

Generalized small cancellation conditions, non-positive curvature and diagrammatic reducibility

Martín Axel Blufstein, Elías Gabriel Minian, and Iván Sadofschí Costa

Departamento de Matemática - IMAS FCEyN, Universidad de Buenos Aires, Buenos Aires, Argentina (mblufstein@dm.uba.ar; gminian@dm.uba.ar; isadofschí@dm.uba.ar)

(Received 27 June 2020; accepted 25 January 2021)

We present a metric condition \mathcal{T}' which describes the geometry of classical small cancellation groups and applies also to other known classes of groups such as two-dimensional Artin groups. We prove that presentations satisfying condition \mathcal{T}' are diagrammatically reducible in the sense of Sieradski and Gersten. In particular, we deduce that the standard presentation of an Artin group is aspherical if and only if it is diagrammatically reducible. We show that, under some extra hypotheses, \mathcal{T}' -groups have quadratic Dehn functions and solvable conjugacy problem. In the spirit of Greendlinger's lemma, we prove that if a presentation $P = \langle X \mid R \rangle$ of group G satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$, the length of any nontrivial word in the free group generated by X representing the trivial element in G is at least that of the shortest relator. We also introduce a strict metric condition $\mathcal{T}'_<$, which implies hyperbolicity.

Keywords: Small cancellation theory; word problem; Dehn function; Artin groups; non-positive curvature; conjugacy problem; hyperbolic groups; isoperimetric inequality

2020 *Mathematics Subject Classification:* 20F65; 20F67; 20F06; 20F10; 08A50; 20M05

1. Introduction

The first ideas behind small cancellation theory appeared more than 100 years ago in the study of M. Dehn. In [9] Dehn formulated the word and conjugacy problems and later he presented an algorithm that solved these problems for the fundamental groups of closed orientable surfaces of genus $g \geq 2$ [10]. The key property of the standard presentations of such groups is that any nontrivial product of two different cyclic permutations of the single relator or its inverse admits only a little cancellation (see [23]). There are currently many variants of small cancellation theory. The development of the classical theory started with the studies of Tartakovskii [34], Schiek [30], Greendlinger [15, 16], Lyndon [22], Schupp [31] (see also [23]), and continued later with the studies of Rips [26–28] and Olshanskii [25]. The graphical variant of the theory started with Gromov [18] and continued

with the studies of Ollivier [24], Gruber [19] and Steenbock [33]. The classical conditions of small cancellation require the pieces to be small with respect to the relators. Concretely, in the metric condition $C'(\lambda)$ the length $\ell(s)$ of any piece s of a relator r is smaller than $\lambda\ell(r)$. The non-metric variant $C(p)$ requires that no relator is a product of fewer than p pieces. In graphical small cancellation the presentations are constructed from a labelled graph and one requires the pieces in the graph to be small. Small cancellation is not only related to the word and conjugacy problems. It has applications to curvature problems, hyperbolicity, asphericity and diagrammatic reducibility. The notion of diagrammatic reducibility first appeared in the study of Sieradski [32] and was developed by Gersten [13]. A presentation P with no proper powers is diagrammatically reducible (DR) if all spherical diagrams over P are reducible. Diagrammatic reducibility, which is a stronger condition than asphericity, is relevant when studying equations over groups. We refer the reader to Gersten's papers [13, 14] for more details on diagrammatic reducibility and applications to equations over groups.

In this paper, we introduce a metric condition \mathcal{T}' and a strict version $\mathcal{T}'_{<}$. These conditions are defined in terms of sums of lengths of pieces incident to interior vertices of reduced diagrams (see definition 2.1). They generalize the classical metric small cancellation conditions and apply also to other known classes of groups. Our motivation is to encompass the geometries of classical small cancellation groups, two-dimensional Artin groups, and one-relator groups satisfying a certain condition (\mathcal{T}') recently introduced in [4], under the same unifying geometry. In the case of Artin groups, we show that condition \mathcal{T}' is equivalent to being two-dimensional.

THEOREM 1.1. *An Artin group A_{Γ} is two-dimensional if and only if its standard presentation P_{Γ} satisfies condition \mathcal{T}' .*

We show that presentations (and groups) satisfying conditions \mathcal{T}' and $\mathcal{T}'_{<}$ have nice properties. We analyse first curvature problems and diagrammatic reducibility. In this direction we obtain the following result.

THEOREM 1.2. *If a presentation P satisfies condition \mathcal{T}' and has no proper powers, then it is DR.*

As an immediate consequence of the above theorem we deduce that the standard presentation of an Artin group is aspherical if and only if it is DR (see remark 2.3). We then prove that groups satisfying the strict condition $\mathcal{T}'_{<}$ are hyperbolic.

THEOREM 1.3. *Let G be a group which admits a finite presentation satisfying conditions $\mathcal{T}'_{<}$ and $C(3)$. Then G is hyperbolic.*

We also prove that \mathcal{T}' -groups have quadratic Dehn functions and solvable conjugacy problem, provided all the relators in the presentation have the same length.

THEOREM 1.4. *Let P be a presentation satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$ and such that all its relators have length r , then P has a quadratic Dehn function. Moreover, if P is finite the group G presented by P has solvable conjugacy problem.*

In fact we believe that the result is still valid without the hypothesis on the length of the relators although we could not prove the general result. At the end of § 4 we discuss a strategy to prove solvability of the word problem for a wider class of groups using this result.

In particular, theorem 4.2 proves the existence of quadratic Dehn functions and the solution of the conjugacy problem for a subclass of two-dimensional Artin groups. The solution of the word problem for two-dimensional Artin groups was proved by Chermak [8]. Recently, Huang and Osajda proved that all two-dimensional Artin groups have quadratic Dehn function and solvable conjugacy problem [20]. Our results provide, although partially, an alternative, direct and simpler proof of these facts.

In the spirit of Greendlinger's lemma, we obtain the following result.

PROPOSITION 1.5. *Let $P = \langle X \mid R \rangle$ be a presentation of a group G , satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$. Let r_{\min} be the length of the shortest relator. Then any nontrivial word W in the free group generated by X representing the trivial element in G has length at least r_{\min} . In particular, if P has a relator of length greater than or equal to 2, then G is nontrivial.*

Conditions \mathcal{T}' and $\mathcal{T}'_{<}$ are defined in terms of the lengths of the pieces and relators incident to interior vertices of reduced diagrams over the presentations. At first glance it may seem that these conditions are difficult to check since the definitions require to analyse all interior vertices of every possible diagram over the presentation P . However we prove below that, for finite presentations, these conditions can be verified by analysing the directed cycles in a finite weighted graph $\Gamma(P)$ associated with P . In this direction, in § 5 we describe an algorithm which decides whether a given finite presentation P satisfies these conditions. This algorithm has been implemented in the GAP [35] package `SmallCancellation` [29]. As an immediate consequence of the construction of the algorithm, we also deduce that for any finite presentation P satisfying the strict condition $\mathcal{T}'_{<}$, the curvature of the interior vertices of any diagram Δ over P is bounded above by a negative constant N , which is independent of the diagram. This fact is used in theorem 3.4 to prove hyperbolicity of the groups satisfying condition $\mathcal{T}'_{<}$.

2. The small cancellation condition \mathcal{T}'

Recall that a cellular map $f : L \rightarrow K$ between CW-complexes is *combinatorial* if its restriction to each open cell of L is a homeomorphism onto a cell of K , and a *combinatorial 2-complex* is a CW-complex for which the attaching map $\psi : S^1 \rightarrow K^{(1)}$ of each 2-cell is combinatorial after a suitable subdivision of S^1 (see [13, 38]). Let K be a combinatorial 2-complex. A diagram Δ in K is a combinatorial map $\varphi : M \rightarrow K$ where M is a combinatorial structure on the sphere to which, perhaps, we remove some open 2-cells. This includes spherical diagrams (when M is the whole sphere), (singular) disk diagrams (when M is a sphere with one 2-cell removed), and annular diagrams (the sphere with two 2-cells removed). As usual, 0-cells, 1-cells and 2-cells will be called respectively vertices, edges and faces. If $P = \langle X \mid R \rangle$ is a presentation of a group G , a diagram over P is a diagram $\varphi : M \rightarrow K_P$ where K_P is

the standard 2-complex associated with the presentation P . Throughout this paper, we will assume that the relators of the presentations are cyclically reduced and no relator is a cyclic permutation of another relator or of the inverse of another relator. Since M is orientable we can fix an orientation in the usual way, so that when traversing the boundaries of the 2-cells the edges in the intersection of two faces f, f' are traversed twice, once in each possible orientation. The map $\varphi: M \rightarrow K_P$ induces a labelling on the edges of M by elements of X and their inverses. The label on the boundary of any oriented face of the diagram (starting at any vertex) is called a boundary label. Note that boundary labels are elements in the set R^* of all cyclic permutations of the elements of R and their inverses. A diagram Δ is *reducible* if it contains two faces f, f' such that the intersection of their boundaries $\partial f \cap \partial f'$ contains an edge such that the boundary labels of f and f' read with opposite orientations and starting at a vertex of this edge coincide, otherwise Δ is called *reduced* (see [23, chapter V, §2] for more details). The degree $d(v)$ of a vertex v in a diagram Δ is the number of edges incident to v (the edges with both boundary vertices at v are counting twice). A vertex v is called interior if $v \notin \partial M$.

Given a reduced diagram Δ over a presentation P , we can remove all interior vertices of degree 2. We obtain a new diagram where the interior edges correspond to *pieces* in R^* . Recall that a piece is a word which is a common prefix of two different elements of R^* . The length $\ell(e)$ of an interior edge e in this new diagram is defined as the length of the corresponding word (equivalently, it is the number of edges of the original diagram that were glued together to obtain e). In what follows we consider diagrams with no interior vertices of degree 2.

Recall that the link of a vertex v is an epsilon sphere about v and the corners of the 2-cells at v correspond to edges in the link. The endpoints of a corner in M (of a 2-cell f) at v correspond to edges in the diagram incident to v (see [13, §2] and [38, definition 2.1]). Given a corner c at an interior vertex v , we denote by $\ell_1(c)$ and $\ell_2(c)$ the lengths of the incident edges and by $\ell_r(c)$ the length of the relator $r \in R$ corresponding to the 2-cell f . Let $d'_F(v) = \sum_{c \ni v} (\ell_1(c) + \ell_2(c)) / \ell_r(c)$, where the sum is taken over all corners at v .

DEFINITION 2.1. *We say that a presentation P satisfies the small cancellation condition \mathcal{T}' if for every interior vertex of any reduced diagram over P (with no interior vertices of degree 2), $d'_F(v) \leq d(v) - 2$. Similarly, P satisfies the strict small cancellation condition $\mathcal{T}'_{<}$ if for every interior vertex of any reduced diagram over P , $d'_F(v) < d(v) - 2$. A group G which admits a presentation P satisfying condition \mathcal{T}' (resp. $\mathcal{T}'_{<}$) is called a \mathcal{T}' -group (resp. $\mathcal{T}'_{<}$ -group).*

We investigate now the first examples of presentations satisfying conditions \mathcal{T}' and $\mathcal{T}'_{<}$.

Classical metric small cancellation conditions

It is easy to verify that the classical metric small cancellation conditions $C'(1/6)$, $C'(1/4) - T(4)$ and $C'(1/3) - T(6)$ imply condition $\mathcal{T}'_{<}$. We will show below that finitely presented $\mathcal{T}'_{<} - C(3)$ -groups are hyperbolic, generalizing the classical result for small cancellation groups (see [5, 17]).

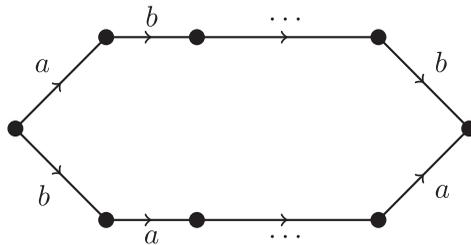


Figure 1. 2-cell of an Artin group.

Two-dimensional Artin groups

Let Γ be a finite simple graph with a labelling on the edges by integers $m \geq 2$. The Artin group defined by Γ is the group A_Γ given by the following presentation P_Γ . The generators of P_Γ are the vertices of Γ , and there is a relation of the form

$$\underbrace{ababa \cdots}_{m \text{ letters}} = \underbrace{babab \cdots}_{m \text{ letters}}$$

for every pair of vertices a and b connected by an edge labelled by m . By results of Charney and Davis [6, 7], it is well-known that an Artin group A_Γ is two-dimensional (i.e. it has geometric dimension 2) if and only if for every triangle in the graph Γ with edges labelled by p, q and r we have $1/p + 1/q + 1/r \leq 1$ (see also [20]).

We will show that an Artin group is two-dimensional if and only if its standard presentation P_Γ satisfies condition \mathcal{T}' . We will also prove below that any group which admits a finite presentation P satisfying conditions \mathcal{T}' and $C'(\frac{1}{2})$ and with all relators of the same length, has quadratic Dehn function and solvable conjugacy problem (see theorem 4.2). These results put together partially recover, with an alternative and simpler proof, similar results for two-dimensional Artin groups recently obtained by Huang and Osajda [20].

THEOREM 2.2. *An Artin group A_Γ is two-dimensional if and only if its standard presentation P_Γ satisfies condition \mathcal{T}' .*

Proof. Let A_Γ be an Artin group and let K be the 2-complex associated with its standard presentation. Note that the 2-cells of K have two distinguished sides in which all edges have the same orientation (see figure 1). If the label of the edge in Γ corresponding to the relator is m , each of these sides has m edges. The terminal vertices of both sides are called initial and final vertices of the relator, according to the orientation of the edges (cf. [20, § 4.1]).

Let $\varphi: M \rightarrow K$ be a reduced diagram. We analyse first the interior vertices of degree 3. It is easy to see that vertices of degree 3 only correspond to intersections of faces in the diagram which are mapped to three different relators that form a triangle in the graph Γ . Since they are three different relators, the length of the three pieces involved is 1. If the labels of the edges in the triangle are p, q and r ,

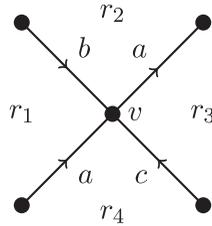


Figure 2. Interior vertex of degree 4.

then the equation for the condition \mathcal{T}' is the following:

$$\frac{1 + 1}{2p} + \frac{1 + 1}{2q} + \frac{1 + 1}{2r} \leq 3 - 2.$$

That is, $1/p + 1/q + 1/r \leq 1$, which is exactly the necessary and sufficient condition for the Artin group to be two-dimensional.

Now we prove that condition \mathcal{T}' is always satisfied in interior vertices of degree greater than or equal to 4 (for any Artin group, not necessarily two-dimensional). Note that such an interior vertex v can be a terminal vertex or it can be inside of one of the sides of the 2-cells containing it. For example, in figure 2, v is a terminal vertex of r_1 and r_4 , and it is on one of the sides of r_2 and r_3 .

If not all of the 2-cells incident to v are mapped to the same relator, at least four of them will share an edge incident to v with a cell corresponding to a different relator. In figure 2, r_1 shares an edge with r_4 , and r_2 with r_3 . Since cells corresponding to different relators intersect at paths of length at most 1, at least four of them will have a piece of length 1. Also, the longest piece in a 2-cell with boundary of length $2n$ is $n - 1$, and therefore the summands in condition \mathcal{T}' can be at most $(2n - 2)/2n$. In conclusion, if there are 2-cells incident to the vertex which are mapped to different relators, there are at least four summands in the equation for condition \mathcal{T}' which are less than or equal to $\frac{1}{2}$, and every other summand is smaller than 1. Then condition \mathcal{T}' is satisfied.

We analyse now the case that the vertex has degree greater than or equal to 4 and all the 2-cells are mapped to the same relator. Observe that if a 2-cell contains v in one of its sides, the summand corresponding to that 2-cell is at most $\frac{1}{2}$. Therefore there are at most three of such 2-cells, and the rest have to contain v as a terminal vertex. If a 2-cell f contains v as a terminal vertex, then the two adjacent 2-cells to f contain the vertex on a side. This implies that we can reduce ourselves to the cases where v has degree 4, 5 or 6.

We look at the orientation of the edges incident to v , traversing them in clockwise order. If we pass by a 2-cell that has v as a terminal vertex, the orientation of these edges is preserved, and if not, it is reversed. Therefore the number of 2-cells having v on one of its sides is even. Therefore, when v has degree 5 or 6, there are at least four 2-cells that have v on a side.

It only remains to check the case where v has degree 4, two of the 2-cells contain it as a terminal vertex and the other two on a side. This situation is illustrated in figure 3. Let l_1, l_2, l_3, l_4 be the lengths of the pieces involved and let $2n$ be the length of the relator.

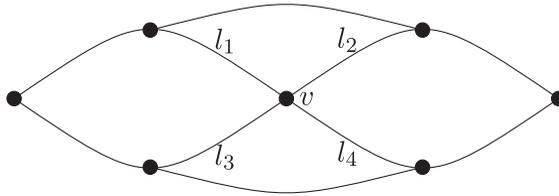


Figure 3. Vertex with degree 4 with the 2-cells mapping to the same relator.

Here condition \mathcal{T}' can be rewritten as:

$$\frac{l_1 + l_2}{2n} + \frac{l_2 + l_4}{2n} + \frac{l_3 + l_4}{2n} + \frac{l_1 + l_3}{2n} = \frac{2l_1 + 2l_2 + 2l_3 + 2l_4}{2n} \leq 2.$$

Since $l_1 + l_2 \leq n$ and $l_3 + l_4 \leq n$, condition \mathcal{T}' is satisfied. □

REMARK 2.3. Combined with theorem 3.2, theorem 2.2 implies that the standard presentation P_Γ of an Artin group is aspherical (i.e. A_Γ is two-dimensional) if and only if P_Γ is diagrammatically reducible (DR, for short). Recall that a presentation P with no proper powers is DR if all spherical diagrams over P are reducible. Note that being DR is in general a stronger condition than being aspherical (see [13,14]).

Let F be a free group on a finite set of generators X . Let $W \subset F$ be a finite subset of words, and let Γ be a finite simple graph with vertex set W and a labelling on the edges by integers $m \geq 2$. We can consider the presentation $P_{\Gamma,F}$ with generators X , and a relation

$$\underbrace{vwvw \dots}_{m \text{ factors}} = \underbrace{wvwv \dots}_{m \text{ factors}}$$

for every pair of words $v, w \in W$ connected by an edge labelled m . We denote by $A_{\Gamma,F}$ the group presented by $P_{\Gamma,F}$. This is obviously a generalization of the definition of an Artin group. The following result generalizes theorem 2.2 for the groups $A_{\Gamma,F}$ under certain restrictions on the words and the labelling of the edges.

THEOREM 2.4. *The presentation $P_{\Gamma,F}$ satisfies \mathcal{T}' provided that all the edges in Γ are labelled by the same integer $m \geq 3$, and the words in W have all the same length, are cyclically reduced and do not share letters (i.e. every generator appears at most in one of the words of W).*

Proof. Let n be the length of the words in W . Let $\varphi: M \rightarrow K$ be a reduced diagram and let v be an interior vertex of degree at least 3. Let $k = d(v)$ and let f_1, \dots, f_k be the 2-cells of M incident to v , numbered clockwise. Note that, since the words in W do not share letters, there is a subset $D(f_i)$ of distinguished vertices of the boundary of f_i which is characterized by the following property: $D(f_i)$ splits the word written in the boundary of f_i into words which belong either to W or to W^{-1} (here W^{-1} denotes the set of the inverses of the words in W). Let x_i (resp. y_i) be the (possibly empty) word read counterclockwise (resp. clockwise) in the boundary of f_i , starting at v and ending at the first occurrence of a vertex which belongs to $D(f_i)$ (see figure 4). Note that $|x_i| + |y_i| \equiv 0 \pmod{n}$. Here $|x|$ denotes the length

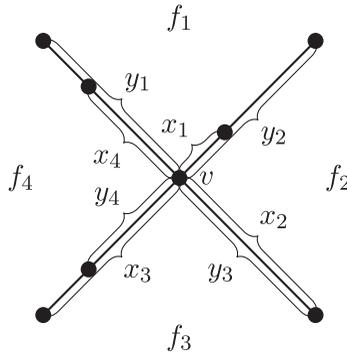


Figure 4. Mismatches around an interior vertex.

of the word x . The *mismatch between f_i and f_{i+1}* is defined as $|y_{i+1}| - |x_i| \in \mathbb{Z}/n\mathbb{Z}$. If the mismatch is nonzero we say that there is a *proper mismatch* between f_i and f_{i+1} . Let s be the number of proper mismatches around v .

If there are no proper mismatches, then the proof that condition \mathcal{T}' is satisfied is analogous to that of theorem 2.2. If there is a proper mismatch between two faces, their intersection has length less than n . Note that the length of a piece is at most $n(m - 1)$, and the length of the relators is $2nm$. Therefore, if $s \geq 2$, condition \mathcal{T}' is satisfied. Therefore, we only need to show that $s \neq 1$. Suppose that there is exactly one proper mismatch, say between f_1 and f_k . Then the following holds:

$$\begin{aligned} |y_1| &\not\equiv |x_k| \pmod{n}, \\ |y_{i+1}| &\equiv |x_i| \pmod{n} \quad \text{for } 1 \leq i \leq k - 1, \\ |x_i| &\equiv -|y_i| \pmod{n} \quad \text{for } 1 \leq i \leq k. \end{aligned}$$

It follows that $|x_1| \not\equiv (-1)^k |x_1| \pmod{n}$. This is a contradiction if k is even. Now we study the case where k is odd and $s = 1$. It is easy to see that if one of the x_i or y_i is empty, then there are at least two proper mismatches, so we can assume that no x_i or y_i is empty. Since the words in W do not share letters, there exists $w \in W$ such that for each $1 \leq i \leq k$, the word $y_i^{-1}x_i$ equals w or w^{-1} . Again, we can assume that the unique proper mismatch is between f_1 and f_k . If there exists $1 \leq i \leq k - 1$ such that $y_i^{-1}x_i = y_{i+1}^{-1}x_{i+1}$, since $x_i = y_{i+1}$, we deduce that $y_i^{-1}x_{i+1} = w^2$ or $y_i^{-1}x_{i+1} = w^{-2}$. This is a contradiction because w is cyclically reduced. Then $y_i^{-1}x_i = (y_{i+1}^{-1}x_{i+1})^{-1}$ for every $1 \leq i \leq k$. However this cannot happen since k is odd. Therefore $s \geq 2$. \square

One-relator groups

In [4] the first two authors introduced a small cancellation condition (T') to study hyperbolicity of one-relator groups. Condition $\mathcal{T}'_{<}$ generalizes condition (T') to any presentation and theorem 3.4 provides an alternative and simpler proof of [4, theorem 3.1] for one-relator groups.

EXAMPLE 2.5. The following one-relator presentation does not satisfy conditions $C(6)$ nor $T(4)$, but it is \mathcal{T}' . It also does not fall under the hypothesis of [4, theorem

3.1] since it is $C'(\frac{1}{2})$, but not $C'(\frac{1}{4})$:

$$\langle a, b \mid a^3b^4a^3b^4(b^4a^3b^4a^3)^{-1} \rangle$$

Cyclic presentations

The claims in the following examples can be verified using the GAP [35] package SmallCancellation [29].

EXAMPLE 2.6. The following cyclic presentation of a superperfect group satisfies condition $\mathcal{T}'_{<}$ but is not $C(6)$ nor $T(4)$:

$$\langle x_0, x_1, x_2, x_3, x_4 \mid x_{i+4}^{-1}x_{i+1}^{-1}x_i^{-1}(x_{i+4}x_{i+1})^2 \text{ for } i = 0, \dots, 4 \rangle$$

EXAMPLE 2.7. The following cyclic presentation of a superperfect group satisfies condition $\mathcal{T}' - C'(\frac{1}{2})$

$$\langle x_0, \dots, x_6 \mid x_{i+1}x_i^{-1}x_{i+6}x_{i+1}^{-1}x_ix_{i+6}^{-1}x_{i+2}^{-1} \text{ for } i = 0, \dots, 6 \rangle$$

Then by theorem 4.2 this group has a quadratic Dehn function. However the group is not $\mathcal{T}'_{<}$, nor $C(6)$, $C(4) - T(4)$ or $C(3) - T(6)$.

3. Non-positive curvature, diagrammatic reducibility and hyperbolicity

We recall first some basic notions on combinatorial curvature. Given a combinatorial 2-complex K , we can assign a real number $w(c)$ to the corners, which we think of as angles. This assignment is a *weight function* for the complex. A finite combinatorial 2-complex together with such a weight function is called an *angled complex* (see [13, 37, 38]).

Let K be an angled complex. If v is a vertex of K , its curvature is defined as

$$\kappa(v) = 2\pi - \pi\chi(\text{lk}_v) - \sum_{c \ni v} w(c).$$

Here $\chi(\text{lk}_v)$ denotes the Euler characteristic of the link of v , and the sum is taken over all corners at v . The curvature of a face f is defined as

$$\kappa(f) = 2\pi - \pi\ell(\partial f) + \sum_{c \in f} w(c),$$

where the sum is taken over all the corners in f and $\ell(\partial f)$ is the number of edges in the boundary of f . The following result can be found in [1, 38].

THEOREM 3.1 (Combinatorial Gauss–Bonnet theorem). *Let K be an angled 2-complex. Then*

$$\sum_{f \in \text{faces}(K)} \kappa(f) + \sum_{v \in \text{vertices}(K)} \kappa(v) = 2\pi\chi(K).$$

Assignment of weight functions

Let P be a presentation satisfying condition \mathcal{T}' or $\mathcal{T}'_<$. Given a reduced diagram $f: M \rightarrow K_P$ over P , we define the following weight function in M . The weight of a corner c at an interior vertex v is $w(c) = \pi - ((\ell_1(c) + \ell_2(c))/\ell_r(c))\pi$ (recall that there are no interior vertices of degree 2). The weight of a corner c at a vertex $v \in \partial M$ of degree 2 is $w(c) = \pi$. If c is a corner at a vertex $v \in \partial M$ of degree greater than 2, we define $\ell_1(c)$ and $\ell_2(c)$ similarly as we did with interior vertices (the lengths of the incident edges obtained if we remove the vertices of degree 2) and $w(c) = \pi - ((\ell_1(c) + \ell_2(c))/\ell_r(c))\pi$.

With this assignment, the curvature of the faces of M is 0, and the curvature of the interior vertices is non-positive if P satisfies condition \mathcal{T}' , and strictly negative if P satisfies condition $\mathcal{T}'_<$.

We will show that presentations satisfying condition \mathcal{T}' and without proper powers are DR and that finitely presented $\mathcal{T}'_<$ - $C(3)$ -groups are hyperbolic. Given a 2-complex M , we denote by $\mathcal{V}(M)$, $\mathcal{E}(M)$ and $\mathcal{F}(M)$ the number of vertices, edges and faces of M respectively.

THEOREM 3.2. *If a presentation P satisfies condition \mathcal{T}' and has no proper powers, then it is DR.*

Proof. Since P has no proper powers, our notion of reduced spherical diagram over P coincides with that of [13]. Therefore, in order to prove that P is DR, we only have to verify that there are no reduced spherical diagrams over P . Suppose $\varphi: M \rightarrow K_P$ is a reduced spherical diagram. We have the following identities:

$$\mathcal{E}(M) = \frac{1}{2} \sum_{v \in \text{vertices}(M)} d(v),$$

$$\mathcal{F}(M) = \frac{1}{2} \sum_{v \in \text{vertices}(M)} d'_F(v).$$

The first one is clear, since every edge is incident to two vertices. The second one is deduced from the fact that in the right-hand side of the second equality we are summing two times the length of each relator, divided by the length of each relator. That is, we are summing 2 for each face. Then

$$\begin{aligned} 2 &= \mathcal{V}(M) - \mathcal{E}(M) + \mathcal{F}(M) \\ &= \mathcal{V}(M) - \frac{1}{2} \sum_{v \in \text{vertices}(M)} d(v) + \frac{1}{2} \sum_{v \in \text{vertices}(M)} d'_F(v) \leq 0, \end{aligned}$$

where the last inequality holds because all the vertices in the sphere are interior vertices. This is a contradiction, and therefore P is diagrammatically reducible. \square

Equations over groups

Recall that a system of equations over a group G with unknowns x_1, x_2, \dots, x_n is a set $\{w_j(x_1, x_2, \dots, x_n)\}_j$ of words in $G * F(x_1, \dots, x_n)$. Here $F(x_1, \dots, x_n)$

is the free group with basis $\{x_1, x_2, \dots, x_n\}$. The letters of w_j which lie in G are the *coefficients* of w_j . The (non-necessarily reduced) word r_j in the alphabet $\{x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}\}$ obtained by deleting the coefficients of w_j is the *shape* of w_j , and the word r_j considered as an element of the free group $F(x_1, x_2, \dots, x_n)$ is called the *content* of w_j . We say that the system has a solution in an overgroup of G if there exists a group H which contains G as a subgroup and elements h_1, h_2, \dots, h_n in H such that

$$w_j(h_1, h_2, \dots, h_n) = 1 \in H$$

for every j . The Kervaire–Laudenbach conjecture states that for any group G , a unique equation w with a unique unknown x has a solution in an overgroup of G if w is non-singular, which means that the total exponent of x is non-zero. The so called Kervaire–Laudenbach–Howie conjecture generalizes this to an arbitrary finite number n of unknowns and a non-singular system of m equations (in this case, non-singular means that the rank of the $m \times n$ matrix of total exponents is equal to m).

Let S be a system of equations w_1, w_2, \dots, w_m over a group G . Let P be the presentation $\langle x_1, x_2, \dots, x_n | r_1, r_2, \dots, r_m \rangle$ whose generators are the unknowns of S and its relators are the shapes of the equations w_j . A well-known result by Gersten [13] states that if P is DR, then S has a solution in an overgroup of G .

In the case of one equation w with many unknowns, by a result attributed to Pride, for any coefficient group G , if the shape of w is cyclically reduced (and non-trivial) then the equation has a solution in an overgroup of G (see [13, corollary 5.7]). Recently Klyachko and Thom [21] proved that one equation in two variables with coefficients in a hyperlinear group G can be solved over G if its content does not lie in $[F_2, [F_2, F_2]]$. Here F_2 denotes the free group generated by the two unknowns. More recently Barmak and Minian analysed some cases which are not covered by the result of [21] by applying a test that is not based on curvature (see [2, §7]).

From the results above we deduce the following.

PROPOSITION 3.3. *Let $k, l, m \in \mathbb{N}$ and $w_1(x_1, \dots, x_k), w_2(y_1, \dots, y_l), w_3(z_1, \dots, z_m)$ be cyclically reduced words of the same length in the unknowns x_i, y_j, z_s . Then for any $p \geq 2$, the presentation*

$$P = \langle x_1, \dots, x_k, y_1, \dots, y_l, z_1, \dots, z_m | (w_1 w_2)^p (w_2 w_1)^{-p}, (w_2 w_3)^p (w_3 w_2)^{-p}, (w_3 w_1)^p (w_1 w_3)^{-p} \rangle$$

is DR. Therefore for any group G , any system of equations modelled by P has a solution in an overgroup of G .

Proof. The presentation P corresponds to a presentation of type $P_{\Gamma, F}$ where Γ is a triangle with vertices w_1, w_2 and w_3 and whose three edges are labelled by $2p$. Now the result follows from theorems 2.4 and 3.2 and from Gersten’s result mentioned above. □

Hyperbolicity

We now show that the strict metric condition $\mathcal{T}'_{<}$ implies hyperbolicity.

THEOREM 3.4. *Let G be a group which admits a finite presentation satisfying conditions $\mathcal{T}'_<$ and $C(3)$. Then G is hyperbolic.*

Proof. We show that a finite presentation P satisfying conditions $\mathcal{T}'_<$ and $C(3)$ has a linear isoperimetric inequality. Note that it suffices to consider non-singular disk diagrams.

Let $\varphi: M \rightarrow K_P$ be a non-singular reduced disk diagram. We assign weights to the corners in M as we did before. Then, by the Combinatorial Gauss–Bonnet theorem,

$$2\pi = \sum_{v \in \text{vertices}(M)} \kappa(v) + \sum_{f \in \text{faces}(M)} \kappa(f) = \sum_{v \in \text{vertices}(M)} \kappa(v).$$

Since P satisfies $\mathcal{T}'_<$, then $\kappa(v) < 0$ for every interior vertex v , and since P is finite, by corollary 5.3 there is a constant $N < 0$, which is independent of the diagram, such that $\kappa(v) \leq N$ for every interior vertex v . Also, for every boundary vertex v , it holds that $\kappa(v) < \pi$ since the weights in M are non-negative. Then

$$\begin{aligned} 2\pi &\leq \mathcal{V}^\circ(M)N + \sum_{v \in \text{vertices}(\partial M)} \kappa(v) \\ &\leq \mathcal{V}^\circ(M)N + \mathcal{V}(\partial M)\pi \\ &= \mathcal{V}(M)N + \ell(\partial M)(\pi - N), \end{aligned}$$

where $\mathcal{V}^\circ(M)$ denotes the number of interior vertices of M , and $\ell(\partial M)$ is the length of the boundary. The last equality holds because M is non-singular. Then,

$$-\mathcal{V}(M)N \leq \ell(\partial M)(\pi - N) - 2\pi$$

and therefore

$$\mathcal{V}(M) \leq \ell(\partial M) \frac{\pi - N}{-N} + \frac{2\pi}{N}.$$

Now, since P satisfies condition $C(3)$, the number of faces can be linearly bounded by the number of vertices in the diagram. Consequently, the number of faces in the diagram is linearly bounded by the length of its boundary. □

4. Quadratic Dehn function and conjugacy problem

In this section, we will show that a finitely presented group which admits a presentation P satisfying conditions \mathcal{T}' and $C'(\frac{1}{2})$ and such that all relators of P have the same length r , has a quadratic Dehn function and solvable conjugacy problem.

Let $\varphi: M \rightarrow K_P$ be a diagram over P . The *boundary layer* L of M consists of every vertex in the boundary of M , every edge incident to a vertex in the boundary, and every open face with a vertex in the boundary. Note that L is usually not a combinatorial complex. Let $M_1 = M \setminus L$ be the complement of the boundary layer. Note that M_1 is a subcomplex of M . The following lemma will be used to prove the main result of this section.

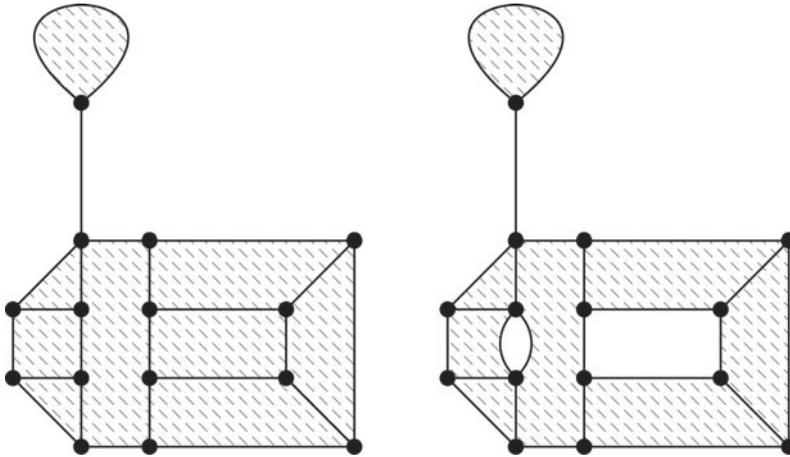


Figure 5. At the left the complex M . At the right the complex B constructed from M in the proof of lemma 4.1.

LEMMA 4.1. *Let P be a presentation satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$ and such that all its relators have length r , and let $\varphi: M \rightarrow K_P$ be an annular or disk diagram over P . Then*

$$\mathcal{V}(\partial M_1) \leq \mathcal{V}(\partial M) - r\chi(M).$$

Proof. We had previously removed interior vertices of degree 2 from the diagrams. We subdivide the boundary of M_1 reintroducing the vertices of degree 2. We still denote this diagram by M .

For the vertices of M of degree greater than 2, we assign weights to the corners as before, and for vertices of degree 2, both weights are equal to π . With this assignment, every face has curvature 0 and all interior vertices have non-positive curvature, since P satisfies \mathcal{T}' (see the paragraph after theorem 3.1). Note that $\kappa(v) = 0$ for interior vertices of degree 2.

In what follows, we can assume without loss of generality that M is non-singular, since we are going to bound the length of the boundary of M_1 in terms of the length of the boundary of M . Since P is $C'(\frac{1}{2})$ we may assume that each boundary 2-cell f has at least two edges which are not on the boundary of M , for otherwise we can remove f decreasing the length of the boundary and without changing M_1 . In particular this reduction allows us to assume that $M_1 \neq \emptyset$.

We consider the complex B constructed by taking the disjoint union of the 0-cells, 1-cells and 2-cells (now closed) of the boundary layer of M and identifying the boundaries of the closed 2-cells but only in the vertices and edges of the boundary layer of M (see figure 5).

We omit vertices of degree 2 in the cell structure of B . Note that $\ell(\partial B) = \ell(\partial M) + \ell(\partial M_1)$. If M is a disk, B is a planar and connected combinatorial complex, so its Euler characteristic is less than or equal to $1 = \chi(M)$. If M is an annulus, B may have more than one connected component, but none of them would be a disk,

since they all have a disconnected complement. Therefore its Euler characteristic is less than or equal to $0 = \chi(M)$.

We separate its vertices into two sets: V_1 will denote the set of vertices of B that are in the boundary of M , and V_2 the set of remaining vertices of B . Since P satisfies condition \mathcal{T}' , by Gauss–Bonnet we have

$$2\pi\chi(M) \leq \sum_{v \in V_1} \kappa(v).$$

Also by Gauss–Bonnet, we have

$$\sum_{v \in \text{vertices}(B)} \kappa(v) = 2\pi\chi(B).$$

Therefore

$$\sum_{v \in V_2} \kappa(v) \leq 0.$$

Now since each boundary 2-cell has at least two edges which are not on the boundary of M we have

$$\mathcal{V}_1 + \sum_{v \in V_2} \sum_{c \ni v} 1 = \mathcal{V}(B) + \mathcal{E}(B) - \ell(\partial B) = \mathcal{V}_2 + \sum_{v \in V_1} \sum_{c \ni v} 1 - \mathcal{V}^\circ(B).$$

Now putting everything together

$$\begin{aligned} 2\pi\chi(M) &\leq \sum_{v \in V_1} \kappa(v) - \sum_{v \in V_2} \kappa(v) \\ &= \sum_{v \in V_1} \left(\pi - \sum_{c \ni v} \left(\pi - \frac{\ell_1(c) + \ell_2(c)}{r} \pi \right) \right) \\ &\quad - \sum_{v \in V_2} \left(\pi - \sum_{c \ni v} \left(\pi - \frac{\ell_1(c) + \ell_2(c)}{r} \pi \right) \right) \\ &= \sum_{v \in V_1} \sum_{c \ni v} \frac{\ell_1(c) + \ell_2(c)}{r} \pi - \sum_{v \in V_2} \sum_{c \ni v} \frac{\ell_1(c) + \ell_2(c)}{r} \pi \\ &= \frac{2\mathcal{V}(\partial M) - 2\mathcal{V}(\partial M_1)}{r} \pi. \end{aligned}$$

It follows that

$$\mathcal{V}(\partial M_1) \leq \mathcal{V}(\partial M) - r\chi(M). \quad \square$$

THEOREM 4.2. *Let P be a presentation satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$ and such that all its relators have length r , then P has a quadratic Dehn function. Moreover, if P is finite the group G presented by P has solvable conjugacy problem.*

Proof of the quadratic Dehn function. Let $\varphi: M \rightarrow K_P$ be a reduced disk diagram. We show that $\mathcal{V}(M) \leq (1/r)\mathcal{V}(\partial M)^2$ by induction in the number of interior vertices of M . We have

$$\begin{aligned} \mathcal{V}(M) &= \mathcal{V}(M_1) + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}\mathcal{V}(\partial M_1)^2 + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}(\mathcal{V}(\partial M) - r)^2 + \mathcal{V}(\partial M) \\ &= \frac{1}{r}\mathcal{V}(\partial M)^2 - 2\mathcal{V}(\partial M) + r + \mathcal{V}(\partial M) \\ &\leq \frac{1}{r}\mathcal{V}(\partial M)^2. \end{aligned}$$

The first inequality follows by induction, and the second one follows from lemma 4.1. Finally, since P satisfies $C'(\frac{1}{2})$, each face of M has at least three sides. Then we can bound the number of faces of M by the number of vertices of M , obtaining the desired quadratic isoperimetric inequality. \square

It follows that the finitely generated groups which admit presentations satisfying the hypotheses of theorem 4.2 have solvable word problem. This will be used to prove that they also have solvable conjugacy problem.

We attack the conjugacy problem following the strategy of [23, § V.7]).

REMARK 4.3. Let $P = \langle X \mid R \rangle$ be a finite presentation of a group G with solvable word problem. Suppose that all the relators have the same length r . Let w_1 and w_2 be words in the free group $F(X)$. We write $w_1 \sim w_2$ if there exists a word b in $F(X)$ with $|b| < r$ such that $bw_1b^{-1}w_2^{-1} = 1$ in G . Here $|b|$ denotes the length of the word b . Since the word problem is solvable, the relation \sim is decidable. Now let u and v be cyclically reduced words in $F(X)$ and let $d = |u| + |v|$. Take $W = \{w \in F(X), |w| \leq d\}$. Note that W is finite since X is finite. Note also that the set W depends on the lengths of u and v and that $u, v \in W$. We write $u \approx v$ if there exist words w_1, \dots, w_k in W such that $u \sim w_1 \sim \dots \sim w_k \sim v$. Equivalently, \approx is the transitive closure in W of the relation \sim . Note that this relation is also decidable since W is finite. In order to prove that the conjugacy problem is solvable it suffices to prove that if two words $u, v \in F(X)$ are conjugate in G , then $u \approx v$.

We will also use the following result of Schupp [23, § V.7]. Let A be an annular diagram and L its boundary layer. The diagram $A_1 = A \setminus L$ (the complement of L in A) may be disconnected, but it has at most one annular component. A simply connected component of A_1 is called a *gap*. Let K_1, \dots, K_n be the gaps. Then $H = A \setminus (L \cup \bigcup_{i=1}^n K_i)$ is the annular component of A_1 , assuming there is any. Let σ and τ be the outer and inner boundaries of H . A pair (D_1, D_2) of faces (not necessarily distinct) in A is called a *boundary linking pair* if $\sigma \cap \partial D_1 \neq \emptyset$, $\partial D_1 \cap \partial D_2 \neq \emptyset$, and $\partial D_2 \cap \tau \neq \emptyset$.

LEMMA 4.4 (Schupp). *Let A be an annular diagram having at least one region, and let H be the diagram obtained by removing its boundary layer and its gaps. If there are no boundary linking pairs, H is an annular diagram.*

Proof of the conjugacy problem. Take $u, v \in F(X)$ cyclically reduced and suppose that they are conjugate in G . Let $d = |u| + |v|$. By remark 4.3, we only have to prove that $u \approx v$. Let A be an annular diagram with u and v^{-1} as inner and outer boundaries. Construct the diagrams $A = H_0, H_1, \dots, H_k$, where H_{i+1} is obtained from H_i by removing its boundary layer and its gaps, and let H_k be the first of such diagrams with a linking pair. By lemma 4.1, $\ell(\partial H_{i+1}) \leq \ell(\partial H_i)$ for each $0 \leq i \leq k - 1$. Therefore, $\ell(\partial H_i) \leq d$ for every i (and so the boundary labels of ∂H_i are in the set W).

Let σ_i and τ_i be the outer and inner boundaries of H_i respectively. Let S_i be the subdiagram of M consisting of σ_i, σ_{i+1} and all the cells of M between these two paths. Define T_i in the same manner with respect to τ_i and τ_{i+1} . It is clear that any boundary face of S_i intersects both boundaries of S_i . So there is a path γ_i from σ_i to σ_{i+1} with a label of length less than or equal to r . Let s_i and s_{i+1}^{-1} be the labels of σ_i and σ_{i+1} starting at a given vertex. Then $s_i \sim s_{i+1}$. Analogously, we have $t_i \sim t_{i+1}$ where t_i^{-1} and t_{i+1} are the labels of T_i .

The last annulus H_k has a boundary linking pair (D_1, D_2) . We have vertices $v_0 \in \sigma_k \cap \partial D_1, v_1 \in \partial D_1 \cap \partial D_2,$ and $v_2 \in \partial D_2 \cap \tau_k$. Therefore there are paths β_1 and β_2 from v_0 to v_1 and from v_1 to v_2 labelled by words b_1 and b_2 of length smaller than or equal to $r/2$. Let $\beta = \beta_1\beta_2$, then its label is a word of length less than r . Let s be the word read in the outer boundary of H_k starting at v_0 , and t^{-1} the word read in the inner boundary of H_k starting at v_2 . We have that $sb_1b_2t^{-1}b_2^{-1}b_1^{-1} = 1$ in G . Then $s \sim t$.

Since s_0 and t_0 are cyclic permutations of u and v respectively, and s and t are cyclic permutations of s_k and t_k respectively, we have

$$u \approx s_0 \sim s_1 \sim \dots \sim s_k \approx s \sim t \approx t_k \sim \dots \sim t_0 \approx v. \quad \square$$

A slight modification in the proof of lemma 4.1 allows one to obtain a lower bound on the length of the words which represent the trivial element in the group G , even if the relators have different lengths.

PROPOSITION 4.5. *Let $P = \langle X \mid R \rangle$ be a presentation of a group G , satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$. Let r_{\min} be the length of the shortest relator. Then any nontrivial word W in the free group generated by X representing the trivial element in G has length at least r_{\min} . In particular, if P has a relator of length greater than or equal to 2, then G is nontrivial.*

Proof. Let M be a reduced disk diagram. We follow the same steps as in the proof of lemma 4.1 and we get that

$$2\pi\chi(M) \leq \sum_{v \in V_1} \sum_{c \ni v} \frac{\ell_1(c) + \ell_2(c)}{l_r(c)} \pi - \sum_{v \in V_2} \sum_{c \ni v} \frac{\ell_1(c) + \ell_2(c)}{l_r(c)} \pi.$$

In particular since the terms $\ell_i(c)/l_r(c)$ in the first sum which do not correspond to edges in the boundary cancel with terms in the second sum, we have $r_{\min}\chi(M) \leq$

$V(\partial M)$. Therefore, since M is a disk, $r_{\min} \leq V(\partial M)$, which implies that words representing the trivial element have length at least r_{\min} .

For the second statement, by removing all the relators of length 1 along with the corresponding generators, we can assume that each relator of P has length at least 2. Note that condition $C'(\frac{1}{2})$ guarantees that each of these generators can appear in only one relator. □

In theorem 4.2 we proved the existence of quadratic Dehn functions and solvability of the conjugacy problem for presentations satisfying conditions $\mathcal{T}' - C'(\frac{1}{2})$, provided the relators have the same length. We believe that the result is still valid without the assumption on the lengths of the relators. We discuss now a strategy to prove solvability of the word problem for a wider class of groups. Given a presentation $P = \langle X \mid R \rangle$ of a group G , our aim is to obtain a new presentation P' of a group H such that G embeds in H , and such that P' satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$ and the relators have the same length. By theorem 4.2, this would imply that H , and therefore G , has solvable word problem. To do so, we will choose a positive integer n_x for some $x \in X$ and replace every occurrence of x in the relators by x^{n_x} . If the element x in the group G has infinite order, this corresponds to adding an n_x -th root, or equivalently, to taking the amalgamated product of G with \mathbb{Z} along the subgroup $n_x\mathbb{Z}$. If we make these replacements for a finite number of $x \in X$, we obtain a new presentation P' of an overgroup H of G . Of course, it is not always possible to choose the n_x so that all the relators in the new presentation have the same length and even if this is possible, the presentation P' obtained may not satisfy conditions $\mathcal{T}' - C'(\frac{1}{2})$ (even if P does). The following example illustrates this technique.

EXAMPLE 4.6. Consider the following presentation:

$$P = \langle a, b, c, s, t \mid tats^{-1}b^{-1}s^{-1}, tbt s^{-1}c^{-2}s^{-1}, tc^2ts^{-1}a^{-1}s^{-1} \rangle.$$

Note that the relators do not have the same length. This presentation does not satisfy conditions $C(5)$, $T(4)$ nor \mathcal{T}' . Now, it is easy to see that a and b have infinite order in the group G presented by P , and by choosing $n_a = 2$ and $n_b = 2$, we obtain the following presentation:

$$P' = \langle a, b, c, s, t \mid ta^2ts^{-1}b^{-2}s^{-1}, tb^2ts^{-1}c^{-2}s^{-1}, tc^2ts^{-1}a^{-2}s^{-1} \rangle.$$

Note that all its relators have the same length. One can verify that P' satisfies conditions $\mathcal{T}' - C'(\frac{1}{2})$ (although it does not satisfy conditions $C(5)$ nor $T(4)$). This implies that G has solvable word problem.

5. Verifying condition \mathcal{T}'

In this section, we give an algorithm to verify if a finite presentation $P = \langle X \mid R \rangle$ satisfies condition \mathcal{T}' . Note that a priori it is not clear that such an algorithm exists, since the definition involves checking something for every possible diagram over P . The algorithm described here has been implemented in the GAP[35] package SmallCancellation [29].

We describe a weighted directed graph $\Gamma(P)$. The vertices of this graph are the tuples (r, p, q) such that

- $r \in R^*$,
- p and q are pieces and
- we can write $r = qsp$ without cancellations.

There is an edge $(r, p, q) \rightarrow (r', p', q')$ if

- $p' = q^{-1}$ and
- $r' \neq r^{-1}$.

The weight of this edge is $1 - (|p| + |q|)/|r|$ (by simplicity, we divide by π the weights that we considered in §3). The weight of a cycle is the sum of the weights of its edges.

Given a diagram $\varphi: M \rightarrow K_P$, we can fix an orientation in M as explained in §2. The corners in the diagram inherit the orientations of the corresponding faces. Note that if c is a corner at an interior vertex $v(c)$, then $(r(c), w_1(c), w_2(c))$ is a vertex in $\Gamma(P)$. Here $r(c)$ denotes the relator read in the boundary of the face, starting from the vertex $v(c)$ and following the orientation of the face, $w_1(c)$ and $w_2(c)$ are the subwords written in the edges of the oriented corner (the first edge being the one oriented towards $v(c)$). This remark and the following proposition make clear why this graph is meaningful.

PROPOSITION 5.1.

- (i) Let v be an interior vertex in a reduced diagram $\varphi: M \rightarrow K_P$. Then there is a directed cycle γ in $\Gamma(P)$ of length at least 3 and weight $d(v) - d'_F(v)$.
- (ii) Let γ be a directed cycle in $\Gamma(P)$ of length at least 3 and weight w . Then there is a reduced diagram over P and an interior vertex v such that $d(v) - d'_F(v) = w$.

Proof. We first prove (i). Let v be an interior vertex in a reduced diagram $\varphi: M \rightarrow K_P$. Let c_1, \dots, c_n be the corners around v , numbered clockwise. Then $w_1(c_{i+1}) = w_2(c_i)^{-1}$ (indices are modulo n). Since the diagram is reduced we have $r(c_{i+1})^{-1} \neq r(c_i)^{-1}$ and therefore there is an edge

$$(r(c_i), w_1(c_i), w_2(c_i)) \xrightarrow{e_i} (r(c_{i+1}), w_1(c_{i+1}), w_2(c_{i+1}))$$

in $\Gamma(P)$ with weight $1 - (\ell_1(c_i) + \ell_2(c_i))/\ell_r(c_i)$. Then the cycle $\gamma = (e_1, \dots, e_n)$ has weight $d(v) - d'_F(v)$.

We now prove (ii). Let $n \geq 3$ and let γ be a cycle in $\Gamma(P)$ of length n . By the first condition for the edges of $\Gamma(P)$, the vertices of γ can be named $(r_1, p_1, p_2^{-1}), (r_2, p_2, p_3^{-1}), \dots, (r_n, p_n, p_1^{-1})$. For each i we consider the word s_i such that $r_i = p_{i+1}^{-1} s_i p_i$ without cancellations. We construct a disk diagram Δ with $n + 1$ vertices, $2n$ edges and n faces as follows. The vertices of Δ will be denoted by v, v_1, \dots, v_n . For each i the diagram has an edge $v_i \xrightarrow{e_i} v$ which reads p_i and an edge $v_{i+1} \xrightarrow{\alpha_i} v_i$

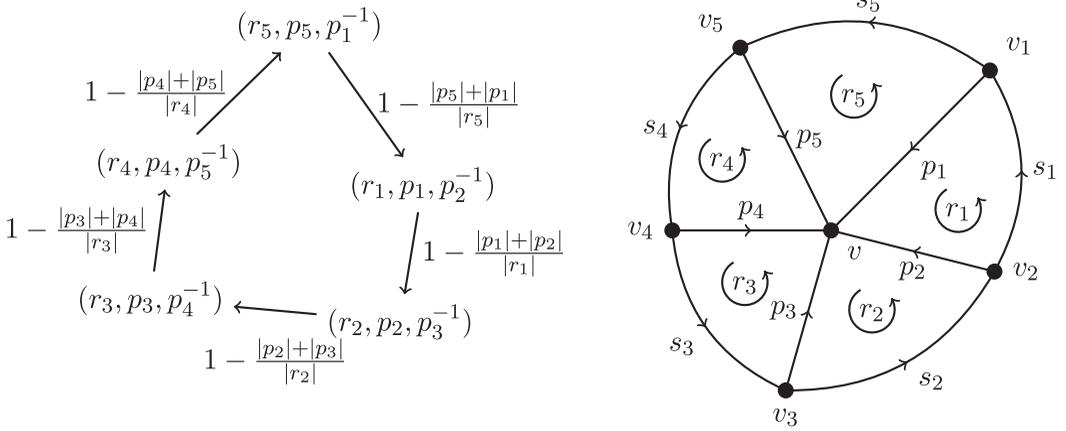


Figure 6. On the left a cycle γ in $\Gamma(P)$, on the right the corresponding diagram constructed in the proof of part(ii) of proposition 5.1.

which reads s_i . For each i there is a face f_i attached with boundary $(e_{i+1}^{-1}, \alpha_i, e_i)$ which reads r_i (starting at v) (see figure 6). By the second condition for an edge in $\Gamma(P)$, the diagram is reduced. Note that by construction, $d(v) - d'_F(v)$ is the weight of γ . □

COROLLARY 5.2. *A presentation P satisfies condition \mathcal{T}' if and only if each directed cycle in $\Gamma(P)$ of length at least 3 has weight greater than or equal to 2. A presentation P satisfies condition $\mathcal{T}'_<$ if and only if each directed cycle in $\Gamma(P)$ of length at least 3 has weight greater than 2.*

Note that corollary 5.2 gives an algorithm to check if a finite presentation satisfies \mathcal{T}' , for it is possible to use Dijkstra’s algorithm to find the least weight of a directed cycle of length at least k in a directed graph with positive edge weights. This can be done by constructing an auxiliary graph having $(k + 1)$ vertices for each vertex in the original graph. For more details on this see the implementation in the GAP package `SmallCancellation` [29].

From proposition 5.1 we deduce the following result, which is used in the proof of theorem 3.4.

COROLLARY 5.3. *If a finite presentation P satisfies condition $\mathcal{T}'_<$ there is a constant $N < 0$ such that $\kappa(v) \leq N$ for every diagram Δ and every interior vertex $v \in \Delta$.*

Proof. Since the weights are positive, we can take N to be $-\pi$ times the minimum weight of a simple directed cycle of length at least 3 in $\Gamma(P)$. Note that, since the graph $\Gamma(P)$ is finite, there is a finite number of such cycles. □

The following examples of groups which do not satisfy \mathcal{T}' are consistent with our conjecture that $\mathcal{T}' - C'(\frac{1}{2})$ implies a quadratic isoperimetric inequality even if the presentation has relators of different lengths.

EXAMPLE 5.4. The presentations A and B from [36] have unsolvable word problem. A GAP computation using SmallCancellation shows that these presentations do not satisfy \mathcal{T}' .

EXAMPLE 5.5. From [3] we know the Baumslag–Solitar group $BS(p, q)$ has exponential Dehn function if $|p| \neq |q|$. Therefore, by theorem 4.2 the groups $BS(n, n+1)$ do not satisfy \mathcal{T}' . It can be seen that the minimum of $d(v) - d'_F(v)$ for v an interior vertex in a diagram for the usual presentation of $BS(n, n+1)$ is $2 - 1/(2n+3)$, which tends to 2 as $n \rightarrow \infty$. Note that $BS(n, n)$ satisfies $\mathcal{T}' - C'(\frac{1}{2})$ so by theorem 4.2 one can verify the well-known fact that these groups have quadratic Dehn function.

EXAMPLE 5.6. In [3, Lemma 11] a family of groups $M_{c,d}$ is considered and it is proved that the Dehn function of $M_{c,d}$ has order n^{c+d} . We have that $M_{1,1}$ (which is a RAAG) satisfies $\mathcal{T}' - C'(\frac{1}{2})$. Some GAP computations suggest that the minimum of $d(v) - d'_F(v)$ for these groups is $\frac{23}{20}$ for any $(c, d) \neq (1, 1)$.

EXAMPLE 5.7. In [3] it is proved that the Dehn function of the group $E = \langle b, s, t \mid s^{-1}bs = b^2, t^{-1}bt = b \rangle$ is at least 2^n . This group does not satisfy \mathcal{T}' (the minimum of $d(v) - d'_F(v)$ is $\frac{8}{5}$).

EXAMPLE 5.8. In [12, chapter 8] it is shown that the Heisenberg group $\langle a, b, z \mid z = [a, b], [a, z], [b, z] \rangle$ is not automatic. In [11] it is shown that the Heisenberg group admits a cyclic presentation

$$P = \langle x_0, x_1 \mid x_0^{-1}x_1x_0x_1^{-1}x_0^{-1}x_1^{-1}x_0x_1, x_1^{-1}x_0x_1x_0^{-1}x_1^{-1}x_0^{-1}x_1x_0 \rangle.$$

For this presentation the minimum of $d(v) - d'_F(v)$ is 0. This implies that the presentation is not DR, there is a reduced spherical diagram with two 0-cells.

It would be interesting to know more about what the minimum of $d(v) - d'_F(v)$ says about a presentation.

Acknowledgments

Researchers of CONICET; partially supported by grants PIP 11220170100357, PICT 2017-2997, PICT-2017-2806 and UBACYT 20020160100081BA.

References

- 1 W. Ballmann and S. Buyalo. Nonpositively curved metrics on 2-polyhedra. *Math. Z.* **222** (1996), 97–134.
- 2 J. A. Barmak and E. G. Minian. A new test for asphericity and diagrammatic reducibility of group presentations. *Proc. R. Soc. Edin. Sect. A* **150** (2020), 871–895.
- 3 G. Baumslag, C. F. III Miller and H. Short. Isoperimetric inequalities and the homology of groups. *Invent. Math.* **113** (1993), 531–560.
- 4 M. A. Blufstein and E. G. Minian. Strictly systolic angled complexes and hyperbolicity of one-relator groups. Preprint, <https://arxiv.org/abs/1907.06738>, 2019.
- 5 M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] (Berlin: Springer-Verlag, 1999).
- 6 R. Charney and M. W. Davis. The $K(\pi, 1)$ -problem for hyperplane complements associated to infinite reflection groups. *J. Am. Math. Soc.* **8** (1995), 597–627.

- 7 R. Charney and M. W. Davis. Finite $K(\pi, 1)$ s for Artin groups. In Frank Quinn (ed.), *Prospects in topology (Princeton, NJ, 1994)*, volume 138 of *Ann. Math. Stud.*, pp. 110–124 (Princeton, NJ: Princeton Univ. Press, 1995).
- 8 A. Chermak. Locally non-spherical Artin groups. *J. Algebra* **200** (1998), 56–98.
- 9 M. Dehn. Über unendliche diskontinuierliche Gruppen. *Math. Ann.* **71** (1911), 116–144.
- 10 M. Dehn. Transformation der Kurven auf zweiseitigen Flächen. *Math. Ann.* **72** (1912), 413–421.
- 11 M. Edjvet. On irreducible cyclic presentations. *J. Group Theory* **6** (2003), 261–270.
- 12 D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. V. F. Levy, M. S. Paterson and W. P. Thurston. *Word processing in groups* (Boston, MA: Jones and Bartlett Publishers, 1992).
- 13 S. M. Gersten. Reducible diagrams and equations over groups. In S. M. Gersten (ed.), *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pp. 15–73 (New York: Springer, 1987).
- 14 S. M. Gersten. Branched coverings of 2-complexes and diagrammatic reducibility. *Trans. Am. Math. Soc.* **303** (1987), 689–706.
- 15 M. Greendlinger. Dehn’s algorithm for the word problem. *Commun. Pure Appl. Math.* **13** (1960), 67–83.
- 16 M. Greendlinger. On Dehn’s algorithms for the conjugacy and word problems, with applications. *Commun. Pure Appl. Math.* **13** (1960), 641–677.
- 17 M. Gromov. Hyperbolic groups. In S. M. Gersten (ed.), *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pp. 75–263 (New York: Springer, 1987).
- 18 M. Gromov. Random walk in random groups. *Geom. Funct. Anal.* **13** (2003), 73–146.
- 19 D. Gruber. Groups with graphical $C(6)$ and $C(7)$ small cancellation presentations. *Trans. Am. Math. Soc.* **367** (2015), 2051–2078.
- 20 J. Huang and D. Osajda. Metric systolicity and two-dimensional Artin groups. *Math. Ann.* **374** (2019), 1311–1352.
- 21 A. Klyachko and A. Thom. New topological methods to solve equations over groups. *Algebr. Geom. Topol.* **17** (2017), 331–353.
- 22 R. C. Lyndon. On Dehn’s algorithm. *Math. Ann.* **166** (1966), 208–228.
- 23 R. C. Lyndon and P. E. Schupp. Classics in Mathematics. *Combinatorial group theory* (Berlin: Springer-Verlag, 2001). Reprint of the 1977 edition.
- 24 Y. Ollivier. On a small cancellation theorem of Gromov. *Bull. Belg. Math. Soc. Simon Stevin* **13** (2006), 75–89.
- 25 A. Y. Olshanskiĭ. *Geometry of defining relations in groups*, volume 70 of *Mathematics and its Applications (Soviet Series)* (Dordrecht: Kluwer Academic Publishers Group, 1991). Translated from the 1989 Russian original by Yu. A. Bakhturin.
- 26 E. Rips. Generalized small cancellation theory and applications. I. The word problem. *Israel J. Math.* **41** (1982), 1–146.
- 27 E. Rips. Another characterization of finitely generated groups with a solvable word problem. *Bull. London Math. Soc.* **14** (1982), 43–44.
- 28 E. Rips. Subgroups of small cancellation groups. *Bull. London Math. Soc.* **14** (1982), 45–47.
- 29 I. Sadofski Costa. SmallCancellation – Metric and nonmetric small cancellation conditions, Version 1.0.4. GAP package, <https://doi.org/10.5281/zenodo.3906472> DOI: 10.5281/zenodo.3906472, 2020.
- 30 H. Schiek. Ähnlichkeitsanalyse von Gruppenrelationen. *Acta Math.* **96** (1956), 157–252.
- 31 P. E. Schupp. On Dehn’s algorithm and the conjugacy problem. *Math. Ann.* **178** (1968), 119–130.
- 32 A. J. Sieradski. A coloring test for asphericity. *Quart. J. Math. Oxford Ser. (2)* **34** (1983), 97–106.
- 33 M. Steenbock. Rips-Segev torsion-free groups without the unique product property. *J. Algebra* **438** (2015), 337–378.
- 34 V. A. Tartakovskiĭ. Solution of the word problem for groups with a k -reduced basis for $k > 6$. *Izvestiya Akad. Nauk SSSR. Ser. Mat.* **13** (1949), 483–494.
- 35 The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.10.1*, 2019.

- 36 X. Wang, G. Li, L. Yang and H. Lin. Groups with two generators having unsolvable word problem and presentations of Mihailova subgroups of braid groups. *Commun. Algebra* **44** (2016), 3020–3037.
- 37 D. T. Wise. Nonpositive immersions, sectional curvature, and subgroup properties. *Electron. Res. Announc. Am. Math. Soc.* **9** (2003), 1–9.
- 38 D. T. Wise. Sectional curvature, compact cores, and local quasiconvexity. *Geom. Funct. Anal.* **14** (2004), 433–468.