\frac{d\bar{Q}}{dP}\right)_t = \exp\left(-\frac{1}{2}\int_0^t ||\lambda^{\bar{Q}}(s)||^2 ds - \int_0^t \lambda^{\bar{Q}}(s)dW^P(s)\right), \tag{4.13}$$

with

$$\lambda^{\bar{Q}}(t) = -a_w(t,2)'(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2). \tag{4.14}$$

Moreover, denote the matrices $A_i(s)$ and $B_i(s)$ by

$$A_i(s) = \sigma_w(s,1)(I-\bar{\sigma}_w(s,2))\sigma_w(s,1)' + 2\sigma_w(s,1)(I-\bar{\sigma}_w(s,2))\sigma_X(s)'\frac{\partial H_i(s)}{\partial X(s)}$$
$$+\frac{\partial H_i(s)'}{\partial X(s)}\sigma_X(s)(I-\bar{\sigma}_w(s,2))\sigma_X(s)'\frac{\partial H_i(s)}{\partial X(s)},$$
$$B_i(s) = -(\sigma_w(s,1) + \frac{\partial H_i(s)'}{\partial X(s)}\sigma_X(s))(I-\bar{\sigma}_w(s,2))\frac{\partial G_i(s)}{\partial X(s)} \tag{4.15}$$

for $s \in [t_i, t_{i+1})$, $i = 1, 2, \cdots, n$. Assume that $E_{t_i}[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^s r(\tau)d\tau}A_i(s)ds]$ is invertible for all i. Under the budget constraints (3.21) *and* (4.2)*, the minimum-*

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variance optimal strategy to the problem (4.11) *is given by*

$$\hat{\boldsymbol{u}}_1^*(t) = \left(E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^{s} r(\tau)d\tau} A_i(s)ds \right] \right)^{-1}$$
$$\times E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^{s} r(\tau)d\tau} B_i(s)ds \right], \quad t \in [t_i, t_{i+1}).$$
$$\hat{\boldsymbol{u}}_2^*(t) = -(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_X(t)'\frac{\partial G(t)}{\partial X(t)}$$
$$-(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_w(t,1)'\boldsymbol{u}_1^*(t_i), \quad t \in [t_i, t_{i+1}) \quad (4.16)$$

for $0 \leq i \leq n$.

Proof. See Appendix. ■

The invertibility assumption in Theorem 1 can be satisfied under reasonable parameterization of the interest rate and mortality rate processes. An illustrating parametric specification is given in Section 5.

4.2.2. *Comparison with the benchmark case.* In this subsection, we compare the constrained optimal strategy in (4.16) with the benchmark optimal strategy given in (4.6).

4.2.3. *Optimal holding of zero-coupon bonds.* The benchmark optimal $\boldsymbol{u}_2^*$ given in (4.6) can be formulated as

$$\boldsymbol{u}_2^*(t) = -(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_X(t)'\frac{\partial G(t)}{\partial X(t)}$$
$$-(\sigma_w(t,2)\sigma_w(t,1)')^{-1}\sigma_w(t,2)\sigma_w(t,2)'\boldsymbol{u}_1^*(t), \quad (4.17)$$

with the $G(t)$ given in (4.7). Therefore, the structure of the optimal holding of zero-coupon bonds is similar in both cases. In fact, comparing (4.16) and (4.17), we see that the holding of the zero-coupon bonds hedges two parts of interest rates risks: one part from the hedger's liabilities under the hedge neutral measure (the first part in (4.17)), and an extra part induced by the holding of longevity bonds (the second part in (4.17)). However, the hedge neutral measure also depends on $\boldsymbol{u}_1$. For the benchmark case, we can write $G(t)$ in (4.7) as

$$G(t, \boldsymbol{u}_1) = E_t^{\bar{Q}}\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau} \boldsymbol{u}_1(s)' a_w(s,1)ds \right] - E_t^{\bar{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)], \quad (4.18)$$

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with $\bar{Q}$ defined in (4.13). When evaluated at the optimal $\boldsymbol{u}_1(t)$, $G(t, \boldsymbol{u}_1(t))$ becomes (By Feynman–Kac Theorem)

$$G(t, \boldsymbol{u}_1^*) = E_t^{\bar{Q}}\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau} \boldsymbol{u}_1^*(t)' a_w(s,1)ds\right] - E_t^{\bar{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)]$$
$$= - E_t^{\tilde{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)], \tag{4.19}$$

with $\tilde{Q}$ defined in Theorem 1. Similarly, for the constrained case, we can see from (4.12) that the constrained hedge neutral measure, denoted by $\tilde{Q}_c$, should satisfy the relation

$$E_t^{\tilde{Q}_c}[e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)]$$
$$= \sum_{j=i+1}^{n} E_t^{\bar{Q}}\left[\int_{t_j}^{t_{j+1}} e^{-\int_t^s r(\tau)d\tau} \hat{\boldsymbol{u}}_1^*(t_j) a_w(s,1)ds\right] - E_t^{\bar{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)]$$
$$+ \hat{\boldsymbol{u}}_1^*(t_i)' E_t^{\bar{Q}}\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau} a_w(s,1)ds\right] \tag{4.20}$$

for $i = 1, 2, \ldots, n$. From the above analysis, we see that the holding of the longevity bond affects the corresponding optimal holding of the zero-coupon bond via two channels: a direct channel by introducing extra interest rate risk; and an indirect channel by changing the hedge neutral measure.

4.2.4. *Optimal holding of longevity bonds.* The benchmark optimal $\boldsymbol{u}_1$ given in (4.6) can be formulated as

$$\boldsymbol{u}_1^*(t) = (\tilde{A}(t))^{-1} \tilde{B}(t), \tag{4.21}$$

with

$$\tilde{A}(t) = \sigma_w(s,1)(I - \bar{\sigma}_w(s,2))\sigma_w(s,1)',$$
$$\tilde{B}(t) = - \sigma_w(s,1)(I - \bar{\sigma}_w(s,2))\sigma_X(s)\frac{\partial G(s)}{\partial X(s)}. \tag{4.22}$$

Compare (4.21) with (4.16), we see that the structure of the optimal holding of longevity bonds is similar in both cases. Intuitively speaking, the benchmark optimal $\boldsymbol{u}_1$ minimizes the portfolio's *instantaneous* sensitivity with respect to the mortality risk under the *benchmark hedge neutral measure*, while the constrained optimal $\hat{\boldsymbol{u}}_1$ minimizes the portfolio's *expected accumulated* sensitivity with respect to the mortality risk in each $[t_i, t_{i+1})$ under the *constrained hedge neutral* *measure*. The difference between the benchmark $\tilde{A}$ and $\tilde{B}$ matrix and the constrained A and B matrix given in (4.15) comes from the difference in the hedge neutral measure. In particular, in both cases, the hedge neutral measure at time t depends on future optimal $\hat{\boldsymbol{u}}_1$. In the benchmark case, the future optimal $\hat{\boldsymbol{u}}_1$ is independent of the optimal $\hat{\boldsymbol{u}}_1$ at time t. However, for the constrained case, for every $t \in [t_i, t_{i+1})$, the future optimal $\hat{\boldsymbol{u}}_1$ up to time t_{i+1} depends on the current optimal $\hat{\boldsymbol{u}}_1$.

4.2.5. *The constrained optimal strategy: a special case.* We have derived the optimal strategy under the trading constraint of longevity bonds. However, from (4.15)–(4.16), we see that, for $i \leq n-1$, the constrained optimal $\hat{\boldsymbol{u}}_1$ at $t \in [t_i, t_{i+1})$ depends explicitly on the future optimal $\hat{\boldsymbol{u}}_1$ (through the term $G_i(t)$). As a result, in the application, we need to solve the optimal $\hat{\boldsymbol{u}}_1$ period by period. However, the optimal $\hat{\boldsymbol{u}}_1$ in each period is a rather complicated expression, which makes the application inconvenient. As an alternative, we consider the optimization problem under an additional constraint, which substantially increase the tractability of the optimal strategies.

The additional constraint is

$$E_t[W^*(T_0)] = E_t[W(T_0, \hat{\boldsymbol{u}})], \quad t \in [0, T_0], \tag{4.23}$$

where $W^*(T_0)$ is the T_0 wealth under the benchmark optimal strategy given in (4.6). The optimal strategy under this additional constraint is given by the next Proposition.

Proposition 2. *Under the budget constraints* (3.21)*,* (4.2)*, and* (4.23)*, the minimum-variance optimal strategy to the problem* (4.11) *is given by*

$$\begin{aligned}
\hat{\boldsymbol{u}}_1^*(t) &= \left(E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^{s} r(\tau)d\tau} A_i(s)ds \right] \right)^{-1} \\
&\quad \times E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^{s} r(\tau)d\tau} B_i(s)ds \right], \quad t \in [t_i, t_{i+1}), \\
\hat{\boldsymbol{u}}_2^*(t) &= -(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_X(t)' \frac{\partial G(t)}{\partial X(t)} \\
&\quad - (\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_w(t,1)'\boldsymbol{u}_1^*(t_i), \quad t \in [t_i, t_{i+1}),
\end{aligned} \tag{4.24}$$

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under the assumption that $E_{t_i}[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^{s} r(\tau)d\tau} A(s, t_i)ds]$ *is invertible for each* $i = 1, \ldots, n-1$. *The matrices* $A(s, t_i)$ *and* $B(s, t_i)$ *are given by*

$$\begin{aligned} A(s, t_i) =& \sigma_w(s, 1)(I - \bar{\sigma}_w(s, 2))\sigma_w(s, 1)', \\ B(s, t_i) =& -\sigma_w(s, 1)(I - \bar{\sigma}_w(s, 2))\sigma_X(s)\frac{\partial G(s)}{\partial X(s)} \end{aligned} \tag{4.25}$$

for all i. *The* $G(t)$ *is given in* (4.7).

Proof. The proof follows Theorem 1 directly, with the $G_i(t)$ replaced by $G(t)$ for every i. ■

Compare (4.25) with (4.22), we see that the constrained optimal $\hat{\boldsymbol{u}}_1$ in this case looks more similar to the benchmark optimal $\hat{\boldsymbol{u}}_1$. Specifically, the constrained optimal $\hat{\boldsymbol{u}}_1$ now only minimizes the portfolio's expected accumulated sensitivity with respect to the mortality risk, without changing the hedge neutral measure. Indeed, under the additional constraint (4.23), we have

$$\begin{aligned} G(i, t) =& E_t\left[\int_{t_j}^{t_{j+1}} e^{-\int_t^s r(\tau)d\tau}\hat{\boldsymbol{u}}^*(s)a_w(s)ds\right] - E_t[e^{-\int_t^{T_0} r(\tau)d\tau} Y_{T_0}], \\ =& E_t\left[\int_{t_j}^{t_{j+1}} e^{-\int_t^s r(\tau)d\tau}\boldsymbol{u}^*(s)a_w(s)ds\right] - E_t[e^{-\int_t^{T_0} r(\tau)d\tau} Y_{T_0}], \\ =& G(t). \end{aligned} \tag{4.26}$$

In other words, under the additional constraint (4.23), the constrained optimal strategy depends on the benchmark hedge neutral measure, and thus the optimal $\hat{\boldsymbol{u}}_1$ does not depend on future $\hat{\boldsymbol{u}}_1$-s anymore.

5. Numerical evaluation of the optimal hedging strategies

In Section 4, we derived optimal hedging strategies using a general HJM framework. In this section, we give a numerical illustration of the optimization problem considered using a specific parameterization of the forward interest rate and mortality rate processes.

5.1. Parametrization

As the focus of this paper is the hedging of longevity risk, we impose a simplified structure for the interest rates. In particular, we define

$$\sigma_f(t, T) = \beta e^{\kappa(T-t)}, \tag{5.1}$$

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i.e., we consider a one-factor Hull–White specification of the interest rate process. Moreover, we consider, for cohort x_0 at time 0, the mortality process as

$$\sigma_\mu(t, T, x_0) = \begin{pmatrix} c_{11}(x_0)e^{\omega_{11}(T-t)} & c_{12}(x_0)e^{\omega_{12}(T-t)} & c_{13}(x_0)e^{\omega_{13}(T-t)} \\ c_{21}(x_0)e^{\omega_{21}(T-t)} & c_{22}(x_0)e^{\omega_{22}(T-t)} & c_{23}(x_0)e^{\omega_{23}(T-t)} \end{pmatrix}, \quad (5.2)$$

where $c_{kj}(x_0) = c_{kj}e^{\omega_{kj}x_0}$, and c_{kj}'s and ω_{kj}'s are parameters to be determined. The derivation of u^* in (4.6) and (4.16) under the specifications (5.1) and (5.2) is rather lengthy, and is available from the author upon request.

5.2. Choice of parameter values

For the one-factor Hull–White interest rate model, we use the estimation results from Driessen *et al.* (2003), who estimate a one-factor Hull–White model from U.S. interest rate data: $\beta = 0.0095$ and $\kappa = -0.009$. For the mortality model, we use the estimation results of the dependent factor model in Blackburn and Sherris (2014), who apply a three-factor two population model to Australian and Swedish males data. The parameter values are given by

$$(\omega_{11}, \omega_{12}, \omega_{13}) = (\omega_{21}, \omega_{22}, \omega_{23}) = (0.1246, 0.08366, 0.1714), \quad (5.3)$$

and

$$\begin{aligned}(c_{11}, c_{12}, c_{13}) &= (0.6657, 2.238, 0.01095) \times 10^{-4}, \\ (c_{21}, c_{22}, c_{23}) &= (0.3435, 3.469, 0.01095) \times 10^{-4}. \quad (5.4)\end{aligned}$$

The third factor has the same effect to both populations, and the time sensitivity parameters, ω_{ij}-s, are population neutral.

Besides the specified parameter values, initial forward curves are also needed to generate future interest and mortality rates. For the interest rate, we generate the forward curve based on the yield curve on September 30, 2014, reported by the U.S. Department of the treasury.[12] For the mortality rates, we generate the initial forward curve using the Lee-Carter 1992 model and mortality data downloaded from the Human Mortality Database.[13] The age groups and sample period chosen to generate the initial forward curve are the ages 21 to 100 and the years 1965 to 2009, respectively. Finally, we use a constant vector of price of risk: $\lambda = (0.05333, 0.3008, 0.2898, 0.2788)'$.[14]

5.3. Optimal hedging strategies

For illustration purpose, we consider the case where the hedger trades only one zero-coupon bond and one longevity bond. In particular, we assume that $\mathcal{T} = \tilde{\mathcal{T}} = \{T_1\}$. In other words, both bonds mature at the same date when the last possible payment in the hedger's liabilities is made. We evaluate the performance of four strategies: the benchmark strategy, a constrained strategy, a static strategy, and an interest-only strategy. The static strategy is derived from

TABLE 1

THE TRADING FREQUENCY OF ZERO-COUPON BOND AND LONGEVITY BOND FOR THE FOUR STRATEGIES.

Strategy	Benchmark	Constrained	Static	Interest Only
Zero-Coupon Bond	Week	Week	Week	Week
Longevity Bond	Week	1/2/5 Year	Buy and Hold	No Trading

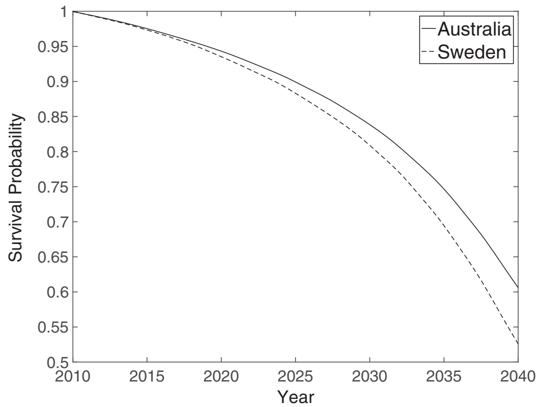

FIGURE 1: The survival probability for the males cohort aged 55 at year 2009.

Proposition 2 with $n = 1$ and $t_1 = 0$. For each strategy, the zero-coupon bond is assumed to be trade weekly, while the longevity bond is assumed to have different trading frequencies as shown in Table 1. Moreover, for the liquidity constraint case, we use the optimal strategy derived in the special case.

The interest-only strategy, where the hedger does not trade the longevity bond at all, is a special case of the benchmark strategy. The optimal interest-only strategy is given by the next proposition.

Proposition 3. *The optimal strategy in the case where the hedger only trades zero-coupon bonds have the same forms as in* (4.6) *with* $u_1(t) = 0$ *and* $\sigma_w(t) = (B(t, T)\sigma_B(t, T), 0_{n_\mu})$ *for* $t \in [0, T_0]$.

For the portfolio, we consider payments to the cohort that is aged 55 in the year 2009 and start 10 years from now. In other words, we let $x_0 = 55$ and $T_0 = 10$. Moreover, we let $T_1 = 30$, i.e., the last possible payment is made when the cohort reaches age 85. Figure 1 reports the best estimated survival probabilities of cohort x_0 for both populations. We see that the estimated survival probabilities are different for each population, which shows the presence of the population basis risk.

We generate 1,000 paths of the state variables, and compute corresponding realizations of the four optimal strategies.[15] The mean optimal strategies for the first three cases are reported in Figure 2, where the left column displays the

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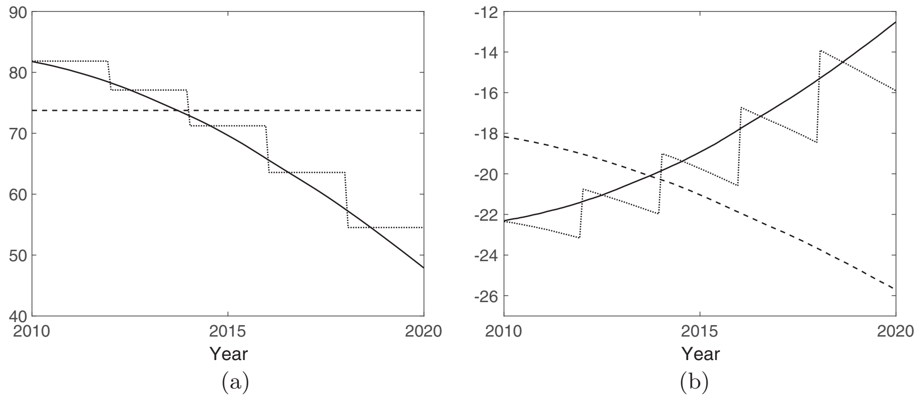

FIGURE 2: The mean optimal strategies for the benchmark case (full line), the constrained case (2-year frequency, dotted line), and the static case (dashed line). (a) Longevity bond. (b) Zero-coupon bond.

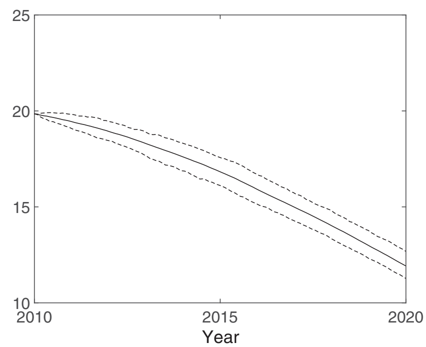

FIGURE 3: The mean, 5% and 95% quantiles of the optimal holding of the zero-coupon bond in the interest-only case.

optimal holdings of the longevity bond, and the right column reports the optimal holding of the zero-coupon bond. For the constrained case, we only report the optimal holdings for the 2-year frequency, since the patterns for the 1- and 5-year-frequency strategy are very similar. In each figure, the solid line, the dashed line, and the dotted line represents the optimal strategy for the benchmark case, the constrained case, and the static case, respectively. Moreover, the mean optimal strategy, as well as the 5% and 95% quantiles, in the interest-only case is reported in Figure 3. We see that, first, for the two dynamic strategies, the optimal holding of the longevity bond is decreasing in time. This observation is intuitive, since as the dates of payment become close, the mortality rates affecting the liabilities become less uncertain. As a result, the longevity risk exposure in the liabilities is decreasing over time, and the hedger would gradually reduce her holding of the longevity bond.

For the zero-coupon bond, the patterns are more complicated. The optimal holding of the zero-coupon bond is increasing in the benchmark case, decreasing in the static case and the interest-only case, and has a decreasing trend with upward jumps in the constrained case. These observations result from the interaction of two opposite forces. First, the duration of the liabilities decreases over time, and so does the interest rate exposure in the liabilities. This effect pushes the hedger to reduce her holding of zero-coupon bonds. Second, the holding of the longevity bond introduces extra interest rate risk, which needs to be hedged by holding zero-coupon bond in the opposite direction. As can be seen in the first three cases, the large holding of the longevity bonds at time 0 introduces extra interest rate risk in the opposite direction, which requires the hedger to short the zero-coupon bond. If the holding of the longevity bonds is unchanged over time, as in the static case and the intervals in the constrained case, the hedger would even short more zero-coupon bonds as the duration of her liabilities decreases. However, if the holding of the longevity bond decreases over time, as in the benchmark case and the trading points of the longevity bond in the constrained case, the hedger would reduce her holdings (in absolute term) of the zero-coupon bond correspondingly.

Besides the mean optimal strategies, we also investigate how sensitive the optimal strategies are to random realizations of the state variables. The 5% and 95% quantiles of the benchmark, the constrained, and the static optimal strategies are reported from top to bottom in Figure 4. In each row, the left column reports the optimal holdings of the longevity bond, and the right column reports the optimal holdings of the zero-coupon bond. Together with Figure 3, we see that the confidence intervals of the optimal strategies in all cases are small. Therefore, it seems that the optimal hedging strategies are insensitive to the noise in the state variables.

Finally, we evaluate the performance of different hedging strategies. In particular, the time 0 standard deviation of the hedger's hedging error under the benchmark strategy is 0.0160. This number can be interpreted using the following example. Assume that a male receives annual pension payment of $46,000 after he retires,[16] then the time 0 standard deviation of the hedger's hedging error corresponding to one pension member is $736, given that she follows the benchmark hedging strategy. In Table 2, we report the ratio of the time 0 standard deviation under the other strategies to the standard deviation of the benchmark strategy.[17] Continuing the above example, the time 0 standard deviation would be about 3.7%, 18%, 53% higher if she could only trade the longevity bond on a 2-year or 5-year or 10-year (static) frequency, respectively. Finally, the standard deviation would be about 7.44 times higher if the hedger does not hedge longevity risk at all.

From the numerical study, we see that lowering the trading frequency of the longevity bond from weekly to a 2-year frequency only leads to a slight decrease of the hedging quality. However, compared with dynamic hedging strategies, even the constrained one, a static hedging strategy would significantly increase the hedger's hedging error. Realistic pension/annuity liabilities typically involve longer planning and payment horizons with more heterogenous portfolios. As

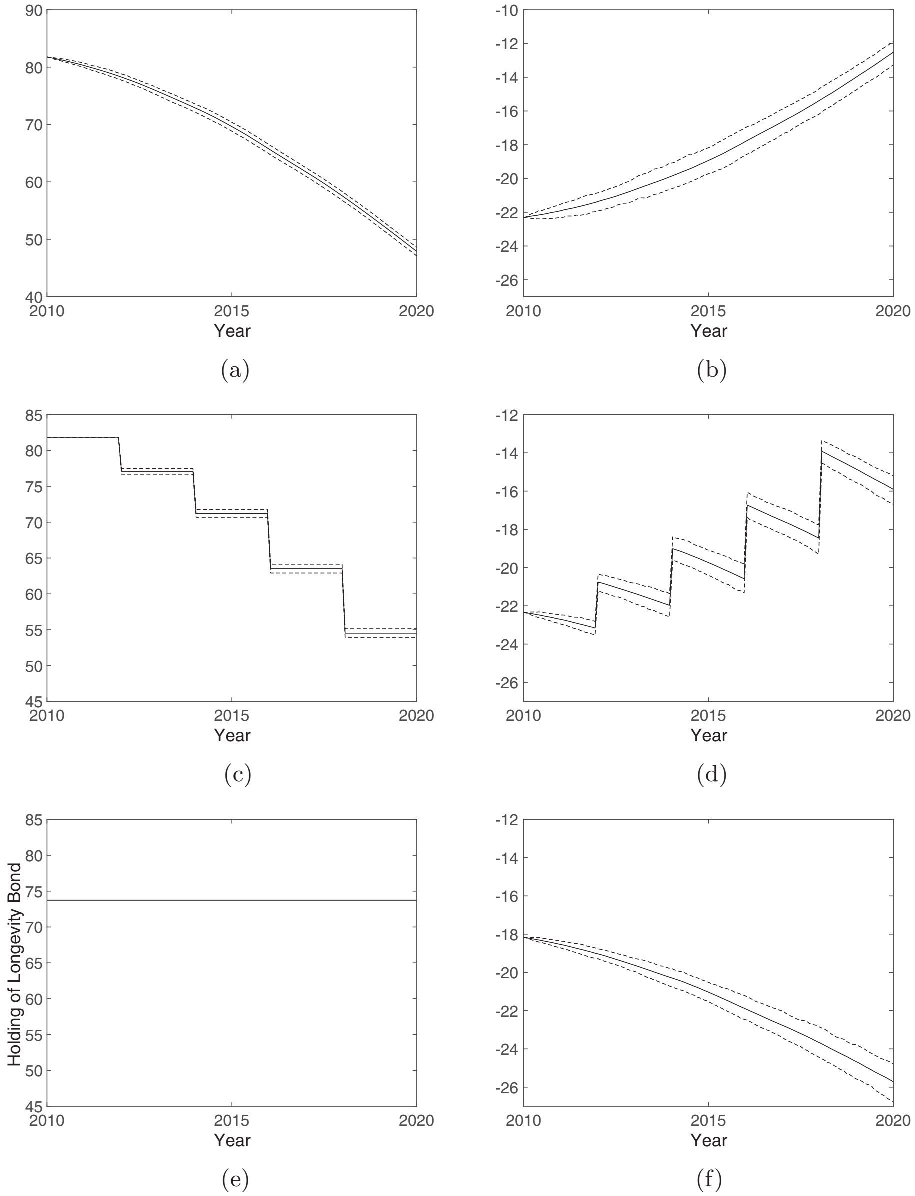

FIGURE 4: The mean, 5% and 95% quantiles of the optimal strategies for the benchmark case, the constrained case (2-year frequency), and the static case (from top to bottom). In each row, the left and right figure displays the optimal holding of the longevity bond and the zero-coupon bond, respectively. (a) u_1^* (benchmark). (b) u_2^* (benchmark). (c) u_1^* (2-year frequency). (d) u_2^* (2-year frequency). (e) u_1^* (static). (f) u_2^* (static).

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TABLE 2

THE RATIO OF THE TIME 0 STANDARD DEVIATION OF THE OPTIMAL HEDGING ERROR RELATIVE TO THE STD. OF THE BENCHMARK CASE. IN EACH CASE, THE RATIO EQUALS $(\text{VAR}_t[e^{-\int_0^{T_0} r(\tau)d\tau}(Y(T_0) - w_s^*(T_0))]/\text{VAR}_t[e^{-\int_0^{T_0} r(\tau)d\tau}(w_b^*(T_0) - Y(T_0))])$, WHERE $w_b^*(T_0)$ AND $w_s^*(T_0)$ ARE THE TIME T_0 ASSETS UNDER THE BENCHMARK STRATEGY AND THE CORRESPONDING STRATEGY, RESPECTIVELY.

	1-Year	2-Year	5-Year	Static	Interest Only
Ratio of the Std.	101.1%	103.7%	118.3%	153.5%	743.9%

a result, the effect of the trading frequency, especially the difference between the constraint frequency and the static trading, is likely to be more larger in practice.

6. SENSITIVITY ANALYSIS

The numerical results in Section 5 depends on assumptions regarding, for example, risk premiums, maturity of the longevity bond, and the cohort in the hedger's portfolio. In this section, we evaluate the sensitivity of the hedging performance with respect to these assumptions.

6.1. Population basis risk

In Section 5, we evaluate the performance of the hedging strategy in the presence of population basis risk. Now we look at the hedging performance of the strategies when both the hedger's portfolio and the reference population are driven by the same mortality process. In particular, we let both the reference population and the portfolio-specific population be the Australian male population, and display the corresponding results in Table 3. The results where both the reference population and the portfolio-specific population is the Swedish male population are similar, and are thus omitted. We see that, the optimal standard deviation in the benchmark case is much smaller in the absence of population basis risk. Moreover, the ratio of the standard deviations increase drastically when trading frequency of the longevity bond decreases. The reason is that the hedging error resulted from the population basis risk is relatively invariant to the trading frequency of the longevity bond. Therefore, the existence of the population basis risk substantially lowers the ratios of the standard deviations. In the situation without population basis risk, the hedging strategies become much more effective, and the effect of trading frequency becomes much more profound.

6.2. Longevity risk premium

In the above analysis, we used the calibrated values of the price of longevity risk in Bauer *et al.* (2010). At the current stage, there exists several methods to calibrate the price of longevity risk from existing market data, such as the the Wang transform (Wang, 2002; Lin and Cox, 2005, 2008), and the Sharpe Ratio

TABLE 3

THE TIME 0 RATIO OF STD. OF THE HEDGING ERROR WITH AND WITHOUT POPULATION BASIS RISK.

	Benchmark	1-Year	2-Year	5-Year	Static	Interest Only
W. p. Risk	0.0160	101.1%	103.7%	118.3%	153.5%	743.9%
W.o. p. Risk	0.0016	145.2%	217.8%	450.8%	821.12%	5307.8%

TABLE 4

THE TIME 0 RATIO OF STD. OF THE HEDGING ERROR UNDER DIFFERENT LONGEVITY RISK PREMIUMS. IN PARTICULAR, $\tilde{\lambda}_{\mu,1} = \frac{1}{4}\lambda_\mu$ AND $\tilde{\lambda}_{\mu,2} = 4\lambda_\mu$.

	Benchmark	1-Year	2-Year	5-Year	Static	Interest Only
$\tilde{\lambda}_{\mu,1}$	0.0159	101.07%	103.65%	118.04%	152.95%	744.4%
λ_μ	0.0160	101.09%	103.70%	118.25%	153.54%	743.9%
$\tilde{\lambda}_{\mu,2}$	0.0166	101.18%	103.96%	119.30%	156.40%	743.6%

method (Cairns *et al.*, 2005; Bayraktar *et al.*, 2009). However, as shown in Bauer *et al.* (2010), when calibrated to the UK annuity quote data, the above methods yield very different risk premiums. Therefore, it is important that the results produced by our model are robust to the values of the price of the longevity risk.

In order to examine the sensitivity of our hedging strategy to the choice of longevity risk premiums, we evaluate the performance of the hedging strategies under two alternative sets of λ_μ-s: $\tilde{\lambda}_{\mu,1} = \frac{1}{4}\lambda_\mu$ and $\tilde{\lambda}_{\mu,2} = 4\lambda_\mu$. The time 0 optimal standard error of the hedger's hedging error in the benchmark case, as well as the ratios under the other strategies are reported in Table 4. We see that, first, the value function and the ratios are robust to the change of the longevity risk premium. For example, the time 0 optimal standard error in the benchmark case changes by only 4.4% when the longevity risk premium becomes 16 times larger. Second, though only slightly, the value function of the benchmark case increases with the longevity risk premium. The intuition is that, as the longevity risk premium increases, the variability of the market value of the hedger's liabilities increases. Moreover, the ratios of the standard deviations of other strategies change in the same direction as the longevity risk premium. Therefore, it seems that the time 0 standard deviation of the hedger's hedging error is more sensitive to the longevity risk premium for the constrained strategies. Finally, the time 0 standard deviation generated by the interest-only strategy is not affected by the change of the longevity risk premium, thus the ratio in the interest-only case decreases slightly as the longevity risk premium increases.

6.3. The undiscounted hedging error

In the above analysis, we consider the hedger's discounted hedging error. Here, we calculate the standard deviations of the hedger's undiscounted hedging error

TABLE 5

THE TIME 0 RATIO OF STD. OF THE DISCOUNTED AND UNDISCOUNTED HEDGING ERROR.

	Benchmark	1-year	2-year	5-year	Static	Interest Only
Dis.	0.0160	101.1%	103.7%	118.2%	153.5%	743.9%
Undis.	0.0192	101.1%	103.7%	118.2%	153.6%	743.9%

under the optimal hedging strategies derived in this section. In other words, we compute

$$\mathrm{Var}_0[w^*(T_0) - Y(T_0)], \tag{6.1}$$

where $w^*(T_0)$ is the hedger's T_0 wealth under the optimal strategy.

In Table 5, we report the optimal standard deviation from the benchmark case, as well as the ratios of the standard deviation, for both the hedger's discounted and undiscounted hedging error. First, we see that the standard deviation in the undiscounted case is larger than the standard deviation in the discounted case. Second, the ratios of the standard deviation are very similar in these two cases. Therefore, although the optimal strategies we derive is to minimize the standard deviation (variance) of the hedger's discounted hedging error, their relative performance remains almost the same when we do not discount.

6.4. Maturity of the longevity bond

In Section 5, we assume that the maturities of both the zero-coupon bond and the longevity bond are equal to the payment horizon of the hedger's portfolio. Now we evaluate the hedging effectiveness when the maturity of the longevity bond is 10 years longer or shorter. The situation where the maturity of the hedging instrument is longer than the portfolio is also studied in Cairns (2013). The maturity of the zero-coupon bond and the payment horizon of the portfolio are unchanged. The results are shown in Table 6. We see that shaving a longer maturity of the longevity bond has little effect on the hedging effectiveness. However, when the maturity of the longevity bond is shorter, the optimal standard deviation in the benchmark case becomes obviously larger, indicating a lower hedging quality of the longevity bond. Moreover, the hedging effectiveness reduces substantially with trading frequency. Therefore, it is important that the hedger has access to longevity hedging instrument with maturities equal to or longer than the payment horizon of her portfolio.

6.5. The target cohort

Finally, we evaluate the impact of the cohort included in the hedger's portfolio and the longevity bond on the hedging effectiveness. Table 7 shows the hedging results when $x_0 = 45$, 55, and 65, respectively. We see that the optimal benchmark standard deviation increase substantially when x_0 increases, which indicates that future mortality rates of older cohorts are more uncertain than

TABLE 6

THE TIME 0 RATIO OF STD. OF THE HEDGING ERROR FOR DIFFERENT MATURITIES OF THE LONGEVITY BOND.

Maturity	Benchmark	1-Year	2-Year	5-Year	Static	Interest Only
20	0.0181	187.1%	248.1%	366.8.7%	202.68%	665.1%
30	0.0160	101.1%	103.7%	118.3%	153.5%	743.9%
40	0.0162	101.3%	105.0%	122.3%	159.6%	739.8%

TABLE 7

THE TIME 0 RATIO OF STD. OF THE HEDGING ERROR FOR DIFFERENT x_0.

x_0	Benchmark	1-Year	2-Year	5-Year	Static	Interest Only
45	0.0056	102.4%	107.6%	137.7%	202.68%	1116.0%
55	0.0160	101.1%	103.7%	118.3%	153.5%	743.9%
65	0.0317	101.0%	102.8%	115.2%	135.4%	513.4%

the younger ones. Moreover, since the optimal benchmark standard deviation is larger for the older cohort, reducing the trading frequency of the longevity bond leads to a lower reduction in hedging effectiveness in this case.

7. CONCLUSION

In this paper, we study the dynamic hedging problem of a portfolio of liabilities exposed to longevity risk. In particular, we consider the case where a pension sponsor or an annuity provider (a hedger) wishes to minimize the variance of her hedging error, defined as the deviation of the market value of her investments from the market value of her liabilities.

Closed-form optimal hedging strategies are obtained for the minimum-variance criterion under a forward mortality and interest rate framework. In particular, time-consistent strategies are derived in a liquidity-constrained case, where the hedger can only trade the longevity-linked assets at a deterministic and lower frequency.

The optimal hedges are evaluated in a numerical analysis with a Hull–White specification and parameter estimates from existing literature. We find that, compared with the benchmark case, limiting the trading of the longevity bond to a 2-year frequency only leads to a slight increase of the variance of the hedging error. Moreover, even when the longevity bond can only be traded at a 5-year frequency, the dynamic hedging strategy still significantly outperform the static one.

There are several directions for future research. First, in this paper, we evaluate the hedging strategy using one parameterization of mortality process. To

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examine the model risk, we may evaluate the hedging quality using different mortality models. Second, we may extend the limitation of trading frequency to more realistic setups, such as stochastic trading times. Third, more general settings, which include life insurance products and more flexible annuity products, such as variable annuities, and small sample risk could be considered. Finally, a mean-variance hedging criterion could be used, if the expected wealth level of the hedger is also of interest (see, for example, Wong *et al.* (2014)).

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NOTES

1. See the report "Longevity risk transfer markets: market structure, growth drivers and impediments, and potential risks", downloaded from http://www.bis.org/publ/joint34.htm at September 30, 2014.
2. Moreover, as noted by Cairns (2013), indemnity contracts are currently only available to large risk holders, e.g., the ones with liabilities exceeding around \$100 million.
3. For a dynamic hedging study under a discrete-time framework, see Cairns (2011) and De Rosa *et al.* (2017).
4. A similar setup of value hedging is considered in Cairns (2013) and Cairns *et al.* (2014) in a static framework.
5. More general cases including multiple cohorts and both genders require obvious extensions.
6. Other popular longevity-linked derivatives, such as longevity swaps and q-forwards (Dawson *et al.*, 2010), can also be incorporated in our framework. For example, a forward can be mathematically regarded as the exchange of the principle repayment of the longevity bond and a preset payment at maturity.
7. As discussed in the Introduction, the reference population could be a national population, or a combination of several national populations.
8. We assume that $T^* \geq \max\{T_1, \tau_M\}$, where τ_M is the longest maturity of the zero-coupon bond (and thus the longevity bond).
9. The case where the zero-coupon bond market is incomplete can be incorporated naturally. When the longevity bond market is incomplete, λ cannot be uniquely determined by market prices. In this case, we assume that λ is determined by the market clearing conditions, together with some underlying no-arbitrage equilibrium conditions. For methods of determining an equivalent martingale measure, see, e.g., Kallsen (2002).
10. We allow initial assets that are not equal to $Y(0)$. In fact, in this paper, we do not impose any solvency constraint on the hedger. Therefore, the hedger's wealth does not need to be above some threshold. Moreover, as will be seen later, the optimal variance-minimizing hedging strategy in both the benchmark case and the constrained case does not depend on $w(t)$ for all t. The only requirement for the optimization to be well defined is that the wealth process is integrable throughout the planning horizon.
11. For notational simplicity, we write $G(t, X(t))$ as $G(t)$.
12. http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yield.

13. http://www.mortality.org/.
14. For the price of longevity risk, we use the calibrated values in Bauer *et al.* (2010). For the price of interest risk, De Jong and Santa-Clara (1999) propose a square root price of risk process, $\lambda_r(t) = \bar{\lambda}\sqrt{\sigma_0 + \sigma_1 r(t)}$, and calibrate the parameter values: $\bar{\lambda} = 47.86$, $\sigma_0 = 0$, and $\sigma_1 = 0.0579$. We impose a constant price of risk which equals to the mean of the $\lambda_r(t)$ process generated using their parameters, i.e., $\lambda_r = \overline{\lambda_r(t)}$.
15. With 1,000 realizations, the mean, 1% and 99% quantiles of the time 0 optimal standard deviation of the hedger's hedging error under the benchmark strategy is 0.0160, 0.0154, and 0.0168. Confidence intervals for the optimal standard deviations under other strategies are also small.
16. The number is calculated from OECD (2013).
17. Other measures of hedging effectiveness, such as variance or volatility reduction, can also be used.
18. Again, for notational simplicity, we subtract the subscript $X(t)$ from the relevant quantities.

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APPENDIX

A. PROOF OF PROPOSITION 1

For any $T \in \tilde{\mathcal{T}}$, we have, from (3.18),

$$dL(t,T) = E_t^Q[{}_Tp(x_0,0,rp)]dB(t,T) + B(t,T)dE_t^Q[{}_Tp(x_0,0,rp)]. \tag{A1}$$

$dB(t,T)$ is given in Equation (3.17). Denote by $\mu(t,s,x_0,1)$, $\hat{\mu}(t,1)$, $a_\mu(u,t,1)$, and $\sigma_\mu(u,t,1)$ the first entry (row) of $\mu(t,s,x_0)$, $\hat{\mu}(t)$, $a_\mu(u,t)$, and $\sigma_\mu(u,t)$ for all $t,s \in [0,T]$.

From (3.5), (3.6), and (3.14), we have

$$\begin{aligned}
&\log(E_t^Q[{}_{T-t}p(x_0+t,t,rp)]) \\
&= -\int_t^T \left[\mu(t,s,x_0,1) - \int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)du\right]ds \\
&= -\int_t^T \mu(0,s,x_0,1)ds - \int_t^T\int_0^t a_\mu(u,s,1)duds - \int_t^T\int_0^t \sigma_\mu(u,s,1)dW_\mu^P(u)ds \\
&\quad + \int_t^T\int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds.
\end{aligned} \tag{A2}$$

(A2) can be further written as

$$\begin{aligned}
&= -\int_0^T \mu(0,s,x_0,1)ds - \int_0^t\int_u^T a_\mu(u,s,1)dsdu - \int_0^t\int_u^T \sigma_\mu(u,s,1)dsdW_\mu^P(u) \\
&\quad + \int_0^t \mu(0,s,x_0,1)ds + \int_0^t\int_u^t a_\mu(u,s,1)dsdu + \int_0^t\int_u^t \sigma_\mu(u,s,1)dsdW_\mu^P(u) \\
&\quad + \int_t^T\int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds,
\end{aligned} \tag{A3}$$

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where we interchange the integration order of $\int_t^T \int_0^t a_\mu(u,s,1)duds$ and $\int_t^T \int_0^t \sigma_\mu(u,s,1)dW_\mu^P(u)ds$, and decompose the inner-integration in two parts. Using (3.12), (3.13), and the fact that

$$\int_0^t \int_u^t \sigma_\mu(u,s,1)dsdW_\mu^P(u) = \int_0^t \int_0^t \sigma_\mu(u,s,1)\mathbf{1}_{u\le s}dsdW_\mu^P(u)$$
$$= \int_0^t \int_0^t \sigma_\mu(u,s,1)\mathbf{1}_{u\le s}dW_\mu^P(u)ds$$
$$= \int_0^t \int_0^s \sigma_\mu(u,s,1)dW_\mu^P(u)ds, \tag{A4}$$

we can further rewrite (A3) as

$$= \log(E_0^Q[{}_Tp(x_0,0,rp)]) - \int_0^T \int_0^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds + \int_t^T \int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds$$
$$- \int_0^t \int_u^T a_\mu(u,s,1)dsdu - \int_0^t \int_u^T \sigma_\mu(u,s,1)dsdW_\mu^P(u)$$
$$+ \int_0^t \mu(0,s,x_0,1)ds + \int_0^t \int_0^s a_\mu(u,s,1)duds + \int_0^t \int_0^s \sigma_\mu(u,s,1)dW_\mu^P(u)ds$$
$$= \log(E_0^Q[{}_Tp(x_0,rp)]) - \int_0^T \int_0^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds + \int_t^T \int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds$$
$$- \int_0^t \int_u^T a_\mu(u,s,1)dsdu - \int_0^t \int_u^T \sigma_\mu(u,s,1)dsdW_\mu^P(u) + \int_0^t \hat{\mu}(s,1)ds. \tag{A5}$$

Denote by

$$\zeta_L(t,T) = -\int_0^T \int_0^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds + \int_t^T \int_t^s \sigma_\mu(u,s,x_0,1)\lambda_\mu(u)duds$$
$$- \int_0^t \int_u^T a_\mu(u,s,1)dsdu - \int_0^t \int_u^T \sigma_\mu(u,s,1)dsdW_\mu^P(u) + \int_0^t \hat{\mu}(s,1)ds, \tag{A6}$$

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then $E_t^Q[{}_{T-t}p(x_0+t, t, rp)]$ can be written as $E_0^Q[{}_Tp(x_0, 0, rp)]\exp(\zeta_L(t, T))$. The differential of $\zeta_L(t, T)$ is given by

$$d\zeta_L(t, T) = \left[\hat{\mu}(t, 1) - \int_t^T \sigma_\mu(t, s, x_0, 1)\lambda_\mu(t)ds - \int_t^T a_\mu(t, s, 1)ds\right]dt - \left\{\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right\}dW_\mu^P(t). \tag{A7}$$

Therefore, we have

$$\begin{aligned}
&dE_t^Q[{}_{T-t}p(x_0+t, t, rp)]\\
&= E_0^Q[{}_Tp(x_0, 0, rp)]d\exp(\zeta_L(t, T))\\
&= E_0^Q[{}_Tp(x_0, 0, rp)]\exp(\zeta_L(t, T))\Bigg\{\hat{\mu}(t, 1) - \int_t^T \sigma_\mu(t, s, x_0, 1)\lambda_\mu(t)ds - \int_t^T a_\mu(t, s, 1)ds\\
&\quad + \frac{1}{2}\left(\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right)\left(\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right)'\Bigg\}dt\\
&\quad - E_0^Q[{}_Tp(x_0, 0, rp)]\exp(\zeta_L(t, T))\left\{\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right\}dW_\mu^P(t)\\
&= E_t^Q[{}_{T-t}p(x_0+t, t, rp)]\Bigg\{\left[\hat{\mu}(t, 1) - \int_t^T \sigma_\mu(t, s, x_0, 1)\lambda_\mu(t)ds\right]dt\\
&\quad - \left[\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right]dW_\mu^P(t)\Bigg\},
\end{aligned} \tag{A8}$$

and the differential of $E_t^Q[{}_Tp(x_0, 0, rp)]$ is given by

$$\begin{aligned}
&dE_t^Q[{}_Tp(x_0, 0, rp)]\\
&= d_tp(x_0, 0, rp)E_t^Q[{}_{T-t}p(x_0+t, t, rp)]\\
&= de^{-\int_0^t \hat{\mu}(s,1)ds}E_t^Q[{}_{T-t}p(x_0+t, t, rp)]\\
&= -\hat{\mu}(t, 1)e^{-\int_0^t \hat{\mu}(s,1)ds}E_t^Q[{}_{T-t}p(x_0+t, t, rp)] + e^{-\int_0^t \hat{\mu}(s,1)ds}dE_t^Q[{}_{T-t}p(x_0+t, t, rp)]\\
&= E_t^Q[{}_Tp(x_0, 0, rp)]\left\{-\left[\int_t^T \sigma_\mu(t, s, x_0, 1)\lambda_\mu(t)ds\right]dt - \left[\int_t^T \sigma_\mu(t, s, x_0, 1)ds\right]dW_\mu^P(t)\right\}.
\end{aligned} \tag{A9}$$

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The differential of $L(t, T)$ can then be obtained by combining (A9) and the dynamics of the zero-coupon bond.

B. PROOF OF THEOREM 1

In the presence of the liquidity constraint, $\hat{\boldsymbol{u}}_1$ is constant in the interval $[t_i, t_{i+1})$. In this case, we solve the optimization problem in two steps: we first solve the optimal $\hat{\boldsymbol{u}}_2$ as a function of $\hat{\boldsymbol{u}}_1$, then we solve the optimal $\hat{\boldsymbol{u}}_1$ recursively for each $[t_i, t_{i+1})$ (with $t_{n+1} = T_0$). For a fixed $\hat{\boldsymbol{u}}_1$, in terms of Wong *et al.* (2014), we have $Y(t) \equiv w(t)$, where $w(t)$ is given in (3.22) with $\hat{\boldsymbol{u}}_1(s) = \hat{\boldsymbol{u}}_1(t_i)$ for $s \in [t_i, t_{i+1})$, $i = 1, 2, \ldots, n$. Moreover, we have[18]

$$\Gamma(t) \equiv G(\hat{\boldsymbol{u}}_1, t) = E_t\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau}\hat{\boldsymbol{u}}' a_w(s)ds - e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)\right],$$

where $G(\hat{\boldsymbol{u}}_1, t)$ emphasizes the dependence of G on $\hat{\boldsymbol{u}}_1$. Again, the risk aversion parameter ϕ in Wong *et al.* (2014) is set to be 0. In the first step, we only need to optimize with respect to $\hat{\boldsymbol{u}}_2$, which is continuously rebalanced. Repeating Proposition 3.1 and 3.2 in Wong *et al.* (2014), we obtain the optimal $\hat{\boldsymbol{u}}_2$:

$$\begin{aligned}\hat{\boldsymbol{u}}_2(t) = &-(\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_X(t)'\frac{\partial G(\hat{\boldsymbol{u}}_1, t)}{\partial X(t)}\\ &- (\sigma_w(t,2)\sigma_w(t,2)')^{-1}\sigma_w(t,2)\sigma_w(t,1)'\hat{\boldsymbol{u}}_1(t_i), \quad t \in [t_i, t_{i+1}),\end{aligned} \tag{B1}$$

with $G(\hat{\boldsymbol{u}}_1, t)$ represented as

$$\begin{aligned}G(\hat{\boldsymbol{u}}_1, t) = &\sum_{j=i+1}^{n} E_t^{\bar{Q}}\left[\int_{t_j}^{t_{j+1}} e^{-\int_t^s r(\tau)d\tau}\hat{\boldsymbol{u}}_1(t_j)a_w(s,1)ds\right] - E_t^{\bar{Q}}[e^{-\int_t^{T_0} r(\tau)d\tau} Y_{T_0}]\\ &+ \hat{\boldsymbol{u}}_1(t_i)' E_t^{\bar{Q}}\left[\int_t^{t_{i+1}} e^{-\int_t^s r(\tau)d\tau} a_w(s,1)ds\right]\\ \equiv\ & G_i(\hat{\boldsymbol{u}}_1, t) + \hat{\boldsymbol{u}}_1(t_i)' H_i(t).\end{aligned} \tag{B2}$$

Next, we proceed to the second step and solve the optimal $\hat{\boldsymbol{u}}_1$. For each $i = 1, \cdots, n$, the value function at t_i is given by

$$J_{t_i} = \min_{\hat{\boldsymbol{u}}_1(t_i)} \mathrm{Var}_{t_i}\left\{e^{-\int_{t_i}^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))\right\}, \tag{B3}$$

where $w^*(T_0)$ is the wealth at T_0 under $\hat{\boldsymbol{u}}^*$ from t_{i+1} to T_0. To derive the optimal $\hat{\boldsymbol{u}}_1^*$, we make use of an alternative formulation of the law of total variance proposed in Proposition 2 from Basak and Chabakauri (2012) (Equations A15–A16). From the law of total variance, with an infinitesimally small time interval ϵ, we obtain the following differential form:

$$0 = E_t[d\ \mathrm{Var}_s\{e^{-\int_s^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))\} + \mathrm{Var}_s\{d\ e^{-\int_t^s r(\tau)d\tau} E_s[e^{-\int_s^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))]\}]. \tag{B4}$$

Letting $t = t_i$ and integrating (B4) from t_i to t_{i+1}, we have (following the notations in Basak and Chabakauri (2012))

$$\begin{aligned}&\mathrm{Var}_{t_i}\{e^{-\int_{t_i}^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))\}\\ &= E_{t_i}[\mathrm{Var}_{t_{i+1}}\{e^{-\int_{t_{i+1}}^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))\}]\\ &\quad + E_{t_i}\left[\int_{t_i}^{t_{i+1}} \frac{\mathrm{Var}_s\{d\ e^{-\int_{t_i}^s r(\tau)d\tau} E_s[e^{-\int_s^{T_0} r(\tau)d\tau}(w(T_0) - Y(T_0))]\}}{ds} ds\right].\end{aligned} \tag{B5}$$

For $t \in [t_i, t_{i+1})$, let $w^*(T_0)$ be the wealth at T_0 under $\hat{\boldsymbol{u}}^*$ from t_{i+1} to T_0. Denote by $\tilde{\boldsymbol{u}}_1$ the strategy where $\tilde{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}_1^*$ for $t \in [t_{i+1}, T_0]$ and $\tilde{\boldsymbol{u}}_1 = \hat{\boldsymbol{u}}_1$ for a fixed $\hat{\boldsymbol{u}}_1$ for $t \in [t_i, t_{i+1})$, and denote by $\tilde{\boldsymbol{u}}_2(\hat{\boldsymbol{u}}_1)$ the strategy where $\tilde{\boldsymbol{u}}_2(\hat{\boldsymbol{u}}_1) = \hat{\boldsymbol{u}}_2^*$ for $t \in [t_{i+1}, T_0]$ and takes the form (B1) for the given $\hat{\boldsymbol{u}}_1$ for $t \in [t_i, t_{i+1})$. Moreover, denote by $\tilde{\boldsymbol{u}}(s) = (\tilde{\boldsymbol{u}}_1(s)', \tilde{\boldsymbol{u}}_2(\hat{\boldsymbol{u}}_1, s)')'$. From the integral form of $w^*(T_0)$,

$$\begin{aligned}w^*(T_0)e^{-\int_t^{T_0} r(\tau)d\tau} &= w(t) + \int_t^{T_0} e^{-\int_t^s r(\tau)d\tau}\tilde{\boldsymbol{u}}(s)' a_w(s)ds\\ &\quad + \int_t^{T_0} e^{-\int_t^s r(\tau)d\tau}\tilde{\boldsymbol{u}}(s)'\sigma_w(s)dW(s),\end{aligned} \tag{B6}$$

we have

$$E_t\left[e^{-\int_t^{T_0} r(\tau)d\tau} w^*(T_0)\right] = w(t) + E_t\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau}\tilde{\boldsymbol{u}}(s)' a_w(s)ds\right].$$

Therefore, applying Ito's lemma, we have

$$\begin{aligned}&d\ e^{-\int_{t_i}^t r(\tau)d\tau} E_t[e^{-\int_t^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))]\\ &= \{\ldots\}dt + e^{-\int_{t_i}^t r(\tau)d\tau}[\sigma_X(t)\frac{\partial \tilde{E}_i(\hat{\boldsymbol{u}}_1, t)}{\partial X(t)} + \tilde{\boldsymbol{u}}(s)'\sigma_w(t)]dW(t),\end{aligned} \tag{B7}$$

where

$$\tilde{E}_i(\hat{\boldsymbol{u}}_1, t) \equiv E_t\left[\int_t^{T_0} e^{-\int_t^s r(\tau)d\tau}\tilde{\boldsymbol{u}}(s)' a_w(s)ds - e^{-\int_t^{T_0} r(\tau)d\tau} Y(T_0)\right]$$

for $t \in [t_i, t_{i+1})$. The term $\tilde{E}_i(\hat{\boldsymbol{u}}_1, t)$ is equal to $G(t)$. To see this, apply the Feynman–Kac Theorem to $\tilde{E}_i(\hat{\boldsymbol{u}}_1, t)$, and substitute $\tilde{\boldsymbol{u}}_2(\hat{\boldsymbol{u}}_1, s)$. Applying again the Feynman–Kac Theorem to the resulting partial differential equation, and recognizing the terminal condition

$\tilde{E}_i(\hat{\boldsymbol{u}}_1, t_{i+1}) = G_i(\hat{\boldsymbol{u}}_1, t_{i+1})$, we have

$$\begin{aligned}\tilde{E}_i(\hat{\boldsymbol{u}}_1, t) =& E_t^{\bar{Q}}\left[G_i(\hat{\boldsymbol{u}}_1, t_{i+1}) + \int_t^{t_{i+1}} e^{-\int_t^s r(\tau)d\tau}\hat{\boldsymbol{u}}_1(t_i)' a_w(s,1)ds\right],\\ =& G_i(\hat{\boldsymbol{u}}_1, t) + \hat{\boldsymbol{u}}_1(t_i)' E_t^{\bar{Q}}\left[\int_t^{t_{i+1}} e^{-\int_t^s r(\tau)d\tau} a_w(s,1)ds\right],\\ =& G(\hat{\boldsymbol{u}}_1, t).\end{aligned} \tag{B8}$$

Substituting (B7) into (B5) and computing $\mathrm{Var}_s\{d\ e^{-\int_{t_i}^s r(\tau)d\tau} E_s[e^{-\int_s^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))]$ yields

$$\begin{aligned}&\mathrm{Var}_{t_i}\{e^{-\int_{t_i}^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))\}\\ &= E_{t_i}[\mathrm{Var}_{t_{i+1}}\{e^{-\int_{t_{i+1}}^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))\}]\\ &+ E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^s r(\tau)d\tau}[\sigma_X(s)\frac{\partial G(\hat{\boldsymbol{u}}_1, s)}{\partial X(s)} + \tilde{\boldsymbol{u}}(s)'\sigma_w(s)]\right.\\ &\left.[\sigma_X(s)\frac{\partial G(\hat{\boldsymbol{u}}_1, s)}{\partial X(s)} + \tilde{\boldsymbol{u}}(s)'\sigma_w(s)]'ds\right].\end{aligned} \tag{B9}$$

Substituting the integral form of $w^*(T_0)$ into the time t_{i+1} value function, $\mathrm{Var}_{t_{i+1}}$ $\{e^{-\int_{t_{i+1}}^{T_0} r(\tau)d\tau}(w^*(T_0) - Y(T_0))\}$, we see that it does not depend on $w(t_{i+1})$, and thus does not depend on the strategy $\hat{\boldsymbol{u}}_1$ on $[t_i, t_{i+1})$. Therefore, we only need to focus on the second part on the right-hand side of (B9). Substituting the optimal $\hat{\boldsymbol{u}}_2^*$ into (B9), and taking the derivative with respect to $\hat{\boldsymbol{u}}_1$, we obtain

$$\hat{\boldsymbol{u}}_1^*(t_i) = \left(E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^s r(\tau)d\tau} A_i(s)ds\right]\right)^{-1} E_{t_i}\left[\int_{t_i}^{t_{i+1}} e^{-2\int_{t_i}^s r(\tau)d\tau} B_i(s)ds\right], \tag{B10}$$

where the $A_i(s)$ and $B_i(s)$ matrices are given in (4.15).