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HANDLEBODY DECOMPOSITIONS FOR G-MANIFOLDS

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We construct handle-bundle decompositions of compact G-manifolds, G a compact Lie group, that are particularly well adapted to the orbit structure of the group action.

1. Introduction

Let G be a compact Lie group of transformations acting smoothly (that is C^{∞}) on the compact manifold M . In this note we show how to construct a particularly nice handlebody decomposition of M , invariant under the action of G . Our result has interesting implications for the stability theory of equivariant dynamical systems; most notably for a generalisation of the C^0 isotopy approximation theorems of Shub and Smale [6], [8] to equivariant maps and we intend to pursue these matters elsewhere.

2. Generalities on G-actions

For the general theory of G-manifolds see Bredon [1]. We follow the notational conventions of Field [4], [5]. Thus if M is a G-manifold and $x \in M$, we let $G(x)$ denote the G-orbit through x and G_x denote the isotropy subgroup of G at x . We say $x, y \in M$ are of the same orbit type if G_x, G_y are conjugate subgroups of G . Equivalence of orbit type partitions M into points of the same orbit type. If M is compact, this partition is finite and we may write

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$$M = \bigcup_{i=1}^{N} M_i ,$$

where the M_i are the equivalence classes of points of the same orbit type. We may label orbit types in such a way that if $\overline{M}_j \cap M_i \neq \emptyset$ then $i < j$. We say M_j is a minimal orbit type if there are no M_i such that $\overline{M}_j \cap M_i \neq \emptyset$. Necessarily M_j will be a closed submanifold of M . We call M_N the principal orbit type and recall that M_N is an open, dense subset of M (we assume here, as elsewhere in this paper, that M is connected).

If ξ is a riemannian metric on M we may average ξ over G using Haar measure to obtain an equivariant riemannian metric on M . We call M , together with an equivariant riemannian metric, a riemannian G-manifold.

3. G-Morse functions

Let $f : M \to \mathbb{R}$ be a smooth G-invariant function on the compact riemannian G-manifold M . We let $\mathrm{grad}(f)$ denote the associated gradient vector field of f and set $\mathrm{crit}(f) = \{x \in M : \mathrm{grad}(f)(x) = 0\}$. Necessarily, $\mathrm{crit}(f)$ is a union of G-orbits. We say that a G-orbit $\alpha \in \mathrm{crit}(f)$ is generic if it is non-degenerate in the sense of Morse theory for $\mathrm{grad}(f)$. That is, if we let ϕ_t denote the flow of $\mathrm{grad}(f)$, then the induced flow $N\phi_t$ on the normal bundle $N\alpha$ of α has spectrum disjoint from the unit circle (see also Field [3, p. 193]). We say that f is a G-Morse function if $\mathrm{crit}(f)$ consists of a, necessarily finite, set of generic G-orbits. We recall from Wasserman [9] that every G-manifold admits a G-Morse function. Suppose that f is a G-Morse function and let α be a G-orbit in $\mathrm{crit}(f)$. We let $W^s(\alpha)$ and $W^u(\alpha)$ denote the stable and unstable manifolds of $\mathrm{grad}(f)$ through α respectively. It is shown in Field [5] that if f is a G-Morse function we can always find a perturbation f' of f , equal to f on some neighbourhood of $\mathrm{crit}(f)$, such that the stable and unstable manifolds of elements of $\mathrm{crit}(f')$ meet G-transversally. (An elementary description of G-transversality may be found in Field [3]; see also Field [2]. As we shall be able to avoid using the deeper results of the theory of G-transversality in the proof of our main result, we refrain from further elaboration of the theory here.)

DEFINITION. Let f be a G-Morse function on the compact riemannian G-manifold M. We say that f is excellent if for all G-orbits $\alpha, \beta \in \mathrm{crit}(f)$, $W^u(\alpha) \pitchfork W^s(\beta)$ and $W^s(\alpha) \pitchfork M_j$ if $\alpha \in M_j$.

PROPOSITION. *Every compact riemannian G-manifold M admits an excellent G-Morse function.*

Proof. Our proof goes by induction on orbit type. As in §2, we write $M = M_1 \cup \ldots \cup M_N$. Suppose that we have constructed an open G-invariant neighbourhood U_r of $M_1 \cup \ldots \cup M_r$, $r < N$, and smooth G-invariant function $f_r : U_r \to \mathbb{R}$ such that

(1) $\mathrm{crit}(f_r) \subset M_1 \cup \ldots \cup M_r$ and no critical orbit of $\mathrm{grad}(f_r)$ is degenerate,

(2) $W^s(\alpha)$ meets M_j transversally for every G-orbit $\alpha \in \mathrm{crit}(f_r) \cap M_j$, $1 \le j \le r$.

Certainly f_r is defined on a neighbourhood of ∂M_{r+1} in M_{r+1}. Let g_{r+1} denote an equivariant smooth extension of $f_r | M_1 \cup \ldots \cup M_r$ to a neighbourhood W_{r+1} of $M_1 \cup \ldots \cup M_{r+1}$ such that $g_{r+1} = f_r$ on a neighbourhood of $M_1 \cup \ldots \cup M_r$ contained in W_{r+1}. Certainly, $g_{r+1} = f_r$ on some neighbourhood of ∂M_{r+1} in M_{r+1}. By Wasserman's approximation theorem [9], we may assume that all critical orbits of g_{r+1} are non-degenerate and hence that g_{r+1} has only finitely many critical orbits on $M_1 \cup \ldots \cup M_{r+1}$. We may clearly do this without changing g_{r+1} on a neighbourhood of $M_1 \cup \ldots \cup M_r$. Let $\alpha \in M_{r+1} \cap \mathrm{crit}(g_{r+1})$. Choose a smooth G-invariant positive bump function θ on M satisfying

(a) $\mathrm{supp}(\theta) \subset W_{r+1}$,

(b) $\theta \equiv 1$ on some neighbourhood of α,

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(c) $\mathrm{supp}(\theta)$ is disjoint from $M_1 \cup \ldots \cup M_r$ and the remaining critical orbits of g_{r+1} on M_{r+1}.

For $\lambda \in \mathbb{R}$, $x \in W_{r+1}$ define

$$g_{r+1}^{\lambda}(x) = \lambda\theta(x)d\left(x, M_{r+1}\right)^2 + g_{r+1}(x) ,$$

where $d\left(x, M_{r+1}\right)$ denotes the riemannian distance of x from M_{r+1}. Certainly g_{r+1} is smooth on some neighbourhood of $M_1 \cup \ldots \cup M_{r+1}$. For sufficiently large negative values of λ, g_{r+1}^{λ} will have a nondegenerate critical orbit at α such that the stable manifold of α for $\mathrm{grad}\left(g_{r+1}^{\lambda}\right)$ meets M_{r+1} transversally. Observe that $g_{r+1}, g_{r+1}^{\lambda}$ have the same critical orbits on $M_1 \cup \ldots \cup M_{r+1}$, though of course we may introduce new critical orbits outside $M_1 \cup \ldots \cup M_{r+1}$. Modifying g_{r+1} in a neighbourhood of each of the critical orbits in M_{r+1} in the manner indicated above, we obtain a smooth G-invariant function f_{r+1} defined on some neighbourhood of $M_1 \cup \ldots \cup M_{r+1}$ such that the critical orbits of f_{r+1} on $M_1 \cup \ldots \cup M_{r+1}$ are non-degenerate and condition (2) of the indictive hypothesis is satisfied. Now choose a neighbourhood U_{r+1} of $M_1 \cup \ldots \cup M_{r+1}$ which is G-invariant and such that $\mathrm{crit}\left(f_{r+1}\right) \subset M_1 \cup \ldots \cup M_{r+1}$. The inductive step is completed. Now although f_N may not be an excellent G-Morse function we may, by Field [5], perturb f_N to obtain a G-Morse function f on M which is equal to f_N on a neighbourhood of $\mathrm{crit}\left(f_N\right)$ and such that the stable and unstable manifolds of critical elements of $\mathrm{grad}(f)$ are G-transversal. But condition (2) guarantees that the stable and unstable manifolds for f are actually transversal. Alternatively, notice that we may use standard transversality theory to perturb f_N to a G-Morse function f such that the stable and unstable manifolds of $\mathrm{grad}(f)$ are transversal within the orbit type components M_j . Again condition (2) implies that the stable and unstable manifolds of f are transversal. //

REMARK. In general a G-Morse function cannot be approximated by an excellent G-Morse function. As an example take the $\mathbb{Z}_2$-action on S^2 defined by the reflection in the (x, y)-plane. The equator of S^2 is then the fixed point set of the $\mathbb{Z}_2$-action. Now choose any $\mathbb{Z}_2$-invariant smooth function f on S^2 which has precisely two non-degenerate critical points on the equator, both of index 1 . The resulting saddle-link cannot be removed by perturbing f .

4. Handle-bundle decompositions of a G-manifold

DEFINITION. Let α be a non-degenerate critical orbit of the G-Morse function f . The index of α , $\operatorname{ind}(\alpha; f)$, is defined to be the dimension of $W^u(\alpha)$.

THEOREM. *Let M be an m-dimensional compact riemannian G-manifold. There exists a G-Morse function f on M such that*

(1) $f \geq 0$,

(2) $f^{-1}([0, j])$ *is a closed neighbourhood of* $M_1 \cup \ldots \cup M_j$, $1 \leq j \leq N$,

(3) $f^{-1}([j, j+1] \cap C_f) \subset M_{j+1}$, $j \geq 0$ (C_f *denotes the set of critical values of* f),

(4) *if* α *is a critical orbit for* f *lying in* M_j , *then* $f(\alpha) = j - 1 + (k+1)/(m+2)$, *where* $k = \operatorname{ind}(\alpha; f)$.

Before giving the proof of the result, we point out some consequences. If we set $W_j = f^{-1}([0, j])$, we see that W_j is a G-invariant submanifold of M containing $M_1 \cup \ldots \cup M_j$. Thus $\{W_j : j = 1, \ldots, N\}$ give a filtration of M compatible with the orbit structure of the group action. We obtain W_{j+1} from W_j by attaching handle-bundles, all of which are associated to critical orbits of grad(f) lying in M_{j+1} (see Wasserman [9, Theorem 4.6]). As in non-equivariant handlebody theory we attach handle-bundles of lowest index first and then successively attach handle-bundles of higher index. Here, of course, we do this process for each j, $1 \leq j \leq N$.

Proof of theorem. Our proof follows Smale [7] closely and we only indicate the modification necessary to perform an induction over orbit type. Choose an excellent G-Morse function F on M. For the first step of the induction we restrict attention to critical orbits of F lying in M_1. Exactly as in Smale [7] we construct a neighbourhood W_1 of M_1 which is a union of handle-bundles associated to the critical orbits of F in M_1. In particular, W_1 will be a G-invariant submanifold of M with smooth boundary, grad(F) will be transversal to ∂W_1 and W_1 will not contain any critical orbits lying in M_j, $j \geq 2$. For the next step of the induction we add handle-bundles to W_1, associated to critical orbits in M_2, to construct a neighbourhood W_2 of $M_1 \cup M_2$ containing only critical orbits lying in $M_1 \cup M_2$. The induction proceeds in the obvious way ending with the addition of handle-bundles associated to critical orbits in M_N. Once we have this handle-bundle decomposition of M, we construct f satisfying the conditions of the theorem as in Smale [7]. //

EXAMPLE. Let $S^1 \times \mathbb{Z}_2$ act on $\mathbb{R}^3$ by rotation about the z-axis and reflection in the (x, y)-plane. The action clearly extends to action on S^3 with two fixed points. In the figure below we have taken a section of $\mathbb{R}^3$ by the (x, z)-plane and have drawn the level surfaces $f^{-1}(j)$, $j = 1, 2, 3$ for a function f satisfying the conditions of the theorem. We also indicate the critical orbits of f. In this example there are 4 critical orbits for grad(f). We have indicated a point on each critical orbit. The corresponding critical values are given by:

$$f(a) = \tfrac{1}{5}\ ;\quad f(b) = 1\tfrac{3}{5}\ ;\quad f(c) = 2\tfrac{2}{5}\ ;\quad f(d) = 3\tfrac{4}{5}\ .$$

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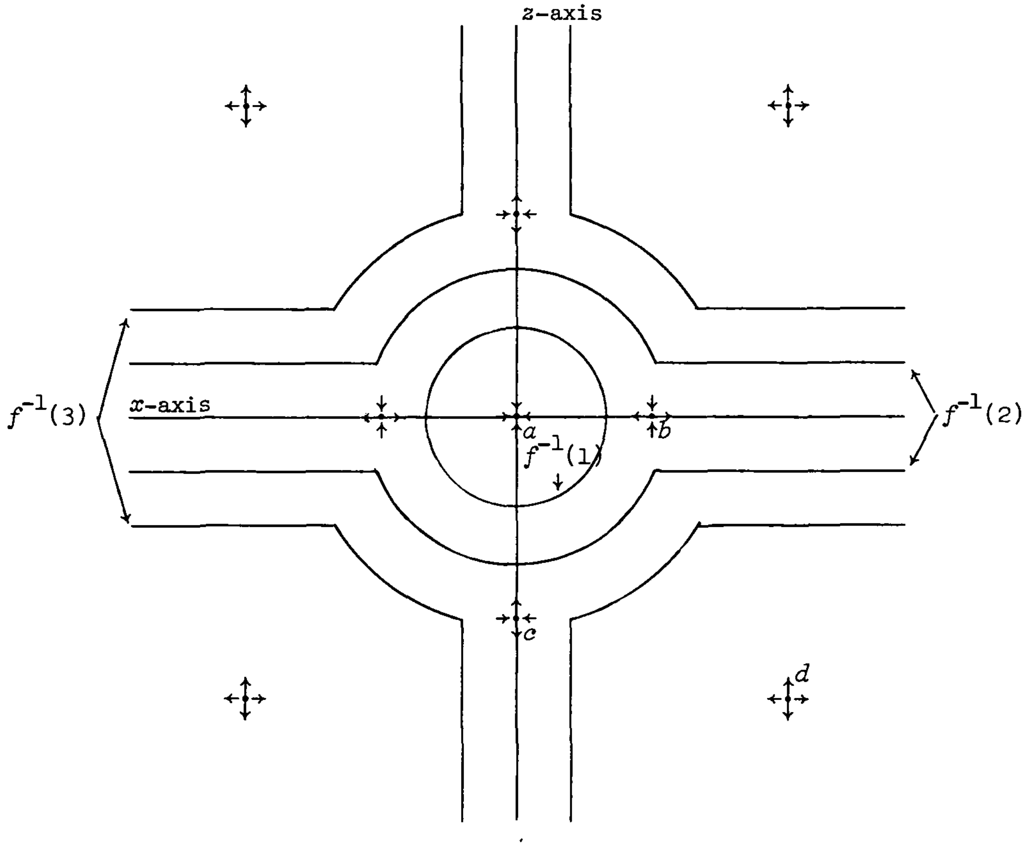

References

[1] Glen E. Bredon, *Introduction to compact transformation groups* (Pure and Applied Mathematics, **46**. Academic Press, New York and London, 1972.

[2] M.J. Field, "Transversality in G-manifolds", *Trans. Amer. Math. Soc.* **231** (1977), 429-450.

[3] M.J. Field, "Stratifications of equivariant varieties", *Bull. Austral. Math. Soc.* **16** (1977), 279-295.

[4] M.J. Field, "Resolving actions of compact Lie groups", *Bull. Austral. Math. Soc.* **18** (1978), 243-254.

[5] M.J. Field, "Equivariant dynamical systems", *Trans. Amer. Math. Soc.* **259** (1980), 185-205.

[6] Michael Shub, "Structurally stable diffeomorphisms are dense", *Bull. Amer. Math. Soc.* **78** (1972), 817-818.

[7] Stephen Smale, "On gradient dynamical systems", *Ann. of Math.* **74** (1961), 199-206.

[8] Steve Smale, "Stability and isotopy in discrete dynamical systems", *Dynamical systems*, 527-530 (Proc. Sympos. University of Bahia, Salvador, Brasil, 1971. Academic Press [Harcourt Brace Jovanovich], New York, (1973).

[9] Arthur G. Wasserman, "Equivariant differential topology", *Topology* **8** (1969), 127-150.

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