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ON NUMERICAL EVALUATION OF FINITE TIME SURVIVAL PROBABILITIES

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ABSTRACT

In this paper we review three algorithms to calculate the probability of ruin/survival in finite time for the classical risk model. We discuss the computational aspects of these algorithms and consider the question of which algorithm should be preferred.

KEYWORDS

Ruin/Survival Probability; Finite Time; Recursive Calculation

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1. INTRODUCTION

In the classical risk model the insurer's surplus at time t is:

$$U_c(t) = u + ct - S_c(t)$$

where u is the initial surplus (i.e. the surplus at time zero), c is the rate of premium income per unit time and $S_c(t)$ denotes aggregate claims up to time t. The aggregate claims process $\{S_c(t)\}_{t\geq 0}$ is a compound Poisson process, and, without loss of generality, we can let the Poisson parameter equal 1. Without loss of generality we can also let the distribution of individual claim amounts, whose distribution function we denote by P, have mean 1. Thus, we can write $c = 1 + \theta$, where θ is the premium loading factor. The probability of survival to time t is defined as:

$$\delta_c(u, t) = \Pr[U_c(\tau) > 0 \quad \text{for all } \tau, \quad 0 < \tau \leq t].$$

In this paper we comment on three methods of approximating $\delta_c(u, t)$ which have appeared in the literature in recent times. Each method is based on the same discrete time risk model. In the discrete model, aggregate claims per unit time have a compound Poisson distribution with Poisson parameter $1/(1+\theta)\beta$, where β is a positive integer, and individual claim amounts are distributed on the non-negative integers with mean β. (In each of our applications in Section 3, we assume a continuous individual claim amount distribution in our classical

model. The individual claim amount distribution in our discrete model approximates that of the continuous model, subject to scaling.) We denote by G and g the distribution function and probability function, respectively, of aggregate claims per unit time. The probability function can be calculated easily using Panjer's (1981) recursion formula. The premium income per unit time in this model is 1. If we let the initial surplus u, be a non-negative integer, then the surplus process moves on the integers, and the probability of survival to (integer) time t is:

$$\delta(u, t) = \Pr[U(\tau) > 0 \quad \text{for all } \tau, \quad \tau = 1, 2, \ldots, t]$$

where $U(\tau)$ denotes the surplus at time τ. Note that under this definition, as in the definition of $\delta_c(u, t)$, ruin does not occur at time 0 if $u = 0$. The methods of calculating $\delta(u, t)$, described in Sections 2.2 and 2.3, require such a definition.

Dickson & Waters (1991) explain why a reasonable approximation to $\delta_c(u,t)$ is:

$$\delta_c(u, t) \approx \delta(\beta u, (1 + \theta)\beta t).$$

In the next section we set out three methods of calculating $\delta(u, t)$. In Section 3 we discuss some computational aspects, and in the final section we consider the question of which method should be preferred.

2. THREE METHODS OF CALCULATING $\delta(u, t)$

2.1 *The Method of De Vylder & Goovaerts*

De Vylder & Goovaerts (1988) use a slightly different definition of survival to the one given in the previous section. However, this does not affect the application of their method to the discrete time model described in the previous section. Their approach, which considers the function $\psi(u, t) = 1 - \delta(u, t)$, is as follows. For $u = 0, 1, 2, \ldots$:

$$\psi(u, 1) = 1 - G(u)$$

that is ruin occurs at the end of the first time period if the aggregate claim amount in that time period is strictly greater than u. For $u = 0, 1, 2, \ldots$ and $t = 2, 3, 4, \ldots$:

$$\psi(u, t) = \psi(u, 1) + \sum_{j=0}^{u} g(j)\psi(u + 1 - j, t - 1)$$

that is if ruin occurs by time t, either the aggregate claim amount in the first time period causes ruin, or else the aggregate claim amount is $j \le u$, and ruin occurs in the next $t - 1$ time periods from the resulting surplus $u + 1 - j$.

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The number of computations involved in this algorithm can be reduced by means of a neat truncation procedure. Let $\varepsilon>0$ be small, and let k_0 be the least integer such that $G(k_0)\geq 1-\varepsilon$. Now define:

$$g^{\varepsilon}(j)=\begin{cases} g(j) & \text{for } j=0,1,2,\ldots,k_0 \\ 0 & \text{for } j=k_0+1,k_0+2,\ldots \end{cases}$$

and let:

$$\psi^{\varepsilon}(u,1)=\begin{cases} \psi(u,1) & \text{if } u\leq k_0 \\ 0 & \text{if } u>k_0. \end{cases}$$

For $t=2, 3, 4, \ldots$ define:

$$\psi^{\varepsilon}(u,t)=\psi^{\varepsilon}(u,1)+\sum_{j=0}^{u} g^{\varepsilon}(j)\psi^{\varepsilon}(u+1-j,t-1)$$

provided that the calculated value is at least ε. Otherwise define $\psi^{\varepsilon}(u,t)$ to be 0. We can easily apply the ideas of De Vylder & Goovaerts (1988) to show that:

$$\psi^{\varepsilon}(u,t)\leq\psi(u,t)\leq\psi^{\varepsilon}(u,t)+3\varepsilon t.$$

In our illustrations in the next section we calculate $\psi^{\varepsilon}(u,t)$, since we can control the error introduced by the truncation. For a given value of t, we always set $\varepsilon=10^{-4}/3t$ so that the error introduced by truncation is no more than 1 in the fourth decimal place. (The rescaling which we introduce in Section 3 produces approximations to $\psi_c(u,t)=1-\delta_c(u,t)$ which are mostly correct to 4 decimal places, so this choice of ε should not have much effect on the accuracy of the approximation.)

We comment that the above prescription is not quite that given by De Vylder & Goovaerts (1988). It is a modification of their approach which produces better approximations to $\psi_c(u,t)$ or $\delta_c(u,t)$. However, all the ideas presented above come from their paper. Details of the modification are given by Dickson & Waters (1991, Section 2).

2.2 *The Method of Dickson & Waters*

Dickson & Waters (1991) also consider the aggregate claim amount in the first time period and apply the formula:

$$\delta(u-1,t+1)=\sum_{j=0}^{u-1} g(j)\delta(u-j,t)$$

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for $u>1$ and $t>0$, to get:

$$\delta(u,t) = g(0)^{-1}\left(\delta(u-1,t+1) - \sum_{j=1}^{u-1} g(j)\delta(u-j,t)\right) \tag{1}$$

with:

$$\delta(1,t) = g(0)^{-1}\delta(0,t+1).$$

Recursive calculation of $\delta(u,t)$ is then possible if values of $\delta(0,\tau)$ can be calculated for $\tau = t+1, t+2, \ldots, t+u$. Dickson & Waters (1991) show that:

$$\delta(0,\tau) = \sum_{j=0}^{\tau-1}\left(1-\frac{j}{\tau}\right)h(j,\tau) \tag{2}$$

where $h(j,\tau)$ is the probability that aggregate claims over a time interval of length τ equal j, $j=0, 1, 2, \ldots$. Note that, for a given value of τ, h is the probability function of a compound Poisson random variable.

This method requires values of $\delta(0,\tau)$ for $\tau = t+1, t+2, \ldots, t+u$. We start by calculating $\delta(0,t+1)$ from formula (2), using Panjer's recursion formula to calculate values of $h(\cdot, t+1)$. At first sight it is tempting to compute values of $\delta(0,\tau)$ for $\tau = t+2, t+3, \ldots, t+u$ in exactly the same way. However, this is computationally inefficient. As noted by Dickson & Egídio dos Reis (1996) who consider a related problem, it is better to use convolutions rather than Panjer's recursion formula in most of the subsequent calculations. Suppose we have calculated $\delta(0,\tau)$ for some τ, $t+1 \le \tau < t+u$. This means that we have values of $h(j,\tau)$ for $j=0, 1, 2, \ldots, \tau-1$. To calculate $\delta(0,\tau+1)$ we require values of $h(j,\tau+1)$ for $j = 0, 1, 2, \ldots, \tau$. For $i=0, 1, 2, \ldots, \tau-1$ we can calculate $h(i,\tau+1)$, as:

$$h(i,\tau+1) = \sum_{k=0}^{i} h(k,1)h(i-k,\tau)$$

and then we can use Panjer's recursion formula to calculate $h(\tau, \tau+1)$. This approach must involve calculating values of $h(\cdot,1)$ (which is the same as $g(\cdot)$), but these can be calculated efficiently by Panjer's recursion formula, and the whole procedure of calculating $h(\cdot,\tau)$ for successive values of τ is much more efficient in terms of the number of operations required to calculate the probability function of aggregate claims over successive time periods.

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A disadvantage of the method of Dickson & Waters is that it is numerically unstable, that is, after a large number of computations, formula (1) produces computed values of survival probabilities outside the interval [0, 1]. Dickson & Waters (1991) propose a pragmatic solution to this problem. They note that:

$$0 \leq \max\{\delta(u, t+1), \delta(u-1, t)\} \leq \delta(u, t) \leq 1 \tag{3}$$

and constrain the calculated survival probabilities to behave in this way. In our numerical illustrations in the next section we apply this constraint to allow a comparison of computer run times. We comment on the effect of this constraint on approximations to $\delta_c(u, t)$ in Section 4.

Unfortunately, there does not appear to be a neat truncation procedure for this algorithm which produces a simple error bound.

2.3 *The Prabhu/Seal Formula*

We can apply the arguments used to derive the Prabhu/Seal formula for the classical risk model (see Prabhu, 1961; and, for example, Seal, 1978) to write down the corresponding formula for our discrete model. If the aggregate claim amount at time t is less than $u+t$ (so that the surplus at time t is greater than 0) then either:

(1) the surplus has been above 0 at time $\tau = 1, 2, \ldots, t$; or else
(2) at some time $j \leq t-1$ the surplus was 0 (the probability of which is $h(u+j, j)$), and in the remaining time period of length $t-j$ the surplus remained above 0 (the probability of which is $\delta(0, t-j)$).

Combining the probabilities of the above events we have:

$$H(u+t-1, t) = \delta(u, t) + \sum_{j=1}^{t-1} h(u+j, j)\delta(0, t-j)$$

where $H(j, t)$ is the probability that aggregate claims over a time interval of length t are less than or equal to j. Hence:

$$\delta(u, t) = H(u+t-1, t) - \sum_{j=1}^{t-1} h(u+j, j)\delta(0, t-j) \tag{4}$$

with $\delta(0, \tau)$ given by (2).

Kling & Goovaerts (1991) approximate $\delta_c(u, t)$ by discretising the continuous time version of (4). However, allowing for rescaling of time units and claim amounts, the formulae from which they calculate their approximations are identical to (2) and (4).

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Table 1. Exponential Claims

		$t=10$	$t=30$	$t=50$	$t=100$	$t=500$
$u=0$	DVG	0	1	3	11	163
	DW	0	0	0	0	3
	P/S	0	0	0	0	3
	$\delta_c(u,t)$	0.2146	0.1480	0.1284	0.1100	0.0925
$u=10$	DVG	0	2	4	13	170
	DW	0	2	4	17	383
	P/S	0	3	11	70	7,230
	$\delta_c(u,t)$	0.9681	0.8758	0.8163	0.7394	0.6435
$u=20$	DVG	0	2	5	14	172
	DW	2	5	11	36	778
	P/S	1	5	16	89	8,075
	$\delta_c(u,t)$	0.9996	0.9908	0.9754	0.9396	0.8629
$u=30$	DVG	1	3	6	16	175
	DW	4	10	20	59	1,163
	P/S	2	9	23	110	8,127
	$\delta_c(u,t)$	1	0.9996	0.9978	0.9890	0.9488
$u=40$	DVG	1	3	6	17	180
	DW	9	18	32	87	1,741
	P/S	3	13	32	133	8,400
	$\delta_c(u,t)$	1	1	0.9999	0.9984	0.9815
$u=50$	DVG	1	3	6	17	183
	DW	17	29	48	121	2,334
	P/S	4	18	42	160	8,972
	$\delta_c(u,t)$	1	1	1	0.9998	0.9936

3. Comparisons

In the previous section we have summarised three algorithms for calculating $\psi(u,t)$ or $\delta(u,t)$. Each of the formulae is exact, and, in principle, each method should give the same solution as the others. (The algorithm for $\psi^{\epsilon}(u,t)$ may, of course, give a different solution.) In Tables 1 and 2 we show approximations to $\delta_c(u,t)$ calculated by the Prabhu/Seal formula. We have chosen this method simply because the calculations are unaffected by truncation or by numerical stability.

Tables 1 and 2 also show computing times in seconds for each of the three algorithms — denoted in the tables by DVG, DW and P/S respectively —for a range of values of u and t. We have chosen computing time as our measure of comparison rather than the number of algebraic operations. Under the method of De Vylder & Goovaerts with truncation, the only way of counting the number of algebraic operations is to run a program, because, due to the definition of ψ^{ϵ}, we do not know at the outset at which points in the calculations truncation will apply. We chose to record computing times rather than incorporate a calculation

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Table 2. Pareto Claims

		$t=10$	$t=30$	$t=50$	$t=100$	$t=500$
$u=0$	DVG	0	1	6	46	6,524
	DW	0	0	0	0	4
	P/S	0	0	0	0	4
	$\delta_c(u,t)$	0.3061	0.2186	0.1886	0.1568	0.1126
$u=10$	DVG	0	3	9	60	6,336
	DW	0	2	4	16	49
	P/S	0	3	9	60	6,228
	$\delta_c(u,t)$	0.9068	0.7826	0.7117	0.6180	0.4595
$u=20$	DVG	1	5	14	76	6,764
	DW	2	5	11	36	756
	P/S	1	5	14	77	6,410
	$\delta_c(u,t)$	0.9722	0.9143	0.8672	0.7878	0.6136
$u=30$	DVG	1	7	20	94	7,274
	DW	4	10	20	59	1,312
	P/S	1	7	20	94	6,728
	$\delta_c(u,t)$	0.9877	0.9591	0.9312	0.8745	0.7127
$u=40$	DVG	2	11	27	115	7,787
	DW	9	18	32	87	1,580
	P/S	3	11	27	115	7,108
	$\delta_c(u,t)$	0.9932	0.9773	0.9605	0.9217	0.7814
$u=50$	DVG	3	15	36	137	7,409
	DW	16	29	47	121	2,010
	P/S	3	15	36	138	7,806
	$\delta_c(u,t)$	0.9957	0.9858	0.9751	0.9484	0.8308

of the number of algebraic operations in our programs. All computer programs were written in Fortran and computations were carried out on a Digital AlphaServer 4000 2/300 MHz CPU. For Table 1 the individual claim amount distribution in the classical risk model is exponential, whereas in Table 2 it is Pareto (2, 1), that is:

$$P(x) = 1 - (1+x)^{-2}, \text{ for } x > 0.$$

We have chosen these two distributions as illustrations of light-tailed and heavy-tailed distributions respectively. See, for example, Embrechts *et al.* (1997, Chapter 1) for a description of other distributions which fall into these categories.

In each set of calculations the value of θ is 0.1. To apply the algorithms of the previous section, we rescaled these continuous individual claim amount distributions to have mean 20, then discretised the rescaled distributions using the method described by De Vylder & Goovaerts (1988). We chose a mean of 20, as this level of rescaling is sufficient to produce very accurate approximations to

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$\delta_c(u,t)$ — see Dickson & Waters (1991, Table 5). We make the following points about the computer run times in Tables 1 and 2:

(1) When $u=0$, it is clearly better to calculate values of $\delta(u,t)$ from formula (2). Even with t as large as 500, the run time is trivial.

(2) There is a marked difference for the run times under the method of De Vylder & Goovaerts in the two tables. The reason for this is simple. The truncation procedure has no impact when the individual claim amount distribution is Pareto. In each calculation, k_0 exceeded the largest value of j for which we had to calculate $g(j)$ in order to apply the method. Recall that we have set $\varepsilon=10^{-4}/3t$. If we increase ε by a factor of 10, the truncation procedure still has no impact, resulting in no change in run time or accuracy of the approximation. If we use the method of De Vylder & Goovaerts without truncation in the situation of Table 1, the run times are broadly similar to those in Table 2 for this method.

(3) In Table 1 it is clear that the method of De Vylder & Goovaerts is the most efficient in terms of run times for $u>0$. However, this is not the case in Table 2. Their method requires run times that are broadly comparable with those for the Prabhu/Seal formula, but are much greater than those required for the method of Dickson & Waters. Indeed, we note that, as t increases from 100 to 500 in Table 2, the relative performance of the method of Dickson & Waters improves considerably.

4. CONCLUSIONS

Based on the numbers presented in the previous section, we can draw the following conclusions:

(1) When $u=0$, calculation of $\delta(u,t)$ by formula (2) is clearly best.

(2) When the individual claim amount distribution has a light tail, the method of De Vylder & Goovaerts will be the most efficient computationally, as the truncation procedure will have a real effect.

(3) There seems to be little reason to use the Prabhu/Seal formula, although it is marginally superior to the method of De Vylder & Goovaerts in Table 2 when $t=500$.

(4) The method of Dickson & Waters appears to be computationally most efficient in Table 2, at least for large values of t. However, Dickson & Waters (1991) observe that the algorithm appears to be unstable for values of u greater than about 30. In Tables 1 and 2 the values shown for $u\geq 30$ arise from the application of the constraint given by (4). When $u=30$ the calculated values are virtually identical to four decimal places to those shown in the tables. However, as u increases, the error can increase, by as much as 2.5% of the true value of $\delta(u,t)$. We would, therefore, recommend the use of this algorithm only when $u<30$. (Recall that we have been working in a framework where the mean individual claim amount is 1.)

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In choosing which algorithm to apply, we should also consider whether we want to know the value of the survival probability for a given combination of u and t, or whether we are interested in a range of values. In calculating $\delta(u,t)$ the method of De Vylder & Goovaerts also gives values of $\delta(\omega,\tau)$ for $1\leq\tau\leq t-1$ and for $1\leq\omega\leq u+t-\tau$, whereas the method of Dickson & Waters gives values of $\delta(\omega,\tau)$ for $0\leq\omega\leq u-1$ and for $t\leq\tau\leq\tau+u-\omega$, excluding $\delta(0,t)$. By contrast, all that the Prabhu/Seal formula offers is values of $\delta(0,\tau)$ for $\tau=1, 2, \ldots, t-1$.

In summary, we would suggest that there is no clear answer to the question of which algorithm should be preferred. The figures in Tables 1 and 2 show that, in terms of computer run times, there is no clear choice. Perhaps the only real conclusion we can draw is a negative one — there appears to be little reason for using the Prabhu/Seal formula when $u>0$.

Finally, we note that the conclusions of this study differ substantially from those of Steenackers & Goovaerts (1991, Section 2.4), who conduct a similar study and conclude that "The best performance is obtained by the method of Kling & Goovaerts", that is what we have called the Prabhu/Seal method. There are three reasons why they reach a different conclusion to ours. First, they take the definition of survival to be:

$$\Pr[U(\tau)\geq 0 \quad \text{for all } \tau, \quad \tau=1, 2, \ldots, t]$$

in applying the method of Dickson & Waters. This method leads to poorer approximations to $\delta_c(u,t)$ — see Dickson & Waters (1991). Second, their analysis applies the algorithm given by De Vylder & Goovaerts (1988), whereas we have used a modified (improved!) version of it. Third, our analysis has been based on computing times, since the algorithms presented in this paper are just different ways of calculating the same quantity. However, their conclusions are based on the numerical accuracy of the algorithms they compare. In this paper, numerical accuracy has been an issue only when an algorithm is unstable.

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