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CHROMATIC SUMS FOR ROOTED PLANAR TRIANGULATIONS II: THE CASE $\lambda = \tau + 1$

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Summary. In an earlier paper [**2**] we defined the chromatic sums g, q, l and h. We determined the derivatives of these sums with respect to the colour-number λ at the special values $\lambda = 1$ and $\lambda = 2$. In the present paper we find parametric equations for h, l and q in the case $\lambda = \tau + 1$, where τ is the golden ratio. We obtain h, l and the basic indeterminate z explicitly in terms of the parameter u, but for q we exhibit only a cubic equation with coefficients depending on u. We obtain an exact formula for the coefficients in h by applying Lagrange's theorem to the parametric equations.

The special value $\lambda = \tau + 1$ seems to be of great importance in the theory of chromials. Some interesting theorems hold in this case that have not yet been generalized to other values of λ, save at the cost of introducing the "constrained chromials" of Birkhoff and Lewis. It is by exploiting these special theorems that we obtain our chromatic sums.

1. The golden ratio. We refer to [**2**] as "I" and we use the definitions of that paper. We write $q(x, z, \lambda) = q(x, z)$, $l(y, z, \lambda) = l(y, z)$.

We make one innovation. We often show the root-face in a diagram as one of the inner or bounded faces, whereas in I the root-face is always on the outside. Evidently a diagram of one kind can be converted into one of the other by a suitable inversion. To justify this greater freedom in diagrams we should make a slight change in the definition of a rooted planar map, but it is a change without real combinatorial significance. The rooting of a map now consists of the choice of an arbitrary face F to be the root-face, followed by the selection of a directed edge in the boundary of F to be the root. The root-vertex V and the root-edge E are defined as before. So is "combinatorial equivalence" except that the homeomorphisms involved are now those of the closed plane, not the open one.

The "golden ratio" τ is the irrational number $(1 + \sqrt{5})/2$. It is one of the roots of the quadratic equation

$$x^2 = x + 1. \tag{1}$$

Most of our calculations with polynomials in τ are made by repeated applications of the basic formula $\tau^{m+2} = \tau^{m+1} + \tau^m$, where m can be any integer. In particular we have $\tau + 1 = \tau^2$, $\tau - 1 = \tau^{-1}$ and $\tau - 2 = -\tau^{-2}$.

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Because the coefficients in chromials are integers we can replace τ by its conjugate irrational number $\tau^* = (1 - \sqrt{5})/2$, the other root of (1), throughout the following theory.

We now state three theorems on chromials, two of them restricted to the case $\lambda = \tau + 1 = \tau^2$. First we have the "Vertex-Elimination Theorem".

1.1. *Let v be a vertex of valency m in a planar triangulation T. Let the sides opposite v of the faces incident with v be those of a circuit C. Let T_v be the near-triangulation obtained from T by replacing v and its incident edges and faces by a single new face W. Then*

$$P(T, \tau^2) = (-1)^m \tau^{1-m} P(T_v, \tau^2).$$

The maps concerned are shown in Figure 1. A proof can be found in [**3**]. It is not really necessary that the edges opposite v should form a circuit, but that is the special case that concerns us here.

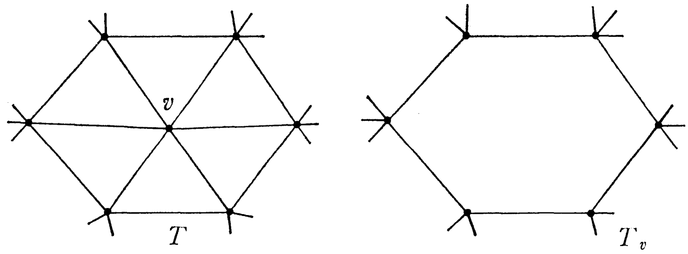

FIGURE 1

The other two theorems are concerned with a "double triangle" in a planar triangulation T. Such a double triangle is a subgraph defined by four distinct vertices V, W, X and Y, and five edges VW, WX, XY, YV and VX. (See the first diagram of Figure 2). We denote VX also by A. The other four edges define a quadrilateral Q in T. The residual domain of Q containing A is to contain no other edge or vertex of T. In diagrams we normally show it as the inside of Q. It contains exactly two faces of T, these being triangles VWX and VXY. In Figure 2 we do not show the detailed structure of the outside of Q.

In Figure 2 we indicate T and some other planar triangulations that can be derived from it by simple operations on A. Thus $\theta_A(T)$ is obtained by replacing A by another diagonal B of the quadrilateral Q, this one joining W and Y. To form the triangulation $\phi_A(T)$ we contract the edge A to a single new vertex, and correspondingly we shrink the faces VWX and VXY mentioned above into single segments. We form $\psi_A(T)$ from $\theta_A(T)$ by similarly contracting the diagonal B.

If V and X are joined by a second edge in T the map $\phi_A(T)$ is not a true triangulation, since it has a loop. In this case $P(\phi_A(T), \lambda)$ is identically zero. Similar remarks apply to $\psi_A(T)$.

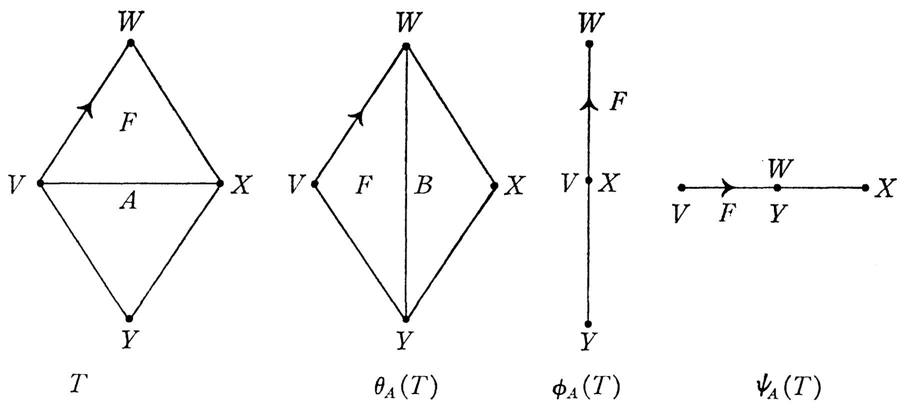

FIGURE 2

Now let N be the near-triangulation obtained from T by deleting A as an edge, that is by combining it with its two incident faces to form a single new quadrangular face. By a familiar recursion formula (Equation (3) of I) we have

$$P(N, \lambda) = P(T, \lambda) + P(\phi_A(T), \lambda)$$

and

$$P(N, \lambda) = P(\theta_A(T), \lambda) + P(\psi_A(T), \lambda).$$

These results are almost obvious when λ is a positive integer and the chromials are numbers of λ-colourings. They extend to other values of λ by the properties of polynomials. From them we obtain our second theorem, a well-known result valid for all λ.

1.2. $P(T, \lambda) - P(\theta_A(T), \lambda) = P(\psi_A(T), \lambda) - P(\phi_A(T), \lambda)$.

The third theorem is similar in form to 1.2, but it applies only when $\lambda = \tau^2$ (or $(\tau^*)^2$).

1.3. $P(T, \tau^2) + P(\theta_A(T), \tau^2) = \tau^{-3}\{P(\psi_A(T), \tau^2) + P(\phi_A(T), \tau^2)\}$.

A proof is given in [**4**]. Moreover Professor D. W. Hall has pointed out to the author that the theorem can be derived from the Birkhoff-Lewis equations for the 4-ring by setting $\lambda = \tau^2$. Putting $\lambda = \tau^2$ in 1.2 and eliminating $P(\psi_A(T), \tau^2)$ between the resulting equation and 1.3 we find that

(2) $$P(T, \tau^2) + \tau P(\theta_A(T), \tau^2) = \tau^{-1}P(\phi_A(T), \tau^2).$$

2. An equation for l**.** We introduce the generating series

(3) $$f(y, z) = \sum_T y^{n(T)} z^{t(T)+1} P(T, \tau^2),$$

where the sum is over all rooted triangulations T. Each such triangulation is derived from a non-degenerate rooted near-triangulation N, satisfying $m(N) = 2$, by deleting the non-root edge of the digon. From this observation we deduce a simple relation between $f(y, z)$ and the chromatic sum $l(y, z)$ defined in I. We observe first that

$$l(y, z) = \tau^3 y + \sum_N y^{n(N)} z^{t(N)} P(N, \tau^2),$$

where the term $\tau^3 y$ represents the contribution to l, in general $\lambda(\lambda - 1)y$, of the degenerate near-triangulation. We replace each N by the corresponding T, and so obtain

$$l(y, z) = y(\tau^3 + f(y, z)). \tag{4}$$

Let S denote the set of all rooted triangulations T of the following kind. T has a double triangle with quadrilateral $Q = VWXY$ and diagonal $A = VX$. The root-vertex is V, the root-edge is the edge VW of Q, and the root-face F is the face VWX incident with A. (See the first diagram of Figure 2.) We denote by S' the set of all remaining rooted triangulations. We write

$$f_1 = \sum_{T \in S} y^{n(T)} z^{t(T)+1} P(T, \tau^2). \tag{5}$$

This is the contribution to $f(y, z)$ of the members of S.

Let us characterize the triangulations T belonging to S'. Let a triangulation T have root-vertex V, root-edge $E = VW$ and root-face $F = VWX$. (We do not mean to imply that F is the only face with vertices V, W and X.) Let the second edge incident with V and F be E_1, joining V and X. Let the second face incident with E_1 be F_1, with vertices V, X and Y. If Y is distinct from W then T belongs to S, but if not it is in S'.

A typical triangulation T of S' is shown in Figure 3. Each of the shaded regions is shown bounded by a digon, but either of them may degenerate into a single segment. Let the shaded regions determine near-triangulations M_1 and M_2, as shown in Figure 3. We take M_1 to have root-vertex V, root-edge E,

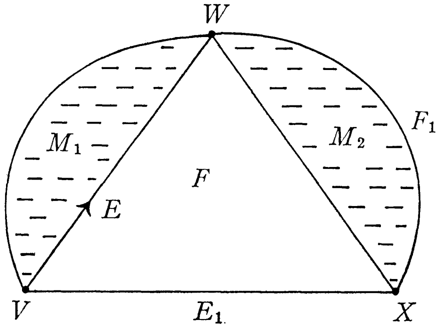

FIGURE 3

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and root-face a digon. We take M_2 to have root-vertex W, the root-edge WX incident with F in T, and a digon as root-face.

The chromials of T, M_1 and M_2, for general λ, are related in the following way

$$P(T, \lambda) = \frac{(\lambda - 2)P(M_1, \lambda)P(M_2, \lambda)}{\lambda(\lambda - 1)}. \tag{6}$$

For if λ is an integer exceeding 2 we have, in λ-colouring T, λ choices for the colour of V, then $\lambda - 1$ for the colour of W, and then $\lambda - 2$ for the colour of X. Since the distinct colours of V and W are now fixed we can complete the colouring of M_1 in $P(M_1, \lambda)/(\lambda(\lambda - 1))$ ways, and similarly we can complete the colouring of M_2 in $P(M_2, \lambda)/(\lambda(\lambda - 1))$ ways. Putting $\lambda = \tau^2$ in (6) we find

$$P(T, \tau^2) = \tau^{-4}P(M_1, \tau^2)P(M_2, \tau^2). \tag{7}$$

We deduce that

$$\sum_{T \in S'} y^{n(T)}z^{t(T)+1}P(T, \tau^2)$$

$$= \sum_{(M_1, M_2)} \tau^{-4}y^{n(M_1)+1}z^{t(M_1)+t(M_2)+2}P(M_1, \tau^2)P(M_2, \tau^2)$$

$$= \tau^{-4}yz^2\left[\sum_{M_1} y^{n(M_1)}z^{t(M_1)}P(M_1, \tau^2)\right]\left[\sum_{M_2} z^{t(M_2)}P(M_2, \tau^2)\right],$$

where M_1 and M_2 are arbitrary rooted near-triangulations with digons as root-faces,

$$= \tau^{-4}yz^2lh.$$

But the expression on the left is the contribution of the triangulations of S' to $f(y, z)$. Hence

$$f_1 = f(y, z) - \tau^{-4}yz^2lh. \tag{8}$$

We now write

$$f_2 = \sum_{M \in S} y^{n(M)}z^{t(M)+1}P(\theta_A(M), \tau^2), \tag{9}$$

$$f_3 = \sum_{M \in S} y^{n(M)}z^{t(M)+1}P(\phi_A(M), \tau^2). \tag{10}$$

We note that, by (2),

$$f_1 + \tau f_2 = \tau^{-1}f_3. \tag{11}$$

We take $\theta_A(M)$ to have the root-vertex V and to have as root-edge the edge VW of the quadrilateral. The root-face F is the face VWY incident with the diagonal B. (See Figure 2.) We can now rewrite (9) as

$$y^{-1}f_2 = \sum_{M \in S} y^{n(\theta_A(M))}z^{t(\theta_A(M))+1}P(\theta_A(M), \tau^2).$$

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Since M and $\theta_A(M)$ uniquely determine one another this expression is equivalent to

$$y^{-1}f_2 = \sum_N y^{n(N)} z^{t(N)+1} P(N, \tau^2), \tag{12}$$

where N runs through the set S_θ of all rooted planar triangulations T_θ such that $T_\theta = \theta_A(T)$ for some $T \in S$.

The rooted triangulations T not belonging to S_θ can be characterized as follows. Let B be the side of the root-face F opposite the root-vertex V. Let F_1 be the second face incident with B, and let X be its vertex opposite B. Then T is outside S_θ if and only if $X = V$. Figure 4 represents such a rooted triangulation outside S_θ.

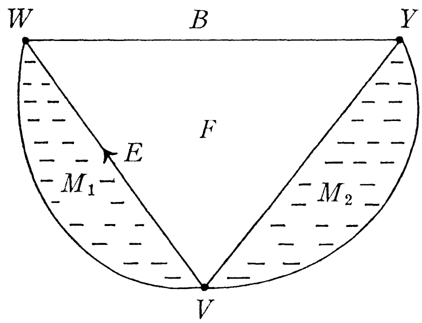

FIGURE 4

The shaded regions, shown bounded by digons, are denoted by M_1 and M_2 as indicated. As with the somewhat similar Figure 3 either of them may degenerate into a single segment. We take V to be the root-vertex of both M_1 and M_2. The root-edges of M_1 and M_2 are the edges VW and VY of F respectively. The root-faces are digons. The contribution of the triangulations outside S_θ to f can be found in the same way as the contribution of those of S'. The only difference is that now we have $n(T) = n(M_1) + n(M_2)$, whereas before we had $n(T) = n(M_1) + 1$. The contribution to f of the triangulations outside S_θ is accordingly found to be $\tau^{-4}l^2z^2$, instead of $\tau^{-4}lhyz^2$ as for the contribution of those outside S. So, by (12), we have

$$f_2 = yf(y, z) - \tau^{-4}yz^2l^2. \tag{13}$$

Let us now go back to Figure 2 and consider the rooting of $\phi_A(T)$. We take V to be the root-vertex, noting that it is now identified with X. The root-edge is the edge VW into which the root-face F of T has been shrunk. We take the root-face of $\phi_A(T)$ to be the face that appears in T as the non-root face incident with the edge WX of the quadrilateral Q. In $\phi_A(T)$ we denote this face by F, as indicated in Figure 2.

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We can now rewrite (10) as

$$f_3 = \sum_{M \in S} y^{n(M)} z^{t(\phi_A(M))+3} P(\phi_A(M), \tau^2), \tag{14}$$

omitting all maps M for which $\phi_A(M)$ has a loop. Any non-degenerate rooted triangulation N can appear as $\phi_A(T)$, for some $T \in S$. For each such N let us determine the set of corresponding triangulations T. We consider how to reverse the contraction of an edge A transforming T into $\phi_A(T)$, that is N. Evidently we must take two distinct edges of N incident with the root-vertex V, one being the root-edge $E = VW$ and the other being an edge E_1 having an end Y distinct from W. We must then expand these edges VW and VY into triangles VWX and VWY respectively, at the same time expanding V into an edge VX. Finally we assign the new face VWX as root-face, the edge VW incident with it as root-edge, and the new vertex V as root-vertex. This operation is illustrated in Figure 5.

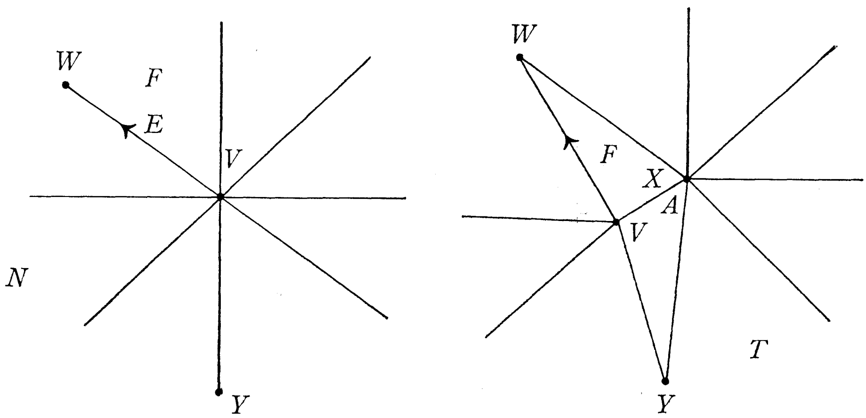

FIGURE 5

The problem is simplest when no two edges of N incident with V have a common end other than V. Then we have $n(N) - 1$ distinct choices for the edge VY, and in the $n(N) - 1$ resulting triangulations T the valency $n(T)$ takes each value from 3 to $n(N) + 1$ once only. In the general case some values of $n(T)$ in the above range may be missing. They correspond to choices of $E_1 = VY$ that make up a digon with the root-edge E. We deduce that

$$f_3 = z^2 \left[\sum_N \left\{ \sum_{j=3}^{n(N)+1} y^j \right\} z^{t(N)+1} P(N, \tau^2) - \sum_{T_0} y^{\beta(T_0)} z^{t(T_0)+1} P(T_0, \tau^2) \right],$$

where T_0 and $\beta(T_0)$ are defined as follows. T_0 is a rooted planar triangulation in which there is a marked digon D through the root-edge E. Both residual domains of D can be interpreted (when closed) as rooted near-triangulations with the root-face a digon. We assign to each the same root-vertex and root-

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edge as for the containing triangulation. Let us denote these near-triangulations by M_1 and M_2, M_1 having the root-face of the triangulation as one of its triangular faces. Both M_1 and M_2 are non-degenerate. The number $\beta(T_0)$ exceeds by 1 the valency of the root-vertex in M_2. We now have

$$P(T_0, \lambda) = \frac{P(M_1, \lambda)P(M_2, \lambda)}{\lambda(\lambda - 1)},$$

$$f_3 = \frac{y^3z^2}{y-1}\sum_N \left\{\left[y^{n(N)-1} - 1\right] z^{t(N)+1}P(N, \tau^2)\right\}$$
$$- \sum_{(M_1,M_2)} \tau^{-3}yz^2y^{n(M_2)}z^{t(M_1)+t(M_2)}P(M_1, \tau^2)P(M_2, \tau^2),$$

(15) $$f_3 = y^3z^2 \bigtriangleup \{y^{-1}f\} - \tau^{-3}yz^2(l - \tau^3y)(h - \tau^3)$$

since the second sum is over all ordered pairs (M_1, M_2) of rooted near-triangulations that are non-degenerate and have digons as root-faces.

Now the operator $\bigtriangleup$, defined by Equation (11) of I, satisfies

(16) $$\bigtriangleup(yF(y)) = F(y) + \bigtriangleup(F(y)).$$

Using (4) we deduce that

$$\begin{aligned}\bigtriangleup (l) &= \tau^3 + \bigtriangleup (yf)\\ &= \tau^3 + f + \bigtriangleup (y \cdot y^{-1}f)\\ &= \tau^3 + f + y^{-1}f + \bigtriangleup (y^{-1}f),\\ \bigtriangleup (y^{-1}f) &= \bigtriangleup (l) - \tau^3 - y^{-1}(l - y\tau^3) - y^{-2}(l - y\tau^3),\end{aligned}$$

(17) $$\bigtriangleup (y^{-1}f) = \bigtriangleup (l) - y^{-1}l - y^{-2}l + y^{-1}\tau^3.$$

We now have

$$\begin{aligned}0 &= y(f_1 + \tau f_2 - \tau^{-1}f_3), \quad \text{by (11)},\\ &= (1 + \tau y)yf - \tau^{-4}y^2z^2lh - \tau^{-3}y^2z^2l^2 - \tau^{-1}y^4z^2 \bigtriangleup (l) + \tau^{-1}y^3z^2l\\ &\quad + \tau^{-1}y^2z^2l - \tau^2y^3z^2 + \tau^{-4}y^2z^2lh - \tau^{-1}y^2z^2l - \tau^{-1}y^3z^2h + \tau^2y^3z^2,\end{aligned}$$

by (8), (13), (15) and (17). But

$$\tau^{-1}y^3z^2l - \tau^{-1}y^3z^2h = \tau^{-1}y^3z^2(y - 1) \bigtriangleup (l).$$

The above equation thus reduces to

$$0 = (1 + \tau y)yf - \tau^{-3}y^2z^2l^2 - \tau^{-1}y^3z^2 \bigtriangleup (l),$$

that is,

(18) $$\tau^3(1 + \tau y)(l - y\tau^3) = y^2z^2l^2 + \tau^2y^3z^2 \bigtriangleup (l).$$

From this difference equation we can calculate coefficients of successive powers of z^2 in l, as polynomials in y. The method has a built-in check; when we obtain the coefficient of $z^{2(n+1)}$ in l we have to divide a polynomial in y by $(1 + \tau y)$, and the division must be exact. The first few coefficients can also easily be checked against a table of triangulations.

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3. Solution of the difference equation. Since $(y - 1) \triangle (l) = l - h$ we can rewrite (18) as

$$(19) \quad y^2z^2(y-1)^2l^2 - l(y-1)\{\tau^3(1+\tau y)(y-1) - \tau^2y^3z^2\} + \tau^6y(1+\tau y)(y-1)^2 - \tau^2y^3z^2(y-1)h = 0.$$

This is equivalent to

$$(20) \quad \{2y^2z^2(y-1)l - \tau^3(1+\tau y)(y-1) + \tau^2y^3z^2\}^2 = D,$$

where

$$(21) \quad D = \{\tau^3(1+\tau y)(y-1) - \tau^2y^3z^2\}^2 - 4\tau^6y^3z^2(1+\tau y)(y-1)^2 + 4\tau^2y^5z^4(y-1)h,$$

a polynomial of the sixth degree in y in which the coefficients are functions of z.

We explore the possibility of substituting for y a power series

$$\xi = \xi_0 + \xi_1 z + \xi_2 z^2 + \ldots$$

in z so as to satisfy the equation

$$(22) \quad 2z^2\xi^2(\xi - 1)l(\xi, z) - \tau^3(1+\tau\xi)(\xi-1) + \tau^2\xi^3z^2 = 0.$$

The expression $l(\xi, z)$, the result of substituting ξ for y in the chromatic sum $l(y, z)$, can be written as

$$l(\xi, z) = L_0 + L_1 z + L_2 z^2 + \ldots ,$$

where the L_j are polynomials in ξ. They can be expanded in powers of z, and so $l(\xi, z)$ can be exhibited as a well-defined power series in z, with coefficients depending on those in the two-variable series $l(y, z)$.

Suppose (22) is satisfied by the series ξ. Considering the coefficient of z^0 on the left we find that,

$$(23) \quad \xi_0 = 1 \quad \text{or} \quad \xi_0 = -\tau^{-1}.$$

Having chosen one of these values for ξ_0 we find that corresponding values of ξ_1, ξ_2 etc. are uniquely determined in terms of the coefficients in $l(y, z)$; if $k \geqq 1$ then by considering the coefficient of z^k in (22) we obtain an equation giving ξ_k explicitly in the form of a polynomial in the numbers ξ_j with $j < k$.

We deduce that (22) has exactly two solutions for ξ, one for each of the possibilities stated in (23). Since a solution ξ has a non-zero constant term it has a reciprocal power series ξ^{-1}. We write u and v for the power series ξ^{-1} corresponding to the cases $\xi_0 = 1$ and $\xi_0 = -\tau^{-1}$ respectively.

Now consider (20) and the identity obtained from it by differentiation with respect to y. In each of these we substitute for y a solution ξ of (22). We thus obtain

$$(24) \quad \{D\}_{y=\xi} = 0, \qquad \{\partial D/\partial y\}_{y=\xi} = 0.$$

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Let us multiply out (21) and then substitute ξ for y. Because of (24) we obtain

$$(25)\quad \tau^6 + 2\tau^5\xi + (2\tau^6 - 3\tau^7)\xi^2 + (-2\tau^6 + (2\tau^7 - 6\tau^6)z^2)\xi^3 + (\tau^8 + 6\tau^4 z^2)\xi^4 + ((8\tau^7 - 6\tau^6)z^2 - 4\tau^2 z^4 h)\xi^5 + (-4\tau^7 z^2 + \tau^4 z^4 + 4\tau^2 z^4 h)\xi^6 = 0.$$

Before substituting for y we can differentiate by y. Again using (24) we find that

$$(26)\quad 2\tau^5 + (4\tau^6 - 6\tau^7)\xi + (-6\tau^6 + (6\tau^7 - 18\tau^6)z^2)\xi^2 + (4\tau^8 + 24\tau^4 z^2)\xi^3 + ((40\tau^7 - 30\tau^6)z^2 - 20\tau^2 z^4 h)\xi^4 + (-24\tau^7 z^2 + 6\tau^4 z^4 + 24\tau^2 z^4 h)\xi^5 = 0.$$

We can solve (25) and (26) for z and h as functions of ξ. It is convenient to begin by multiplying (25) by 6 and (26) by ξ and then subtracting.

$$(27)\quad 6\tau^6 + 10\tau^5\xi + (8\tau^6 - 12\tau^7)\xi^2 + (-6\tau^6 + (6\tau^7 - 18\tau^6)z^2)\xi^3 + (2\tau^8 + 12\tau^4 z^2)\xi^4 + ((8\tau^7 - 6\tau^6)z^2 - 4\tau^2 z^4 h)\xi^5 = 0.$$

We now eliminate h by multiplying (27) by $\xi - 1$ and adding the result to (25). The result is

$$(28)\quad [-5\tau^6 + (2\tau^4 + 2\tau^2)\xi + (3\tau^7 + 16\tau^5)\xi^2 + (-12\tau^5)\xi^3 + (-\tau^8 - 6\tau^6)\xi^4 + 2\tau^8\xi^5] + z^2\xi^3[(4\tau^5 + 8\tau^4) + (-6\tau^5 - 18\tau^4)\xi + 12\tau^4\xi^2 + 2\tau^3\xi^3] + \tau^4 z^4 \xi^6 = 0.$$

It can be verified by multiplication that (28) is equivalent to

$$(29)\quad [\xi^3 z^2 - \tau^{-2}(1 - \xi)(1 + \tau\xi)(2\xi + \sqrt{5})] \times [\xi^3 z^2 - \tau^4(1 - \xi)(\tau\xi - \sqrt{5})] = 0.$$

Hence any solution ξ of (22) satisfies either

$$(30)\quad \xi^3 z^2 = \tau^4(1 - \xi)(\tau\xi - \sqrt{5})$$

or

$$(31)\quad \xi^3 z^2 = \tau^{-2}(1 - \xi)(1 + \tau\xi)(2\xi + \sqrt{5}).$$

By (21) and (24) we have

$$(32)\quad 4\tau^{-4}\xi^5(\xi - 1)^{-1} z^4 h = 4\xi^3 z^2(1 + \tau\xi) - \{(1 + \tau\xi) - \tau^{-1}\xi^3(\xi - 1)^{-1} z^2\}^2.$$

If ξ satisfies (30) it follows that

$$\begin{aligned} 4\tau^{-4}\xi^5(\xi - 1)^{-1} z^4 h &= 4\tau^4(1 + \tau\xi)(1 - \xi)(\tau\xi - \sqrt{5}) - \{1 + \tau\xi + \tau^3(\tau\xi - \sqrt{5})\}^2 \\ &= 4\tau^4(1 - \xi)(\tau^2\xi^2 + (\tau - 1 - \tau^2)\xi - \tau - \tau^{-1}) - 4\tau^6(1 - \xi)^2, \end{aligned}$$

$$(33)\quad \xi^5 z^4 h = -\tau^8(1 - \xi)^2(\tau^2\xi^2 + \tau^{-1}\xi - 3\tau).$$

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In the remaining case, when ξ satisfies (31) we have instead

$$\begin{aligned}4\tau^{-4}\xi^5(\xi-1)^{-1}z^4h &= 4(1+\tau\xi)^2\tau^{-2}(1-\xi)(2\xi+\sqrt{5}) \\ &\qquad - \{1+\tau\xi+\tau^{-3}(1+\tau\xi)(2\xi+\sqrt{5})\}^2 \\ &= 4\tau^{-2}(1+\tau\xi)^2(1-\xi)(2\xi+\sqrt{5}) \\ &\qquad - (1+\tau\xi)^2(2\tau^{-3}\xi+4\tau^{-2})^2 \\ &= 4\tau^{-2}(1+\tau\xi)^2(-(2+\tau^{-4})\xi^2-5\tau^{-3}\xi+3\tau^{-3}),\end{aligned}$$

$$\xi^5 z^4 h = \tau^2(1-\xi)(1+\tau\xi)^2((2+\tau^{-4})\xi^2+5\tau^{-3}\xi-3\tau^{-3}). \tag{34}$$

Let us consider the two possible solutions u^{-1} and v^{-1} for ξ. Considering powers of z^0 we find that v does not satisfy (30); it must therefore be a solution of (31). If ξ satisfies (31) it satisfies (34) also, and we can derive the identity

$$\xi^2(2\xi+\sqrt{5})z^4h = \tau^4z^2(1+\tau\xi)((2+\tau^{-4})\xi^2+5\tau^{-3}\xi-3\tau^{-3}).$$

If we substitute $\xi = u^{-1}$ in this we obtain a contradiction. The coefficient of z^2 is zero on the left and non-zero on the right. We conclude that u satisfies (30). We have now established the following four identities.

$$z^2 = -\tau^4u(1-u)(\tau-u\sqrt{5}), \tag{35}$$

$$z^4h = \tau^8u(1-u)^2(3\tau u^2-\tau^{-1}u-\tau^2), \tag{36}$$

$$z^2 = -\tau^{-2}(1-v)(\tau+v)(2+v\sqrt{5}), \tag{37}$$

$$z^4h = \tau^2(1-v)(\tau+v)^2(3\tau^{-3}v^2-5\tau^{-3}v-2-\tau^{-4}). \tag{38}$$

We know the constant terms in u and v. Lagrange's theorem enables us to determine their other coefficients. We can use this fact to establish a simple linear relation between u and v.

First we write

$$v = 1-\tau^2w. \tag{39}$$

We deduce from this that $1-v=\tau^2w$, $\tau+v=\tau^2(1-w)$ and $2+v\sqrt{5} = \tau^2(\tau-w\sqrt{5})$. So if we substitute from (39) in (37) we obtain Equation (35) with u replaced by w. Moreover the coefficient of z^0 in w is $\tau^{-2}(1+\tau)=1$, the same as in u. Since (35) and the condition that the constant term is 1 uniquely determine u it follows that

$$v = 1-\tau^2u. \tag{40}$$

We note that h can be found as a power series in z^2 by eliminating u from (35) and (36), and that this elimination can be effected by an application of Lagrange's theorem. We could use (37) and (38) instead, but an application of (40) shows that these equations are equivalent to (35) and (36) respectively.

We can substitute our series for h in Equation (21) for D, and then determine the double power series $l(y,z)$ from (20). However this computation

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can be simplified by first factorizing D. We deduce from (24) that D, as a polynomial in y, divides by $(y - \xi)^2$ if ξ satisfies (22), that is if ξ is u^{-1} or v^{-1}. Thus D divides by $(1 - uy)^2$ and $(1 - (1 - \tau^2 u)y)^2$. The remaining quadratic factor can be found by division. Except as a check it is not necessary in this operation to consider powers of y higher than the second. We find that

$$(41)\quad D = \tau^6(1 - uy)^2(1 - (1 - \tau^2 u)y)^2 \times (1 + 2\tau(1 - u)y + \tau^2(1 - u)(1 - 5u)y^2).$$

This is perhaps more conveniently written as

$$(42)\quad D = \tau^6(1 - uy)^2(1 - (1 - \tau^2 u)y)^2(1 + \tau(1 - u)y)^2 \times \left\{1 - \frac{4\tau^2 u(1 - u)y^2}{(1 + \tau(1 - u)y)^2}\right\}.$$

We can now deduce from (20) that

$$(43)\quad l(y, z) = \frac{\tau^3(1 + \tau y)}{2y^2 z^2} + \frac{\tau^2 y}{2(1 - y)} - \left\{\frac{\tau^3(1 - uy)(1 - (1 - \tau^2 u)y)(1 + \tau(1 - u)y)}{2y^2 z^2(1 - y)} \times \left[1 - \frac{4\tau^2 u(1 - u)y^2}{(1 + \tau(1 - u)y)^2}\right]^{\frac{1}{2}}\right\}.$$

Here the square root is to be interpreted as a power series in u and y with constant term $+1$.

4. The chromatic sum $q(x, z)$**.** In the case $\lambda = \tau + 1$ the chromatic sum $q(x, z)$ defined in I can be related to $l(y, z)$ through the Vertex-Elimination Theorem (1.1).

First we introduce an auxiliary power series

$$(44)\quad S(y, z) = \sum_M y^{n(M)} z^{t(M)+1} P(M, \tau^2),$$

where M can be any rooted triangulation in which no two edges incident with the root-vertex V have the same other end.

From such a triangulation M we can construct a triangulation T of the kind contributing to the sum $f(y, z)$ of Equation (3) as follows. We expand some (possibly none) of the edges incident with V into two-sided faces, and we triangulate these faces (see Figure 6). We retain the original root-vertex V and root-face F, and these determine the new root-edge E.

It is not difficult to see that each T can be derived from exactly one M in this way. We are thus led to the following relation between the generating functions S and f.

$$(45)\quad f(y, z) = S(\tau^{-3} l(y, z), z).$$

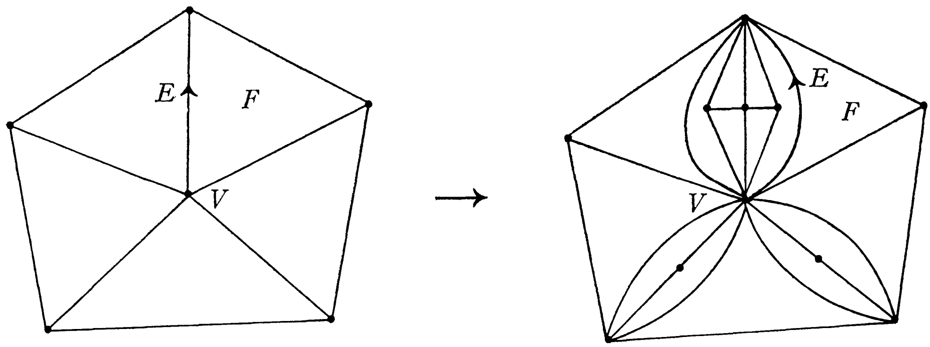

FIGURE 6

In one of the triangulations M let W be the end of E other than V, and let A be the edge of F opposite V. We form a near-triangulation N from M by deleting V and its incident edges and faces, replacing them by a single new face H of $n(M)$ edges. We take W, A and H to be the root-vertex, root-edge and root-face of N respectively. Conversely, if we are given a near-triangulation N we can reverse the above process by taking a new root-vertex V inside the root-face H, triangulating H by joining V to the vertices of the polygon H, and rooting the resulting planar triangulation in the obvious way. By (44) we have

$$S(y, z) = \sum_N y^{m(N)} z^{t(N)+m(N)} P(M, \tau^2),$$

where M and N are related as described above,

$$= \sum_N (yz)^{m(N)} z^{t(N)} (-1)^{m(N)} \tau^{1-m(N)} P(N, \tau^2),$$

by the Vertex-Elimination Theorem. Thus

$$S(y, z) = \tau q(-\tau^{-1} yz, z). \tag{46}$$

If we write

$$x = -\tau^{-4} z l(y, z) \tag{47}$$

we have

$$q(x, z) = \tau^{-1} S(\tau^{-3} l(y, z), z), \quad \text{by (46)},$$

$$= \tau^{-1} f(y, z) \quad \text{by (45)};$$

$$q(x, z) = \tau^{-1} y^{-1} l(y, z) - \tau^2, \quad \text{by (4)}. \tag{48}$$

If we can use (19) to express y^{-1} as a power series in l and z we can substitute the result in (48), substitute for l from (47), and so obtain q as a power series in x and z.

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Write $y^{-1} = s$. Dividing (19) by $y^3(y-1)$ we find that

$$(49) \quad z^2(1-s)l^2 - (\tau^3 s(1-s)(\tau+s) - \tau^2 z^2)l + \tau^6(1-s)(\tau+s) - \tau^2 z^2 h = 0,$$

a cubic equation for s in terms of l and z, since h can now be regarded as a known function of z.

In principle we can solve (49) for s as a power series and so obtain $q(x, z)$. This would appear to be more difficult than obtaining $l(y, z)$, as a power series in y and z, from (43). The next step, presumably, is to obtain $g(x, y, z, \tau^2)$ by substituting for q and l in the chromatic equation (Equation (13) of I). However in this paper we attempt the explicit determination of coefficients only for h, when they can be interpreted as sums over the rooted planar triangulations with a given number of faces.

5. The coefficients in h. One method whereby we can obtain explicit formulae for the coefficients in h is as follows. We take $\xi = u^{-1}$. Then (30) can be rewritten as

$$(50) \qquad \xi = 1 - \frac{\tau^{-4} z^2 \xi^3}{(\tau\xi - \sqrt{5})}.$$

From (33) we have

$$\xi^5 z^4 h = -\tau^8 (1-\xi)^2 (\tau^2 \xi^2 + \tau^{-1}\xi - 3\tau),$$

$$(51) \qquad \xi^5 z^4 h = \tau^9 \{(1-\xi)^2 + (\tau^2+1)(1-\xi)^3 - \tau(1-\xi)^4\}.$$

Let us now use the symbol D to denote the operation of differentiation with respect to ξ. Applying it to (51) we have

$$5\xi^4(z^4 h) + \xi^5 D(z^4 h) = \tau^9\{-2(1-\xi) - 3(\tau^2+1)(1-\xi)^2 + 4\tau(1-\xi)^3\}.$$

On multiplying this by $\xi = 1 - (1-\xi)$, multiplying (51) by 5 and subtracting we find that

$$(52) \quad D(z^4 h) = \tau^9 \xi^{-6}\{-2(1-\xi) - 3(\tau^2+2)(1-\xi)^2 - 2\tau^{-2}(1-\xi)^3 + \tau(1-\xi)^4\}.$$

From (50) and (52), by Lagrange's theorem,

$$z^4 h = \tau^9 \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \tau^{-4n} z^{2n}}{n!} \left[D^{n-1} \left\{ \frac{\xi^{3n-6}}{(\tau\xi - \sqrt{5})^n} \times \{(-2(1-\xi) - 3(\tau^2+2)(1-\xi)^2 - 2\tau^{-2}(1-\xi)^3 + (1-\xi)^4\} \right\} \right]_{\xi=1} \right\}$$

$$= \tau^9 \sum_{n=1}^{\infty} \left\{ \frac{(-1)^n \tau^{-4n} z^{2n}}{n!} \left[\left\{ 2(n-1)D^{n-2} - 3(\tau^2+2)(n-1)(n-2)D^{n-3} + 2\tau^{-2}(n-1)(n-2)(n-3)D^{n-4} + \tau(n-1)(n-2)(n-3)(n-4)D^{n-5} \right\} \left\{ \frac{\xi^{3n-6}}{(\tau\xi - \sqrt{5})^n} \right\} \right]_{\xi=1} \right\}.$$

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Now for $r = 2, 3, 4$ or 5, and $n \geqq r$ we have

$$\left[D^{n-r}\left\{\frac{\xi^{3n-6}}{(\tau\xi - \sqrt{5})^n}\right\}\right]_{\xi=1}$$
$$= \sum_{j=0}^{n-r}\left\{\frac{(n-r)!}{r!(n-r-j)!}\,[D^{n-r-j}(\xi^{3n-6})\cdot D^j((\tau\xi-\sqrt{5})^{-n})]_{\xi=1}\right\}$$
$$= \sum_{j=0}^{n-r}\left\{\frac{(n-r)!}{r!(n-r-j)!}\cdot\frac{(3n-6)!}{(2n+r+j-6)!}\cdot\frac{(n+j-1)!(-\tau)^j}{(n-1)!(-\tau^{-1})^{n+j}}\right\}$$
$$= \sum_{j=0}^{n-r}\left\{\frac{(n-r)!(3n-6)!(n+j-1)!(-1)^n\tau^{n+2j}}{(n-1)!j!(n-r-j)!(2n+r+j-6)!}\right\}.$$

Substituting this into our equation for z^4h we find that

$$(53)\quad z^4h = \tau^9\sum_{n=1}^{\infty}\left\{\frac{\tau^{-3n}z^{2n}(3n-6)!}{n!}\sum_{j=0}^{n-2}\left[\frac{(n+j-1)!\tau^{2j}}{j!}\right.\right.$$
$$\times\left\{\frac{2}{(2n+j-4)!(n-j-2)!}-\frac{3(\tau^2+2)}{(2n+j-3)!(n-j-3)!}\right.$$
$$\left.\left.\left.+\frac{2\tau^{-2}}{(2n+j-2)!(n-j-4)!}+\frac{\tau}{(2n+j-1)!(n-j-5)!}\right\}\right]\right\}.$$

Here the reciprocal of the factorial of a negative number is to be interpreted as zero.

Consider now the coefficient h_{2n} of z^{2n} in h. For $n > 0$ it is the sum of the numbers $P(T, \tau^2)$ over all rooted planar triangulations T with $2n$ faces. Since it is the coefficient of z^{2n+4} in z^4h we deduce that

$$(54)\quad h_{2n} = \tau^{-3n+3}\sum_{j=0}^{\infty}\frac{(3n)!(n+j+1)!\tau^{2j}}{(n+2)!j!}\left\{\frac{2}{(n-j)!(2n+j)!}\right.$$
$$-\frac{3(\tau^2+2)}{(n-j-1)!(2n+j+1)!}+\frac{2\tau^{-2}}{(n-j-2)!(2n+j+2)!}$$
$$\left.+\frac{\tau}{(n-j-3)!(2n+j+3)!}\right\}.$$

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