



Higher dimensional algebraic fiberings for pro- p groups

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Abstract. We prove some conditions for higher-dimensional algebraic fibering of pro- p group extensions, and we establish corollaries about incoherence of pro- p groups. In particular, if $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ is a short exact sequence of pro- p groups, such that Γ contains a finitely generated, non-abelian, free pro- p subgroup, K a finitely presented pro- p group with N a normal pro- p subgroup of K such that $K/N \simeq \mathbb{Z}_p$ and N not finitely generated as a pro- p group, then G is incoherent (in the category of pro- p groups). Furthermore, we show that if K is a finitely generated, free pro- p group with $d(K) \geq 2$, then either $\text{Aut}_0(K)$ is incoherent (in the category of pro- p groups) or there is a finitely presented pro- p group, without non-procyclic free pro- p subgroups, that has a metabelian pro- p quotient that is not finitely presented, i.e., a pro- p version of a result of Bieri–Strebel does not hold.

1 Introduction

For a pro- p group G , we denote by $K[[G]]$ the completed group algebra of G over the ring K , where K is the field with p elements \mathbb{F}_p or the ring of the p -adic numbers \mathbb{Z}_p . By definition a pro- p group G is of type FP_m if the trivial $\mathbb{Z}_p[[G]]$ -module \mathbb{Z}_p has a projective resolution where all projectives in dimension $\leq m$ are finitely generated $\mathbb{Z}_p[[G]]$ -modules. Note that G is of type FP_1 if and only if G is finitely generated as a pro- p group. And, G is of type FP_2 if and only if G is finitely presented as a pro- p group, i.e., $G \simeq F/R$, where F is a free pro- p group with a finite free basis X and R is the smallest normal pro- p subgroup of F that contains some fixed finite set of relations of G . It is interesting to note that for abstract (discrete) groups, the abstract versions of the properties FP_2 and finite presentability do not coincide [3].

In this paper, we develop results on algebraic fibering and coherence of pro- p groups. The case of abstract groups was considered by Kochloukova and Vidussi in [13], where the authors used specific techniques from geometric group theory, namely the Bieri–Neumann–Renz–Strebel Σ -invariants. We will use the pro- p version of the Σ^1 -invariant, suggested in [10] for pro- p metabelian groups, only in the proof of Proposition 3.4 and most of the results in this paper would have purely homological proofs. We note that the results on incoherence we obtain are quite general and in their full generality are not known for abstract groups (see Corollaries 1.3 and 1.4).

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Theorem 1.1 Let $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ be a short exact sequence of pro- p groups and let $n_0 \geq 1$ be an integer such that:

- 1) G and K are of type FP_{n_0} ,
- 2) Γ^{ab} is infinite,
- 3) there is a normal pro- p subgroup N of K such that $G' \cap K \subseteq N$, $K/N \simeq \mathbb{Z}_p$ and N is of type FP_{n_0-1} .

Then there is a normal pro- p subgroup M of G such that $G/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and M is of type FP_{n_0} . Furthermore, if K , G , and N are of type FP_∞ , then M can be chosen of type FP_∞ .

We call a discrete pro- p character of G a nontrivial homomorphism of pro- p groups $\alpha : G \rightarrow H$ such that $H \simeq \mathbb{Z}_p$. Then Theorem 1.1 could be restated as: assume that G and K are of type FP_{n_0} , Γ^{ab} is infinite, and there is a discrete pro- p character α of G such that $\alpha|_K \neq 0$, $\text{Ker}(\alpha) \cap K = N$ is of type FP_{n_0-1} . Then there exists a discrete pro- p character μ of G such that $M = \text{Ker}(\mu)$ is of type FP_{n_0} and $\mu|_K = \alpha|_K$, in particular $M \cap K = N$.

There is a lot in the literature on coherent abstract groups (see, for example, [27]), but very little is known for coherent pro- p groups. Similar to the abstract case, a pro- p group G is coherent (in the category of pro- p groups) if every finitely generated pro- p subgroup of G is finitely presented as a pro- p group, i.e., is of type FP_2 . We generalize this concept and define that a pro- p group G is n -coherent if any pro- p subgroup of G that is of type FP_n is of type FP_{n+1} . Thus, a pro- p group is 1-coherent if and only if it is coherent (in the category of pro- p groups).

Corollary 1.2 Let K , Γ , and $G = K \rtimes \Gamma$ be pro- p groups and let $n_0 \geq 1$ be an integer such that:

- 1) Γ is finitely generated free pro- p but not pro- p cyclic,
- 2) K is of type FP_{n_0} ,
- 3) there is a normal pro- p subgroup N of K such that $G' \cap K \subseteq N$, $K/N \simeq \mathbb{Z}_p$ and N is of type FP_{n_0-1} but is not of type FP_{n_0} .

Then there is a normal pro- p subgroup M of G such that $G/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and M is of type FP_{n_0} but is not of type FP_{n_0+1} . In particular, G is not n_0 -coherent.

As in the case of Theorem 1.1, Corollary 1.2 can be restated in terms of discrete pro- p characters.

We say that a group is incoherent if it is not coherent. The following result can be deduced from Theorem 3.6, that follows from Corollary 1.2.

Corollary 1.3 Let $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ be a short exact sequence of pro- p groups such that:

- 1) K is a finitely generated pro- p group,
 - 2) there is a normal pro- p subgroup N of K with $K/N \simeq \mathbb{Z}_p$ and N is not finitely generated,
 - 3) Γ contains a non-abelian free pro- p subgroup.
- Then G is incoherent (in the category of pro- p groups).

The class of pro- p groups \mathcal{L} was first considered by Kochloukova and Zalesskii in [14]. It contains all finitely generated free pro- p groups, and its profinite version

was considered by Zalesskii and Zapata in [28]. A pro- p group from \mathcal{L} shares many properties with an abstract limit group, in particular it is defined using extensions of centralizers. Still there are many open questions about the class of pro- p groups \mathcal{L} . For example, by Wilton's result from [26], every finitely generated subgroup of an abstract limit group is a virtual retract, but the pro- p version of this result is still an open problem.

For a pro- p group K , we write $d(K)$ for the cardinality of a minimal set of (topological) generators, i.e., $d(K) = \dim_{\mathbb{F}_p} H_1(K, \mathbb{F}_p)$.

Corollary 1.4 *Let $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ be a short exact sequence of pro- p groups such that:*

- 1) K is a non-abelian pro- p group from the class \mathcal{L} ,
- 2) Γ contains a non-abelian, free pro- p subgroup.

Then G is incoherent (in the category of pro- p groups). In particular, if K is a finitely generated free pro- p group with $d(K) \geq 2$, then G is incoherent (in the category of pro- p groups).

We note that the version of Corollary 1.4 for abstract groups is still open even when K and Γ are free, non-abelian with K of rank at least 3. The same holds for Corollary 1.3.

It is known that abstract (free finite rank)-by- \mathbb{Z} groups are coherent [8]. There is a conjecture suggested by Wise and independently by Kropholler and Walsh that an abstract (free of finite rank ≥ 2)-by-(free of finite rank ≥ 2) group is incoherent (see [15]). The conjecture was proved in [15] for a (free of rank 2)-by-(free of finite rank ≥ 2) abstract group, with a proof that cannot be modified for pro- p groups. By Corollary 1.4, a pro- p version of this result holds too.

A pro- p right angled Artin group (pro- p RAAG) associated with a finite simplicial graph X can be defined either as the pro- p completion of the abstract RAAG associated with X or by the same presentation as the abstract RAAG associated with X but in the category of pro- p groups.

Demushkin groups are some special, finitely generated, 1-related pro- p groups. The pro- p completion of an orientable surface group is a Demushkin pro- p group, but there are Demushkin pro- p groups that are not obtained this way. There are several types of Demushkin pro- p groups completely described in terms of presentations in [5, 6, 17, 23]. The following corollary provides many examples where Theorem 1.1 and Corollary 1.3 apply.

Corollary 1.5 *Let $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ be a short exact sequence of pro- p groups such that:*

- 1) K is a non-abelian pro- p RAAG or a non-soluble Demushkin group,
- 2) Γ is a non-abelian pro- p RAAG or a non-soluble Demushkin group.

Then G is incoherent (in the category of pro- p groups).

For a finite rank free pro- p group F , the structure of $\text{Aut}(F)$ was studied first by Lubotsky in [18]. $\text{Aut}(F)$ is a topological group with a pro- p subgroup of finite index. In [9], Gordon proved that the automorphism group of an abstract free group of rank 2 is incoherent. Unfortunately we could not prove a pro- p version of this result, but still it would hold if the group of outer pro- p automorphisms of a free pro- p group of rank 2 contains a free non-procyclic pro- p subgroup. For a free abstract group F_2 of

rank 2, we have that $\text{Out}(F_2) \simeq GL_2(\mathbb{Z})$; hence, $\text{Out}(F_2)$ contains a free non-cyclic abstract group. Nevertheless, the group $GL_2^1(\mathbb{Z}_p) = \text{Ker}(GL_2(\mathbb{Z}_p) \rightarrow GL_2(\mathbb{F}_p))$ does not contain a free pro- p non-procyclic pro- p subgroup, since it is p -adic analytic and so there is an upper limit on the number of generators of finitely generated pro- p subgroups [7]. For related results on non-existence of free pro- p subgroups in matrix groups, see [1, 2, 29].

Let G be a finitely generated pro- p group. Define $\text{Aut}_0(G) = \text{Ker}(\text{Aut}(G) \rightarrow \text{Aut}(G/G^*))$, where G^* is the Frattini subgroup of G . Then $\text{Aut}_0(G)$ is a pro- p subgroup of $\text{Aut}(G)$ of finite index.

Corollary 1.6 *Suppose that K is a finitely generated free pro- p group with $d(K) \geq 2$. If $\text{Out}(K)$ contains a pro- p free non-procyclic subgroup, then $\text{Aut}_0(K)$ is incoherent (in the category of pro- p groups).*

By the Bieri–Strebel results in [4] for a finitely presented abstract group H that does not contain free non-cyclic abstract subgroups, every metabelian quotient of H is finitely presented. It is an open question whether a pro- p version of the Bieri–Strebel result holds, i.e., whether if G is a finitely presented pro- p group without free non-procyclic pro- p subgroups, then every metabelian pro- p quotient of G is finitely presented as a pro- p group. Note that by the King classification of the finitely presented metabelian pro- p groups in [11], every pro- p quotient of a finitely presented metabelian pro- p group is finitely presented pro- p . Using Corollary 1.6 and some ideas introduced by Romankov in [21, 22], we prove the following result.

Corollary 1.7 *Suppose that K is a finitely generated free pro- p group with $d(K) \geq 2$. Then either $\text{Aut}_0(K)$ is incoherent (in the category of pro- p groups) or the pro- p version of the Bieri–Strebel result does not hold.*

2 Preliminaries

2.1 Homological finiteness properties of pro- p groups

Let G be a pro- p group. By definition,

$$\mathbb{Z}_p[[G]] = \varprojlim \frac{\mathbb{Z}}{p^i \mathbb{Z}}[[G/U]],$$

where the inverse limit is over all $i \geq 1$ and U open subgroups of G . And,

$$\mathbb{F}_p[[G]] = \mathbb{Z}_p[[G]]/p\mathbb{Z}_p[[G]] = \varprojlim \mathbb{F}_p[[G/U]],$$

where the inverse limit is over all open subgroups U of G .

By definition, the pro- p group G is of type FP_m if the trivial $\mathbb{Z}_p[[G]]$ -module \mathbb{Z}_p has a projective resolution where all projectives in dimension $\leq m$ are finitely generated $\mathbb{Z}_p[[G]]$ -modules, i.e., there is an exact complex of pro- p $\mathbb{Z}_p[[G]]$ -modules

$$\mathcal{P}: \cdots \rightarrow P_i \rightarrow P_{i-1} \rightarrow \cdots \rightarrow P_0 \rightarrow \mathbb{Z}_p \rightarrow 0,$$

where each P_i projective and for $i \leq m$ we have that P_i is finitely generated.

Such resolutions can be used to compute the pro- p homology groups $H_i(G, -)$. Suppose V is a left pro- p $\mathbb{Z}_p[[G]]$ -module and \mathcal{P} is a complex of right pro- p $\mathbb{Z}_p[[G]]$ -modules. Then the pro- p homology group $H_i(G, V)$ is defined as $H_i(\mathcal{P}^{\text{del}} \otimes_{\mathbb{Z}_p[[G]]} V)$. If W is a discrete right G -module, the cohomology group $H^i(G, W)$ is defined as $H^i(\text{Hom}_G(\mathcal{P}^{\text{del}}, W))$. Here, \mathcal{P}^{del} denotes the deleted resolution obtained from \mathcal{P} by deleting the module \mathbb{Z}_p from dimension -1 , i.e., substituting it with the zero module and Hom_G denotes continuous G -homomorphisms. For more on homology and cohomology of pro- p groups, see [20, 25].

By [10], for a pro- p group, the following conditions are equivalent:

- 1) G is of type FP_m ;
- 2) $H_i(G, \mathbb{Z}_p)$ is a finitely generated (abelian) pro- p group for $i \leq m$;
- 3) $H_i(G, \mathbb{F}_p)$ is finite for $i \leq m$;
- 4) for K either \mathbb{F}_p or \mathbb{Z}_p and N a normal pro- p subgroup of G such that $K[[G/N]]$ is left and right Noetherian, the homology groups $H_i(N, K)$ are finitely generated as pro- p $K[[G/N]]$ -modules for all $i \leq m$, where the G/N action is induced by the conjugation action of G on N .

The equivalence of the above conditions is a corollary of the fact that $\mathbb{Z}_p[[G]]$ and $\mathbb{F}_p[[G]]$ are local rings. Furthermore, in 3) $H_i(G, \mathbb{F}_p)$ could be substituted with $H^i(G, \mathbb{F}_p)$.

2.2 The King invariant

Let Q be a finitely generated abelian pro- p group, and let \mathbb{F} be the algebraic closure of \mathbb{F}_p . Denote by $\mathbb{F}[[t]]^\times$ the multiplicative group of invertible elements in $\mathbb{F}[[t]]$. Consider

$$T(Q) = \{ \chi : Q \rightarrow \mathbb{F}[[t]]^\times \mid \chi \text{ is a continuous homomorphism} \},$$

where $\mathbb{F}[[t]]^\times$ is a topological group with topology induced by the topology of the ring $\mathbb{F}[[t]]$, given by the sequence of ideals $(t) \supseteq (t^2) \supseteq \dots \supseteq (t^i) \supseteq \dots$. Note that since χ is continuous, we have that

$$\chi(Q) \subset 1 + t\mathbb{F}[[t]].$$

For $\chi \in T(Q)$, there is a unique continuous ring homomorphism

$$\bar{\chi} : \mathbb{Z}_p[[Q]] \rightarrow \mathbb{F}[[t]]$$

that extends χ .

Let A be a finitely generated pro- p $\mathbb{Z}_p[[Q]]$ -module. In [11], King defined the following invariant:

$$\Delta(A) = \{ \chi \in T(Q) \mid \text{ann}_{\mathbb{Z}_p[[Q]]}(A) \subseteq \text{Ker}(\bar{\chi}) \}.$$

In [11], King used the notation $\Xi(A)$, that we here substitute by $\Delta(A)$.

Let P be a pro- p subgroup of Q . Define

$$T(Q, P) = \{ \chi \in T(Q) \mid \chi(P) = 1 \}.$$

Theorem 2.1 [11, Theorem B], [11, Lemma 2.5] *Let Q be a finitely generated abelian pro- p group. Let A be a finitely generated pro- p $\mathbb{Z}_p[[Q]]$ -module.*

- a) *Then A is finitely generated as an abelian pro- p group if and only if $\Delta(A) = \{1\}$.*
 b) *If P is a pro- p subgroup of Q , then*

$$T(Q, P) \cap \Delta(A) = \Delta(A/[A, P]).$$

In particular, A is finitely generated as a pro- p $\mathbb{Z}_p[[P]]$ -module if and only if $T(Q, P) \cap \Delta(A) = \{1\}$.

We state the classification of the finitely presented metabelian pro- p groups given by King in [11].

Theorem 2.2 [11] *Let $1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$ be a short exact sequence of pro- p groups, where G is a finitely generated pro- p group and A and Q are abelian. Then G is a finitely presented pro- p group if and only if $\Delta(A) \cap \Delta(A)^{-1} = \{1\}$.*

Example Let $A = \mathbb{F}_p[[s]]$, $Q = \mathbb{Z}_p$, $G = A \rtimes Q$, where $Q = \mathbb{Z}_p$ has a generator b and b acts via conjugation on A by multiplication with $1 + s$. Since

$$\text{ann}_{\mathbb{Z}_p[[Q]]}(A) = p\mathbb{Z}_p[[Q]] \subseteq \text{Ker}(\bar{\chi}) \text{ for any } \chi \in T(Q),$$

we conclude that $\Delta(A) = T(Q) = \Delta(A)^{-1}$. Hence, by Theorem 2.2, G is not finitely presented (as a pro- p group).

Alternatively, it could be shown by a homological argument that if $1 \rightarrow A \rightarrow G \rightarrow Q \rightarrow 1$ is a short exact sequence of pro- p groups, where G is finitely presented and A and Q are abelian, then the pro- p homology group $H_2(A, \mathbb{Z}_p) \simeq A \widehat{\wedge}_{\mathbb{Z}_p} A$ is finitely generated as $\mathbb{Z}_p[[Q]]$ -module, where $\widehat{\wedge}$ denotes completed exterior product. In our example, the last condition fails, so G is not finitely presented (as a pro- p group).

2.3 Demushkin pro- p groups

Following [24], a Demushkin group G is a Poincaré duality group of dimension 2, i.e., $H^i(G, \mathbb{F}_p)$ is finite for all i , $\dim H^2(G, \mathbb{F}_p) = 1$ and the cup product

$$H^i(G, \mathbb{F}_p) \cup H^{2-i}(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p)$$

is a non-degenerated bilinear form for all $i \geq 0$. In particular, the cohomological dimension of G is $\text{cd}(G) = 2$.

There are two invariants associated with a Demushkin pro- p group: the minimal number of (topological) generators d and q that is either a power of the prime p or ∞ . We state several results from [5, 6, 17, 23] that classify the Demushkin pro- p groups. Other excellent reference for Demushkin pro- p groups is [25].

Theorem 2.3 [5, 6] *Let D be a Demushkin group with invariants d, q and suppose that $q \neq 2$. Then d is even and D is isomorphic to F/R , where F is a free pro- p group with basis x_1, \dots, x_d and R is generated as a normal closed subgroup by*

$$x_1^q [x_1, x_2] \dots [x_{d-1}, x_d],$$

where for $q = \infty$ we define $x_1^\infty = 1$. Furthermore, all groups having such presentations are Demushkin.

Theorem 2.4 [23] *Let D be a Demushkin pro-2 group with invariants d, q and suppose that $q = 2$ and d is odd. Then D is isomorphic to F/R , where F is a free pro-2 group with basis x_1, \dots, x_d and R is generated as a normal closed subgroup by*

$$x_1^2 x_2^{2^f} [x_2, x_3] \dots [x_{d-1}, x_d]$$

for some integer $f \geq 2$ or ∞ . Furthermore, all groups having such presentations are Demushkin.

Theorem 2.5 [17] *Let D be a Demushkin pro-2 group with d even and $q = 2$. Then D is isomorphic to F/R , where F is a free pro-2 group with basis x_1, \dots, x_d and R is generated as a normal closed subgroup either by*

$$x_1^{2^f+2} [x_1, x_2] [x_3, x_4] \dots [x_{d-1}, x_d]$$

for some integer $f \geq 2$ or $f = \infty$, or by

$$x_1^2 [x_1, x_2] x_3^{2^f} [x_3, x_4] \dots [x_{d-1}, x_d]$$

for some integer $f \geq 2$ or $f = \infty$, $d \geq 4$. Furthermore, all groups having such presentations are Demushkin.

3 Proofs of the main results

The following result is a pro- p version of results from [12, 16], where homotopical and homological versions for discrete groups are considered.

Lemma 3.1 *Let $n \geq 1$ be a natural number,*

$$A \hookrightarrow B \twoheadrightarrow C$$

a short exact sequence of pro- p groups with A of type FP_n and C of type FP_{n+1} . Assume that there is another short exact sequence of pro- p groups

$$A \hookrightarrow B_0 \twoheadrightarrow C_0$$

with B_0 of type FP_{n+1} and that there is a homomorphism of pro- p groups $\theta : B_0 \rightarrow B$ such that $\theta|_A = id_A$, i.e., there is a commutative diagram of homomorphisms of pro- p groups

$$\begin{array}{ccccc} A & \hookrightarrow & B_0 & \xrightarrow{\pi_0} & C_0 \\ \downarrow id_A & & \downarrow \theta & & \downarrow \nu \\ A & \hookrightarrow & B & \xrightarrow{\pi} & C \end{array}$$

Then B is of type FP_{n+1} .

Remark θ does not need to be injective or surjective.

Proof Consider the LHS spectral sequence

$$E_{i,j}^2 = H_i(C_0, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B_0, \mathbb{F}_p).$$

Similarly there is an LHS spectral sequence

$$\widehat{E}_{i,j}^2 = H_i(C, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B, \mathbb{F}_p).$$

Since A is of type FP_n , we have that $H_j(A, \mathbb{F}_p)$ is finite for all $j \leq n$. Then there is a pro- p subgroup C_1 of finite index in C such that C_1 acts trivially on $H_j(A, \mathbb{F}_p)$ for every $j \leq n$. Since C is of type FP_{n+1} , we have that C_1 is of type FP_{n+1} . Then

$$H_i(C_1, H_j(A, \mathbb{F}_p)) \simeq \oplus H_i(C_1, \mathbb{F}_p) \text{ is finite for } j \leq n, i \leq n + 1,$$

where we have $\dim_{\mathbb{F}_p} H_j(A, \mathbb{F}_p)$ direct summands in the right-hand side of the above isomorphism. Since C_1 has finite index in C , we deduce that

$$\widehat{E}_{i,j}^2 = H_i(C, H_j(A, \mathbb{F}_p)) \text{ is finite for } j \leq n, i \leq n + 1,$$

hence by the convergence of the second spectral sequence, we obtain that

$$H_k(B, \mathbb{F}_p) \text{ is finite for } k \leq n.$$

Note that we have shown that if $i + j = n + 1, i \neq 0$, then $\widehat{E}_{i,j}^2$ is finite, and hence $\widehat{E}_{i,j}^\infty$ is finite. By the convergence of the spectral sequence, there is a filtration of $H_{n+1}(B, \mathbb{F}_p)$

$$\begin{aligned} 0 &= F_{-1}(H_{n+1}(B, \mathbb{F}_p)) \subseteq \dots \subseteq F_i(H_{n+1}(B, \mathbb{F}_p)) \subseteq F_{i+1}(H_{n+1}(B, \mathbb{F}_p)) \\ &\subseteq \dots \subseteq F_{n+1}(H_{n+1}(B, \mathbb{F}_p)) = H_{n+1}(B, \mathbb{F}_p), \end{aligned}$$

where $F_i(H_{n+1}(B, \mathbb{F}_p))/F_{i-1}(H_{n+1}(B, \mathbb{F}_p)) \simeq \widehat{E}_{i,n+1-i}^\infty$. Thus,

$$H_{n+1}(B, \mathbb{F}_p) \text{ is finite if and only if } \widehat{E}_{0,n+1}^\infty \text{ is finite.}$$

Note that since any differential that comes out from $\widehat{E}_{0,n+1}^\infty$ is zero, we have that $\widehat{E}_{0,n+1}^\infty$ is a quotient of $\widehat{E}_{0,n+1}^2 = H_0(C, H_{n+1}(A, \mathbb{F}_p))$, and thus there is a map

$$\mu : H_0(C, H_{n+1}(A, \mathbb{F}_p)) \rightarrow H_{n+1}(B, \mathbb{F}_p)$$

with image that equals $\widehat{E}_{0,n+1}^\infty$. Thus,

$$B \text{ is of type } FP_{n+1} \text{ if and only if } \text{Im}(\mu) \text{ is finite.}$$

Similarly, there is a map

$$\mu_0 : H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) \rightarrow H_{n+1}(B_0, \mathbb{F}_p)$$

with image that equals $E_{0,n+1}^\infty$ and such that B_0 is of type FP_{n+1} if and only if $\text{Im}(\mu_0)$ is finite. Since B_0 is of type FP_{n+1} , we conclude that $\text{Im}(\mu_0)$ is finite.

The naturality of the LHS spectral sequence implies that we have the commutative diagram

$$\begin{array}{ccc}
 H_0(C_0, H_{n+1}(A, \mathbb{F}_p)) & \xrightarrow{\rho} & H_0(C, H_{n+1}(A, \mathbb{F}_p)) \\
 \mu_0 \downarrow & & \downarrow \mu \\
 H_{n+1}(B_0, \mathbb{F}_p) & \xrightarrow{\rho_0} & H_{n+1}(B, \mathbb{F}_p)
 \end{array}$$

where the maps ρ and ρ_0 are induced by $\nu : C_0 \rightarrow C$ and by θ .

Recall that the action of B_0 on A via conjugation induces an action of B_0 on $H_{n+1}(A, \mathbb{F}_p)$ where A acts trivially and this induces the action of C_0 on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C_0, H_{n+1}(A, \mathbb{F}_p))$. Similarly, the action of B on A via conjugation induces an action of B on $H_{n+1}(A, \mathbb{F}_p)$ where A acts trivially and this induces the action of C on $H_{n+1}(A, \mathbb{F}_p)$ that is used to define $H_0(C, H_{n+1}(A, \mathbb{F}_p))$. If ν is surjective, then ρ is an isomorphism; if ν is injective, then ρ is surjective. Since every homomorphism ν is a composition of one epimorphism followed by one monomorphism, we conclude that ρ is always surjective. Then

$$\text{Im}(\mu) = \text{Im}(\mu \circ \rho) = \text{Im}(\rho_0 \circ \mu_0) \text{ is a quotient of } \text{Im}(\mu_0).$$

Since $\text{Im}(\mu_0)$ is finite, we conclude that $\text{Im}(\mu)$ is finite. Hence, B is of type FP_{n+1} as required. ■

Lemma 3.2 *Let $1 \rightarrow A \rightarrow B \rightarrow C \rightarrow 1$ be a short exact sequence of pro- p groups such that for some $m \geq 1$ we have that A is of type FP_{m-1} and B is of type FP_m . Then C is of type FP_m .*

Proof We induct on $m \geq 1$. The case $m = 1$ is obvious since a pro- p group is of type FP_1 if and only if the group is finitely generated (as a pro- p group).

Assume that $m > 1$ and that the result holds for $m - 1$; hence, C is of type FP_{m-1} . Since A is FP_{m-1} , we have that $H_j(A, \mathbb{F}_p)$ is finite for $0 \leq j \leq m - 1$. Consider the LHS spectral sequence

$$E^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(B, \mathbb{F}_p).$$

By substituting if necessary C with a subgroup of finite index, we can assume that C acts trivially on $H_j(A, \mathbb{F}_p)$ for $0 \leq j \leq m - 1$. Then

$$E^2_{i,j} = H_i(C, H_j(A, \mathbb{F}_p)) \simeq H_i(C, \mathbb{F}_p) \widehat{\otimes} H_j(A, \mathbb{F}_p)$$

is finite for $0 \leq i, j \leq m - 1$. Then, for $r \geq 2$, consider the differential

$$d^r_{m,0} : E^r_{m,0} \rightarrow E^r_{m-r,r-1}$$

and note that either $m - r < 0$; hence, $E^2_{m-r,r-1} = 0$ or $m - r \leq m - 1, r - 1 \leq m - 1$. In all cases, $E^2_{m-r,r-1}$ is finite, and hence $E^r_{m-r,r-1}$ is finite and so $E^{r+1}_{m,0} = \text{Ker}(d^r_{m,0})$ is finite if and only if $E^r_{m,0}$ is finite. Thus,

$$E^\infty_{m,0} \text{ is finite if and only if } E^2_{m,0} = H_m(C, \mathbb{F}_p) \text{ is finite.}$$

Finally, since B is FP_m , we have that $H_m(B, \mathbb{F}_p)$ is finite and by the convergence of the spectral sequence $E_{m,0}^\infty$ is finite. Thus, we conclude that $H_m(C, \mathbb{F}_p)$ is finite. And, this together with C is of type FP_{m-1} implies that C is FP_m . ■

Recall that a pro- p HNN extension is called proper if the canonical map from the base group to the pro- p HNN extension is injective.

Lemma 3.3 *Let $G = \langle A, t \mid K^t = K \rangle$ be a proper pro- p HNN extension and m is a positive integer. Suppose that M is a normal pro- p subgroup of G such that $G/M \simeq \mathbb{Z}_p$, $K \not\subseteq M$, and $M \cap A$ is of type FP_m . Then the following conditions hold:*

- a) M is of type FP_m if and only if $M \cap K$ is of type FP_{m-1} ;
- b) if M is of type FP_{m+1} , then $M \cap K$ is of type FP_m .

Proof By [19, Theorem 4.1], the proper pro- p HNN extension gives rise to the exact sequence of $\mathbb{F}_p[[G]]$ -modules

$$(3.1) \quad 0 \rightarrow \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \rightarrow \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \rightarrow \mathbb{F}_p \rightarrow 0.$$

Note that since $K \not\subseteq M$, we have that $M \setminus G/K = G/MK$ is a proper pro- p quotient of $G/M \simeq \mathbb{Z}_p$, hence is finite. Similarly, $M \setminus G/A = G/MA$ is finite.

Note that there is an isomorphism of (left) $\mathbb{F}_p[[M]]$ -modules

$$\begin{aligned} \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p &\simeq (\oplus_{t \in M \setminus G/K} \mathbb{F}_p[[M]] t \mathbb{F}_p[[K]]) \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p \simeq \\ &\oplus_{t \in M \setminus G/K} \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M \cap tKt^{-1}]]} \mathbb{F}_p. \end{aligned}$$

Similarly, there is an isomorphism of (left) $\mathbb{F}_p[[M]]$ -modules

$$\begin{aligned} \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p &\simeq (\oplus_{t \in M \setminus G/A} \mathbb{F}_p[[M]] t \mathbb{F}_p[[A]]) \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p \simeq \\ &\oplus_{t \in M \setminus G/A} \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M \cap tAt^{-1}]]} \mathbb{F}_p. \end{aligned}$$

The short exact sequence (3.1) gives rise to a long exact sequence in pro- p homology

$$\begin{aligned} \cdots \rightarrow H_{m+1}(M, \mathbb{F}_p) &\rightarrow H_m(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) \rightarrow \\ H_m(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) &\rightarrow H_m(M, \mathbb{F}_p) \rightarrow H_{m-1}(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) \\ \rightarrow \cdots \rightarrow H_1(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) &\rightarrow H_1(M, \mathbb{F}_p) \rightarrow \\ H_0(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) &\rightarrow H_0(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \rightarrow H_0(M, \mathbb{F}_p) \rightarrow 0. \end{aligned}$$

Note that

$$\begin{aligned} H_i(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[K]]} \mathbb{F}_p) &\simeq H_i(M, \oplus_{t \in M \setminus G/K} \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M \cap tKt^{-1}]]} \mathbb{F}_p) \simeq \\ \oplus_{t \in M \setminus G/K} H_i(M, \mathbb{F}_p[[M]] \otimes_{\mathbb{F}_p[[M \cap tKt^{-1}]]} \mathbb{F}_p) &\simeq \oplus_{t \in M \setminus G/K} H_i(M \cap tKt^{-1}, \mathbb{F}_p) = \\ \oplus_{t \in M \setminus G/K} H_i(t(M \cap K)t^{-1}, \mathbb{F}_p) &\simeq \oplus_{t \in M \setminus G/K} H_i(M \cap K, \mathbb{F}_p). \end{aligned}$$

Similarly,

$$H_i(M, \mathbb{F}_p[[G]] \otimes_{\mathbb{F}_p[[A]]} \mathbb{F}_p) \simeq \oplus_{t \in M \setminus G/A} H_i(M \cap A, \mathbb{F}_p).$$

Then the long exact sequence could be rewritten as

$$\begin{aligned} \cdots \rightarrow H_{m+1}(M, \mathbb{F}_p) \rightarrow \oplus_{t \in M \setminus G/K} H_m(M \cap K, \mathbb{F}_p) \rightarrow \oplus_{t \in M \setminus G/A} H_m(M \cap A, \mathbb{F}_p) \rightarrow \\ H_m(M, \mathbb{F}_p) \rightarrow \oplus_{t \in M \setminus G/K} H_{m-1}(M \cap K, \mathbb{F}_p) \rightarrow \cdots \rightarrow \oplus_{t \in M \setminus G/A} H_1(M \cap A, \mathbb{F}_p) \rightarrow \\ H_1(M, \mathbb{F}_p) \rightarrow \oplus_{t \in M \setminus G/K} H_0(M \cap K, \mathbb{F}_p) \rightarrow \\ \oplus_{t \in M \setminus G/A} H_0(M \cap A, \mathbb{F}_p) \rightarrow H_0(M, \mathbb{F}_p) \rightarrow 0. \end{aligned}$$

Since $M \cap A$ is of type FP_m , we have that $H_i(M \cap A, \mathbb{F}_p)$ is finite for $i \leq m$. Combining with $M \setminus G/A$ is finite, we conclude that

$$\oplus_{t \in M \setminus G/A} H_i(M \cap A, \mathbb{F}_p) \text{ is finite for } i \leq m.$$

a) Note that M is of type FP_m if and only if $H_i(M, \mathbb{F}_p)$ is finite for $i \leq m$. By the above long exact sequence together with the fact that $M \setminus G/K$ is finite, $H_i(M, \mathbb{F}_p)$ is finite for $i \leq m$ if and only if $\oplus_{t \in M \setminus G/K} H_i(M \cap K, \mathbb{F}_p)$ is finite for $i \leq m - 1$, i.e., $M \cap K$ is of type FP_{m-1} .

b) If M is of type FP_{m+1} , then $H_{m+1}(M, \mathbb{F}_p)$ is finite and since $H_m(M \cap A, \mathbb{Z}_p)$ is finite by the long exact sequence $H_m(M \cap K, \mathbb{F}_p)$ is finite. We already know by a) that $M \cap K$ is of type FP_{m-1} , and hence $M \cap K$ is of type FP_m . ■

Next, we prove a technical lemma that will be used in the proof of Proposition 3.5. For a pro- p group G with a subset S , denote by $\langle S \rangle$ the pro- p subgroup of G generated by S .

Proposition 3.4 *Let $Q = \langle x, y \rangle \simeq \mathbb{Z}_p^2$ and A be a finitely generated pro- $p \mathbb{Z}_p[[Q]]$ -module. Suppose that for $H = \langle x \rangle$, we have that A is finitely generated as a pro- $p \mathbb{Z}_p[[H]]$ -module. Let $H_j = \langle xy^{-p^j} \rangle$. Then there is $j_0 > 0$ such that for every $j \geq j_0$, we have that A is finitely generated as a pro- $p \mathbb{Z}_p[[H_j]]$ -module.*

Proof By Theorem 2.1, if P is a pro- p subgroup of Q , then A is finitely generated as pro- $p \mathbb{Z}_p[[P]]$ -module if and only if $T(Q, P) \cap \Delta(A) = \{1\}$.

Let

$$J = \text{ann}_{\mathbb{Z}_p[[Q]]}(A).$$

Since A is finitely generated as a pro- $p \mathbb{Z}_p[[H]]$ -module for every $\chi \in T(Q, H) \setminus \{1\}$, we have that $J \not\subseteq \text{Ker}(\bar{\chi})$. Since $\mathbb{Z}_p[[Q]]$ is a Noetherian ring, there is a finite subset Λ of J that generates J as an ideal (abstractly or topologically is the same).

We aim to show that for sufficiently big j , we have $T(Q, H_j) \cap \Delta(A) = \{1\}$. Let $\mu_j \in T(Q, H_j) \setminus \{1\}$; thus, we aim to show that $\mu_j \notin \Delta(A)$. Then, by Theorem 2.1, A is finitely generated as a pro- $p \mathbb{Z}_p[[H_j]]$ -module.

Let

$$\bar{\mu}_j : \mathbb{Z}_p[[Q]] \rightarrow \mathbb{F}[[t]]$$

be the continuous ring homomorphism induced by μ_j . Since $\bar{\mu}_j(H_j) = 1$, we have $\mu_j(x) = \mu_j(y^{p^j}) \neq 1$. Let

$$\chi \in T(Q, H) \setminus \{1\} \text{ be such that } \chi(y) = \mu_j(y), \chi(x) = 1$$

and

$$\bar{\chi} : \mathbb{Z}_p[[Q]] \rightarrow \mathbb{F}[[t]]$$

be the continuous ring homomorphism induced by χ . Let

$$\lambda \in \mathbb{Z}_p[[Q]] = \mathbb{Z}_p[[t_1, t_2]], \text{ where } x = 1 + t_1, y = 1 + t_2.$$

Then, since $\chi(y) = \mu_j(y)$, $\chi(x) = 1$, we have $\chi(t_2) = 0$, and hence

$$\bar{\chi}(\lambda) = \bar{\chi}(\lambda|_{t_1=0}) = \bar{\mu}_j(\lambda|_{t_1=0}),$$

where $\lambda = \sum_{i,k \geq 0} z_{i,k} t_1^i t_2^k$, $z_{i,k} \in \mathbb{Z}_p$ and $\lambda|_{t_1=0} = \sum_{k \geq 0} z_{0,k} t_2^k$. Note that

$$\bar{\mu}_j(t_2) = \bar{\mu}_j(1 + t_2) - \bar{\mu}_j(1) = \bar{\mu}_j(y) - 1 \in 1 + t\mathbb{F}[[t]] - 1 = t\mathbb{F}[[t]].$$

Note that since \mathbb{F} has characteristic $p > 0$, we have

$$\begin{aligned} 1 + \bar{\mu}_j(t_1) &= \bar{\mu}_j(1 + t_1) = \bar{\mu}_j(x) = \mu_j(x) = \\ \mu_j(y^{p^j}) &= \bar{\mu}_j((1 + t_2)^{p^j}) = \bar{\mu}_j(1 + t_2^{p^j}) = 1 + \bar{\mu}_j(t_2^{p^j}). \end{aligned}$$

Consider

$$\lambda|_{t_1=t_2^{p^j}} := \sum_{i,k \geq 0} z_{i,k} t_2^{i p^j + k} = \sum_{s \geq 0} \left(\sum_{i p^j + k = s} z_{i,k} \right) t_2^s.$$

Then, using that $\bar{\mu}_j(t_1) = \bar{\mu}_j(t_2^{p^j})$, we conclude that

$$\begin{aligned} \bar{\mu}_j(\lambda) &= \bar{\mu}_j(\lambda|_{t_1=t_2^{p^j}}) = \bar{\mu}_j \left(\sum_{i,k \geq 0} z_{i,k} t_2^{i p^j + k} \right) = \bar{\mu}_j \left(\sum_{k \geq 0} z_{0,k} t_2^k \right) + \bar{\mu}_j \left(\sum_{i \geq 1, k \geq 0} z_{i,k} t_2^{i p^j + k} \right) \\ (3.2) \quad &= \bar{\mu}_j(\lambda|_{t_1=0}) + \bar{\mu}_j \left(\sum_{i \geq 1, k \geq 0} z_{i,k} t_2^{i p^j + k} \right) = \bar{\chi}(\lambda) + \bar{\mu}_j \left(\sum_{i \geq 1, k \geq 0} z_{i,k} t_2^{i p^j + k} \right). \end{aligned}$$

Suppose now that $\mu_j \in \Delta(A)$. Then $\bar{\mu}_j(J) = 0$, in particular $\bar{\mu}_j(\Lambda) = 0$. On the other hand, $\chi \notin \Delta(A)$; hence, $\bar{\chi}(J) \neq 0$. This is equivalent with $\bar{\chi}(\Lambda) \neq 0$. So there is $\lambda_0 \in \Lambda$ such that

$$\bar{\chi}(\lambda_0) \neq 0 = \bar{\mu}_j(\lambda_0).$$

Write as before $\lambda_0 = \sum_{i,k \geq 0} z_{i,k} t_1^i t_2^k$ where $z_{i,k} \in \mathbb{Z}_p$. Then, by (3.2),

$$0 = \bar{\mu}_j(\lambda_0) = \bar{\chi}(\lambda_0) + \bar{\mu}_j \left(\sum_{i \geq 1, k \geq 0} z_{i,k} t_2^{i p^j + k} \right).$$

So, for

$$\lambda_0|_{t_1=0} = \sum_{i \geq 0} z_i t_2^i,$$

where $z_i \in \mathbb{Z}_p$ and using that $\bar{\chi}(\lambda_0) = \bar{\chi}(\lambda_0|_{t_1=0})$, we have

$$(3.3) \quad -\bar{\chi}\left(\sum_{i \geq 0} z_i t_2^i\right) = \bar{\mu}_j \left(\sum_{i \geq 1, k \geq 0} z_{i,k} t_2^{ip^j+k} \right).$$

Let

$$f_0 := \bar{\mu}_j(y) - 1 = \bar{\mu}_j(t_2 + 1) - 1 = \bar{\mu}_j(t_2) = \mu_j(t_2) \in t\mathbb{F}[[t]] \setminus \{0\};$$

hence,

$$\bar{\chi}(t_2) = \bar{\chi}(t_2 + 1) - 1 = \bar{\chi}(y) - 1 = \chi(y) - 1 = \mu_j(y) - 1 = f_0.$$

Then, by (3.3), we conclude that

$$(3.4) \quad 0 \neq -\sum_{i \geq 0} \bar{z}_i f_0^i = \sum_{i \geq 1, k \geq 0} \bar{z}_{i,k} f_0^{ip^j+k},$$

where for $z \in \mathbb{Z}_p$ we denote by \bar{z} the image of z in $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$.

Consider the map

$$o : \mathbb{F}[[t]] \setminus \{0\} \rightarrow \mathbb{Z}_{\geq 0}$$

that sends $\sum_{i \geq 0} a_i t^i$ to the smallest i_0 such that $a_{i_0} \neq 0$, where each $a_i \in \mathbb{F}$. Define

$$d = o(f_0) \geq 1 \text{ and } k_0 = o\left(\sum_i \bar{z}_i t^i\right) \geq 0.$$

Then

$$(3.5) \quad o\left(-\sum_{i \geq 0} \bar{z}_i f_0^i\right) = dk_0.$$

By (3.4) and (3.5), there is $\bar{z}_{i,k} \neq 0$ for some $i \geq 1, k \geq 0$ such that

$$dp^j \leq d(ip^j + k) = o(f_0^{ip^j+k}) \leq dk_0, \text{ hence } p^j \leq k_0.$$

Note that k_0 depends only on $\lambda_0 \in \Lambda$, where Λ is a finite set, hence it does not depend on j . From the very beginning, we can choose $j_0 \in \mathbb{Z}_{>0}$ such that

$$p^{j_0} > \max\{\tilde{k} = o\left(\sum_i \bar{m}_i t^i\right) \mid \tilde{\lambda}|_{t_1=0} = \sum_i m_i t_2^i \neq 0, \tilde{\lambda} \in \Lambda\} \geq k_0.$$

Then, for $j \geq j_0$, we get a contradiction, so $\mu_j \notin \Delta(A)$ as required. ■

Proposition 3.5 *Let G be a pro- p group with a normal pro- p subgroup G_0 such that $G/G_0 \simeq \mathbb{Z}_p^2$. Let S be a normal pro- p subgroup of G such that $G/S \simeq \mathbb{Z}_p$, $G_0 \subseteq S$, and S is of type FP_m for some $m \geq 1$. Then there is a normal pro- p subgroup S_0 of G such that $G/S_0 \simeq \mathbb{Z}_p$, $S \neq S_0$, $G_0 \subseteq S_0$, and S_0 is of type FP_m .*

Proof Note that since S is a pro- p group of type FP_m and $G/S \simeq \mathbb{Z}_p$ is a pro- p group of type FP_∞ , hence of type FP_m , we can conclude that G is a pro- p group of type FP_m . Set

$$Q = G/G_0 = \langle x, y \rangle \text{ and } H = S/G_0 = \langle x \rangle.$$

Since $Q = G/G_0$ is a finitely generated abelian pro- p group, hence $\mathbb{Z}_p[[Q]]$ is left and right Noetherian and G is of type FP_m , we conclude that

$$A_i = H_i(G_0, \mathbb{Z}_p) \text{ is finitely generated as a pro-}p \mathbb{Z}_p[[Q]]\text{-module for } i \leq m.$$

Since S is a pro- p group of type FP_m , we conclude that

$$A_i \text{ is finitely generated as a pro-}p \mathbb{Z}_p[[H]]\text{-module for } i \leq m.$$

Then, by Proposition 3.4, for sufficiently big j , we have that A_i is finitely generated as a pro- $p \mathbb{Z}_p[[H_j]]$ -module, where $H_j = \langle xy^{-p^j} \rangle \leq Q$, for every $i \leq m$. Then we define S_0 as the preimage in G of one such H_j . ■

We recall the statement of Theorem 1.1.

Theorem 1.1 *Let $1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$ be a short exact sequence of pro- p groups and $n_0 \geq 1$ be an integer such that:*

- 1) G and K are of type FP_{n_0} ,
- 2) Γ^{ab} is infinite,
- 3) *there is a normal pro- p subgroup N of K such that $G' \cap K \subseteq N$, $K/N \simeq \mathbb{Z}_p$ and N is of type FP_{n_0-1} .*

Then there is a normal pro- p subgroup M of G such that $G/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and M is of type FP_{n_0} . Furthermore, if K , G , and N are of type FP_∞ , then M can be chosen of type FP_∞ .

Proof of Theorem 1.1 Consider a commutative diagram

$$\begin{array}{ccccc} K & \hookrightarrow & \Pi & \twoheadrightarrow & F_n \\ \text{id}_K \downarrow & & \pi \downarrow & & \downarrow \\ K & \hookrightarrow & G & \twoheadrightarrow & \Gamma \end{array}$$

where the lines are short exact sequences of pro- p groups, F_n is the free pro- p group with a free basis s_1, \dots, s_n , and the vertical maps are surjective homomorphisms of pro- p groups with the most left map being the identity map.

Define

$$\Pi = \Pi_1 \amalg_K \Pi_2 \amalg_K \dots \amalg_K \Pi_n,$$

where \amalg_K is the amalgamated free product with amalgam K in the category of pro- p groups, and each

$$\Pi_i = K \rtimes \langle s_i \rangle, \langle s_i \rangle \simeq \mathbb{Z}_p.$$

Note that since K is normal in Π and $\Pi/K \simeq \Pi_1/K \amalg \Pi_2/K \amalg \dots \amalg \Pi_n/K$ is a free pro- p product, we conclude that $\Pi_1 \amalg_K \Pi_2 \amalg_K \dots \amalg_K \Pi_i$ embeds in Π for every $1 \leq i \leq n$.

Recall that Γ^{ab} is infinite; hence, the image in Γ^{ab} of at least one $\pi(s_i)$ has infinite order. Without loss of generality, we can assume that the image of $\pi(s_1)$ in Γ^{ab} has infinite order. In particular, the restriction map

$$\pi|_{\Pi_1} : \Pi_1 \rightarrow \pi(\Pi_1)$$

is an isomorphism.

Note that $[K, s_1] \subseteq G' \cap K \subseteq N$, hence $\Pi'_1 \subseteq N$. We have $N \subseteq K \subseteq \Pi_1$ where $K/N \simeq \mathbb{Z}_p$, $\Pi_1/K \simeq \mathbb{Z}_p$, this together with the inclusion $\Pi'_1 \subseteq N$ implies that $\Pi_1/N \simeq \mathbb{Z}_p^2$.

By assumption, K is of type FP_{n_0} . By Proposition 3.5, there is S_0 a normal pro- p subgroup of Π_1 such that

$$N \subseteq S_0, S_0 \text{ is of type } FP_{n_0}, S_0 \neq K \text{ and } \Pi_1/S_0 \simeq \mathbb{Z}_p.$$

Let

$$\mu : G \rightarrow \mathbb{Z}_p$$

be a homomorphism of pro- p groups such that

$$\text{Ker}(\mu \circ \pi) \cap \Pi_1 = S_0, \text{ i.e., } \text{Ker}(\mu) \cap \pi(\Pi_1) = \pi(S_0).$$

This is possible since $\Pi_1/N \simeq \mathbb{Z}_p^2$ is abelian and $G' \cap K \subseteq N \subseteq S_0$. Note that $K \not\subseteq S_0$, hence $\mu(K) \neq 0$.

Consider the epimorphism of pro- p groups

$$\chi = \mu \circ \pi : \Pi \rightarrow \mathbb{Z}_p.$$

Note that

$$\chi(K) \neq 0, \text{Ker}(\chi) \cap \Pi_1 = S_0 \text{ is of type } FP_{n_0}$$

and

$$\text{Ker}(\chi) \cap K = S_0 \cap K = N \text{ is of type } FP_{n_0-1}.$$

Then we view $\Pi_1 \amalg_K \Pi_2$ as a proper HNN extension

$$\langle \Pi_1, s_2 \mid K^{s_2} = K \rangle$$

with a pro- p base group Π_1 , associated pro- p subgroup K and stable letter s_2 . Then, by Lemma 3.3(a),

$$\text{Ker}(\chi) \cap (\Pi_1 \amalg_K \Pi_2) \text{ is of type } FP_{n_0}.$$

We view $\Pi_1 \amalg_K \Pi_2 \amalg_K \Pi_3$ as a proper HNN extension with a base pro- p group $\Pi_1 \amalg_K \Pi_2$, associated pro- p subgroup K and stable letter s_3 then by Lemma 3.3(a)

$$\text{Ker}(\chi) \cap \left(\Pi_1 \amalg_K \Pi_2 \amalg_K \Pi_3 \right) \text{ is of type } FP_{n_0}.$$

Then, repeating this argument several times, we deduce that $\text{Ker}(\chi)$ is of type FP_{n_0} .

By construction, $\text{Ker}(\mu)$ is a quotient of $\text{Ker}(\chi)$. If $n_0 = 1$, then $\text{Ker}(\chi)$ is finitely generated (as a pro- p group), then any pro- p quotient of $\text{Ker}(\chi)$ is finitely

generated (as a pro- p group). In particular, $\text{Ker}(\mu)$ is finitely generated (as a pro- p group).

Now, for the general case, i.e., $n_0 \geq 2$, we will apply Lemma 3.1. Write $\widetilde{\text{Ker}}(\chi)$ for the image of $\text{Ker}(\chi)$ in F_n and $\widetilde{\text{Ker}}(\mu)$ for the image of $\text{Ker}(\mu)$ in Γ . By construction,

$$\text{Ker}(\chi) \cap K = N = \text{Ker}(\mu) \cap K.$$

By assumption, N is of type FP_{n_0-1} and we have already shown that $\text{Ker}(\chi)$ is of type FP_{n_0} . By construction, $\mu(K) \neq 0$, hence $K \cdot \text{Ker}(\mu) \neq \text{Ker}(\mu)$ and since $G/\text{Ker}(\mu) \simeq \mathbb{Z}_p$, we deduce that $K \cdot \text{Ker}(\mu)$ has finite index in G and so $\widetilde{\text{Ker}}(\mu)$ has finite index in Γ .

By Lemma 3.2, since in the short exact sequence of pro- p groups

$$1 \rightarrow K \rightarrow G \rightarrow \Gamma \rightarrow 1$$

the pro- p groups G and K are of type FP_{n_0} (it suffices that K is of type FP_{n_0-1}), we deduce that Γ is of type FP_{n_0} . Then $\widetilde{\text{Ker}}(\mu)$ is a pro- p group of type FP_{n_0} . Then we can apply Lemma 3.1 for the commutative diagram

$$\begin{array}{ccccc} N = \text{Ker}(\chi) \cap K & \hookrightarrow & \text{Ker}(\chi) & \twoheadrightarrow & \widetilde{\text{Ker}}(\chi) \\ \downarrow id_N & & \downarrow \pi|_{\text{Ker}(\chi)} & & \downarrow \\ N = \text{Ker}(\mu) \cap K & \hookrightarrow & \text{Ker}(\mu) & \twoheadrightarrow & \widetilde{\text{Ker}}(\mu) \end{array}$$

to deduce that $\text{Ker}(\mu)$ is a pro- p group of type FP_{n_0} . Finally, we set $M = \text{Ker}(\mu)$. ■

Proof of Corollary 1.2 We define M as in the proof of Theorem 1.1 for $\Gamma = F_n$ and π the identity map, $\mu = \chi$. Thus, $M = \text{Ker}(\chi) = \text{Ker}(\mu)$ is a normal pro- p subgroup of G , $G/M \simeq \mathbb{Z}_p$ and M is of type FP_{n_0} . We view

$$G = \Pi = \Pi_1 \coprod_K \Pi_2 \coprod_K \dots \coprod_K \Pi_n$$

as a proper HNN extension with a base pro- p subgroup

$$A = \Pi_1 \coprod_K \Pi_2 \coprod_K \dots \coprod_K \Pi_{n-1},$$

associated pro- p subgroup K and stable letter s_n . By the proof of Theorem 1.1,

$$A \cap M = A \cap \text{Ker}(\chi) \text{ is of type } FP_{n_0}.$$

Suppose that M is of type FP_{n_0+1} . By Lemma 3.3(b), $N = M \cap K$ is of type FP_{n_0} , a contradiction. Hence, M is not of type FP_{n_0+1} . This completes the proof of the corollary. ■

Theorem 3.6 Let $G = K \rtimes \Gamma$ be a pro- p group such that:

- 1) K is a finitely generated pro- p group, i.e., is of type FP_1 ,
- 2) there is a normal pro- p subgroup N of K with $K/N \simeq \mathbb{Z}_p$ and N is not finitely generated,
- 3) Γ a finitely generated free pro- p group with $d(\Gamma) \geq 2$.

Then G is incoherent (in the category of pro- p groups).

Proof We claim that there is a finitely generated non-procyclic pro- p subgroup Γ_0 of Γ such that Γ_0 acts trivially on the abelianization $K^{ab} = K/K'$ via conjugation. Indeed, let $T = \text{tor}(K/K')$ be the torsion pro- p subgroup of K^{ab} . Then $V = K^{ab}/T \simeq \mathbb{Z}_p^d$, where $d \geq 1$. Note that the conjugation action of Γ on $V \simeq \mathbb{Z}_p^d$ induces a homomorphism

$$\rho : \Gamma \rightarrow GL_d(\mathbb{Z}_p).$$

Note that $\text{Im}(\rho)$ is a pro- p subgroup of $GL_d(\mathbb{Z}_p)$, hence is p -adic analytic and there is an upper bound on the number of generators of any finitely generated pro- p subgroup of $\text{Im}(\rho)$ [7]. Hence, ρ is not injective. Alternatively, we can use the main result of [1] to deduce that ρ is not injective. Thus, $\text{Ker}(\rho)$ is a nontrivial normal pro- p subgroup of Γ and we can choose Γ_0 any non-procyclic finitely generated pro- p subgroup of $\text{Ker}(\rho)$.

Set $G_0 = K \rtimes \Gamma_0$. Then, by Corollary 1.2, there is a normal pro- p subgroup M of G_0 such that $G_0/M \simeq \mathbb{Z}_p$, $M \cap K = N$, and M is FP_1 but not FP_2 , i.e., it is finitely generated as a pro- p group, but is not finitely presented as a pro- p group. Thus, G_0 and hence G are incoherent (in the category of pro- p groups). This completes the proof. ■

Proof of Corollary 1.3 Let Γ_0 be a finitely generated, free non-abelian pro- p subgroup of Γ . Consider the preimage G_0 of Γ_0 in G , i.e., there is a short exact sequence

$$1 \rightarrow K \rightarrow G_0 \rightarrow \Gamma_0 \rightarrow 1.$$

Note that $G_0 = K \rtimes \Gamma_0$, then by Theorem 3.6, G_0 is not coherent. ■

4 More results on coherence

In this section, we show some applications of Corollary 1.3.

We recall the definition of the class of pro- p groups \mathcal{L} . It uses the extension of centralizer construction. We define inductively the class \mathcal{G}_n of pro- p groups by setting \mathcal{G}_0 as the class of all finitely generated free pro- p groups and a group $G_n \in \mathcal{G}_n$ if there is a decomposition

$$G_n = G_{n-1} \coprod_C A,$$

where $G_{n-1} \in \mathcal{G}_{n-1}$, C is self-centralized procyclic subgroup of G_{n-1} , and A is a finitely generated free abelian pro- p group such that C is a direct summand of A . The class \mathcal{L} is defined as the class of all finitely generated pro- p subgroups G of G_n , where G_n runs through all pro- p groups in \mathcal{G}_n for $n \geq 0$. The minimal n such that $G \leq G_n \in \mathcal{G}_n$ is called the weight of G .

Proposition 4.1 [14] *Let $K \in \mathcal{L}$ be a nontrivial pro- p group. Then $K^{ab} = K/K'$ is infinite.*

Proof of Corollary 1.4 By Proposition 4.1, K^{ab} is infinite. Let N be a normal pro- p subgroup of K such that $K/N \simeq \mathbb{Z}_p$. By part (4) from the main theorem of [14], we have that N is not finitely generated as a pro- p group. Then we can apply Corollary 1.3. This completes the proof. ■

Definition Given a finite simplicial graph X , the pro- p RAAG associated with X is the pro- p group defined by the presentation in the category of pro- p groups

$$\langle V(X) \mid [v, w] = 1 \text{ if } v, w \text{ are adjacent in } X \rangle,$$

where $V(X)$ is the set of vertices of X .

Lemma 4.2 Let G be a Demushkin pro- p group such that $d(G) = 2$. Then G is soluble and has Euler characteristic 0.

Proof The classification of Demushkin groups has several cases described in Theorems 2.3–2.5. In the case of 2-generated Demushkin group, we have a 1-relation presentation with a relation of the type $[x_1, x_2]$ or $x_1^q[x_1, x_2]$, where q is a power of p or of the type $2^f + 2$, $p = 2$. In all these cases, the group is soluble, since it is $\langle x_1 \rangle \rtimes \langle x_2 \rangle$ and has zero Euler characteristic. ■

Proof of Corollary 1.5 We claim that Γ has a free non-abelian pro- p subgroup F . Suppose first that Γ is a pro- p RAAG. Let v_1, v_2 be vertices of the graph that defines the pro- p RAAG Γ that are not adjacent. Then the pro- p subgroup F of Γ generated by v_1 and v_2 , is a retract of Γ , hence it is non-abelian free pro- p .

If Γ is a non-soluble Demushkin group, then every pro- p subgroup of infinite index in Γ has cohomological dimension 1, so is free pro- p [24, Ex. 5b), p. 44]. Furthermore, the abelianization of Γ is infinite, so we can set F to be a normal pro- p subgroup of Γ such that $\Gamma/F \simeq \mathbb{Z}_p$.

We claim that there is a normal pro- p subgroup N of K such that N is not finitely generated (as a pro- p group) and $K/N \simeq \mathbb{Z}_p$. Suppose first that K is a non-abelian pro- p RAAG. Let w_1 and w_2 be vertices of the graph that defines the pro- p RAAG K that are not adjacent. Then the pro- p subgroup F_0 of K generated by w_1 and w_2 is non-abelian free pro- p and it is a retract of K . Note that any normal pro- p subgroup S of F_0 such that $F_0/S \simeq \mathbb{Z}_p$ is not finitely generated (as a pro- p group), hence any preimage N of S in K is not finitely generated (as a pro- p group).

Suppose that K is a non-soluble Demushkin group. Note that K/K' is infinite and let N be a normal pro- p subgroup of K such that $K/N \simeq \mathbb{Z}_p$. If N is finitely generated (as a pro- p subgroup), together with the fact that N is free pro- p , we conclude that N has finite Euler characteristic $\chi(N)$, $\chi(K/N) = \chi(\mathbb{Z}_p) = 0$ and

$$0 = \chi(N)\chi(K/N) = \chi(K) = 1 - d(K) + 1 = 2 - d(K),$$

so $d(K) = 2$. But by Lemma 4.2 2-generated Demushkin groups are soluble, a contradiction.

Finally, by Corollary 1.3, G is not coherent. ■

Proposition 4.3 Let $1 \rightarrow K \rightarrow G \rightarrow \mathbb{Z}_p^m \rightarrow 1$ be a short exact sequence of pro- p groups with K free pro- p or Demushkin pro- p group. Then G is coherent (as a pro- p group).

Proof Let H be a finitely generated, pro- p subgroup of G . We aim to prove that H is FP_∞ , in particular is finitely presented.

Consider the short exact sequence of pro- p groups

$$1 \rightarrow K_0 \rightarrow H \rightarrow Q \rightarrow 1,$$

where Q is a pro- p subgroup of \mathbb{Z}_p^m , so is finitely generated, abelian, and $K_0 = K \cap H$. Recall that an infinite index subgroup in a Demushkin group is free pro- p , in particular K_0 is either free pro- p or Demushkin. In the latter, K_0 is FP_∞ , hence H is FP_∞ . Thus, we can assume from now on that K_0 is free pro- p .

Consider the LHS spectral sequence

$$E_{i,j}^2 = H_i(Q, H_j(K_0, \mathbb{F}_p)) \text{ that converges to } H_{i+j}(H, \mathbb{F}_p).$$

Note that $E_{i,j}^2 = 0$ for $j \geq 2$ and

$$d_{i,j}^k : E_{i,j}^k \rightarrow E_{i-k, j+k-1}^k \text{ for } k \geq 2.$$

Thus, $E_{i,j}^3 = E_{i,j}^\infty$ and

$$E_{i,0}^3 = \text{Ker}(d_{i,0}^2) \text{ is finite}$$

since $E_{i,0}^2$ is finite for all i . Note that

$$E_{i,1}^3 = \text{Coker}(d_{i+2,0}^2) = E_{i,1}^2 / \text{Im}(d_{i+2,0}^2) \text{ is finite if and only if } E_{i,1}^2 \text{ is finite,}$$

since $E_{i+2,0}^2$ is finite for all i .

Since H is finitely generated and by the convergence of the spectral sequence, we have that

$$E_{0,1}^3 = E_{0,1}^\infty \text{ is finite,}$$

hence

$$E_{0,1}^2 = H_0(Q, H_1(K_0, \mathbb{F}_p)) \text{ is finite.}$$

Since $\mathbb{F}_p[[Q]]$ is a local ring, this implies that $H_1(K_0, \mathbb{F}_p)$ is finitely generated as a pro- p $\mathbb{F}_p[[Q]]$ -module. Then $E_{j,1}^2$ is finite for $j \geq 0$.

By the convergence of the spectral sequence, there is an exact sequence for $i \geq 2$

$$0 \rightarrow E_{i-1,1}^\infty \rightarrow H_i(H, \mathbb{F}_p) \rightarrow E_{i,0}^\infty \rightarrow 0$$

with both $E_{i-1,1}^\infty$ and $E_{i,0}^\infty$ finite. Hence, $H_i(H, \mathbb{F}_p)$ is finite for $i \geq 2$ and H is of type FP_∞ , in particular is finitely presented as a pro- p group. ■

5 Proofs of Corollaries 1.6 and 1.7

Proof of Corollary 1.6 Let F be a finitely generated free non-procyclic pro- p group that embeds as a closed subgroup of $\text{Out}(K)$. Note that $G = K \rtimes F$ is a pro- p group that embeds as a closed subgroup of $\text{Aut}(K)$ and by Theorem 3.6 G is incoherent (in the category of pro- p groups). Finally, $G_0 = G \cap \text{Aut}_0(G)$ is a pro- p subgroup of finite index in G , hence G_0 is incoherent (in the category of pro- p groups). ■

Proof of Corollary 1.7 We recall first some results from [18]. Let G be a finitely generated pro- p group and $\text{Aut}(G)$ denote all continuous automorphisms of G (which coincide with the abstract automorphisms of G). Denote $\text{Inn}(G)$ the group of the internal automorphisms. The group $\text{Aut}(G)$ is a profinite group. ■

Lemma 5.1 [18] a) Let G be a finitely generated pro- p group and G^* be the Frattini subgroup of G , i.e., the intersection of all maximal open subgroups of G . Then $\text{Ker}(\text{Aut}(G) \rightarrow \text{Aut}(G/G^*))$ is a pro- p subgroup of $\text{Aut}(G)$ of finite index.

b) Let F be a finitely generated free pro- p group and N be a characteristic pro- p subgroup of F . Then the map $\text{Aut}(F) \rightarrow \text{Aut}(F/N)$, obtained by taking the induced automorphisms, is surjective.

We set $\text{Aut}_0(G) = \text{Ker}(\text{Aut}(G) \rightarrow \text{Aut}(G/G^*))$ and $\text{Out}_0(G) = \text{Aut}_0(G)/\text{Inn}(G)$.

Lemma 5.2 Suppose K is a finitely generated, free pro- p group, $d(K) = n \geq 2$, and M is the maximal pro- p metabelian quotient of K . Then $\text{Out}(M)$ contains a finitely generated pro- p subgroup H such that H has a metabelian pro- p quotient that is not finitely presented (as a pro- p group).

Lemma 5.2 implies Corollary 1.7: If $\text{Out}(K)$ contains a pro- p free non-procyclic subgroup, we can apply Corollary 1.6. Then we can assume that $\text{Out}(K)$ does not contain a pro- p free non-procyclic subgroup. We can further assume that the pro- p version of the Bieri–Strebel result holds; otherwise, Corollary 1.7 holds, i.e., if a finitely presented pro- p group does not contain a free non-procyclic pro- p subgroup, then any metabelian pro- p quotient of that group is a finitely presented pro- p group.

Let H be a pro- p subgroup of $\text{Out}(M)$ as in Lemma 5.2. Since $\text{Aut}_0(M)$ has finite index in $\text{Aut}(M)$, without loss of generality, we can assume that $H \subseteq \text{Out}_0(M)$. The epimorphism of pro- p groups $\text{Aut}_0(K) \rightarrow \text{Aut}_0(M)$ induces an epimorphism of pro- p groups $\text{Out}_0(K) \rightarrow \text{Out}_0(M)$. Then there is a finitely generated pro- p subgroup \tilde{H} of $\text{Out}_0(K)$ that maps surjectively to H , in particular \tilde{H} has a metabelian pro- p quotient that is not finitely presented (as a pro- p group). Then, by the previous considerations, \tilde{H} is not a finitely presented pro- p group.

Note that $\text{Inn}(K) \simeq K$. Consider the short exact sequence

$$1 \rightarrow K \rightarrow \text{Aut}_0(K) \rightarrow \text{Out}_0(K) \rightarrow 1,$$

and let H_0 be the preimage of \tilde{H} in $\text{Aut}_0(K)$. Then there is a short exact sequence

$$1 \rightarrow K \rightarrow H_0 \rightarrow \tilde{H} \rightarrow 1$$

of pro- p groups. Since K is a finitely generated pro- p group, we have that H_0 is a finitely generated pro- p group and H_0 is not finitely presented; otherwise, \tilde{H} would be a finitely presented pro- p group, a contradiction. Thus, $\text{Aut}_0(K)$ is incoherent (in the category of pro- p groups).

Proof of Lemma 5.2 Here, we use significantly ideas introduced in [21]. We fix x_1, x_2, \dots, x_n a generating set of M . Define

$$\text{IAut}(M) = \{ \varphi \in \text{Aut}(M) \mid \varphi \text{ induces on } M/M' \text{ the identity map} \},$$

where $\text{Aut}(M)$ denotes continuous automorphisms of M . In fact, every abstract automorphism of a finitely generated pro- p group is a continuous one. Then there is a short exact sequence of profinite groups

$$1 \rightarrow \text{IAut}(M) \rightarrow \text{Aut}(M) \rightarrow \text{Aut}(M^{ab}) = \text{GL}_n(\mathbb{Z}_p) \rightarrow 1.$$

By [21], there is a Bachmut embedding β of $\text{IAut}(M)$ in $GL_n(\mathbb{Z}_p[[M^{ab}]])$, where M^{ab} is the maximal abelian pro- p quotient of M . By [21], where $\text{Aut}(M)$ acts on the right,

$$\beta(\varphi) = \left(\frac{\partial}{\partial x_j}(x_i^\varphi) \right) \text{ and } \frac{\partial}{\partial x_j} : M \rightarrow \mathbb{Z}_p[[M^{ab}]] \text{ are the Fox derivatives defined by}$$

$$\frac{\partial}{\partial x_j}(1) = 0, \quad \frac{\partial}{\partial x_j}(g_1 g_2) = \frac{\partial}{\partial x_j}(g_1) + \bar{g}_1 \frac{\partial}{\partial x_j}(g_2), \quad \frac{\partial}{\partial x_j}(x_i) = \delta_{i,j},$$

where \bar{g}_1 is the image of $g_1 \in M$ in M^{ab} , $\delta_{i,j}$ is the Kronecker symbol.

Set s_i for the image of $x_i - 1$ in $\mathbb{Z}_p[[M^{ab}]]$; thus,

$$\mathbb{Z}_p[[M^{ab}]] \simeq \mathbb{Z}_p[[s_1, s_2, \dots, s_n]].$$

Define

$$\det(\varphi) = \det(\beta(\varphi)).$$

By [21],

$$\det(\text{IAut}(M)) = 1 + \Delta =: P$$

is a multiplicative abelian group, where Δ is the unique maximal ideal of $\mathbb{Z}_p[[M^{ab}]]$, and the $GL_n(\mathbb{Z}_p)$ -action via conjugation on the abelianization of $\text{IAut}(M)$ induces an action on $\det(\text{IAut}(M)) = P$. Then we have a short exact sequence of profinite groups

$$1 \rightarrow P \rightarrow \text{Aut}(M)/\text{Ker}(\det) \rightarrow GL_n(\mathbb{Z}_p) \rightarrow 1.$$

Consider the pro- p group

$$GL_n^1(\mathbb{Z}_p) = \text{Ker}(GL_n(\mathbb{Z}_p) \rightarrow GL_n(\mathbb{F}_p)).$$

Let Q be the maximal pro- p quotient of P that has exponent p . Then there is a pro- p subgroup T of $\text{Aut}(M)/\text{Ker}(\det)$ and a short exact sequence of pro- p groups

$$1 \rightarrow P \rightarrow T \rightarrow GL_n^1(\mathbb{Z}_p) \rightarrow 1$$

and a pro- p quotient T_0 of T together with a short exact sequence of pro- p groups

$$1 \rightarrow Q \rightarrow T_0 \rightarrow GL_n^1(\mathbb{Z}_p) \rightarrow 1.$$

By [22, (7)],

$$P^p \cap (1 + p\Delta) = 1 + p^2\Delta,$$

and for $\delta \in \Delta$ using $[\delta]$ for the image of $1 + p\delta$ in Q , we have that

$$[\delta_1][\delta_2] = [\delta_1 + \delta_2].$$

Thus, the multiplicative subgroup of Q generated by $\{[\delta] \mid \delta \in \Delta\}$ could be identified with the additive group $\Delta/p\Delta$. Furthermore, using the long exact sequence in homology for the short exact sequence

$$0 \rightarrow \Delta \rightarrow \mathbb{Z}_p[[s_1, s_2, \dots, s_n]] \rightarrow \mathbb{F}_p \rightarrow 0,$$

we have a long exact sequence

$$0 = \text{Tor}_1^{\mathbb{Z}_p}(\mathbb{Z}_p[[s_1, s_2, \dots, s_n]], \mathbb{F}_p) \rightarrow \text{Tor}_1^{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p) \rightarrow \Delta/p\Delta \rightarrow \mathbb{F}_p[[s_1, s_2, \dots, s_n]] \rightarrow \mathbb{F}_p \rightarrow 0.$$

Note that $\text{Tor}_1^{\mathbb{Z}_p}(\mathbb{F}_p, \mathbb{F}_p) \simeq \mathbb{F}_p$ and thus we have a short exact sequence of additive pro- p groups

$$0 \rightarrow \mathbb{F}_p \rightarrow \Delta/p\Delta \xrightarrow{\nu} \Omega \rightarrow 0,$$

where Ω is the augmentation ideal of $\mathbb{F}_p[[s_1, s_2, \dots, s_n]]$ and for the canonical projection

$$\pi : \Delta \rightarrow \Delta/p\Delta,$$

the composition map $\nu \circ \pi : \Delta \rightarrow \Omega$ is the restriction of the map $\mathbb{Z}_p[[s_1, s_2, \dots, s_n]] \rightarrow \mathbb{F}_p[[s_1, s_2, \dots, s_n]]$ that reduces coefficients mod p . Actually, $\text{Ker}(\nu) = \mathbb{F}_p$ is generated as an additive group by $p + p\Delta$.

Consider now $\varphi_2 \in \text{Aut}(M)$ given by

$$\varphi_2 = \rho^p, \text{ where } \rho(x_1) = x_1x_2, \rho(x_k) = x_k \text{ for } 2 \leq k \leq n$$

and $\varphi_1 \in \text{IAut}(M)$ such that

$$\det(\beta(\varphi_1)) = 1 + ps_1.$$

Note that φ_1 is not uniquely determined and that the image of φ_2 in $GL_n(\mathbb{Z}_p)$ is in $GL_n^1(\mathbb{Z}_p)$. Hence, the profinite subgroup Γ of $\text{Aut}(M)$ generated by φ_1, φ_2 is in fact a pro- p group. Let

$$\Gamma_0 = \langle \psi_1, \psi_2 \rangle$$

be the image of Γ in T_0 , where ψ_i is the image of φ_i in T_0 . Thus, Γ_0 is a pro- p group.

By [21, Proposition 4.4], for every $\varphi \in \text{IAut}(M)$ for $\varphi' = \rho^{-1}\varphi\rho$, $h' = \det(\beta(\varphi'))$ and $h = \det(\beta(\varphi))$, we have that h' is obtained from h applying the substitution $s_1 \rightarrow s_1 + s_2 + s_1s_2$. Then the action of ψ_2 on $\psi_1 = [s_1]$ by conjugations is induced by applying the substitution $s_1 \rightarrow s_1 + s_2 + s_1s_2$ exactly p -times, i.e., we apply the substitution

$$s_1 \rightarrow (1 + s_1)(1 + s_2)^p - 1.$$

Similarly, the action of ψ_2^k on $\psi_1 = [s_1]$ by conjugation is induced by applying the substitution $s_1 \rightarrow s_1 + s_2 + s_1s_2$ exactly pk -times, and thus gives the substitution $s_1 \rightarrow (1 + s_1)(1 + s_2)^{pk} - 1$.

Let A be the normal pro- p subgroup of Γ_0 generated by ψ_1 . Thus, A can be identified with an additive subgroup of $\Delta/p\Delta$ and

$$\nu(A) \subseteq \nu(\Delta/p\Delta) = \Omega =$$

$$s_1\mathbb{F}_p[[s_1, s_2, \dots, s_n]] + s_2\mathbb{F}_p[[s_1, s_2, \dots, s_n]] + \dots + s_n\mathbb{F}_p[[s_1, s_2, \dots, s_n]].$$

The previous paragraph shows that

$$\{(1 + s_1)(1 + s_2)^{pk} - 1 \mid k \geq 0\} \subseteq \nu(A),$$

in particular $\nu(A)$ and A are infinite.

Note that $\Gamma_0 \simeq A \rtimes D$, where $D \simeq \mathbb{Z}_p$ is generated by ψ_2 . We view A as an $\mathbb{F}_p[[t]]$ -module via the conjugation action of $\psi_2 = 1 + t$, where $\mathbb{F}_p[[t]] \simeq \mathbb{F}_p[[D]]$. Furthermore, A is a pro- p cyclic $\mathbb{F}_p[[t]]$ -module, with a generator ψ_1 . Since every proper $\mathbb{F}_p[[t]]$ -module quotient of $\mathbb{F}_p[[t]]$ is a finite additive group and A is infinite, we deduce that $A \simeq \mathbb{F}_p[[t]]$. Then, by the example after Theorem 2.2, Γ_0 is a metabelian pro- p group that is not finitely presented.

Note that the image W of $M \simeq \text{Inn}(M)$ in T_0 is inside Q and since M is a finitely generated pro- p group and Q is an abelian pro- p group of finite exponent p , then W and consequently $\Gamma_0 \cap W$ are finite. Since $\Gamma_0 \cap W$ is finite, $\Gamma_0/(\Gamma_0 \cap W)$ is not a finitely presented pro- p group. Actually examining the structure of Γ_0 , it is easy to see that any finite normal subgroup of Γ_0 is trivial, in particular $\Gamma_0 \cap W = 1$. Finally, $\Gamma_0 \simeq \Gamma_0/(\Gamma_0 \cap W)$ is a metabelian pro- p quotient of a 2-generated pro- p group $H \leq \text{Out}(M)$. This completes the proof of the lemma. ■

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