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Entire functions of slow growth whose Julia set coincides with the plane

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Abstract. We construct a transcendental entire function f with $J(f) = \mathbb{C}$ such that f has arbitrarily slow growth; that is, $\log|f(z)| \le \phi(|z|)\log|z|$ for $|z| > r_0$, where ϕ is an arbitrary prescribed function tending to infinity.

For an entire function f we denote the Julia set by $J(f)$. By definition, it is the complement of the maximal open set $F(f)$, the set of normality, where the iterates f^n form a normal family.

While for polynomials the Julia set always has empty interior, for transcendental functions it may coincide with the whole complex plane $\mathbb{C}$. The first example with this property was given by Baker [**1**] and later Misiurewicz [**16**] showed that this is the case for the exponential function. There are several methods of constructing such examples (besides [**1**] and [**16**] we refer to [**3**, p. 74; **4**, pp. 155, 172; **7**, pp. 167–168; **8**, p. 225; **9**, p. 625; **10**, p. 610; **12**]) but none of them seems to be applicable to entire functions of arbitrarily slow growth, the main problem being to exclude the possibility of a wandering component of the set of normality where the iterates tend to infinity. That such a wandering component may indeed occur for functions of arbitrarily slow growth was shown by Baker [**2**] and Hinkkanen [**11**]. Notice that for entire functions of order less than one-half there is always a sequence of critical values tending to infinity (see [**13**, p. 1788]). This makes the usual arguments for the proof of the absence of wandering domains hard to apply.

THEOREM 1. *Let $t \mapsto \phi(t) : [0,\infty) \to [1,\infty)$ be an arbitrary increasing function tending to ∞ as $t \to \infty$. Then there exists an entire function f and $r_0 > 0$ with the properties $J(f) = \mathbb{C}$ and*

$$\log|f(z)| \le \phi(|z|)\log|z|, \quad |z| > r_0.$$

We use the following notation: $D(R) = \{z : |z| < R\}$, $\Delta(R) = \{z \in \mathbb{C} : |z| > R\}$ and $A(R, R') = \{z : R < |z| < R'\}$, where $0 < R < R' < \infty$. The sequence $(P^n(z))_{n=0}^{\infty}$ is called the P-orbit of the point z.

The proof of Theorem 1 is based on the following.

PROPOSITION 1. *Let P be a polynomial, $P(0) = 0$, $P(1) = 1$, $\deg P \geq 2$. Assume that the P-orbits of all the critical points of P tend to infinity. Let $z_1, \dots, z_{k-1} \in \mathbb{C}$ and $m_1, \dots, m_{k-1} \in \mathbb{N}$ and suppose that $P^{m_j}(z_j) = 0$ for $1 \leq j \leq k-1$. Let $z_k \in \mathbb{C}$, $\epsilon > 0$ and $R > 0$ be given.*

Then there exists a polynomial Q, $Q(0) = 0$, $Q(1) = 1$, such that the Q-orbits of all the critical points of Q tend to infinity, and there exist $z'_1, \dots, z'_k \in \mathbb{C}$ and $m_k \in \mathbb{N}$ such that $|z_j - z'_j| < \epsilon$ and $Q^{m_j}(z_j) = 0$ for $1 \leq j \leq k$. Moreover, $|P(z) - Q(z)| < \epsilon$ for $z \in D(R)$, $\deg Q = \deg P + 1$ and if $a_1, \dots, a_d$ are the zeros of P, then Q has a zero in each disk $|z - a_j| < \epsilon$, and a zero in $\Delta(R)$.

For the proof of Proposition 1 we need the following two lemmas. In these lemmas, we shall use some concepts from the theory of quasiconformal (and quasiregular) maps; see [**15**] for a general introduction to quasiconformal maps, and [**5, 6**] for a discussion of their role in complex dynamics.

LEMMA 1. *For every $\delta > 0$ and $\hat{R} > 0$ there exists $\eta > 0$ such that every quasiconformal homeomorphism $\phi : \mathbb{C} \to \mathbb{C}$ fixing 0 and 1 with Beltrami coefficient $\|\mu\|_\infty < \eta$ satisfies*

$$|\phi(z) - z| < \delta, \quad \textit{for } z \in D(\hat{R}).$$

Proof. Assume that the lemma is incorrect. Then there is a sequence of quasiconformal homeomorphisms (ϕ_n), each fixing 0 and 1, such that the corresponding Beltrami coefficients μ_n satisfy $\|\mu_n\|_\infty \to 0$, but

$$|\phi_n(z_n) - z_n| \geq \delta > 0$$

for some $z_n \in D(\hat{R})$. As a family of quasiconformal maps with uniformly bounded distortion fixing 0 and 1 is normal [**15**, §II.5], we may assume that $\phi_n \to \phi$ as $n \to \infty$, uniformly on compacta in $\mathbb{C}$, and ϕ is a conformal homeomorphism. Our normalization implies that $\phi(z) = z$ and we obtain a contradiction. □

LEMMA 2. *For every positive integer d and $\eta > 0$ there exists $\gamma \in (0, 1/2)$ with the following property:*

Let h_1 and h_2 be holomorphic functions in $A(r/2, 4r)$ such that $\|h_i\|_\infty < \gamma$, $i = 1, 2$. Then there exists a quasiregular local homeomorphism $\phi : A(r, 2r) \to \mathbb{C}$ with boundary values

$$\phi(z) = z^d(1 + h_1(z)), \quad |z| = r$$

and

$$\phi(z) = z^d(1 + h_2(z)), \quad |z| = 2r$$

and the Beltrami coefficient μ of ϕ satisfies $\|\mu\|_\infty < \eta$.

Proof. We define $h(z) := (2-|z|/r)h_1(z)+(|z|/r-1)h_2(z)$. This function is smooth in the ring $A(r, 2r)$ and has boundary values $h_1(z)$, $|z| = r$, and $h_2(z)$, $|z| = 2r$. The sup-norm of the derivative $Dh : A(r, 2r) \to \mathbb{R}^2$ tends to 0 when $\gamma \to 0$. Thus, $\phi(z) := z^d(1+h(z))$ has all the required properties when γ is small enough. □

Proof of Proposition 1. It follows from our hypotheses on the critical points of P that $J(P)$ is totally disconnected and $P^n(z) \to \infty$ for all $z \in \mathbb{C}\setminus J(P)$, see, for example, [**6**, p. 67].

Let $d := \deg P$. Recall (see [**6**, p. 34] or [**18**, pp. 63, 147]) that the limit

$$u := \lim_{n\to\infty} \frac{1}{d^n} \log |P^n| \tag{1}$$

exists uniformly on compacta in $\mathbb{C}\setminus J(P)$ and u is a positive harmonic function there, satisfying

$$u(z) \sim \log |z|, \quad z \to \infty. \tag{2}$$

If we extend u by setting $u(z) = 0$ for $z \in J(P)$, the resulting function is continuous, and we have $u(z) > 0$ if and only if $z \in \mathbb{C}\setminus J(P)$.

We may assume without loss of generality that $z_k \in \mathbb{C}\setminus J(f)$, because this can be achieved by a small shift of z_k, using the fact that $J(f)$ is totally disconnected. Performing another small shift of z_k if necessary, we may also assume that

$$0 < u(z_k) \neq d^j u(c) \quad \text{for all } c \in \mathrm{crit}(P) \text{ and } j \in \mathbb{Z}, \tag{3}$$

where $\mathrm{crit}(P)$ denotes the set of critical points of P. It follows from (3) that there exists $\kappa > 0$ with the property

$$|d^n u(z_k) - d^j u(c)| > \kappa d^n \quad \text{for all } c \in \mathrm{crit}(P) \text{ and } n \in \mathbb{N},\ j \in \mathbb{Z},$$

from which it follows, in view of (1), that

$$\min_{j\in\mathbb{N}} \left| \log \frac{|P^n(z_k)|}{|P^j(c)|} \right| \to \infty \quad \text{as } n \to \infty \text{ and } c \in \mathrm{crit}(P). \tag{4}$$

Similarly

$$\min_{0\le j<n} \frac{|P^n(z_k)|}{|P^j(z_k)|} \to \infty \quad \text{as } n \to \infty. \tag{5}$$

We fix arbitrary $\delta > 0$ and apply Lemma 1 for some $\hat{R}$ satisfying $\hat{R} \geq R + 1$, $\hat{R} \geq 1 + \max_{1\le j\le d} |a_j|$, $\hat{R} \geq 1 + \max_{1\le j\le k} |z_j|$, and $\hat{R} \geq 1 + \max_{|z|=R+1} |P(z)|$. Then, using η obtained from Lemma 1 and d, we apply Lemma 2 to obtain $\gamma \in (0, 1/2)$.

Now we are going to find a large integer n so that the following conditions (6)–(11) are satisfied:

$$|P^n(z_k)| > \frac{4}{\gamma}(R+1), \tag{6}$$

$$r := \frac{\gamma |P^n(z_k)|}{4} > \frac{16}{\gamma}, \tag{7}$$

$$|z^{-d}P(z) - 1| < \gamma, \quad \text{for } z \in \Delta(r/2), \tag{8}$$

$$\min_{j\in\mathbb{N}} \left| \log \frac{|P^n(z_k)|}{|P^j(c)|} \right| > \log \frac{4}{\gamma}, \quad c \in \mathrm{crit}(P), \tag{9}$$

$$\min_{0\le j<n} \frac{|P^n(z_k)|}{|P^j(z_k)|} > \frac{4}{\gamma}, \tag{10}$$

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and

$$\text{the } P\text{-orbits of all points } z_1 \dots, z_{k-1} \text{ are contained in } D(r). \tag{11}$$

Conditions (9) and (10) can be satisfied in view of (4) and (5), respectively.

We define a quasiregular map $Q_1 : \mathbb{C} \to \mathbb{C}$ in the following way:

$$Q_1(z) = P(z), \quad z \in D(r), \tag{12}$$

$$Q_1(z) = z^d \left(1 - \frac{z}{P^n(z_k)}\right), \quad z \in \Delta(2r), \tag{13}$$

and in the annulus $A(r, 2r)$ we interpolate using Lemma 2 with $h_1(z) = z^{-d}P(z) - 1$ and $h_2(z) = -z/P^n(z_k)$. The conditions of Lemma 2 are satisfied in view of (7) and (8).

If $U := \Delta(2|P^n(z_k)|)$ then U is Q_1-invariant and all Q_1-orbits in U tend to infinity. The map Q_1 has the following properties.

(i) The Q_1-orbits of the critical points of Q_1 tend to infinity. Indeed, the critical set of Q_1 consists of the critical set of P and one additional point $w := dP^n(z_k)/(d+1)$. The P-orbits of the critical points of P do not intersect the annulus $A(r, 2|P^n(z_k)|)$ in view of (9), so their Q_1-orbits also do not intersect this annulus, but do intersect the set U, and thus tend to infinity. Furthermore, $Q_1(w) = w^d/(d+1) \in U$, so the Q_1-orbit of w also tends to infinity.

(ii) $(Q_1)^{n+1}(z_k) = 0$. Indeed,

$$(Q_1)^2((Q_1)^{n-1}(z_k)) = (Q_1)^2(P^{n-1}(z_k)) = Q_1(P^n(z_k)) = 0,$$

because $P^j(z_k) \in D(r)$ for $j < n$ in view of (10) and $P(z) = Q_1(z)$ for $z \in D(r)$ by definition.

(iii) $Q_1^{m_j}(z_j) = 0$ for $1 \le j \le k-1$. This follows from (11) since $Q_1(z) = P(z)$ for $z \in D(r)$.

Thus Q_1 has all the required properties, except that it is not holomorphic in the annulus $A(r, 2r)$. To make it holomorphic we use a method of Shishikura [**17**]; see also [**5**, §§8–9] for an account of Shishikura's method. The image of the annulus $A(r, 2r)$ is contained in the invariant domain U, which is disjoint from $A(r, 2r)$. This permits us to define a new conformal structure σ in $\mathbb{C}$ such that it coincides with the standard conformal structure σ_0 in U, and $Q_1 : (\mathbb{C}, \sigma) \to (\mathbb{C}, \sigma)$ is holomorphic. The distortion of this structure with respect to the standard one is measured by the sup-norm of the Beltrami coefficient which is the same as that of Q_1, namely at most η (see Lemmas 1 and 2). By the basic existence theorem for quasiconformal mappings [**15**, Ch. 5], there exists a conformal homeomorphism $\psi : (\mathbb{C}, \sigma_0) \to (\mathbb{C}, \sigma)$. We can normalize it by $\psi(0) = 0$ and $\psi(1) = 1$. Then $Q := \psi^{-1} \circ Q_1 \circ \psi$ is easily seen to be a polynomial. The dynamics of Q are similar to those of Q_1, namely from (i)–(iii) it follows that the Q-orbits of the critical points of Q tend to infinity, and with $z'_j := \psi^{-1}(z_j)$, $1 \le j \le k$, and $m_k = n+1$ we have $Q^{m_j}(z'_j) = 0$ for $1 \le j \le k$.

Finally, we notice that $\psi : (\mathbb{C}, \sigma_0) \to (\mathbb{C}, \sigma_0)$ is quasiconformal and the sup-norm of its Beltrami coefficient is at most η. The same is true for ψ^{-1} and so by Lemma 1 we have

$$|\psi(z) - z| < \delta \quad \text{and} \quad |\psi^{-1}(z) - z| < \delta \quad \text{for } z \in D(\hat{R}).$$

If $\delta < 1$ and $|z| \leq R$, then $|\psi(z)| \leq R + \delta < r$ and hence $|Q_1(\psi(z))| = |P(\psi(z))| \leq \hat{R} - 1$. We deduce that if $\delta \to 0$, then

$$Q(z) = \psi^{-1}(Q_1(\psi(z))) = \psi^{-1}(P(\psi(z))) \to P(z),$$

uniformly for $z \in D(R)$. This implies that Q and z_j' have all the required properties for sufficiently small δ. □

Proof of Theorem 1. We fix a dense sequence $(z_j)_{j=1}^{\infty}$ in $\mathbb{C}$ with $z_1 = 3/4$, a sequence of positive numbers (ϵ_j) with the property

$$\sum_{j=1}^{\infty} \epsilon_j < 1, \tag{14}$$

and an increasing sequence $(R_j) \to \infty$ with the property

$$\sum_{j=1}^{\infty} \frac{1}{R_j} < \infty. \tag{15}$$

Starting with $k = 2$, $P_2(z) = 4z^2 - 3z$, $m_1 = 1$, and $z_1 = z_{1,2} = 3/4$, we apply Proposition 1 repeatedly, and obtain a sequence (P_k) of polynomials and a sequence (m_k) of positive integers with the following properties: $\deg P_k = k$, $P_k(0) = 0$, $P_k(1) = 1$, and, for every $j \in \mathbb{N}$ and $k > j$, there is a point $z_{j,k}$ satisfying

$$|z_k - z_{k,k+1}| < \epsilon_{k+1} \quad \text{and} \quad |z_{j,k} - z_{j,k+1}| < \epsilon_{k+1} \quad \text{for } j < k$$

such that

$$P_k^{m_j}(z_{j,k}) = 0. \tag{16}$$

In addition, the zeros $a_{j,k}$ of P_k satisfy

$$|a_{k,k}| > R_k \quad \text{for } k \geq 3 \quad \text{and} \quad |a_{j,k} - a_{j,k+1}| < \epsilon_{k+1} \quad \text{for } k \geq 2,\ j \leq k,$$

and the sequence (P_k) converges uniformly on compacta in $\mathbb{C}$ to an entire function f.

It follows that the limits $w_j := \lim_{k\to\infty} z_{j,k}$ exist for all $j \in \mathbb{N}$ and $|z_j - w_j| \to 0$ as $j \to \infty$. Thus, the sequence (w_j) is dense in $\mathbb{C}$. Passing to the limit as $k \to \infty$ in (16), we conclude that $f^{m_j}(w_j) = 0$. This means that the preimages of zero are dense in $\mathbb{C}$. Thus, $J(f) = \mathbb{C}$.

Finally, we have to estimate the growth. We have

$$P_k(z+1) = \prod_{j=1}^{k} \left(1 - \frac{z}{c_{j,k}}\right),$$

with $c_{j,k} = a_{j,k} - 1$. Thus, $|c_{j,k} - c_{j,k+1}| < \epsilon_{k+1}$ for $k \geq 2$, $j \leq 2$ and $|c_{k,k}| > R_k - 1$ for $k \geq 3$. Passing to the limit when $k \to \infty$ and taking (15) into account we conclude that

$$f(z+1) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{c_j}\right)$$

where $|c_j| = |\lim_{k\to\infty} c_{j,k}| > R_j - 1 - \sum_{n=j+1}^{\infty} \epsilon_n > R_j - 2$. Thus, f is an entire function of genus zero. Using standard estimates for canonical products (see, for example, [**14**]) we can choose (R_j) so that the growth of f is arbitrarily slow. □

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