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Decay measures on locally compact abelian topological groups

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We show that the Banach space $\mathfrak{M}$ of regular σ-additive finite Borel complex-valued measures on a non-discrete locally compact Hausdorff topological Abelian group is the direct sum of two linear closed subspaces $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$, where $\mathfrak{M}^{\mathrm{D}}$ is the set of measures $\mu \in \mathfrak{M}$ whose Fourier transform vanishes at infinity and $\mathfrak{M}^{\mathrm{ND}}$ is the set of measures $\mu \in \mathfrak{M}$ such that $\nu \notin \mathfrak{M}^{\mathrm{D}}$ for any $\nu \in \mathfrak{M} \setminus \{0\}$ absolutely continuous with respect to the variation $|\mu|$. For any corresponding decomposition $\mu = \mu^{\mathrm{D}} + \mu^{\mathrm{ND}}$ ($\mu^{\mathrm{D}} \in \mathfrak{M}^{\mathrm{D}}$ and $\mu^{\mathrm{ND}} \in \mathfrak{M}^{\mathrm{ND}}$) there exist a Borel set $A = A(\mu)$ such that μ^{D} is the restriction of μ to A, therefore the measures μ^{D} and μ^{ND} are singular with respect to each other. The measures μ^{D} and μ^{ND} are real if μ is real and positive if μ is positive. In the case of singular continuous measures we have a refinement of Jordan's decomposition theorem. We provide series of examples of different behaviour of convolutions of measures from $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$.

1. Introduction

Decay (or decaying) measures over the real line are measures with Fourier transform vanishing at ∞. We say that a measure μ on a real line is purely non-decaying if any non-zero measure absolutely continuous with respect to the variation $|\mu|$ is not a decay measure. The decay measures on the real line describe mixing flows [5] and decay in quantum physics [7, 8], since they correspond to the spectral measures of the self-adjoint generator of evolution. In the case of mixing flows, the generator is the Liouville operator [5, 13], while in the case of quantum systems, the generator is the Hamiltonian or the Liouville–von Neumann operator [15, 21].

We have proven [1] that any measure on a real line admits a unique decomposition into a sum of a decaying measure and a purely non-decaying measure, which leads to a refinement of the Jordan decomposition theorem. Evidently, discrete measures are purely non-decaying and absolutely continuous measures are decaying according to the Riemann–Lebesgue theorem. The non-triviality of the above decomposition appears for singular continuous measures. For example,

(i) the standard Cantor measure μ_1 [12] is a purely non-decaying singular continuous measure on $\mathbb{R}$;

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(ii) according to [6], the measure μ_2 with the Fourier transform

$$\tilde{\mu}_2(x) = \prod_{n=1}^{\infty} \cos\biggl(\frac{2^n x}{5^n}\biggr)$$

is a decaying singular continuous measure on $\mathbb{R}$;

(iii) however, the sum $\mu_3 = \mu_1 + \mu_2$ of the two above measures is an example of a singular continuous measure on $\mathbb{R}$ that is neither decaying nor purely non-decaying.

Note that all three above measures have compact support of zero Lebesgue measure and are positive.

In order to discuss decay for operators with complex spectra, appearing in more general systems like ergodic endomorphisms or stochastic processes, we have to extend the results of [1] for measures on the complex plane and especially on the unit circle. We realized, however (theorem 2.1), that this extension can be easily achieved for measures over any non-discrete locally compact topological abelian group.

The spectral measures of the Liouville–von Neumann operator [15,21], describing the statistical evolution in quantum mechanics, are convolutions $\mu * \mu^R$ [3] of the spectral measures μ of the underlying Hamiltonian operator with their reflections $\mu^R(A) = \mu(-A)$. Therefore, in order to characterize the decaying statistical states, we have to study the decay properties of the convolutions of decaying, as well as of purely non-decaying, measures. This study is summarized in proposition 2.6.

First, we introduce the key concepts and notations, and formulate the Lebesgue theorem.

We say that a function f on a locally compact topological space X *vanishes at infinity* if and only if, for any $\varepsilon > 0$, there exists a compact set $K \subset X$ such that $f(x) < \varepsilon$ for any $x \in X \setminus K$. We denote the space of all continuous complex-valued functions on X vanishing at infinity by $C_0(X)$. This space is a Banach space with respect to the norm

$$\|f\|_c = \sup |f|. \tag{1.1}$$

Let G be a non-discrete Hausdorff locally compact topological abelian group (we denote the group operation by $+$), $G^\times$ be the dual group for G, i.e. $G^\times$ is the group of all continuous homomorphisms $h : G \to \mathbb{T}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ from G to the circle $\mathbb{T}^1$, which is also (with the natural so-called compact-open topology [9,16]) a Hausdorff locally compact topological abelian group. Also let $\mathfrak{M} = \mathfrak{M}(G)$ be the space of all regular σ-additive finite Borel complex-valued measures on G (see [16] for definitions), $\mathfrak{M}^+ = \mathfrak{M}^+(G)$ be the set of positive measures from $\mathfrak{M}$, $\mathcal{P} = \mathcal{P}(G)$ be the set of probability measures from $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{c}} = \mathfrak{M}_{\mathrm{c}}(G)$ be the subspace of continuous measures from $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{p}} = \mathfrak{M}_{\mathrm{p}}(G)$ be the subspace of measures with countable support (i.e. regular measures equivalent to Borel measures with countable support), $\mathfrak{M}_{\mathrm{sc}} = \mathfrak{M}_{\mathrm{sc}}(G)$ be the subspace of continuous measures singular with respect to the Haar measure m on G (i.e. a positive locally finite σ-additive translation invariant measure; m exists and is unique up to multiplication over a positive constant) and $\mathfrak{M}_{\mathrm{ac}} = \mathfrak{M}_{\mathrm{ac}}(G)$ be the subspace of measures absolutely continuous with respect to

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m (see [12] for definitions). For $\mu \in \mathfrak{M}$, denote the variation or absolute value of μ by $|\mu|$ ($|\mu|$ is an element of $\mathcal{P}$), i.e.

$$|\mu|(A) = \sup\biggl\{\sum_n |\mu(A_n)| : A = \bigcup_n A_n,\ A_n \cap A_m = \emptyset \text{ if } m \neq n\biggr\}.$$

The space $\mathfrak{M}$ is a Banach space with the norm

$$\|\mu\|_v = |\mu|(G) \tag{1.2}$$

and an abelian Banach algebra with respect to the convolution

$$(\mu * \nu)(A) = \int_G \nu(A - g)\,\mathrm{d}\mu(g),$$

where $A - g = \{h - g : h \in A\}$.

Let μ be a σ-additive complex-valued measure on some measurable space $(X, \mathcal{F})$ and ν be a positive (not necessarily finite) σ-additive measure on $(X, \mathcal{F})$. We write $\mu \prec \nu$ if μ is absolutely continuous with respect to ν and we write $\mu \perp \nu$ if μ is singular with respect to ν. If $A \in \mathcal{F}$, then let μ_A be the restriction of μ to A, i.e. the measure defined by the equality

$$\mu_A(B) = \mu(A \cap B). \tag{1.3}$$

Lebesgue theorem (see [18]). Let μ be a σ-additive complex-valued measure on some measurable space $(X, \mathcal{F})$ and ν be a positive (not necessarily finite) σ-additive measure on $(X, \mathcal{F})$. Then there exist unique measures μ_1 and μ_2 such that $\mu_1 \prec \nu$, $\mu_2 \perp \nu$ and $\mu = \mu_1 + \mu_2$. Moreover, there exists $A \in \mathcal{F}$ such that $\mu_1 = \mu_A$ and $\mu_2 = \mu_{X \setminus A}$.

According to this theorem $\mathfrak{M}_{\mathrm{p}}$, $\mathfrak{M}_{\mathrm{c}}$, $\mathfrak{M}_{\mathrm{sc}}$ and $\mathfrak{M}_{\mathrm{ac}}$ are closed linear subspaces of $\mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}_{\mathrm{p}} \oplus \mathfrak{M}_{\mathrm{c}}$ and $\mathfrak{M}_{\mathrm{c}} = \mathfrak{M}_{\mathrm{sc}} \oplus \mathfrak{M}_{\mathrm{ac}}$, where $\oplus$ is the direct sum in the category of Banach spaces.

We present here a new decomposition of $\mathfrak{M}$ based on the asymptotic behaviour of the Fourier transform (the characteristic functional) of a measure,

$$\tilde{\mu}(h) = \int_G \mathrm{e}^{\mathrm{i}(g|h)}\,\mathrm{d}\mu(g), \tag{1.4}$$

where $(g \mid h) = h(g)$.

It is well known that $\tilde{\mu}$ is a bounded uniformly continuous complex-valued function on $G^{\times}$. We say that a measure $\mu \in \mathfrak{M}$ is *decaying* if its Fourier transform vanishes at infinity and denote by $\mathfrak{M}^{\mathrm{D}} = \mathfrak{M}^{\mathrm{D}}(G)$ the set of all decaying measures from $\mathfrak{M}(G)$. This interesting class of measures has been studied by many authors (see, for example, [4, 11]). In particular, it has been shown in [11] that the space $\mathfrak{M}_{\mathrm{sc}} \cap \mathfrak{M}^{\mathrm{D}}$ is non-zero for any non-discrete locally compact topological abelian group.

We say that a measure $\mu \in \mathfrak{M}$ is *purely non-decaying* if $\nu \notin \mathfrak{M}^{\mathrm{D}}$ for any $\nu \in \mathfrak{M}$, $\nu \prec |\mu|$, $\nu \neq 0$ and denote by $\mathfrak{M}^{\mathrm{ND}} = \mathfrak{M}^{\mathrm{ND}}(G)$ the set of all purely non-decaying measures from $\mathfrak{M}(G)$.

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We show that $\mathfrak{M}^{\mathrm{ND}}$ is a closed linear subspace of the Banach space $\mathfrak{M}$ and that $\mathfrak{M}$ is the direct sum of $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$. We prove that subspace $\mathfrak{M}_{\mathrm{bs}}$ of $\mathfrak{M}(\mathbb{R})$ consisting of measures with bounded support can not be decomposed as a direct sum of $\mathfrak{M}^{\mathrm{D}} \cap \mathfrak{M}_{\mathrm{bs}}$ and any subalgebra of $\mathfrak{M}(\mathbb{R})$ with respect to the convolution $*$.

Obviously, $\mathfrak{M}^{\mathrm{D}}$ is a linear subspace of $\mathfrak{M}$, $\mathfrak{M}_{\mathrm{p}} \subseteq \mathfrak{M}^{\mathrm{ND}}$ and the generalized Riemann–Lebesgue theorem [16] implies that $\mathfrak{M}_{\mathrm{ac}} \subseteq \mathfrak{M}^{\mathrm{D}}$.

After the statement of the main results in § 2, we prove the lemmas in §§ 3 and 5 and the main results in §§ 4, 6 and 7.

2. Main results

The main results of the present paper are summarized by the following two theorems.

Theorem 2.1.

(i) *For any measure $\mu \in \mathfrak{M}$, there exists a unique decomposition*

$$\mu = \mu^{\mathrm{D}} + \mu^{\mathrm{ND}}, \quad \textit{where } \mu^{\mathrm{D}} \in \mathfrak{M}^{\mathrm{D}} \quad \textit{and} \quad \mu^{\mathrm{ND}} \in \mathfrak{M}^{\mathrm{ND}}. \tag{2.1}$$

Moreover, there exists a unique (up to coincidence almost everywhere with respect to the measure $|\mu|$) Borel set $A \subseteq G$ such that $\mu^{\mathrm{D}} = \mu_A$ and $\mu^{\mathrm{ND}} = \mu_{G\setminus A}$ (in the sense of (1.3)). In particular, if $\mu \in \mathfrak{M}^+$, then $\mu^{\mathrm{D}}, \mu^{\mathrm{ND}} \in \mathfrak{M}^+$ and if $\mu \in \mathfrak{M}_{\mathrm{sc}}$, then $\mu^{\mathrm{D}}, \mu^{\mathrm{ND}} \in \mathfrak{M}_{\mathrm{sc}}$.

(ii) *The sets $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$ are closed linear subspaces of the Banach space $\mathfrak{M}$,*

$$\mathfrak{M} = \mathfrak{M}^{\mathrm{D}} \oplus \mathfrak{M}^{\mathrm{ND}}, \tag{2.2}$$

and the norms of the projections $P^{\mathrm{D}} : \mu \mapsto \mu^{\mathrm{D}}$ onto $\mathfrak{M}^{\mathrm{D}}$ along $\mathfrak{M}^{\mathrm{ND}}$ and $P^{\mathrm{ND}} : \mu \mapsto \mu^{\mathrm{ND}}$ onto $\mathfrak{M}^{\mathrm{ND}}$ along $\mathfrak{M}^{\mathrm{D}}$ are equal to 1.

(iii) *If X is a closed linear subspace of $\mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}^{\mathrm{D}} \oplus X$ and the norm of the projection onto X along $\mathfrak{M}^{\mathrm{D}}$ is equal to 1, then $X = \mathfrak{M}^{\mathrm{ND}}$.*

This theorem implies the following result.

Corollary 2.2. *The spaces $\mathfrak{M}_{\mathrm{sc}}$, $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$ admit the decompositions*

$$\mathfrak{M}_{\mathrm{sc}} = \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}, \tag{2.3}$$

$$\mathfrak{M}^{\mathrm{D}} = \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathfrak{M}_{\mathrm{ac}}, \tag{2.4}$$

$$\mathfrak{M}^{\mathrm{ND}} = \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}} \oplus \mathfrak{M}_{\mathrm{p}}, \tag{2.5}$$

where

$$\mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}} = \mathfrak{M}_{\mathrm{sc}} \cap \mathfrak{M}^{\mathrm{D}} \quad \textit{and} \quad \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}} = \mathfrak{M}_{\mathrm{sc}} \cap \mathfrak{M}^{\mathrm{ND}}.$$

This corollary and the generalized Riemann–Lebesgue lemma immediately imply the next result.

Corollary 2.3. *The complete decomposition of the space $\mathfrak{M}$ in terms of the four spaces*

$$\mathfrak{M}_{\mathrm{p}}, \quad \mathfrak{M}_{\mathrm{ac}}, \quad \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}, \quad \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}} \tag{2.6}$$

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has the form

$$\mathfrak{M} = \mathfrak{M}_{\mathrm{p}} \oplus \mathfrak{M}_{\mathrm{ac}} \oplus \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}} \oplus \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}. \tag{2.7}$$

The decay properties of convolutions of measures are presented in the following theorems 2.4 and 2.5. Pick two classes (they may be identical) among the four classes of measures listed in (2.6). The question is whether convolutions of measures from these two classes belong to a certain class among (2.6). Theorem 2.4 summarizes the cases where these convolutions belong to precisely one space from (2.6). The examples in theorem 2.5 show cases when these convolutions are not in precisely one class from (2.6). These examples involve measures over the real line; however, they can be easily generalized for measures on $G = \mathbb{R}^n \times \mathbb{T}^m$.

THEOREM 2.4. *For any $\mu, \nu \in \mathfrak{M}$, we have*

(i) *if $\mu \in \mathfrak{M}_{\mathrm{p}}$ and $\nu \in \mathfrak{M}_{\mathrm{ac}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{ac}}$;*

(ii) *if $\mu \in \mathfrak{M}_{\mathrm{p}}$ and $\nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$;*

(iii) *if $\mu \in \mathfrak{M}_{\mathrm{p}}$ and $\nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$;*

(iv) *if $\mu \in \mathfrak{M}_{\mathrm{p}}$ and $\nu \in \mathfrak{M}_{\mathrm{p}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{p}}$;*

(v) *if $\mu \in \mathfrak{M}_{\mathrm{ac}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{ac}}$;*

(vi) *if $\mu \in \mathfrak{M}_{\mathrm{c}}$, then $\mu * \nu \in \mathfrak{M}_{\mathrm{c}}$;*

(vii) *if $\mu \in \mathfrak{M}^{\mathrm{D}}$, then $\mu * \nu \in \mathfrak{M}^{\mathrm{D}}$.*

Theorem 2.4 immediately follows from elementary properties of Fourier transforms and convolutions of measures (see, for example, [16]); except for the third part, which follows from theorem 2.1. According to theorem 2.4, the subspaces $\mathfrak{M}_{\mathrm{c}}$, $\mathfrak{M}_{\mathrm{ac}}$ and $\mathfrak{M}^{\mathrm{D}}$ are ideals in the Banach algebra $\mathfrak{M}$. Subspace $\mathfrak{M}_{\mathrm{p}}$ is a subalgebra of $\mathfrak{M}$.

THEOREM 2.5.

(i) *There exists a measure $\mu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ such that*

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}} \in \mathfrak{M}^{\mathrm{ND}}(\mathbb{R}) \quad \text{for any } n \in \mathbb{N}.$$

(ii) *There exist measures $\mu, \nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ such that $\mu * \nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}(\mathbb{R})$.*

(iii) *There exist measures $\mu, \nu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ such that $\mu * \nu \in \mathfrak{M}_{\mathrm{ac}}(\mathbb{R})$.*

(iv) *Let $\mu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R})$, $\mu \neq 0$. Then there exists a unique representation*

$$\mu * \mu = \mu_1 + \mu_2 + \mu_3,$$

where $\mu_1 \in \mathfrak{M}_{\mathrm{ac}}$, $\mu_2 \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$, $\mu_3 \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$ and $\mu_3 \neq 0$. Moreover,

(a) *there exists a measure $\mu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ such that $\mu_1 \neq 0$ and $\mu_2 = 0$;*

(b) *there exists a measure $\mu \in \mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ such that $\mu_2 \neq 0$ and $\mu_1 = 0$;*

(c) *there exists a measure* $\mu \in \mathfrak{M}^{\mathrm{ND}}_{\mathrm{sc}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *such that* $\mu_2 \neq 0$ *and* $\mu_1 \neq 0$.

(v) *There exists a measure* $\mu \in \mathfrak{M}^{\mathrm{D}}_{\mathrm{sc}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *such that*

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}} \in \mathfrak{M}^{\mathrm{D}}_{\mathrm{sc}}(\mathbb{R}) \quad \text{for any } n \in \mathbb{N}.$$

(vi) *There exists a measure* $\mu \in \mathfrak{M}^{\mathrm{D}}_{\mathrm{sc}}(\mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ *such that* $\mu * \mu \in \mathfrak{M}_{\mathrm{ac}}(\mathbb{R})$.

Proposition 2.6. *Let the linear space* $\mathfrak{M}_{\mathrm{bs}}(\mathbb{R})$ *be an algebraic direct sum of* $\mathfrak{M}^{\mathrm{D}}(\mathbb{R}) \cap \mathfrak{M}_{\mathrm{bs}}$ *and some linear subspace* X. *Then* X *is not a subalgebra of* $\mathfrak{M}$.

Note that in the case of discrete G (equivalently, compact $G^{\times}$), we have

$$\mathfrak{M} = \mathfrak{M}^{\mathrm{D}} = \mathfrak{M}_{\mathrm{ac}} = \mathfrak{M}_{\mathrm{c}} = \mathfrak{M}_{\mathrm{p}}, \qquad \mathfrak{M}_{\mathrm{sc}} = \mathfrak{M}^{\mathrm{ND}} = \{0\}.$$

So, in this case, some of the formulated results are trivial, others are wrong.

3. Lemmas for the proof of theorem 2.1

The following lemma is an easy exercise.

Lemma 3.1. *Let* $f \in C_0(G^{\times})$ *and* $g \in L_1(G^{\times}, m)$ *(*m *is the Haar measure). Then* $f * g \in C_0(G^{\times})$, *where*

$$(f * g)(x) = \int_{G^{\times}} f(u) g(x-u) \, \mathrm{d}m(u)$$

is the convolution of f *and* g.

Lemma 3.2. *Let* $\mu \in \mathfrak{M}^{+}$. *Then the set* $\{f \in C_0(G) : \hat{f} \in L_1(G^{\times}, m)\}$ *is a dense linear subspace of the Banach space* $L_1(G, \mu)$, *where*

$$\hat{f}(h) = \int_{G} f(g) \mathrm{e}^{\mathrm{i}(g|h)} \, \mathrm{d}m(g)$$

is the Fourier transform of f.

Proof. The inclusion $C_0(G) \subseteq L_1(G, \mu)$ is obvious since all elements of $C_0(G)$ are bounded and measurable. Clearly, $C_0(G)$ is dense in $L_1(G, \mu)$. To complete the proof, it suffices to verify that the set $A = \{f \in C_0(G) : \hat{f} \in L_1(G^{\times}, m)\}$ is dense in $C_0(G)$ with respect to the uniform convergence topology. Since A is the image of Banach algebra $L_1(G^{\times})$ under the Gelfand transform (see [16]), A is a subalgebra of the Banach algebra $C_0(G)$ (with pointwise multiplication), separating points. The desired density follows from the Stone–Weierstrass theorem [10]. □

The following lemma is a generalization of the theorem 10.9 of [23, chapter XII].

Lemma 3.3. *Let* $\mu \in \mathfrak{M}^{\mathrm{D}}$ *and* $\nu \in \mathfrak{M}$, $\nu \prec |\mu|$. *Then* $\nu \in \mathfrak{M}^{\mathrm{D}}$.

Proof. Let f be the density of ν with respect to $|\mu|$ and g be the density of μ with respect to $|\mu|$. Then $|g| = 1$ almost everywhere with respect to $|\mu|$ and $f \in L_1(G, |\mu|)$. According to lemma 3.2, there exists a sequence $f_n \in C_0(G)$ such that $\|f_n - fg^{-1}\|_{L_1(G,\mu)} \to 0$ for $n \to \infty$ and $\hat{f}_n \in L_1(G^\times, m)$. For any $n \in \mathbb{N}$, let ν_n be the measure with density $f_n g$ with respect to the measure $|\mu|$. Since

$$\tilde{\nu}_n(h) = \int_G f_n(x)g(x)\mathrm{e}^{\mathrm{i}(x|h)}\,\mathrm{d}|\mu(x)| = \int_G f_n(x)\mathrm{e}^{\mathrm{i}(x|h)}\,\mathrm{d}\mu(x),$$

we have that $\tilde{\nu}_n$ is equal (up to multiplication by a positive constant) to the convolution $\tilde{\mu} * \hat{f}_n$. From the assumptions, $\tilde{\mu} \in C_0(G^\times)$. Since $\hat{f}_n \in L_1(G^\times)$, lemma 3.1 implies that $\tilde{\nu}_n \in C_0(G^\times)$. Since the sequence $f_n g$ converges to f with respect to the norm of $L_1(G, |\mu|)$, we have that $\|\nu_n - \nu\|_v \to 0$ for $n \to \infty$. The inequality

$$\|\hat{f}\|_c \leqslant c\|f\|_{L_1} \tag{3.1}$$

(the constant c depends on normalizations of Haar measures on G and $G^\times$) implies uniform convergence of $\tilde{\nu}_n$ to $\tilde{\nu}$. Since the uniform limit of the elements of $C_0(G^\times)$ is again an element of $C_0(G^\times)$, we have that $\tilde{\nu} \in C_0(G^\times)$. Lemma 3.3 is proved. □

Lemma 3.4. *Let $\mu \in \mathfrak{M}^{\mathrm{D}}$ and $\nu \in \mathfrak{M}^{\mathrm{ND}}$. Then μ is singular with respect to $|\nu|$.*

Proof. From the Lebesgue theorem, there exist unique $\gamma, \eta \in \mathfrak{M}$ such that $\mu = \gamma + \eta$, $\gamma \prec |\nu|$ and $\eta \perp |\nu|$. Moreover, there exists a Borel set $A \subseteq G$ such that $\gamma = \mu_A$, $\eta = \mu_{G\setminus A}$. Then $\gamma \prec |\mu|$ and therefore, according to lemma 3.3, $\tilde{\gamma} \in C_0(G^\times)$. If $\gamma \neq 0$, then, since $\gamma \prec |\nu|$, we have that $\tilde{\gamma} \notin C_0(G^\times)$. Therefore, $\gamma = 0$. Hence $\mu = \eta \perp |\nu|$. The lemma is proved. □

Corollary 3.5. *Let $\mu_1, \dots, \mu_n \in \mathfrak{M}^{\mathrm{ND}}$, $c_1, \dots, c_n \in \mathbb{C}$, $\nu = c_1\mu_1 + \dots + c_n\mu_n \neq 0$. Then $\nu \notin \mathfrak{M}^{\mathrm{D}}$.*

Proof. Suppose that $\nu \in \mathfrak{M}^{\mathrm{D}}$. Lemma 3.4 implies that, for any $j = 1, \dots, n$, there exists a Borel set A_j such that $|\nu|(G \setminus A_j) = 0$ and $|\mu_j|(A_j) = 0$. Let $A = \bigcap_{j=1}^n A_j$. Then $|\nu|(G \setminus A) = 0$ and $|\mu_j|(A_j) = 0$ for all $j = 1, \dots, n$. Hence

$$|\nu(A)| \leqslant \sum_{j=1}^n |c_j||\mu_j|(A) = 0.$$

Therefore, $|\nu|(G) = |\nu|(A) + |\nu|(G \setminus A) = 0$. Thus $\nu = 0$. This contradiction completes the proof of the corollary. □

4. Proof of theorem 2.1

Part I. Let $\mathcal{A}^{\mathrm{D}}$ be the set of all Borel subsets A of G for which $|\mu|_A \in \mathfrak{M}^{\mathrm{D}}$. The strategy of the proof of the first part of the theorem is to find the maximal element M of $\mathcal{A}^{\mathrm{D}}$ and to show that the decaying part of the measure μ coincides with the restriction of μ to M.

First, let us show that $A \cup B \in \mathcal{A}^{\mathrm{D}}$ for any $A, B \in \mathcal{A}^{\mathrm{D}}$. Obviously,

$$|\mu|_{A\cup B} \prec \nu = |\mu|_A + |\mu|_B.$$

From the definition of $\mathcal{A}^{\mathrm{D}}$, we have that $\nu \in \mathfrak{M}^{\mathrm{D}}$. From lemma 3.3, $|\mu|_{A\cup B} \in \mathfrak{M}^{\mathrm{D}}$. Hence $A \cup B \in \mathcal{A}^{\mathrm{D}}$.

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Let us show now that there exists $M \in \mathcal{A}^{\mathrm{D}}$ such that

$$|\mu|(M \setminus A) = 0 \quad \text{for any } A \in \mathcal{A}^{\mathrm{D}}. \tag{4.1}$$

First, choose $A_n \in \mathcal{A}^{\mathrm{D}}$ for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} |\mu|(A_n) = \sup_{A\in\mathcal{A}^{\mathrm{D}}} |\mu|(A)$$

and put $B_n = \bigcup_{j=1}^{n} A_j$. Since the class $\mathcal{A}^{\mathrm{D}}$ is closed (as we just proved) with respect to finite unions, $B_n \in \mathcal{A}^{\mathrm{D}}$. Let

$$M = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n.$$

We shall prove that M is the desired set. The equality $\| |\mu|_M - |\mu|_{B_n} \|_v = |\mu|(M \setminus B_n)$ and σ-additivity of the measure $|\mu|$ imply that $\| |\mu|_M - |\mu|_{B_n} \|_v \to 0$ for $n \to \infty$. Inequality (3.1) implies uniform convergence of the sequence

$$\widetilde{|\mu|_{B_n}} \quad \text{to} \quad \widetilde{|\mu|_M}.$$

Therefore,

$$\widetilde{|\mu|_M} \in C_0(G^{\times}).$$

Hence $M \in \mathcal{A}^{\mathrm{D}}$. Let $A \in \mathcal{A}^{\mathrm{D}}$. Then, as we already showed, $M \cup A \in \mathcal{A}^{\mathrm{D}}$. Consequently,

$$|\mu|(M \cup A) \geqslant |\mu|(M) \geqslant \lim_{n\to\infty} |\mu|(B_n) \geqslant \lim_{n\to\infty} |\mu|(A_n) \geqslant |\mu|(M \cup A).$$

Since the first and the last terms in the previous formula are identical, all inequalities in this formula are actually equalities. So $|\mu|(M \cup A) = |\mu|(M)$. Therefore, $|\mu|(M \setminus A) = 0$. Thus the set M has all the desired properties.

Now let $\mu^{\mathrm{D}} = \mu_M$ and $\mu^{\mathrm{ND}} = \mu - \mu^{\mathrm{D}} = \mu_{G\setminus M}$. From the definition of $\mathcal{A}^{\mathrm{D}}$, it follows that $|\mu|_M \in \mathfrak{M}^{\mathrm{D}}$. Since $\mu^{\mathrm{D}} \prec |\mu|_M$, lemma 3.3 implies that $\mu^{\mathrm{D}} \in \mathfrak{M}^{\mathrm{D}}$. Let the measure $\gamma \in \mathfrak{M}^{\mathrm{D}}$ and $\gamma \prec |\mu^{\mathrm{ND}}| = |\mu|_{G\setminus M}$. To prove that $\mu^{\mathrm{ND}} \in \mathfrak{M}^{\mathrm{ND}}$, it suffices to show that $\gamma = 0$. Suppose that $\gamma \neq 0$. Then the density f of γ with respect to $|\mu|_{G\setminus M}$ is a non-zero element of $L_1(G, |\mu|_{G\setminus M})$. Then there exists $\varepsilon > 0$ such that $|\mu|(C) > 0$, where $C = \{x \in G \setminus M : |f(x)| \geqslant \varepsilon\}$. Since the density of $|\mu|_C$ with respect to $|\mu|$ is the indicator of the set C and the absolute value of the density of γ with respect to $|\mu|$ is greater than ε on C, $|\mu|_C \prec |\gamma|$. Since $\gamma \in \mathfrak{M}^{\mathrm{D}}$, lemma 3.3 implies that $|\mu|_C \in \mathfrak{M}^{\mathrm{D}}$. Therefore, $C \in \mathcal{A}^{\mathrm{D}}$. Formula (4.1) and the inclusion $C \subseteq G \setminus M$ imply that $|\mu|(C) = |\mu|(C \setminus M) = 0$. This contradiction proves that $\mu^{\mathrm{ND}} \in \mathfrak{M}^{\mathrm{ND}}$.

It remains to verify the uniqueness of the decomposition (2.1) of μ. Suppose $\mu = \nu + \eta$ and $\nu \in \mathfrak{M}^{\mathrm{D}}$, $\eta \in \mathfrak{M}^{\mathrm{ND}}$. Then $\nu - \mu^{\mathrm{D}} = \mu^{\mathrm{ND}} - \eta$. Since $\nu - \mu^{\mathrm{D}} \in \mathfrak{M}^{\mathrm{D}}$ and $\eta, \mu^{\mathrm{ND}} \in \mathfrak{M}^{\mathrm{ND}}$, corollary 3.5 implies that $\nu - \mu^{\mathrm{D}} = 0$. Therefore, $\nu = \mu^{\mathrm{D}}$ and $\eta = \mu^{\mathrm{ND}}$.

The last statement of the first part of theorem 2.1 follows from the fact that singularity, continuity and positivity are preserved under the restriction of a measure to a subset. Part (i) of theorem 2.1 is proved.

Part II. Let us show now that $\mathfrak{M}^{\mathrm{ND}}$ is a linear subspace of $\mathfrak{M}$ (we already mentioned that $\mathfrak{M}^{\mathrm{D}}$ is a linear subspace of $\mathfrak{M}$). The inclusion $c\mu \in \mathfrak{M}^{\mathrm{ND}}$ for any $\mu \in \mathfrak{M}^{\mathrm{ND}}$ and $c \in \mathbb{C}$ is trivial. Let $\mu, \nu \in \mathfrak{M}^{\mathrm{ND}}$. From the already proved first part of theorem 2.1, we have that there exist $\eta \in \mathfrak{M}^{\mathrm{ND}}$ and $\gamma \in \mathfrak{M}^{\mathrm{D}}$ such that $\mu + \nu = \eta + \gamma$. Then $\mu + \nu - \eta = \gamma$. From corollary 3.5, we have that $\gamma = 0$. Therefore, $\mu + \nu \in \mathfrak{M}^{\mathrm{ND}}$. So $\mathfrak{M}^{\mathrm{ND}}$ is a linear subspace of $\mathfrak{M}$. The first part of theorem 2.1 now implies that the linear space $\mathfrak{M}$ is the direct sum of the linear subspaces $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$ and that the projections P^{D} onto $\mathfrak{M}^{\mathrm{D}}$ along $\mathfrak{M}^{\mathrm{ND}}$ and P^{ND} onto $\mathfrak{M}^{\mathrm{ND}}$ along $\mathfrak{M}^{\mathrm{D}}$ map each measure to some restriction of this measure to some Borel subset. Therefore, $\|P^{\mathrm{D}}\| = \|P^{\mathrm{ND}}\| = 1$. In particular, P^{D} is continuous. Consequently, the subspaces $\mathfrak{M}^{\mathrm{D}}$ and $\mathfrak{M}^{\mathrm{ND}}$ of the Banach space $\mathfrak{M}$ are closed. The second part of theorem 2.1 is proved.

Part III. Suppose that X is a closed linear subspace of $\mathfrak{M}$, $\mathfrak{M} = \mathfrak{M}^{\mathrm{D}} \oplus X$ and the norm of the projection T onto X along $\mathfrak{M}^{\mathrm{D}}$ is equal to 1. It remains to show that $X = \mathfrak{M}^{\mathrm{ND}}$. Suppose $X \neq \mathfrak{M}^{\mathrm{ND}}$. Then there exists a measure $\mu \in \mathfrak{M}^{\mathrm{ND}}$ such that $\nu = \mu - T\mu \neq 0$. Since $T\nu = 0$, we have that $\nu \in \mathfrak{M}^{\mathrm{D}}$. Lemma 3.4 implies that $\nu \perp |\mu|$. Therefore,

$$\|T\mu\|_v = \|\mu - \nu\|_v = \|\mu\|_v + \|\nu\|_v > \|\mu\|_v.$$

Hence $\|T\| > 1$, which is a contradiction.

5. Lemmas for the proof of theorem 2.5

In this section we consider only measures on real line, i.e. here, $G = \mathbb{R}$. Let us consider the a following seminorm on $\mathfrak{M}$:

$$\|\mu\|_{\mathrm{ls}} = \varlimsup_{x\to\infty} \tilde{\mu}(x). \tag{5.1}$$

Inequality (3.1) implies that

$$\|\mu\|_{\mathrm{ls}} \leqslant \|\tilde{\mu}\|_c \leqslant \|\mu\|_v. \tag{5.2}$$

We denote the set of all $\mu \in \mathfrak{M}(\mathbb{R})$ such that $\|\mu\|_v = \|\mu\|_{\mathrm{ls}}$ by $\mathfrak{M}_{\mathrm{uls}}$.

Lemma 5.1. *The following inclusions are valid:*

$$\mathfrak{M}_{\mathrm{p}} \subseteq \mathfrak{M}_{\mathrm{uls}} \subseteq \mathfrak{M}^{\mathrm{ND}}.$$

Moreover,

(i) *if $\mu \in \mathfrak{M}_{\mathrm{uls}}$, then*

$$\underbrace{\mu * \mu * \cdots * \mu}_{n\ \text{times}} \in \mathfrak{M}_{\mathrm{uls}} \quad \text{for any } n \in \mathbb{N};$$

(ii) *if $\mu \in \mathfrak{M}^{+}$, $n \in \mathbb{N}$ and*

$$\underbrace{\mu * \mu * \cdots * \mu}_{n\ \text{times}} \in \mathfrak{M}_{\mathrm{uls}},$$

then $\mu \in \mathfrak{M}_{\mathrm{uls}}$.

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Proof. The inclusion $\mathfrak{M}_{\mathrm{p}} \subseteq \mathfrak{M}_{\mathrm{uls}}$ is obvious. Let $\mu \in \mathfrak{M}_{\mathrm{uls}}$. According to theorem 2.1, $\mu = \mu_1 + \mu_2$, where $\mu_1 \in \mathfrak{M}^{\mathrm{D}}$ and $\mu_2 \in \mathfrak{M}^{\mathrm{ND}}$ and $\mu_1 \perp \mu_2$. Since $\|\mu_1\|_{\mathrm{ls}} = 0$, we have that

$$\|\mu_1\|_{\mathrm{ls}} = \|\mu\|_{\mathrm{ls}} = \|\mu\|_v = \|\mu_1\|_v + \|\mu_2\|_v \leqslant \|\mu_1\|_v \leqslant \|\mu_1\|_{\mathrm{ls}}.$$

Since the first and last terms in this formula are identical, all inequalities in it are actually equalities. Therefore, $\|\mu_1\|_v + \|\mu_2\|_v = \|\mu_1\|_v$, i.e. $\|\mu_2\|_v = 0$. Hence $\mu_2 = 0$. Thus $\mu = \mu_1 \in \mathfrak{M}^{\mathrm{ND}}$.

Let $\mu \in \mathfrak{M}_{\mathrm{uls}}$ and

$$\nu = \underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}}.$$

Then

$$\|\nu\|_{\mathrm{ls}} = \varlimsup_{x\to\infty} |\tilde{\nu}(x)| = \varlimsup_{x\to\infty} |\tilde{\mu}(x)|^n = \|\mu\|_{\mathrm{ls}}^n = \|\mu\|_v^n \geqslant \|\nu\|_v.$$

This inequality, together with (5.2), implies that $\|\nu\|_{\mathrm{ls}} = \|\nu\|_v$. Thus $\nu \in \mathfrak{M}_{\mathrm{uls}}$.

Suppose now that $\mu \in \mathfrak{M}^+$ and

$$\nu = \underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}} \in \mathfrak{M}_{\mathrm{uls}}.$$

Since

$$\|\nu\|_v = \nu(\mathbb{R}) = (\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}})(\mathbb{R}) = \mu(\mathbb{R})^n = \|\mu\|_v^n$$

and

$$\|\nu\|_{\mathrm{ls}} = \varlimsup_{x\to\infty} |\tilde{\nu}(x)| = \varlimsup_{x\to\infty} |\tilde{\mu}(x)|^n = \|\mu\|_{\mathrm{ls}}^n,$$

we have

$$\|\mu\|_{\mathrm{ls}} = (\|\nu\|_{\mathrm{ls}})^{1/n} = \|\nu\|_v^{1/n} = \|\mu\|_v.$$

Thus $\mu \in \mathfrak{M}_{\mathrm{uls}}$ and the lemma is proved. □

Recall [16] that a Borel set $A \subseteq \mathbb{R}$ is called a Kronecker set if and only if, for any continuous function $f : A \to \mathbb{C}$ such that $|f| \equiv 1$ and any $\varepsilon > 0$, there exists a $t \in \mathbb{R}$ such that $|f(x) - \mathrm{e}^{itx}| \leqslant \varepsilon$ for all $x \in A$. Theorem 5.2.2 of [16] shows the existence of a Cantor-type (i.e. perfect and nowhere dense) compact subset K_R of $\mathbb{R}$ that is a Kronecker set.

LEMMA 5.2. *Let $\mu \in \mathfrak{M}$ be a measure concentrated on a compact Kronecker set. Then $\mu \in \mathfrak{M}_{\mathrm{uls}}$.*

Proof. Lemma 5.2 follows immediately from theorem 5.5.2 of [16]. □

We will need below a well-known class of continuous measures with Fourier transform identical to a Kronecker product with parameter $\theta > 1$,

$$\tilde{\mu}_\theta(x) = \prod_{n=1}^{\infty} \cos\biggl(\frac{x}{\theta^n}\biggr). \tag{5.3}$$

The measure μ_θ is the weak limit of the sequence of measures $\nu_{\theta^{-1}} * \cdots * \nu_{\theta^{-n}}$, where ν_α is the measure on the real line concentrated in the two point set $\{-\alpha, \alpha\}$ such that $\nu_\alpha(\{-\alpha\}) = \nu_\alpha(\{\alpha\}) = \frac{1}{2}$. Note that

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(i) $\mu_\theta \in \mathcal{P}$ for all $\theta > 1$;

(ii) the standard Cantor measure (up to the transformation $x \mapsto x - \frac{1}{2}$) coincides with μ_3;

(iii) μ_2 is the normalized Lebesgue measure on the segment $[-1, 1]$.

LEMMA 5.3. *For any $\theta \in \mathbb{N}$, $\theta \geqslant 3$, the measure $\mu = \mu_\theta$ is an element of $\mathfrak{M}^{\mathrm{ND}}_{\mathrm{sc}} \setminus \mathfrak{M}_{\mathrm{uls}}$.*

Proof. Obviously, measure μ is concentrated in the Cantor-type compact set

$$K_\theta = \left\{ \sum_{j=1}^{\infty} \frac{\varepsilon_j}{\theta^j} : \varepsilon_j \in \{-1, 1\} \right\}, \tag{5.4}$$

which has Lebesgue measure zero. Therefore, $\mu \in \mathfrak{M}_{\mathrm{sc}}$. Let

$$c = \left| \prod_{n=1}^{\infty} \cos\left(\frac{2\pi}{\theta^n}\right) \right|.$$

Clearly, $c > 0$ and $|\tilde{\mu}(2\pi\theta^k)| = c$ for all $k \in \mathbb{N}$.

Suppose that $\mu \notin \mathfrak{M}^{\mathrm{ND}}$. According to theorem 2.1, there exists a subset A of K_θ such that $\mu(A) > 0$ and $\tilde{\mu}_A \in C_0(\mathbb{R})$. Since $\mu(A) > 0$, there exists an element x_0 of K such that

$$\lim_{\substack{\varepsilon \to 0,\ \delta \to 0 \\ \varepsilon, \delta > 0}} \frac{\mu((x_0 - \delta, x_0 + \varepsilon) \cap A)}{\mu(x_0 - \delta, x_0 + \varepsilon)} = 1.$$

Therefore, there exist $n \in \mathbb{N}$ and $\alpha \in \{0, 1\}^n$ such that

$$\frac{\mu(I^n_\alpha \setminus A)}{\mu(I^n_\alpha)} < \tfrac{1}{2} c, \quad \text{where } I^n_\alpha = \left[a^n_\alpha - \frac{\theta^{-n}}{\theta - 1}, a^n_\alpha + \frac{\theta^{-n}}{\theta - 1} \right], \quad a^n_\alpha = \sum_{j=1}^{n} \alpha_j \theta^{-j}.$$

From lemma 3.3,

$$\widetilde{\mu_{I^n_\alpha \cap A}} \in C_0(\mathbb{R}). \tag{5.5}$$

Inequality (3.1) implies that

$$\|\widetilde{\mu_{I^n_\alpha \setminus A}}\|_c \leqslant \|\mu_{I^n_\alpha \setminus A}\|_v = \mu(I^n_\alpha \setminus A) \leqslant \tfrac{1}{2} c \mu(I^n_\alpha) = c 2^{-n-1}. \tag{5.6}$$

Using (5.5) and (5.6), we arrive at

$$\varlimsup_{x \to +\infty} |\widetilde{\mu_{I^n_\alpha}}(x)| \leqslant c 2^{-n-1}. \tag{5.7}$$

As the measure $\mu_{I^n_\alpha}$ can be obtained from measure μ by contraction (with factor θ^n), shift and multiplication by 2^{-n}, we have

$$\varlimsup_{x \to +\infty} |\widetilde{\mu_{I^n_\alpha}}(x)| = 2^{-n} \varlimsup_{x \to +\infty} |\tilde{\mu}(x)| \geqslant 2^{-n} c. \tag{5.8}$$

The inequalities (5.7) and (5.8) contradict each other. Therefore, $\mu \in \mathfrak{M}^{\mathrm{ND}}$. The inequality $\|\mu_\theta\|_{\mathrm{ls}} < \|\mu_\theta\|_v$ can be easily verified. Therefore, $\mu_\theta \notin \mathfrak{M}_{\mathrm{uls}}$. Lemma 5.3 is proved. □

LEMMA 5.4 (see [6, 23]). *Let θ be a rational number, $\theta > 2$ and $\theta \notin \mathbb{Z}$. Then $\mu_\theta \in \mathfrak{M}^{\mathrm{D}}_{\mathrm{sc}}(\mathbb{R})$.*

6. Proof of theorem 2.5

Part I. Let $\mu \in \mathfrak{M}$ be a probability continuous measure concentrated on the above perfect Kronecker compact set K_R. Then, from lemma 5.2, it follows that $\mu \in \mathfrak{M}_{\rm uls}$ and from lemma 5.1 it follows that

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}} \in \mathfrak{M}_{\rm uls} \subseteq \mathfrak{M}^{\rm ND}(\mathbb{R}) \quad \text{for any } n \in \mathbb{N}.$$

Singularity of μ follows from the Riemann–Lebesgue theorem.

Part II. Let μ be the measure μ_5 of (5.3) and ν be the measure obtained from μ by expansion with the factor

$$\mathfrak{a} = \sum_{n=0}^{\infty} 5^{-n^2}.$$

According to (5.3),

$$\tilde{\mu}(x) = \prod_{n=1}^{\infty} \cos(x5^{-n}), \qquad \tilde{\nu}(x) = \prod_{n=1}^{\infty} \cos(\mathfrak{a}x5^{-n}). \tag{6.1}$$

Lemma 5.3 implies that $\mu, \nu \in \mathfrak{M}_{\rm sc}^{\rm ND}$. It suffices to show that $\mu * \nu \in \mathfrak{M}_{\rm sc}^{\rm D}$. One can easily verify, using (5.4), that (see, for example, [23]) the Hausdorff dimension of supports of μ and ν is $\log_5 2$ and the Hausdorff dimension of the support of $\mu * \nu$ is at most $\log_5 4 < 1$. Hence measure $\mu * \nu$ is singular. In view of (6.1), it remains to verify that $F \in C_0(\mathbb{R})$, where

$$F(x) = \prod_{n=1}^{\infty} (\cos(x5^{-n}) \cos(\mathfrak{a}x5^{-n})).$$

For this purpose, let us consider the following notion. Suppose $m \in \mathbb{N}$ is represented in pentadic system,

$$m = \sum_{j=0}^{l} m_j 5^j, \quad \text{where } m_j \in \{0, 1, 2, 3, 4\}.$$

Denote by $\varkappa(m)$ the number of $j \in \overline{1, l}$ for which either $m_j \in \{1, 2, 3\}$ or $m_{j-1} \neq m_j$. It is easy to see that

$$\left| \prod_{n=1}^{\infty} \cos(x5^{-n}) \right| \leqslant q^{[\varkappa(x/2\pi)]}$$

for some $q \in (0, 1)$, where $[y]$ is the integer part of the number y. That is why, in order to prove that F vanishes at infinity, it suffices to show that $\varkappa(m) + \varkappa([\mathfrak{a}m]) \to +\infty$ as $m \to +\infty$ ($m \in \mathbb{N}$). The last statement follows from the definition of $\mathfrak{a}$, namely it is ensured by the form of the pentadic representation of $\mathfrak{a}$.

Part III. Let λ be the normalized Lebesgue measure on the segment $[-1, 1]$. One can directly verify (see [23]) that

$$\tilde{\lambda}(x) = \frac{\sin x}{x} = \prod_{n=1}^{\infty} \cos\left(\frac{x}{2^n}\right) = F(2x)F(x),$$

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where

$$F(x)=\prod_{n=0}^{\infty}\cos\biggl(\frac{x}{4^n}\biggr).$$

According to lemma 5.3, the function F is the Fourier transform of a singular continuous probability measure $\mu=\mu_4\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$ of (5.3). Then the measure ν, obtained from μ by expansion with factor 2, is also an element of $\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$ and has Fourier transform $F(2x)$. Therefore, $\lambda=\mu*\nu$. Thus the Lebesgue measure on a segment is a convolution of two measures from $\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$.

Part IV. Let $\mu\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R})$. Corollary 2.3 implies that $\mu*\mu=\mu_1+\mu_2+\mu_3$, where $\mu_1\in\mathfrak{M}_{\mathrm{ac}}$, $\mu_2\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$ and $\mu_3\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$. Since $\tilde{\mu}\notin C_0(\mathbb{R})$ and $|\widetilde{\mu*\mu}|=|\tilde{\mu}|^2$, we have that $\widetilde{\mu*\mu}\notin C_0(\mathbb{R})$. Therefore, $\mu_3\neq 0$.

Part IVA. Follows easily from the (already proved) first part of theorem 2.5.

Part IVB. Let μ and ν be measures, constructed in the proof of the second part of theorem 2.5, and let $\gamma=\frac{1}{2}(\mu+\nu)$. Theorem 2.1 implies that $\gamma\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}(\mathbb{R})$. Since the Hausdorff dimension of the support of $\gamma*\gamma$ is less than 1, we have $\gamma*\gamma\in\mathfrak{M}_{\mathrm{sc}}$. Evidently,

$$\gamma*\gamma=\tfrac{1}{4}(\mu*\mu+\nu*\nu)+\tfrac{1}{2}(\mu*\nu).$$

Similarly to the proof of lemma 5.3, one can show that $\frac{1}{4}(\mu*\mu+\nu*\nu)\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{ND}}$, while in the proof of the third part of theorem 2.5 it have been shown that $\frac{1}{2}(\mu*\nu)\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$. Therefore, γ is the desired probability measure.

Part IVC. Consider four measures μ_j, $j\in\{1,2,3,4\}$, where μ_1 and μ_2 are measures μ and ν, constructed in the proof of the second part of theorem 2.5 and μ_3, μ_4 are measures μ and ν, constructed in the proof of the third part of theorem 2.5. The desired measure is

$$\gamma=\frac{1}{4}\sum_{j=1}^{4}\mu_j.$$

The proof is similar to the proof of part (iv) b.

Part V. The main point of the proof is to find a measure $\mu\in\mathfrak{M}_{\mathrm{sc}}^{\mathrm{D}}$ with the compact support K such that the Hausdorff dimension of the set

$$\underbrace{K+K+\cdots+K}_{n\ \text{times}}$$

is zero for any $n\in\mathbb{N}$. Let $\theta=\frac{5}{2}$. We shall show that there exists a strictly increasing sequence n_k of non-negative integers such that $n_0=0$ and

$$\lim_{t\to\infty}\prod_{m=1}^{\infty}\prod_{j=n_{m-1}+1}^{n_m}\cos\biggl(\frac{t}{\theta^{j2^m}}\biggr)=0. \tag{6.2}$$

According to lemma 5.4, $\lim_{t\to\infty}F_1(t)=0$, where

$$F_1(t)=\prod_{j=1}^{\infty}\cos\biggl(\frac{t}{\theta^j}\biggr).$$

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Thus there exists $A_1 > 1$ such that $|F_1(t)| < \frac{1}{3}$ for all t such that $|t| \geqslant A_1$. Again, using lemma 5.4, we have that $\lim_{t\to\infty} G_1(t) = 0$, where

$$G_1(t) = \prod_{j=1}^{\infty} \cos\biggl(\frac{t}{\theta^{2j}}\biggr).$$

Therefore, there exists $A_2 > \max\{2, A_1\}$ such that $|G_1(t)| < \frac{1}{4}$ for all t such that $|t| \geqslant A_2$. Using uniform convergence of the product and defining F_1 on the segment $[-A_2, A_2]$, we obtain the existence of $n_1 \in \mathbb{N}$ such that

$$\prod_{j=1}^{n_1} \cos\biggl(\frac{t}{\theta^{j}}\biggr) \leqslant \tfrac{1}{2} \quad \text{for all } t \in \mathbb{R} \text{ such that } A_1 \leqslant |t| \leqslant A_2. \tag{6.3}$$

Consider now

$$F_2(t) = \lim_{t\to\infty} \prod_{j=1}^{n_1} \cos\biggl(\frac{t}{\theta^{j}}\biggr) \prod_{j=n_1+1}^{\infty} \cos\biggl(\frac{t}{\theta^{2j}}\biggr).$$

The obvious inequality $|F_2(t)| \leqslant |G_1(t)|$ implies that $|F_2(t)| < \frac{1}{4}$ for all t such that $|t| > A_2$. Let

$$G_2(t) = \lim_{t\to\infty} \prod_{j=1}^{n_1} \cos\biggl(\frac{t}{\theta^{j}}\biggr) \prod_{j=n_1+1}^{\infty} \cos\biggl(\frac{t}{\theta^{4j}}\biggr).$$

Lemma 5.4 implies that $\lim_{t\to\infty} G_2(t) = 0$. Therefore, there exists $A_3 \geqslant \max\{A_2, 3\}$ such that $|G_2(t)| \leqslant \frac{1}{5}$ for all t such that $|t| > A_3$. Using uniform convergence of the product and defining F_2 on the segment $[-A_3, A_3]$ and inequality (6.3), we obtain the existence of $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and

$$\prod_{j=1}^{n_1} \cos\biggl(\frac{t}{\theta^{j}}\biggr) \prod_{j=n_1+1}^{n_2} \cos\biggl(\frac{t}{\theta^{2j}}\biggr) \leqslant \tfrac{1}{2} \quad \text{for all } t \in \mathbb{R} \text{ such that } A_1 \leqslant |t| \leqslant A_2,$$

$$\prod_{j=1}^{n_1} \cos\biggl(\frac{t}{\theta^{j}}\biggr) \prod_{j=n_1+1}^{n_2} \cos\biggl(\frac{t}{\theta^{2j}}\biggr) \leqslant \tfrac{1}{3} \quad \text{for all } t \in \mathbb{R} \text{ such that } A_2 \leqslant |t| \leqslant A_3.$$

Proceeding in the described way, we can obtain a strictly increasing sequence of integers n_k and a converging to $+\infty$ sequence of positive real numbers A_k such that

$$\prod_{m=1}^{l} \prod_{j=n_{m-1}+1}^{n_m} \cos\biggl(\frac{t}{\theta^{j2^m}}\biggr) \leqslant \frac{1}{l+1} \quad \text{for all } t \in \mathbb{R} \text{ such that } A_l \leqslant |t| \leqslant A_{l+1}. \tag{6.4}$$

Formula (6.4) implies (6.2). But, obviously,

$$F(t) = \prod_{m=1}^{\infty} \prod_{j=n_{m-1}+1}^{n_m} \cos\biggl(\frac{t}{\theta^{j2^m}}\biggr)$$

is the Fourier transform of the measure μ, which is the (infinite) convolution of measures $\nu_{\theta^{-j2^m}}$ $(j = 1, 2, \dots,\ m = 0, 1, \dots)$. Therefore, $\mu \in \mathfrak{M}^{\mathrm{D}}(\mathbb{R})$. Thus the

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measure μ is continuous and its support is included in the perfect compact set

$$K = \left\{ \sum_{j=1}^{\infty} \sum_{m=1}^{\infty} \varepsilon_{j,m} \theta^{-j2^m} : \varepsilon_{j,m} \in \{-1, 1\} \right\}.$$

Using the same argument as for the calculation of the Hausdorff dimension of the standard Cantor set, one can easily verify that the set

$$\underbrace{K + K + \cdots + K}_{n \text{ times}}$$

has zero Hausdorff dimension for any $n \in \mathbb{N}$. Therefore, the support of

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}}$$

has zero Hausdorff dimension. Hence the measure

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}}$$

is singular. Thus

$$\underbrace{\mu * \mu * \cdots * \mu}_{n \text{ times}} \in \mathfrak{M}_{\text{sc}}^{\text{D}}(\mathbb{R}) \quad \text{for any } n \in \mathbb{N}.$$

Part VI. Wiener and Wintner [22], for any $\varepsilon \in (0, \frac{1}{2})$, constructed an example of a measure $\mu \in (\mathfrak{M}_{\text{sc}} \cap \mathcal{P})$ with compact support such that

$$\tilde{\mu}(t) = O(t^{-1/2+\varepsilon})$$

for $|t| \to +\infty$. Therefore, $|\widetilde{\mu * \mu}(t)|^2 = O(t^{-2+4\varepsilon})$ and $|\widetilde{\mu * \mu}(t)|^2$ is integrable if $\varepsilon < \frac{1}{4}$. In this case, $\mu * \mu \in \mathfrak{M}_{\text{ac}}$ (with square integrable density).

7. Proof of proposition 2.6

If μ_1 and μ_2 are the measures μ and ν from the proof of the third part of theorem 2.5, then $\mu_1, \mu_2 \in \mathfrak{M}^{\text{ND}}(\mathbb{R}) \cap \mathfrak{M}_{\text{bs}}$ and $\mu_1 * \mu_2 \in \mathfrak{M}_{\text{ac}}(\mathbb{R}) \subset \mathfrak{M}^{\text{D}}(\mathbb{R})$. Then $\mu_1 = \nu_1 + \nu_2$ and $\mu_2 = \nu_3 + \nu_4$, $\nu_1, \nu_3 \in \mathfrak{M}^{\text{D}}$ and $\nu_2, \nu_4 \in X$. Suppose that X is a subalgebra of $\mathfrak{M}$. Since $\mu_1, \mu_2 \notin \mathfrak{M}^{\text{D}}$, we have $\nu_2 \neq 0$ and $\nu_4 \neq 0$. Since $\mathfrak{M}^{\text{D}}$ is an ideal in the algebra $\mathfrak{M}$ and $\mu_1 * \mu_2 \in \mathfrak{M}^{\text{D}}$, $\nu_2 * \nu_4 = \mu * \nu - \nu_1 * \nu_3 - \nu_1 * \nu_4 - \nu_2 * \nu_3$ belongs $\mathfrak{M}^{\text{D}}$. On the other hand, since X is a subalgebra of $\mathfrak{M}$, $\nu_2 * \nu_4 \in X$. Hence $\nu_2 * \nu_4 = 0$. Therefore, $\tilde{\nu}_2 \tilde{\nu}_4 \equiv 0$. Since ν_2 and ν_4 have bounded support, $\tilde{\nu}_2$ and $\tilde{\nu}_4$ are analytic. By the uniqueness theorem for analytic functions, either $\tilde{\nu}_2 \equiv 0$ or $\tilde{\nu}_4 \equiv 0$, which is a contradiction. The proposition is proved.

8. Concluding remarks

(1) It seems interesting to investigate further properties of the set $\mathfrak{M}_{\text{uls}}$, defined here in § 3. One can verify that a measure absolutely continuous with respect to an element of $\mathfrak{M}_{\text{uls}}$ is again an element of $\mathfrak{M}_{\text{uls}}$. On the other hand, it is not clear whether the sum or convolution of different elements of $\mathfrak{M}_{\text{uls}}$ is necessarily an element of $\mathfrak{M}_{\text{uls}}$.

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(2) The decomposition from theorem 2.1 provides a decomposition of the space $\mathcal{F}$ of Fourier transforms of measures from $\mathfrak{M}(\mathbb{R})$ as the direct sum of two closed (with respect to the uniform convergence topology) subspaces, one of which consists of all elements of $\mathcal{F}$, converging to zero at infinity. The corresponding decomposition of the space of all continuous bounded functions is impossible. This follows from the fact that the space c_0 of sequences converging to zero is non-complementable (see [17]) in the Banach space l_∞ of bounded sequences.

(3) For any self-adjoint operator on a Hilbert space, our refinement of the Jordan decomposition of any probability measure $\mu \in \mathfrak{M}$ leads to a corresponding invariant decomposition of the Hilbert space [2]. This fact allows us to characterize completely [3] the mixing and decaying states of classical and quantum systems [5, 7, 8]. In particular, classical chaotic systems are mixing and they give rise to decay of the correlation functions. In the case of strange attractors which give rise to singular measures [13, 14, 20], our decomposition of singular measures (2.3) allows us to characterize mixing which so far has been achieved only for absolutely continuous measures [5, 13]. Decaying quantum systems appear in quantum complex systems like unstable atoms, particles and nano-electronic devices.

In the case of quantum systems, the spectral measure of the Liouville–von Neumann operator generating the statistical evolution is the convolution of the spectral measure of the Hamiltonian with its reflection [3]. Thus the decay properties of the statistical evolution of quantum systems motivated the study of convolutions (theorem 2.5) of decaying and purely non-decaying measures. In [3], we corrected some erroneous statements [19] on the spectrum of the Liouville–von Neumann operator in the Hilbert–Schmidt space.

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