Ergod. Th. & Dynam. Sys. (2009), **29**, 1881–1905
doi:10.1017/S0143385709000704

Continuity of the Lyapunov exponent for analytic quasiperiodic cocycles

S. JITOMIRSKAYA†, D. A. KOSLOVER†‡ and M. S. SCHULTEIS†§

† *Department of mathematics, University of California, Irvine, CA 92697, USA*
(e-mail: szhitomi@math.uci.edu)
‡ *Department of mathematics, University of Texas at Tyler, Tyler, TX 75799, USA*
(e-mail: dkoslover@uttyler.edu)
§ *Department of mathematics, Concordia University, Irvine, CA 92612, USA*
(e-mail: melinda.schulteis@cui.edu)

(*Received* 29 *June* 2005 *and accepted in revised form* 2 *May* 2006)

Abstract. It is known that the Lyapunov exponent is not continuous at certain points in the space of continuous quasiperiodic cocycles. In this paper we show that it is continuous in the analytic category. Our corollaries include continuity of the Lyapunov exponent associated with general quasiperiodic Jacobi matrices or orthogonal polynomials on the unit circle, in various parameters, and applications to the study of quantum dynamics.

1. *Introduction*

Let $M_2(\mathbb{C})$ be the set of 2×2 matrices with complex entries. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, $S : \mathbb{T} \to \mathbb{T}$, $Sx = x + \omega$. A one-dimensional quasiperiodic cocycle is a pair $(B, \omega) \in C(\mathbb{T}, M_2(\mathbb{C})) \times \mathbb{R}$ understood as a linear skew product

$$(B, \omega) : \mathbb{C}^2 \times \mathbb{T} \to \mathbb{C}^2 \times \mathbb{T} \quad \text{with } (w, x) \to (B(x)w, Sx).$$

For $N \geq 1$, set

$$B_N(x, \omega) = \prod_{j=N-1}^{0} B(S^j x). \tag{1}$$

The Lyapunov exponent (which exists under certain conditions that are discussed below) is defined by

$$L'(B, \omega) = \lim_{N\to\infty} \frac{1}{N} \log \|B_N(x, \omega)\|.$$

Note that L' is not necessarily non-negative. It is known that $L' : C(\mathbb{T}, M_2(\mathbb{C})) \to \mathbb{R}$ is discontinuous at certain $B \in C(\mathbb{T}, M_2(\mathbb{C}))$ (see [**Bo, F, Kn, Th**]). Namely, it is discontinuous at all non-uniformly hyperbolic cocycles [**Bo, Bo1, BV, F**]. In particular, such are the cocycles of transfer matrices of one-dimensional Schrödinger operators on the spectrum and in the region where the Lyapunov exponent is positive. In this paper, we study continuity properties of L' in the analytic category. This issue has been addressed previously in the setting of transfer matrices of quasiperiodic Schrödinger operators and continuity in energy. Namely, those cocycles are defined using

$$B_E(x) = \begin{pmatrix} E - v(x) & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

In [**GS**] Goldstein–Schlag developed an important tool, the avalanche principle, and showed that for analytic v with ω satisfying a strong Diophantine condition, $L'(B_E, \omega)$ has Hölder regularity in E. Similar results for underlying dynamics, with S being a shift or a skew-shift of a higher-dimensional torus, were obtained in [**BGS, GS**]. It was shown in [**BJ**] that for analytic v, $L'(B_E, \omega)$ is a jointly continuous function of (E, ω) at every irrational ω. This was appropriately extended to shifts of higher-dimensional tori in [**B2**]. The main result of our paper is that for general quasiperiodic cocycles, $L'(B, \omega)$ is jointly continuous in (B, ω) in the analytic category. As far as we know, this is the first result in this direction for general (rather than Schrödinger) cocycles. It is an interesting open question as to what the weakest regularity assumption on B is for continuity to hold. The current techniques cannot handle anything between C^0 and (quasi-)analyticity. There are some indications (see [**Y**]) that at least for cocycles with hyperbolic underlying dynamics the continuity may start already in the C^1 category, although for cocycles over rotations of the torus the situation could be different.

We now build up the terminology to state our main result precisely. Define

$$M_2^d = \{A \in M_2(\mathbb{C}), \det A = d\}$$

and note that $M_2^1 = \mathrm{SL}_2(\mathbb{C})$. Fix $d : \mathbb{T} \to \mathbb{C}$. Define $C_\rho(\mathbb{T}, M_2^{d(x)})$ to be the set of maps of $\mathbb{R} \to M_2^{d(x)}$ which are one-periodic and analytic on a strip $|\mathrm{Im}\, z| \le \rho$, with

$$\mathrm{dist}(B^1, B^2) = \sup_{|\mathrm{Im}\, z| \le \rho} \|B^1(z) - B^2(z)\|.$$

Fix $\omega \in \mathbb{R}$. Given $B \in C_\rho(\mathbb{T}, M_2^{d(x)})$, define for x with $d(x) \neq 0$

$$M_N(x, \omega) = \frac{1}{|\det B_N(x, \omega)|^{1/2}} B_N(x, \omega). \tag{3}$$

Assume that $d(x)$ is not identically zero. Since $\|M_N(x, \omega)\| \ge 1$,

$$L_N(B, \omega) = \frac{1}{N} \int \log \|M_N(x, \omega)\|\, dx \ge 0 \quad \text{for any } N. \tag{4}$$

Set

$$L'_N(B, \omega) = \frac{1}{N} \int \log \|B_N(x, \omega)\|\, dx. \tag{5}$$

Define $D = \int \log |d(x)|\, dx = \int \log |\det B(x)|\, dx$. It is easy to check that

$$L'_N(B, \omega) = L_N(B, \omega) + \tfrac{1}{2} D. \tag{6}$$

By the subadditive ergodic theorem, we can define the Lyapunov exponents

$$L'(B,\omega) = \lim_{N\to\infty} \frac{1}{N} \log \|B_N(x,\omega)\| = \lim_{N\to\infty} L'_N(B,\omega) = \inf_N L'_N(B,\omega), \tag{7}$$

$$L(B,\omega) = \lim_{N\to\infty} \frac{1}{N} \log \|M_N(x,\omega)\| = \lim_{N\to\infty} L_N(B,\omega) = \inf_N L_N(B,\omega) \geq 0. \tag{8}$$

We now consider L, L' as maps of $C_\rho(\mathbb{T}, M_2^{d(x)})$ into $\mathbb{R}$. We comment here that in order to establish the existence of the Lyapunov exponent (using the theorem of Furstenberg–Kesten) we must assume that $D > -\infty$, so that $\log \|B(x,\omega)\| \in L_1$ and $\log \|M_1(x,\omega)\| \in L_1$.

We say that ω is Diophantine if there exists $b(\omega) > 0$ and $1 < r(\omega) < \infty$ such that, for all $j \neq 0$,

$$|\sin 2\pi j\omega| > \frac{b(\omega)}{|j|^{r(\omega)}}. \tag{9}$$

It is well known that almost every ω is Diophantine.

THEOREM 1. *Let d be analytic with $|\int \ln |d(x)|\, dx| < \infty$. Fix ω Diophantine and $\rho > 0$. Then L is a continuous function on $C_\rho(\mathbb{T}, M_2^{d(x)})$.*

Remarks.

(1) By (6), L' is then also a continuous function.

(2) In case d does not extend analytically to $|\mathrm{Im}\, z| < \rho$, the statement is vacuously true.

An important corollary is the case of Jacobi matrices. Specifically, we consider the Lyapunov exponent associated with general quasiperiodic Jacobi matrices defined by the following one-dimensional Hamiltonian $H_{x,\omega}$ acting on $\ell^2(\mathbb{Z})$

$$(H_{x,\omega}\Psi)(n) = c_n(x)\Psi(n+1) + \overline{c_{n-1}(x)}\Psi(n-1) + \lambda v_n(x)\Psi(n),$$

where $v : \mathbb{T} \to \mathbb{R}$ and $c : \mathbb{T} \to \mathbb{C}$ are analytic functions on $\mathbb{T}$, and we define $c_n(x) = c(x + (n-1)\omega)$ (similarly for $v_n(x)$). If $c(x) \equiv 1$, this is a discrete one-dimensional Schrödinger operator with quasiperiodic potential, $v(x)$. When $c(x)$ is not identically one, we call this a Jacobi operator with quasiperiodic potential. The one-step transfer matrices are defined by

$$A(E,x) = \frac{1}{c_1(x)} \begin{pmatrix} E - \lambda v_1(x) & -\overline{c_0}(x) \\ c_1(x) & 0 \end{pmatrix}. \tag{10}$$

Note that $\overline{c}$ is also analytic on $\mathbb{T}$.

The N-step transfer matrix for the eigenvalue equation $H_{x,\omega}\Psi = E\Psi$ is given by

$$B_N(E,x,\omega) = \prod_{j=N-1}^{0} A(E, S^j x).$$

Set $d(x) = \overline{c_0}(x)c_1(x)$. For $\rho(c,v) > 0$, we define $B_E \in C_\rho(\mathbb{T}, M_2^{d(x)})$ by

$$B_E(x) = c_1(x)A(E,x). \tag{11}$$

Thus, transfer matrices naturally form a quasiperiodic cocycle. Set $L = L(B_E, \omega)$. Since $\int \log |d(x)|\, dx = 2\int \log |c(x)|\, dx$, we have the following corollary.

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

COROLLARY 1. *Suppose that* $|\int \log |c(x)|\, dx| < \infty$, *with* ω *Diophantine. Then* L *is continuous on* $C_\rho(\mathbb{T}, M_2^{d(x)})$ *for* $\rho = \rho(c, v) > 0$. *In particular,* L *is jointly continuous in* (E, λ).

In the case when $c(x) \equiv 1$, we have the following results.

COROLLARY 2.
(1) *For any* ω, L *is continuous on* $C_\rho(\mathbb{T}, \mathrm{SL}_2(\mathbb{C}))$ *for* $\rho = \rho(v)$. *In particular,* L *is jointly continuous in* (E, λ).
(2) *For any irrational* ω *and any analytic* v, L *is jointly continuous at* (v, ω) *on* $C_\rho(\mathbb{T}, \mathrm{SL}_2(\mathbb{C})) \times \mathbb{R}$ *for* $\rho = \rho(v)$.

Many ingredients needed to establish Corollary 2 for any irrational ω are present in [**BJ**], albeit not formulated there in such generality. The main achievement of the present paper is that we can consider general rather than just Schrödinger (or, more specifically, $\mathrm{SL}_2(\mathbb{C})$) cocycles. This generality is important for our application to the case of transfer matrices of quasiperiodic Jacobi operators. In fact, as should be clear from the formulation of Corollary 2, no Diophantine restriction on ω is needed for the $\mathrm{SL}_2(\mathbb{C})$ case (or, more generally, when $d(x)$ is bounded away from zero). We believe such a restriction is not necessary in general either, however treating the case of non-Diophantine frequencies would require a series of additional arguments. This will be done in a future work.

We are particularly interested in the quasiperiodic Jacobi matrix representing the following one-dimensional Hamiltonian, $H_{x,\lambda_1-\lambda_4,\omega}$ acting on $\ell^2(\mathbb{Z})$, and given by

$$\begin{aligned}(H_{x,\lambda_1-\lambda_4,\omega}\Psi)(n) &= (\lambda_3 + \lambda_4 \exp(2\pi\imath(x + (n + \tfrac{1}{2})\omega)) \\ &\quad + \lambda_2 \exp(-2\pi\imath(x + (n + \tfrac{1}{2})\omega)))\Psi(n+1) \\ &\quad + (\lambda_3 + \lambda_2 \exp(2\pi\imath(x + (n - \tfrac{1}{2})\omega)) \\ &\quad + \lambda_4 \exp(-2\pi\imath(x + (n - \tfrac{1}{2})\omega)))\Psi(n-1) \\ &\quad + (2\lambda_1 \cos 2\pi(x + n\omega))\Psi(n) \\ &\triangleq c_n(x)\Psi(n+1) + \overline{c_{n-1}(x)}\Psi(n-1) + v_n(x)\Psi(n).\end{aligned}$$

Using the Poisson–Jensen formula, one can show that $\int \log |c_1(x)|\, dx > -\infty$ (see e.g. [**JKS**]). Thus L is jointly continuous in $(E, \lambda_1, \dots, \lambda_4)$. This model, called the second-nearest-neighbors model of a Bloch electron in a perpendicular magnetic field, has been studied in physics literature and was first introduced by Thouless in 1983 [**T**]. It describes an electron moving in an infinite two-dimensional crystalline lattice with translational invariance, subjected to a perpendicular magnetic field, where the electron is confined to hop to nearest neighbors (north, south, east, west) and second nearest neighbors (the diagonals on the lattice). The λ_i represent normalized hopping terms which are proportional to the probability an electron will hop to one of these neighboring sites and ω is the magnetic flux. In the case $\lambda_2 = \lambda_4 = 0$, this reduces to the one-dimensional Schrödinger case with quasiperiodic potential $v(x) = \lambda \cos 2\pi(x + n\omega)$. This model, called the almost Mathieu operator, has been studied extensively over the past 25 years (see [**L**]). Our main result, continuity for general Jacobi matrices is an essential ingredient in establishing strong dynamical localization for the second-nearest-neighbors operator [**JKS**]. The continuity of L is expected to also play a crucial role in the proof of Cantor spectrum for this model, similarly to how it was used in [**AJ**] for the almost Mathieu operator.

Another important immediate application is to transfer matrices associated with orthogonal polynomials on the unit circle with quasiperiodic Verblunsky coefficients (see [**S**]). Those transfer matrices are related to a cocycle with

$$B_z(x) = \begin{pmatrix} \sqrt{z} & -\dfrac{\overline{f(x)}}{\sqrt{z}} \\ -\sqrt{z}f(x) & \dfrac{1}{\sqrt{z}} \end{pmatrix}, \tag{12}$$

where $f : \mathbb{T} \to \overline{\mathbb{D}}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \leq 1\}$, is such that the $\text{meas}\{x \in \mathbb{T} : |f(x)| = 1\} = 0$, and $\sqrt{\ }$ means any fixed branch. The corresponding Verblunsky coefficients are then defined by $\alpha_n(x) = f(x + n\omega)$ for an irrational ω and almost every $x \in \mathbb{T}$. We therefore have the following.

COROLLARY 3. *Let f as above be analytic with $|\int \ln|1 - |f(x)|^2|\,dx| < \infty$. Then:*

(1) *for ω Diophantine, L is a continuous function on $C_\rho(\mathbb{T}, M_2^{1-|f(x)|^2})$ for $\rho = \rho(f) > 0$. In particular, for a fixed f, $L = L(z)$ is a continuous function of z on $\overline{\mathbb{D}}\backslash 0$;*

(2) *if $f : \mathbb{T} \to \mathbb{D}$, then for any irrational ω L is jointly continuous at (f, ω) on $C_\rho(\mathbb{T}, M_2^{1-|f(x)|^2}) \times \mathbb{R}$ for $\rho = \rho(f)$.*

It is easy to show that at least some regularity of f is needed for this statement to hold. As pointed out in [**S**] the theory of orthogonal polynomials on the unit circle with quasiperiodic Verblunsky coefficients is largely undeveloped. Continuity of L is expected to play an important role in various aspects of this theory, as it does for Schrödinger operators. In particular, continuity in ω is a statement that should enable one to prove various facts through approximations by periodic Verblunsky coefficients, the theory of which has been well developed [**S**].

Compared with existing proofs of regularity for Schrödinger cocycles, we face certain technical difficulties related to possible small values of $d(x)$. A key technique underlying most recent developments in the area of quasiperiodic Schrödinger operators with analytic potential is based on the fact that $(1/N)\log\|B_N(x, \omega)\|$ for B as in (12) have uniformly (in N) bounded subharmonic extensions (see [**B1**]). Situations where such subharmonic extensions cannot be uniformly bounded in the same strip therefore present certain challenges. Such difficulties appeared and were treated successfully in the Schrödinger transfer-matrix models where either quasiperiodicity [**BGS**] or analyticity [**K**] were relaxed. We note that in our case for $d(x)$ not bounded away from zero, the subharmonic extensions are not bounded at all. The techniques we develop to overcome this problem are different from those of [**BGS, K**], and we believe can be used in certain other situations where similar difficulties appear. Some other problems created by d not being bounded away from zero include the absence of almost invariance of $(1/N)\log\|B_N(x, \omega)\|$ with respect to the shift of x by ω, which is crucially used in all proofs for the Schrödinger case, and the absence of continuity of $\log\|B_N(x, \omega)\|$ in x.

The rest of this paper is organized as follows. In §2 we establish large deviation bounds for L_N and L'_N. This is one of the main places where we develop techniques to deal with unboundedness of $\log \|B_N(x, \omega)\|$ that allow us to treat general non-SL(2, $\mathbb{R}$) cocycles. Sections 3 and 4 are devoted to the main induction argument, mostly following that of [**BJ**] but developed in more detail. We would like to note that there is an important necessary arithmetic condition on the scales in the induction argument (see (56)) that was overlooked in [**BJ**]. In §5 the proof is completed and an additional result, Theorem 2, is stated and proved. We note that in our setting, unlike in the proofs of continuity in E for Schrödinger cocycles, we cannot use subharmonicity to establish continuity at zeros (or other minima) of the Lyapunov exponent, thus requiring a separate argument, based on the quantitative estimates from the induction procedure.

In order to avoid cumbersome notation, we will omit some variables that remain fixed throughout an argument.

2. *Large deviation bound for non-SL*(2, $\mathbb{R}$) *cocycles*

We use $\langle f \rangle$ for $\int_{\mathbb{T}} f(x)\,dx$.

LEMMA 1. (Large deviation bound (LDB)) *Let* $B \in C_\rho(\mathbb{T}, M_2^{d(x)})$, $\rho > 0$, d *not identically zero. Consider* ω *satisfying (9). Let* $|\omega - (a/q)| < (1/q^2)$, $(a, q) = 1$. *Let* $0 < \kappa < 1$. *Then for appropriate* $c > 0$ *and* $C < \infty$, *for* $N > (C\kappa^{-2}q)^\eta$, *where* $\eta = \eta(\omega) > 1$, *and sufficiently large* q *we have*

$$\operatorname{meas}\left\{x : \left|\frac{1}{N}\log\|B_N(x, \omega)\| - L'_N\right| > \kappa\right\} < e^{-c\kappa q}.$$

Remarks.

(1) For any $0 < \delta < r(\omega)^{-1} \le 1$ with $r(\omega)$ from (9), we can choose $\eta = \eta(\omega) = (\min(\delta, 1-\delta))^{-1}$ so that the statement of Lemma 1 holds with appropriately adjusted C.

(2) For any $\tau > 0$, constants c, C, as well as the largeness condition on q can be chosen uniformly for B' with $\operatorname{dist}(B, B') < \tau$.

(3) The main difficulty here, compared with [**BJ**], is that the operator studied there had $\log\|M_N(x)\|$ bounded below, which allowed for a bounded subharmonic extension of $(1/N)\log\|M_N(x)\|$. In our case, $\log\|B_N\|$ is unbounded from below.

LEMMA 2. (Large deviation bound for M_N) *Under the hypotheses of Lemma 1, for appropriate* $c_1 > 0$ *and* $C < \infty$, *for* $N > (C\kappa^{-2}q)^\eta$, *where* $\eta = \eta(\omega) > 1$, *and sufficiently large* q *we have*

$$\operatorname{meas}\left\{x : \left|\frac{1}{N}\log\|M_N(x, \omega)\| - L_N\right| > \kappa\right\} < e^{-c_1\kappa q}.$$

Proof of Lemma 1. Let

$$u_N(x) = u_N(B, x, \omega) = \frac{1}{N}\log\|B_N(x)\|. \tag{13}$$

The strategy of the proof is to first introduce a truncation of $u_N(x)$, called $w_N(x)$, which has a bounded subharmonic extension, and thus will make it possible to use a

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

modified version of the proof of the large deviation bound shown in [**BJ**]. The adaptation is not straightforward because w_N is not uniformly close to its shifts. This is addressed in Proposition A. We then show in Proposition B that the measure of the set where the original function differs from the truncation is exponentially small in N, which yields the desired result.

Note that $B_N(x)$ has an analytic extension, $B_N(z)$, to a strip $|\text{Im } z| < \rho$, for some $\rho > 0$, satisfying $\|B_N(z)\| < C^N$. Fix $A > -D/2$ (see (6)). Define

$$w_N(z) = w_N(B, z, \omega) = \max\left(\frac{1}{N} \log \|B_N(z)\|, -A\right). \tag{14}$$

Thus, $w_N(z)$ is a bounded subharmonic function on $|\text{Im } z| < \rho$. This will be essential later on. □

Next we would like to bound $|w_N(x) - w_N(x+\omega)|$.

PROPOSITION A. *There exists* $c > 0$ *such that for any* $0 < \delta < 1$, *there exists* $C < \infty$ *such that:*

$$\text{meas}\left\{|w_N(x) - w_N(x+\omega)| > \frac{C}{N^{1-\delta}}\right\} < e^{-cN^\delta}. \tag{15}$$

Remarks.

(1) For Schrödinger cocycles, a stronger inequality, $|w_N(x) - w_N(x+\omega)| < CN^{-1}$, is easily established to hold for all x (see [**BG**]). This does not hold in our case, however, due to the unboundedness of $\|B(x)^{-1}\|$.

(2) Here c depends on $d(x)$ only.

Proof. Note that

$$|w_N(x) - w_N(x+\omega)| < \left|\frac{1}{N} \log \|B_N(x)\| - \frac{1}{N} \log \|B_N(x+\omega)\|\right|.$$

There exists $C_B < \infty$ such that:

$$\|B(S^j x)\| < C_B \tag{16}$$

$$\|B(S^j x)^{-1}\| \le \frac{1}{|d(S^j x)|} C_B. \tag{17}$$

For ease of notation, let $d_j(x) = d(S^j x)$. Therefore, recalling (1), we have

$$\|B_N(x+\omega)\| \le \frac{1}{|d(x)|} C_B^2 \|B_N(x)\|.$$

Similarly,

$$\|B_N(x)\| \le \frac{1}{|d_N(x)|} C_B^2 \|B_N(x+\omega)\|.$$

Set $0 < \delta < 1$. Consider the two cases: (a) $|d_j| \ge \exp\{-N^\delta\}$, $j = 0, N$ and (b) $|d_j| < \exp\{-N^\delta\}$, for some $j \in \{0, N\}$.

If we are in case (a), then the above calculation gives

$$\max\left(\frac{\|B_N(x)\|}{\|B_N(x+\omega)\|}, \frac{\|B_N(x+\omega)\|}{\|B_N(x)\|}\right) < e^{N^\delta} C_B^2$$

and hence,

$$\left|\frac{1}{N}\log\|B_N(x)\| - \frac{1}{N}\log\|B_N(x+\omega)\|\right| < \frac{C}{N^{1-\delta}}.$$

We now need to bound the measure for case (b). Let $S = \{x \in \mathbb{T} : |d_j| < \exp\{-N^\delta\}$, for some $j \in \{0, N\}\}$. By the Lojasiewicz inequality [**Lo**],

$$\text{meas}\{x \in \mathbb{T} : |d(x)| < \epsilon\} < \epsilon^\alpha \tag{18}$$

for any sufficiently small ϵ and α depending only on d. Therefore,

$$\text{meas}(S) < 2e^{-\alpha N^\delta} < e^{-cN^\delta}$$

for $c < \alpha$ and sufficiently large N. □

We now write

$$\begin{aligned}
&|w_N(x) - \langle w_N\rangle| \\
&\quad\le \left|w_N(x) - \sum_{|j|<R}\frac{R-|j|}{R^2}w_N(x+j\omega)\right| + \left|\sum_{|j|<R}\frac{R-|j|}{R^2}w_N(x+j\omega) - \langle w_N\rangle\right| \\
&\quad= (\text{I}) + (\text{II}).
\end{aligned}$$

In order to estimate (II), we use the following.

Large deviation bound for subharmonic functions. [**BJ**] Let $v(x)$ be a bounded one-periodic subharmonic function defined on a neighborhood of $\mathbb{R}$. Let $|\omega - (a/q)| < (1/q^2)$, $(a, q) = 1$. Let $0 < \kappa < 1$. Then for appropriate $c_1 > 0$, $C < \infty$, for $R > C\kappa^{-1}q$,

$$\text{meas}\left\{x : \left|\sum_{|j|<R}\frac{R-|j|}{R^2}v(x+j\omega) - \langle v\rangle\right| > \kappa\right\} < e^{-c_1\kappa q}.$$

Remarks.

(1) This result in this form is essentially shown in [**BJ**] (see also [**B1**, Ch. 5]). We provide a proof for the reader's convenience at the end of the section.

(2) This statement will be applied to w_N. Note that the constants c_1, C will be uniform in N since $w_N(z)$ are uniformly (in N) bounded on $|\text{Im}\, z| < \rho$ (see (14)).

Since w_N satisfies the conditions of the large deviation bound for subharmonic functions, uniformly in N, we have $\text{meas}\{x : (\text{II}) > \kappa/2\} < e^{-c_1(\kappa/2)q}$.

Let $X = \{x : |w_N(x) - w_N(x+\omega)| > C/N^{1-\delta}\}$, which by Proposition A has exponentially small measure. If $\bigcup_{j=-R+1}^{R} S^j x \subset \mathbb{T}\backslash X$, we have

$$(\text{I}) = \left|w_N(x) - \sum_{|j|<R}\frac{R-|j|}{R^2}w_N(x+j\omega)\right| < \frac{RC}{N^{1-\delta}}. \tag{19}$$

We can choose $R < \kappa N^{1-\delta}(2C)^{-1}$, so that $|(\text{I})| < \kappa/2$. If $N > (C_1\kappa^{-2}q)^{1/(1-\delta)}$, R can also be chosen to satisfy the conditions of the large deviation bound for subharmonic functions. Therefore, by Proposition A, we have

$$\text{meas}\{x : |w_N(x) - \langle w_N\rangle| > \kappa\} < 2\,Re^{-cN^\delta}.$$

Choosing $R \sim C\kappa^{-1}q$, we obtain

$$\operatorname{meas}\{x : |w_N(x) - \langle w_N\rangle| > \kappa\} < e^{-c_1\kappa q} \tag{20}$$

for $N > \max((C\kappa^{-2}q)^{1/(1-\delta)}, (C\kappa q)^{1/\delta})$.

We now want to bound $|\langle w_N\rangle - \langle u_N\rangle|$ (see (13) and (14)).

PROPOSITION B. *Fix $0 < \delta < r(\omega)^{-1} \leq 1$. There exists $c > 0$ such that for N large enough*

$$|\langle w_N\rangle - \langle u_N\rangle| < e^{-cN^\delta}. \tag{21}$$

Proof. Set $X = \{x \in \mathbb{T} : \|B_N(x)\| < e^{-NA}\} = \{x : w_N(x) \neq u_N(x)\}$. Then

$$|\langle w_N\rangle - \langle u_N\rangle| \leq \frac{1}{N}\int_X \left|\log \frac{e^{-NA}}{\|B_N(x)\|}\right| dx. \tag{22}$$

Since $\|M\|^2 > |\det M|$, recalling (1), we have

$$\|B_N(x)\|^2 \geq \prod_{j=0}^{N-1} |d_j(x)|. \tag{23}$$

Hence, if $x \in X$, then

$$\prod_{j=0}^{N-1} |d_j(x)| < e^{-2NA}.$$

The following is a combination of [**J**, Lemmas 11 and 12]. □

PROPOSITION C. [**J**] *For analytic $d(x)$ and Diophantine ω satisfying (9), for every $\varepsilon > 0$ and N sufficiently large, we have*

$$-\varepsilon + CN^{-r(\omega)^{-1}} \log N \log \min_{j=1,\ldots,N} |d_j(x)| \leq \frac{1}{N}\sum_{j=1}^{N} \log|d_j(x)| - D \leq \varepsilon \tag{24}$$

where $C = C(\omega, d) > 0$.

Using the left inequality and (14), we get for $x \in X$

$$\begin{aligned}\min_{j=1,\ldots,N} |d_j(x)| &< \exp\left(\frac{-(2A + D - \varepsilon)}{C \log N} N^{r(\omega)^{-1}}\right)\\ &\leq e^{-N^\delta}\end{aligned} \tag{25}$$

where $0 < \delta < r(\omega)^{-1} \leq 1$. Thus,

$$\operatorname{meas}(X) < \operatorname{meas}\left\{x \in \mathbb{T} : \min_{j=1,\ldots,N} |d_j(x)| < e^{-N^\delta}\right\}. \tag{26}$$

In addition, by (23), we have for $x \in X$

$$0 \leq \log \frac{e^{-NA}}{\|B_N(x)\|} \leq \log \frac{e^{-NA}}{\left(\prod_{j=0}^{N-1} |d_j(x)|\right)^{1/2}}. \tag{27}$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

PROPOSITION D. *There exists $C < \infty$ such that for sufficiently small ε,*

$$\left|\int_{|d_i(x)|<\varepsilon} \log|d_j(x)|\,dx\right| < C\varepsilon^{\alpha}|\log\varepsilon|, \quad i, j \in \{1, \ldots, N\}.$$

Remark. The constant $\alpha = \alpha(d)$ is the same as in (18).

Proof. Set $A_i = \{x : d_i(x) < \varepsilon\}$. Fix $j \in \{1, \ldots, N\}$. Set

$$B_k = \{x : \varepsilon/2^k \le |d_j(x)| < \varepsilon/2^{k-1}\}.$$

Then

$$\left|\int_{|d_i(x)|<\varepsilon} \log|d_j(x)|\,dx\right| = \left|\sum_{k=1}^{\infty}\int_{B_k\cap A_i} \log|d_j(x)|\,dx + \int_{A_i\setminus\bigcup_{k=1}^{\infty} B_k} \log|d_j(x)|\,dx\right|$$
$$\le \sum_{k=1}^{\infty} \operatorname{meas}(B_k)\left|\log\frac{\varepsilon}{2^k}\right| + \operatorname{meas}(A_i)|\log\varepsilon|.$$

By (18), we have $\operatorname{meas}(B_k) < (\varepsilon/2^{k-1})^{\alpha}$, $\operatorname{meas}(A_i) < \varepsilon^{\alpha}$. Therefore,

$$\left|\int_{A_i} \log|d_j(x)|\,dx\right| < C\varepsilon^{\alpha}|\log\varepsilon|.$$

To finish the proof of Proposition B, we use (22), (26), (27), Proposition D and (18), to obtain:

$$|\langle w_N\rangle - \langle u_N\rangle| \le \sum_{i=0}^{N} \frac{1}{N}\int_{|d_i(x)|<\exp(-N^{\delta})}\left(-NA - \frac{1}{2}\sum_{j=0}^{N-1}\log|d_j(x)|\right)dx$$
$$\le CN^{1+\delta}e^{-\alpha N^{\delta}} + NAe^{-\alpha N^{\delta}} < e^{-cN^{\delta}} \tag{28}$$

for $c < \alpha$ and sufficiently large N. □

We now finish the proof of Lemma 1. For any $\beta > 0$, using (26) and (18):

$$\operatorname{meas}\{x \in \mathbb{T} : |u_N(x) - w_N(x)| > \beta\} < \operatorname{meas}(X) \tag{29}$$
$$< e^{-cN^{\delta}} \quad \text{for sufficiently large } N.$$

Thus, overall we can combine Proposition B, (20), and (29) to obtain, for N large enough so that Proposition C holds, $N \gtrsim \max((\kappa^{-2}q)^{1/(1-\delta)}, (\kappa q)^{1/\delta})$, and for large q:

$$\operatorname{meas}\{x \in \mathbb{T} : |u_N(x) - \langle u_N\rangle| > \kappa\}$$
$$< \operatorname{meas}\left\{x \in \mathbb{T} : |u_N(x) - w_N(x)| > \frac{\kappa}{2} - e^{-cN^{\delta}}\right\}$$
$$+ \operatorname{meas}\left\{x \in \mathbb{T} : |w_N(x) - \langle w_N\rangle| > \frac{\kappa}{2}\right\}$$
$$< e^{-cN^{\delta}} + e^{-c_1\kappa q/2} < e^{-c\kappa q}.$$

This proves Lemma 1 with $\eta = (\min(\delta, 1-\delta))^{-1}$. □

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Proof of Lemma 2. From Proposition C, selecting $\varepsilon < \kappa/2$ (see (24)), and recalling (25), we have

$$\begin{aligned}
&\text{meas}\left\{x : \left|\frac{1}{N}\sum_{j=0}^{N_1}\log|d_j(x)| - D\right| > \kappa\right\}\\
&< \text{meas}\left\{x : CN^{-r(\omega)^{-1}}\log N \log \min_{1\dots N}|d_j(x)| < -\frac{\kappa}{2}\right\}\\
&\le \text{meas}\left\{x : \min_{1\dots N}|d_j(x)| < e^{-c_2 N^{\delta}}\right\}. \qquad (30)
\end{aligned}$$

Using (18) and Lemma 1, and recalling (3), (4), the result is established. □

Proof of the large deviation bound for subharmonic functions. Set $\|\beta\| = \text{dist}(\beta, 2\pi\mathbb{Z})$. Using Fourier expansion

$$v(x) = \sum_{k\in\mathbb{Z}} \hat{v}(k)\exp(2\pi ikx) \quad \text{and} \quad \hat{v}(0) = \langle v\rangle$$

we have, for any K,

$$\begin{aligned}
&\left|\sum_{|j|<R}\frac{R-|j|}{R^2}v(x+j\omega) - \langle v\rangle\right|\\
&= \left|\sum_{|j|<R}\frac{R-|j|}{R^2}\left(\sum_{0\le|k|<K}\hat{v}(k)\exp(2\pi ik(x+j\omega))\right.\right.\\
&\quad \left.\left. + \sum_{|k|\ge K}\hat{v}(k)\exp(2\pi ik(x+j\omega))\right) - \hat{v}(0)\right|\\
&\le \left|\sum_{0<|k|<K}\hat{v}(k)\exp(2\pi ikx)\sum_{|j|<R}\frac{R-|j|}{R^2}\exp(2\pi ikj\omega)\right.\\
&\quad \left. + \hat{v}(0)\left(\left(\sum_{|j|<R}\frac{R-|j|}{R^2}\right) - 1\right)\right|\\
&\quad + \left|\sum_{|k|\ge K}\hat{v}(k)\exp(2\pi ikx)\left(\sum_{|j|<R}\frac{R-|j|}{R^2}\exp(2\pi ikj\omega)\right)\right|\\
&= \text{(I)} + \text{(II)}. \qquad (31)
\end{aligned}$$

We have

$$\begin{aligned}
\text{(I)} &= \left|\sum_{0<|k|<K}\hat{v}(k)\exp(2\pi ikx)\sum_{|j|<R}\frac{R-|j|}{R^2}\exp(2\pi ikj\omega)\right|\\
&\le C_0\sum_{0<|k|<K}|\hat{v}(k)|\frac{1}{(1+R^2\|k\omega\|^2)}\\
&= C_0\left(\sum_{0<|k|<q/4}|\hat{v}(k)|\frac{1}{(1+R^2\|k\omega\|^2)} + \sum_{q/4<|k|<K}|\hat{v}(k)|\frac{1}{(1+R^2\|k\omega\|^2)}\right)\\
&= C_0(\text{(III)} + \text{(IV)}). \qquad (32)
\end{aligned}$$

The inequality here is standard and follows from the properties of Fejer's kernel. We show that (III) and (IV) can be uniformly bounded in x by $\kappa/10$. We estimate (II) in the L^2 norm.

Consider (III). We first make the following observation: as $|\omega - a/q| < 1/q^2$, for $|k| \le q/2$, $k \ne 0$, $|k\omega - (ka/q)| < 1/2q \implies \|k\omega\| \ge 1/2q$. Denote by $\alpha_1, \ldots, \alpha_{q/4}$ the decreasing arrangement of $(\|k\omega\|^{-1})_{k=1,\ldots,q/4}$. Then $\alpha_i \le 2q/i$. Also, note that the same result will hold for $\{\|k\omega\|^{-1}\}_{k\in I}$ if I is any interval of length $q/4$, where we exclude at most one value of k (for the possibility of a value close to zero).

Therefore, we have the following inequality

$$\sum_{0<|k|<q/4} \frac{1}{k\|k\omega\|} < C_1 q.$$

It is shown in [**BG**] (see also [**B1**, Ch. 4]) that the following bound for the Fourier coefficients holds

$$|\hat{v}(k)| < \frac{C}{|k|} \quad \text{for all } k \ne 0. \tag{33}$$

The constant C depends only on B, ρ for $|v(z)| < B$ on $|\mathrm{Im}\, z| < \rho$. Thus,

$$\text{(III)} < \sum_{0<|k|<q/4} \frac{C}{|k|} \frac{1}{R\|k\omega\|} < \frac{C_2 q}{R}. \tag{34}$$

We define R appropriately at the end to achieve the desired uniform bound.

Set $I_l = [lq/4, (l+1)q/4)$. Then we have by (33)

$$\begin{aligned}\text{(IV)} &\le \sum_{l=1}^{4Kq^{-1}} \sum_{k\in I_l} |\hat{v}(k)| \frac{1}{(1+R^2\|k\omega\|^2)} \le \sum_{l=1}^{4Kq^{-1}} \frac{4C}{lq} \sum_{k\in I_l} \frac{1}{1+R^2\|k\omega\|^2} \\ &\le \sum_{l=1}^{4Kq^{-1}} \frac{4C}{lq}\left(1 + \sum_{0<|j|<q/4} \frac{(2q)^2}{R^2 j^2}\right) < C_3 \sum_{l=1}^{4Kq^{-1}} \frac{1}{lq} < \frac{C_3}{q} \log K \end{aligned} \tag{35}$$

for large K and q. We define the cutoff K at the end.

We now bound (II). We have by (33)

$$\|\text{(II)}\|_2^2 = \sum_{|k|\ge K} |\hat{v}(k)|^2 \left|\sum_{|j|<R} \frac{R-|j|}{R^2} \exp(2\pi i k j\omega)\right|^2 < \sum_{|k|\ge K} |\hat{v}(k)|^2 < \frac{C_4}{K}. \tag{36}$$

Now putting (31)–(36) together, and letting

$$\frac{10 C_0 C_2 q}{\kappa} < R \quad \text{and} \quad \log K = \frac{\kappa q}{10 C_0 C_3}$$

we have

$$\left|\sum_{|j|<R} \frac{R-|j|}{R^2} v(x+j\omega) - \langle v\rangle\right| < C_0(\text{(III)} + \text{(IV)}) + \text{(II)} < \frac{2\kappa}{10} + \text{(II)}.$$

Hence, overall we have, by (36) and the definition of K,

$$\begin{aligned} &\mathrm{meas}\left\{x \in \mathbb{T} : \left|\sum_{|j|<R} \frac{R-|j|}{R^2} v(x+j\omega) - \langle v\rangle\right| > \kappa\right\} \\ &\quad < \mathrm{meas}\left\{x \in \mathbb{T} : |\text{(II)}| > \frac{4\kappa}{5}\right\} \le \left(\frac{5}{4\kappa}\right)^2 \|\text{(II)}\|^2 \le \frac{C_5}{K\kappa^2} < \frac{C_5}{\kappa^2} e^{-c\kappa q}. \end{aligned}$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Thus, for q large enough, there exists $c_1 > 0$ such that

$$\operatorname{meas}\left\{x : \left|\sum_{|j|<R} \frac{R-|j|}{R^2} v(x+j\omega) - \langle v\rangle\right| > \kappa\right\} < e^{-c_1\kappa q} \tag{37}$$

□

3. *Preparation for the induction argument*

Let L, L_N be as in (8), (4). Note that in general, $\|M_N(x, \omega)\|$ are not uniformly bounded from above. This creates a source of difficulties compared with previous works.

LEMMA 3. *Suppose that $|D| < \infty$. Consider ω satisfying (9). Let $|\omega - (a/q)| < 1/q^2$, $(a, q) = 1$. Let $L(B, \omega) > 100\kappa > 0$. Let $N > (C\kappa^{-2}q)^\eta$ with η from Lemma 1. Assume that $L_{2N}(B, \omega) > \frac{9}{10} L_N(B, \omega)$. Then for N_1 such that $N|N_1$ and $N_1N^{-1} = m \lesssim e^{c'\kappa q}$, we have*

$$\left|L_{N_1} + \frac{m-2}{m} L_N - \frac{2(m-1)}{m} L_{2N}\right| < Ce^{-c'\kappa q}$$

where $c' = c/2$, c from the large deviation bound of Lemma 1.

Remark. We show the lemma for L_N, although our final result, Theorem 1, will hold for both L and L'. We utilize the fact that $L \geq 0$ for M_N. At the end of the argument, we reintroduce the truncation function from (14) and use the result of Proposition B in order to adapt the argument used in [**BJ**].

Proof. All integrals will be with respect to x in this argument, so we suppress it in the notation. We use the following result introduced as the 'avalanche principle' for $\mathrm{SL}_2(\mathbb{R})$ in [**GS**] (see also [**B1**, Ch. 6]). We formulate and use it for $\mathrm{SL}_2(\mathbb{C})$ since the proof is the same. □

Avalanche principle. [**GS**] Let $A_1, \ldots, A_n$ be a sequence in $\mathrm{SL}_2(\mathbb{C})$ satisfying the conditions:

(A) $\min_{1\leq j\leq n} \|A_j\| \geq \mu > n$;

(B) $\max_{1\leq j\leq n} |\log\|A_j\| + \log\|A_{j+1}\| - \log\|A_{j+1}A_j\|| < \frac{1}{2}\log\mu$.

Then there exists $C_A < \infty$ such that

$$\left|\log\left\|\prod_{j=n}^{1} A_j\right\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\right| < C_A\frac{n}{\mu}.$$

We use the avalanche principle on $A_j^N(x) = M_N(x + jN\omega)$ with x restricted to the set $\Lambda \subset \mathbb{T}$, defined by $2m$ conditions:

$$\left|\frac{1}{N}\log\|A_j^N(x)\| - L_N\right| < \kappa$$

$$\left|\frac{1}{2N}\log\|A_j^{2N}(x)\| - L_{2N}\right| < \kappa \qquad \text{for all } j \leq m. \tag{38}$$

We have Lemma 2, the large deviation bound for M_N, holding, hence for any j:

$$\operatorname{meas}\left\{x : \left|\frac{1}{N}\log\|A_j^N(x)\| - L_N\right| > \kappa\right\} < e^{-c\kappa q}.$$

Similarly for $2N$, so that

$$\text{meas}(\mathbb{T}\setminus\Lambda) < 2me^{-c\kappa q}. \tag{39}$$

From (38), we have for each $A_j^N(x)$ with $x \in \Lambda$,

$$e^{N(L_N-\kappa)} < \|A_j^N(x)\| < e^{N(L_N+\kappa)}. \tag{40}$$

Note that since $L_N(E, \omega) > L(E, \omega) > 100\kappa$, we have

$$\|A_j^N(x)\| > \exp((99/100)NL_N) \triangleq \mu.$$

For large enough q, and hence N by hypothesis, we have $\mu > 2m$. Also for $j < m$, by (38) and the fact that $A_{j+1}^N A_j^N = A_j^{2N}$,

$$\begin{aligned}
&|\log\|A_j^N(x)\| + \log\|A_{j+1}^N(x)\| - \log\|A_{j+1}^N(x)A_j^N(x)\||\\
&\quad < 4N\kappa + 2N|L_N - L_{2N}|\\
&\quad < \frac{1}{25}NL_N + 2N\left(\frac{1}{10}L_N\right)\\
&\quad < \frac{1}{2}\log\mu = \frac{99}{200}NL_N.
\end{aligned}$$

Thus, we can apply the avalanche principle for $x \in \Lambda$, to obtain, for sufficiently large N,

$$\begin{aligned}
&\left|\log\left\|\prod_{j=m}^{1} A_j^N(x)\right\| + \sum_{j=2}^{m-1}\log\|A_j^N(x)\| - \sum_{j=1}^{m-1}\log\|A_{j+1}^N(x)A_j^N(x)\|\right|\\
&\quad < C_A\frac{m}{\mu}\\
&\quad < me^{-\frac{1}{2}NL_N}.
\end{aligned} \tag{41}$$

Integrating on Λ, from (41) we obtain

$$\begin{aligned}
&\left|\int_\Lambda \log\|M_{N_1}(x+N\omega)\| + \sum_{j=2}^{m-1}\int_\Lambda \log\|M_N(x+jN\omega)\|\right.\\
&\quad \left. - \sum_{j=1}^{m-1}\int_\Lambda \log\|M_{2N}(x+jN\omega)\|\right| < me^{-\frac{1}{2}NL_N}.
\end{aligned} \tag{42}$$

Recalling how M_N are defined (see (3)), observe that

$$\begin{aligned}
&\log\left\|\prod_{j=m}^{1} M_N(x+jN\omega)\right\| + \sum_{j=2}^{m-1}\log\|M_N(x+jN\omega)\| - \sum_{j=1}^{m-1}\log\|M_{2N}(x+jN\omega)\|\\
&= \log\left\|\prod_{j=m}^{1} B_N(x+jN\omega)\right\| + \sum_{j=2}^{m-1}\log\|B_N(x+jN\omega)\| - \sum_{j=1}^{m-1}\log\|B_{2N}(x+jN\omega)\|
\end{aligned}$$

since the determinant terms cancel. Using this and (42),

$$\begin{aligned}
&\left|L_{N_1} + \frac{m-2}{m} L_N - \frac{2(m-1)}{m} L_{2N}\right| \\
&= \left|\frac{1}{N_1}\int_{\mathbb{T}} \log \|M_{N_1}(x+N\omega)\|\right. \\
&\quad \left. + \frac{1}{N_1}\sum_{j=2}^{m-1}\int_{\mathbb{T}} \log\|M_N(x+jN\omega)\| - \frac{1}{N_1}\sum_{j=1}^{m-1}\int_{\mathbb{T}} \log\|M_{2N}(x+jN\omega)\|\right| \\
&\le \frac{m}{N_1} e^{-\frac{1}{2}NL_N} + \left|\frac{1}{N_1}\int_{\mathbb{T}\setminus\Lambda} \log\|B_{N_1}(x+N\omega)\|\right. \\
&\quad \left. + \frac{1}{N_1}\sum_{j=2}^{m-1}\int_{\mathbb{T}\setminus\Lambda} \log\|B_N(x+jN\omega)\| - \frac{1}{N_1}\sum_{j=1}^{m-1}\int_{\mathbb{T}\setminus\Lambda} \log\|B_{2N}(x+jN\omega)\|\right| \\
&= \frac{m}{N_1} e^{-\frac{1}{2}NL_N} + (\mathrm{I}).
\end{aligned} \tag{43}$$

With $u_N(x)$ and $w_N(x)$, defined in (13), (14) respectively, and noting that $u_N(x) \le w_N(x)$, we have by (21), for sufficiently large N,

$$\begin{aligned}
(\mathrm{I}) \le &\left|\int_{\mathbb{T}\setminus\Lambda} (u_{N_1}(x+N\omega) - w_{N_1}(x+N\omega))\right. \\
&+ \frac{1}{m}\sum_{j=2}^{m-1}\int_{\mathbb{T}\setminus\Lambda} (u_N(x+jN\omega) - w_N(x+jN\omega)) \\
&\left. - \frac{2}{m}\sum_{j=1}^{m-1}\int_{\mathbb{T}\setminus\Lambda} (u_{2N}(x+jN\omega) - w_{2N}(x+jN\omega))\right| \\
&+ \left|\int_{\mathbb{T}\setminus\Lambda} (w_{N_1}(x+N\omega)) + \frac{1}{m}\sum_{j=2}^{m-1}\int_{\mathbb{T}\setminus\Lambda} (w_N(x+jN\omega))\right. \\
&\left. - \frac{2}{m}\sum_{j=1}^{m-1}\int_{\mathbb{T}\setminus\Lambda} (w_{2N}(x+jN\omega))\right| \\
< &\; e^{-cN^{\delta}} + (\mathrm{II}).
\end{aligned} \tag{44}$$

It remains to bound (II). Recall that by (16) and (1), for all x,

$$\|B_N(x)\| \le C_B^N \tag{45}$$

and, hence, for any N and any x,

$$|w_N(x)| < \max(\log C_B, A) \triangleq C_w$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

and thus, by (39) and our condition on m, we have

$$\begin{aligned}(\mathrm{II}) &< \int_{\mathbb{T}\setminus\Lambda} |w_{N_1}(x+N\omega)| + \frac{1}{m}\sum_{j=2}^{m-1}\int_{\mathbb{T}\setminus\Lambda} |w_N(x+jN\omega)| \\ &\quad + \frac{2}{m}\sum_{j=1}^{m-1}\int_{\mathbb{T}\setminus\Lambda} |w_{2N}(x+jN\omega)| \\ &< 4C_w \mathrm{meas}(\mathbb{T}\setminus\Lambda) \le Ce^{-(c/2)\kappa q}. \end{aligned} \tag{46}$$

Therefore, combining (43), (44), and (46) we have, since $\eta > \delta^{-1}$ (see remark after Lemma 1), after appropriately adjusting a constant,

$$\begin{aligned}\left|L_{N_1} + \frac{m-2}{m}L_N - \frac{2(m-1)}{m}L_{2N}\right| &< \frac{m}{N_1}e^{-\frac{1}{2}NL_N} + e^{-cN^\delta} + Ce^{-(c/2)\kappa q} \\ &< Ce^{-(c/2)\kappa q}.\end{aligned}$$

□

LEMMA 4. *Let N' be such that $m = N'/N$ can be represented as $m = \prod_{i=1}^s a_i$ with $Ke^{c'\kappa q} < a_i < K_1 e^{c'\kappa q}$, $a_i \in \mathbb{N}$, $i = 1 \ldots s$, and $s < e^{(c'/2)\kappa q}$, for some $0 < K, K_1 < \infty$. Then under the hypotheses of Lemma 3, there exists $c'' > 0$ and $\tilde{C} < \infty$ such that*

$$|L_{N'} + L_N - 2L_{2N}| < e^{-c''\kappa q} + \tilde{C}\frac{N}{N'}.$$

Remark. If $m < K_1 e^{c'\kappa q}$ $(s = 1)$, then $c'' = c' = c/2$ with c from Lemma 2, and if $m > Ke^{c'\kappa q}$, then any choice of $c'' < c'/2 = c/4$ with c from Lemma 2 will work.

Proof. The case $s = 1$ follows directly from Lemma 3 since

$$\begin{aligned}|L_{N'} + L_N - 2L_{2N}| &< Ce^{-c'\kappa q} + \frac{2}{m}|L_N - L_{2N}| < Ce^{-c'\kappa q} + \frac{NL_N}{5N'} \\ &< e^{-c'\kappa q} + \tilde{C}\frac{N}{N'}.\end{aligned}$$

For $s > 1$, set $N_j = N\prod_{i=1}^j a_i$. We have $Ke^{c'\kappa q}N \le N_1 \le K_1 e^{c'\kappa q}N$. By Lemma 3,

$$\begin{aligned}|L_{N_1} + L_N - 2L_{2N}| &< Ce^{-c'\kappa q} + \frac{NL_N}{5N_1} \\ &< Ce^{-c'\kappa q} + \frac{L_N}{5K}e^{-c'\kappa q} \\ &\le C_1 e^{-c'\kappa q}.\end{aligned} \tag{47}$$

Similarly we can apply Lemma 3 to $2Ke^{c'\kappa q} \le 2N_1 \le 2K_1 Ne^{c'\kappa q}$ to obtain

$$\begin{aligned}|L_{2N_1} + L_N - 2L_{2N}| &< Ce^{-c'\kappa q} + \frac{NL_N}{10N_1} \\ &\le C_1 e^{-c'\kappa q}.\end{aligned}$$

Hence,

$$|L_{2N_1} - L_{N_1}| < 2C_1 e^{-c'\kappa q}.$$

For q large enough this implies

$$L_{2N_1} > \tfrac{9}{10}L_{N_1}.$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Thus, we can apply Lemma 3 again, this time for N_1 and $Ke^{c'\kappa q}N_1 \le N_2 \le K_1 e^{c'\kappa q}N_1$ (with the same K, K_1 as before):

$$\begin{aligned}|L_{N_2} + L_{N_1} - 2L_{2N_1}| &< Ce^{-c'\kappa q} + \frac{N_1 L_{N_1}}{5N_2} \\ &< Ce^{-c'\kappa q} + \frac{L_{N_1}}{5K}e^{-c'\kappa q} \le C_1 e^{-c'\kappa q}\end{aligned}$$

$$|L_{2N_2} + L_{N_1} - 2L_{2N_1}| \le C_1 e^{-c'\kappa q}$$

$$|L_{2N_2} - L_{N_2}| < 2C_1 e^{-c'\kappa q}.$$

So

$$\begin{aligned}|L_{N_2} - L_{N_1}| &\le |L_{N_2} - L_{2N_2}| + |L_{2N_2} + L_{N_1} - 2L_{2N_1}| + 2|L_{2N_1} - L_{N_1}| \\ &\le 7C_1 e^{-c'\kappa q}.\end{aligned}$$

In general, for $Ke^{c'\kappa q}N_{s-1} \le N_s \le K_1 e^{c'\kappa q}N_{s-1}$:

$$|L_{N_s} + L_{N_{s-1}} - 2L_{2N_{s-1}}| \le C_1 e^{-c'\kappa q}$$

$$|L_{2N_s} - L_{N_s}| < 2C_1 e^{-c'\kappa q}$$

$$|L_{N_s} - L_{N_{s-1}}| \le 7C_1 e^{-c'\kappa q}.$$

Consequently,

$$|L_{N_s} - L_{N_1}| < 7C_1(s-1)e^{-c'\kappa q}. \tag{48}$$

From (47) and (48), we obtain

$$|L_{N_s} + L_N - 2L_{2N}| < 7C_1 s e^{-c'\kappa q}.$$

As $s < e^{(c'/2)\kappa q}$, we have

$$|L_{N_s} + L_N - 2L_{2N}| < 7C_1 e^{-(c'/2)\kappa q}.$$

4. *Induction argument*

LEMMA 5. *Suppose that $|D| < \infty$. Let ω_i, $i = 1, 2$, satisfying (9) with $r(\omega_1) = r(\omega_2)$, have the same approximant a/q, so $|\omega_i - a/q| < 1/q^2$, $(a, q) = 1$. Assume that $0 < \kappa < 1$, $q > C\kappa^{-2}$. Let $L(B^i, \omega_i) > 100\kappa > 0$, $i = 1, 2$. Then there exists $c > 0$, $0 < C_2 < \infty$ and $N_0 < (\kappa^{-C_2}q)^\eta$, with η from Lemma 1, such that*

$$|L(B^i, \omega_i) + L_{N_0}(B^i, \omega_i) - 2L_{2N_0}(B^i, \omega_i)| < e^{-c\kappa q}, \quad i = 1, 2.$$

Remarks.

(1) Note that $L(B^i)$ is defined by (8), as opposed to (7), as it is the Lyapunov exponent of the corresponding $SL_2(\mathbb{C})$ cocycle.

(2) This proposition follows in a similar manner to [**BJ**, Proposition 9], but without the need for an extra lemma since we are working with Diophantine ω. Our dependence on Diophantine conditions is mirrored in the appearance of $\eta = \eta(\omega)$ throughout the proof. We note that the induction argument in [**BJ**] was not correct without an extra condition on the specific form of consecutive scales (see (56)). Our proof contains this important correction.

(3) At several points in the proof, we make calculations that are valid for large initial q_0. We may do this, as we make this claim only finitely many times and, at the end, we take $q_0 \to \infty$ in Theorem 1.

(4) The c in Lemma 5 can be chosen to be any number less than $c/4$ with c from the large deviation bound.

Proof. Take $q_0 = q$ as in Lemma 5. Note that for any n, by subadditivity $L_{2n} \le L_n$. So that, by (45) and (6),

$$100\kappa < L(B, \omega) = \inf L_n \le L_{2n} \le L_n < C_1. \tag{49}$$

Consider the sequence $\{2^l n\}$, $l \in \mathbb{N}$. If for each $1 \le l_1 < l_2 < \cdots < l_j$ we have $L_{2^{l_i+1}n} \le \frac{9}{10} L_{2^{l_i}n}$, then $100\kappa \le L_{2^{l_j+1}n} \le (\frac{9}{10})^j C_1$. Therefore, we must have

$$j < j_0 = \left(\log \frac{9}{10}\right)^{-1} \log \frac{100\kappa}{C_1}. \tag{50}$$

Take $N = [(C\kappa^{-2} q_0)^\eta] + 1$, with C from Lemma 2. Then for some $1 \le j \le 2j_0$, we have $L_{2^{j+1}N}(B^i, \omega_i) > \frac{9}{10} L_{2^j N}(B^i, \omega_i)$, $i = 1, 2$. Therefore, we can find

$$(C\kappa^{-2} q_0)^\eta < N_0 < 2^{2j_0}(C\kappa^{-2} q_0)^\eta \tag{51}$$

with

$$L_{2N_0}(B^i, \omega_i) > \tfrac{9}{10} L_{N_0}(B^i, \omega_i), \quad i = 1, 2. \tag{52}$$

The right-hand side of (51) can be bounded by $(\kappa^{-C_2} q_0)^\eta$ for an appropriate $0 < C_2 < \infty$. From now on we fix $i \in \{1, 2\}$ and set $L_N = L_N(B^i, \omega_i)$, $L = L(B^i, \omega_i)$. □

Define a sequence

$$q_0 < N_0 < q_1 < N_1 < \cdots < N_s < q_{s+1} < N_{s+1} < \cdots$$

such that

$$\left|\omega - \frac{a_i}{q_i}\right| < \frac{1}{q_i^2}, \quad (a_i, q_i) = 1 \tag{53}$$

$$q_{s+1} \text{ is the smallest } q_j \text{ satisfying (53) such that } q_j > e^{q_s} \tag{54}$$

$$(C\kappa^{-2} q_s)^\eta < N_s < (\kappa^{-C_2} q_s)^{2\eta} \tag{55}$$

$$N_s = N_{s-1} \prod_{i=1}^{k_s} a_i^s \quad \text{with } K e^{c'\kappa q_{s-1}} \le a_i^s \le K_1 e^{c'\kappa q_{s-1}}. \tag{56}$$

Clearly we can select our sequence so that (53)–(55) hold. For sufficiently large q_0, we can split the set of N satisfying (56) into a disjoint union of $A_p \subset \mathbb{Z}$ so that for $N \in A_p$, $N = N_{s-1} \prod_{i=1}^{p} a_i^s$ with $K e^{c'\kappa q_{s-1}} < a_i^s < K_1 e^{c'\kappa q_{s-1}}$. If N_s satisfying both (55) and (56) cannot be found, then for some p, we have

$$\max A_p = N_{s-1} K_1^p e^{c' p\kappa q_{s-1}} < (C\kappa^{-2} q_s)^\eta,$$
$$\min A_{p+1} = N_{s-1} K^{p+1} e^{c'(p+1)\kappa q_{s-1}} > (\kappa^{-C_2} q_s)^{2\eta}.$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

This implies that

$$(\kappa^{-C_2} q_s)^{2\eta} < \frac{K^{p+1}}{K_1^p} e^{c'\kappa q_{s-1}} (C\kappa^{-2} q_s)^\eta,$$

which is in contradiction to (54). Thus, such a sequence can be selected. Let c'' be the constant from Lemma 4. Fix a constant C such that $2C_1 < C < \infty$ where C_1 is defined in (49). Fix another three constants $0 < c_3 < c_2 < c_1 < c''$.

Set $q_{-1} = 0$. We use induction to show that for sufficiently large q_0 the sequence additionally satisfies, for $s \geq 0$,

$$|L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| < e^{-c_1 \kappa q_s}, \tag{57}$$

$$|L_{2N_s} - L_{N_s}| < C e^{-c_2 \kappa q_{s-1}}, \tag{58}$$

$$|L_{N_{s+1}} - L_{N_s}| < C e^{-c_3 \kappa q_{s-1}}. \tag{59}$$

We first check the case $s = 0$. Let q_1 be the first approximant satisfying (54). Since we consider ω Diophantine, for $M \gtrsim r(\omega)$ with $r(\omega)$ from (9), we have

$$q_1 < e^{M q_0}. \tag{60}$$

Fix N_1 satisfying (55)–(56). We want to show

$$|L_{N_1} + L_{N_0} - 2L_{2N_0}| < e^{-c_1 \kappa q_0}, \tag{61}$$

$$|L_{2N_0} - L_{N_0}| < C e^{-c_2 \kappa q_{-1}} = C, \tag{62}$$

$$|L_{N_1} - L_{N_0}| < C e^{-c_3 \kappa q_{-1}} = C. \tag{63}$$

Clearly the second and third inequalities are true for any $C > 2C_1$, so it remains to show (61). By (52), (60), and (55), (56) with $s = 1$, we have that for sufficiently large q_0, the conditions of Lemma 4 with $N' = N_1$ and $N = N_0$ are satisfied. Therefore, by Lemma 4, (55) and (54), for large q_0, we have

$$|L_{N_1} + L_{N_0} - 2L_{2N_0}| < e^{-c''\kappa q_0} + \tilde{C} \frac{N_0}{N_1} < e^{-c''\kappa q_0} + C_3 \kappa^{-C_4} q_0^{2\eta} e^{-\eta q_0} < e^{-c_1 \kappa q_0}.$$

Thus, we have verified the initial case.

Now assume that we have (57)–(59) for

$$q_0 < N_0 < q_1 < N_1 < \cdots < N_s < q_s < N_s.$$

Let q_{s+1} be the smallest approximant of ω satisfying (54). Pick N_{s+1} satisfying (55)–(56).

By inductive assumption we have that $|L_{2N_s} - L_{N_s}| < C e^{-c_2 \kappa q_{s-1}}$. For large enough q_0 this implies $L_{2N_s} > \frac{9}{10} L_{N_s}$. The other conditions of Lemma 4 follow from the construction and (60).

Thus, by Lemma 4, with $N' = N_{s+1}$ and $N = N_s$,

$$\begin{aligned} |L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| &< e^{-c''\kappa q_s} + \tilde{C} \frac{N_s}{N_{s+1}} \\ &< e^{-c''\kappa q_s} + C_3 \kappa^{-C_4} q_s^{2\eta} e^{-\eta q_s} < e^{-c_1 \kappa q_s}. \end{aligned} \tag{64}$$

Thus, (57) holds for q_{s+1}, N_{s+1}. By the same argument

$$|L_{2N_{s+1}} + L_{N_s} - 2L_{2N_s}| < e^{-c''\kappa q_s} + \tilde{C} \frac{N_s}{2N_{s+1}} < e^{-c_1 \kappa q_s}.$$

Hence, we obtain

$$|L_{2N_{s+1}} - L_{N_{s+1}}| < 2e^{-c_1\kappa q_s} < e^{-c_2\kappa q_s}.$$

Also by (64) and (58)

$$\begin{aligned}|L_{N_{s+1}} - L_{N_s}| &< |L_{N_{s+1}} + L_{N_s} - 2L_{2N_s}| + |2L_{2N_s} - 2L_{N_s}| \\ &< e^{-c_1\kappa q_s} + 2Ce^{-c_2\kappa q_{s-1}} < e^{-c_3\kappa q_{s-1}}.\end{aligned}$$

Therefore, (57)–(59) hold by induction. Thus, by (57) and (59), for $c_4 < c_3$ and for sufficiently large q_0,

$$\begin{aligned}|L + L_{N_0} - 2L_{2N_0}| &= \left|\lim_{s\to\infty} L_{N_{s+1}} + L_{N_0} - 2L_{2N_0}\right| \\ &= \left|\sum_{s\geq 1}(L_{N_{s+1}} - L_{N_s}) + L_{N_1} + L_{N_0} - 2L_{2N_0}\right| \\ &\leq |L_{N_1} + L_{N_0} - 2L_{2N_0}| + \sum_{s\geq 1}|L_{N_{s+1}} - L_{N_s}| \\ &< e^{-c_1\kappa q_0} + \sum_{s\geq 1} e^{-c_3\kappa q_{s-1}} < e^{-c_4\kappa q_0}.\end{aligned}$$

□

Note. The only reason we needed the condition $L(B^i, \omega_i) > 100\kappa$ was to establish (50) and therefore (51) and (52). Equations (50)–(52) would follow equally well under the assumption $L_N(B^i, \omega_i) > 100\kappa$ for some $N > (\kappa^{-C_2}q_0)^{2\eta}$ of the form

$$N = 2^k n, \quad k > 2j_0 \quad \text{and} \quad n > (C\kappa^{-2}q_0)^\eta. \tag{65}$$

Hence, after rescaling κ, Lemma 5 can be reformulated as follows.

LEMMA 5'. *Suppose that $|D| < \infty$. Let ω_i, $i = 1, 2$, satisfying (9) with $r(\omega_1) = r(\omega_2)$, have the same approximant a/q, so $|\omega_i - a/q| < 1/q^2$, $(a, q) = 1$. Assume that $0 < \kappa < 1$, $q > C\kappa^{-2}$, and $L_N(B^i, \omega_i) > \kappa > 0$, $i = 1, 2$, for some $N > (\kappa^{-C_2}q)^\eta$ of the form (65) with $\eta = \eta(\omega) > 1$ from Lemma 1. Then there exists $0 < C_2 < \infty$, $(C\kappa^{-2}q_0)^\eta < N_0 < (\kappa^{-C_2}q)^\eta$ and $c_4 > 0$ such that*

$$|L(B^i, \omega_i) + L_{N_0}(B^i, \omega_i) - 2L_{2N_0}(B^i, \omega_i)| < e^{-c_4\kappa q}, \quad i = 1, 2. \tag{66}$$

We are ready to prove our main theorem.

5. *Proof of Theorem 1*

The strategy of the proof is to use Lemma 5' in order to approximate L with L_N and combine it with continuity of L_N. The quantitative bounds required for the argument are not as straightforward as in the Schrödinger case as $\|M_N\|$ are unbounded from above. We address this issue by converting back to the $\|B_N\|$ and using a truncation argument similar to that used in Lemmas 1 and 3 in order to control the regime when $\|B_N\|$ is small. Namely we need a continuity property of w_N as defined in (14).

PROPOSITION E. *There exists $0 < C < \infty$ such that*

$$|w_{N_0}(B, x) - w_{N_0}(B^\alpha, x)| < C^{N_0} \operatorname{dist}(B, B^\alpha). \tag{67}$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Proof. We assume without loss of generality that $\operatorname{dist}(B, B^\alpha) < 1$. There exists $C_B < \infty$ such that whenever $\operatorname{dist}(B, B^\alpha) < 1$,

$$\begin{aligned} \|B^\alpha(S^j x)\| &< C_B, \\ \|B^\alpha(S^j x)^{-1}\| &\le \frac{1}{|d(S^j x)|} C_B. \end{aligned} \tag{68}$$

Set

$$\begin{aligned} X_1 &= \{x : \|B_{N_0}(x)\| < e^{-N_0 A};\ \|B^\alpha_{N_0}(x)\| < e^{-N_0 A}\}, \\ X_2 &= \{x : \|B_{N_0}(x)\| \ge e^{-N_0 A};\ \|B^\alpha_{N_0}(x)\| \ge e^{-N_0 A}\}, \\ X_3 &= \{x : \|B_{N_0}(x)\| \ge e^{-N_0 A};\ \|B^\alpha_{N_0}(x)\| < e^{-N_0 A}\}, \\ X_4 &= \{x : \|B_{N_0}(x)\| < e^{-N_0 A};\ \|B^\alpha_{N_0}(x)\| \ge e^{-N_0 A}\}. \end{aligned}$$

For $x \in X_1$, we have $w_{N_0}(B, x) = w_{N_0}(B^\alpha, x) = -A$ so (67) holds.

For the other three cases we use

$$|\|B^\alpha_N(x)\| - \|B_N(x)\|| < \|B^\alpha_N(x) - B_N(x)\| < N C_B^N \operatorname{dist}(B, B^\alpha), \tag{69}$$

which can be easily verified by the Trotter product formula and (1), (68). Consider $x \in X_2$. Without loss of generality, assume that $\|B^\alpha_{N_0}(x)\| < \|B_{N_0}(x)\|$. Then we have by (69)

$$\begin{aligned} |w_{N_0}(B, x) - w_{N_0}(B^\alpha, x)| &= \left| \frac{1}{N_0} \log \|B_{N_0}(x)\| - \frac{1}{N_0} \log \|B^\alpha_{N_0}(x)\| \right| \\ &= \left| \frac{1}{N_0} \log\left(1 + \frac{\|B_{N_0}(x)\| - \|B^\alpha_{N_0}(x)\|}{\|B^\alpha_{N_0}(x)\|}\right)\right| \\ &< \frac{1}{N_0} \left| \frac{\|B_{N_0}(x)\| - \|B^\alpha_{N_0}(x)\|}{\|B^\alpha_{N_0}(x)\|} \right| \\ &< \frac{1}{N_0} e^{N_0 A} N_0 C_B^{N_0} \operatorname{dist}(B, B^\alpha) \\ &= (e^A C_B)^{N_0} \operatorname{dist}(B, B^\alpha). \end{aligned}$$

For $x \in X_3$, noting that $\|B^\alpha_{N_0}(x)\| < e^{-N_0 A} \le \|B_{N_0}(x)\|$ we have by a similar argument

$$\begin{aligned} |w_{N_0}(B, x) - w_{N_0}(B^\alpha, x)| &= \left| \frac{1}{N_0} \log\left(1 + \frac{\|B_{N_0}(x)\| - e^{-N_0 A}}{e^{-N_0 A}}\right)\right| \\ &< \frac{1}{N_0} e^{N_0 A} |\|B_{N_0}(x)\| - \|B^\alpha_{N_0}(x)\|| \\ &< (e^A C_B)^{N_0} \operatorname{dist}(B, B^\alpha). \end{aligned}$$

Case X_4 follows exactly as X_3. □

Remark. For B^α with $\operatorname{dist}(B, B^\alpha) < 1$, C from Proposition E can be taken to be $e^A C_B$ with C_B from (68), and A from (14).

Proof of Theorem 1. Fix ω satisfying (9). Suppose that $B^\alpha \to B$. Assume first that $L(B, \omega) > \kappa > 0$. Recall that $L(B, \omega)$, $L_N(B, \omega)$ are defined by (8), (4), respectively.

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Let $q > C\kappa^{-2}$ be an approximant of ω, that is $|\omega - a/q| < 1/q^2$, $(a, q) = 1$. Let $N > (\kappa^{-C_2}q)^\eta$ with $\eta = \eta(\omega)$ from Lemma 2 and C_2 from Lemma 5'. We have

$$L_N(B, \omega) > L(B, \omega) > \kappa > 0$$

and since it can easily be shown using (69) and (6) that L_N is continuous in B, we have

$$L_N(B^\alpha, \omega) > \kappa \quad \text{for } \alpha > \alpha_0.$$

As we can assume that $\kappa < \frac{1}{100}$, Lemma 5' applies and we obtain that there exists

$$(C\kappa^{-2}q)^\eta < N_0 < (\kappa^{-C_2}q)^\eta$$

such that

$$|L(B^i, \omega) + L_{N_0}(B^i, \omega) - 2L_{2N_0}(B^i, \omega)| < e^{-c_4\kappa q} \quad \text{for } B^i = B, B^\alpha. \tag{70}$$

Recall that $L'_N = \langle u_N \rangle$ with u_N defined in (13). Recall also that $\eta > 1/\delta$. Then by (5), (70), (6), Propositions B and E, for sufficiently large q we obtain

$$\begin{aligned}
&|L(B) - L(B^\alpha)| \\
&\quad \le |L(B) + L_{N_0}(B) - 2L_{2N_0}(B)| + |L_{N_0}(B) - L_{N_0}(B^\alpha)| \\
&\qquad + |L(B^\alpha) + L_{N_0}(B^\alpha) - 2L_{2N_0}(B^\alpha)| + 2|L_{2N_0}(B) - L_{2N_0}(B^\alpha)| \\
&\quad \le 2e^{-c_4\kappa q} + |L'_{N_0}(B) - L'_{N_0}(B^\alpha)| + 2|L'_{2N_0}(B) - L'_{2N_0}(B^\alpha)| \\
&\quad \le 2e^{-c_4\kappa q} + |\langle u_{N_0}(B)\rangle - \langle w_{N_0}(B)\rangle| + |\langle u_{N_0}(B^\alpha)\rangle - \langle w_{N_0}(B^\alpha)\rangle| \\
&\qquad + 2|\langle u_{2N_0}(B)\rangle - \langle w_{2N_0}(B)\rangle| + 2|\langle u_{2N_0}(B^\alpha)\rangle - \langle w_{2N_0}(B^\alpha)\rangle| \\
&\qquad + |\langle w_{N_0}(B)\rangle - \langle w_{N_0}(B^\alpha)\rangle| + 2|\langle w_{2N_0}(B)\rangle - \langle w_{2N_0}(B^\alpha)\rangle| \\
&\quad \le 2e^{-c_4\kappa q} + 2e^{-cN_0^\delta} + |\langle w_{N_0}(B)\rangle - \langle w_{N_0}(B^\alpha)\rangle| \\
&\qquad + 4e^{-c(2N_0)^\delta} + 2|\langle w_{2N_0}(B)\rangle - \langle w_{2N_0}(B^\alpha)\rangle| \\
&\quad \le 3e^{-c_4\kappa q} + 3C^{2N_0}\,\mathrm{dist}(B, B^\alpha) < 3e^{-c_4\kappa q} + C(\kappa)^{q^\eta}\,\mathrm{dist}(B, B^\alpha).
\end{aligned} \tag{71}$$

Thus, we have $\limsup_\alpha |L(B, \omega) - L(B^\alpha, \omega)| \le 3e^{-c_4\kappa q}$. Letting $q \to \infty$, we obtain the desired continuity for the case when $L(B) > 0$.

Assume that $L(B) = 0$. Let $B^\alpha \to B$. Assume that L is not continuous at B. For κ small enough, for any $\delta > 0$, $|L(B^\alpha)| > 2\kappa$ for infinitely many α with $\mathrm{dist}(B^\alpha, B) < \delta$. For q as in Lemma 5', let $\overline{N}$ be the minimum $N > (\kappa^{-C_2}q)^\eta$ satisfying (65). Since L_N is continuous for each N, we can find δ_1 such that $\mathrm{dist}(B^\alpha, B) < \delta_1$ implies

$$|L_{\overline{N}}(B) - L_{\overline{N}}(B^\alpha)| < \kappa. \tag{72}$$

Let $\delta_0 = \min(C(\kappa)^{-q^\eta} e^{-c_4\kappa q}, \delta_1)$. Pick B^α with $\mathrm{dist}(B^\alpha, B) < \delta_0$, $L(B^\alpha) > 2\kappa$. Then by (72), $L_{\overline{N}}(B) > \kappa$. Thus, we can apply Lemma 5' to $B^1 = B$ and $B^2 = B^\alpha$, $\omega_{1,2} = \omega$. Let N_0 be as given in Lemma 5'. We then have (66). Thus, the argument used to arrive at (71) applies and by (71) we have

$$2\kappa < |L(B^\alpha) - L(B)| < 4e^{-c_4\kappa q} \tag{73}$$

a contradiction for large q. Hence, $L(B)$ is continuous. □

Fix $r > 1$. We define a Diophantine condition, call it $DC(r)$, to be ω such that there exists $b(\omega) > 0$ such that for all $j \neq 0$,

$$|\sin 2\pi j\omega| > \frac{b(\omega)}{|j|^r}.$$

It can be shown that $DC(r)$ for $r > 1$ is a full measure set. Our analysis also leads to the following.

THEOREM 2. *Under the assumptions of Theorem 1, we have:*
(1) $L'(B, \omega) : C_\rho(\mathbb{T}, M_2^{d(x)}) \times DC(r) \to \mathbb{R}$ *is jointly continuous;*
(2) $L(B, \omega) : C_\rho(\mathbb{T}, M_2^{d(x)}) \times DC(r) \to \mathbb{R}$ *is jointly continuous.*

Proof. Assume that $(B^\alpha, \omega_\alpha) \to (B, \omega)$ in $C_\rho(\mathbb{T}, M_2^{d(x)}) \times DC(r)$. As before assume first that $L(B, \omega) > \kappa > 0$. Let $q > C\kappa^{-2}$ be an approximant of ω, hence

$$\left|\omega - \frac{a}{q}\right| < \frac{1}{q^2}.$$

Let $N > (\kappa^{-C} q)^\eta$. The value of $\eta(\omega)$ in Lemma 1, only depends on the $r(\omega)$ from (9) (see the remark after Lemma 1). As we have a fixed value $r > 1$ for the class $DC(r)$, we see that all of our estimates will hold for both ω and ω_α. Therefore, $L_N(B^\alpha, \omega_\alpha) > \kappa$ and $|\omega_\alpha - a/q| < 1/q^2$ for $\alpha > \alpha_0$. Fix an $\alpha > \alpha_0$. As above, by Lemma 5', we can find an $N_0 < (\kappa^{-C_2} q)^\eta$ such that

$$|L(B, \omega) + L_{N_0}(B, \omega) - 2L_{2N_0}(B, \omega)| < e^{-c_4\kappa q}$$

and

$$|L(B^\alpha, \omega_\alpha) + L_{N_0}(B^\alpha, \omega_\alpha) - 2L_{2N_0}(B^\alpha, \omega_\alpha)| < e^{-c_4\kappa q}.$$

Then we note that the proof of Theorem 1 holds with only minor changes. Let $Sx = x + \omega$ as before, and let $S_\alpha x = x + \omega_\alpha$. It is easy to see that

$$\|B^\alpha(S_\alpha^j x) - B(S^j x)\| < \mathrm{dist}(B, B^\alpha) + Cj|\omega - \omega_\alpha|,$$

where C depends on the Lipshitz constant for the analytic function B and is uniform in a neighborhood of B. Thus, (69) can be replaced by

$$|\|B_N^\alpha(x)\| - \|B_N^\alpha(x)\|| < N^2 C^N(|\omega - \omega_\alpha| + \mathrm{dist}(B, B^\alpha)).$$

Therefore, instead of (71), we obtain

$$|L(B, \omega) - L(B^\alpha, \omega_\alpha)| < C(\kappa)^{q^\eta}(|\,\omega - \omega_\alpha| + \mathrm{dist}(B, B^\alpha)) + 3e^{-c_4\kappa q}. \tag{74}$$

Hence, $\limsup_\alpha |L(B, \omega) - L(B^\alpha, \omega_\alpha)| < 3e^{-c_4\kappa q}$. Letting $q \to \infty$, the result follows for the case $L > 0$.

Assume now that $L(B, \omega) = 0$. Let $(B^\alpha, \omega_\alpha) \to (B, \omega)$. Assume that L is not continuous at (B, ω). For κ small enough, for any $\delta > 0$, $|L(B^\alpha, \omega_\alpha)| > 2\kappa$ for infinitely many α with $\mathrm{dist}(B^\alpha, B) + |\omega_\alpha - \omega| < \delta$. For a/q as in Lemma 5', let $\overline{N}$ be the minimum $N > (\kappa^{-C_2} q)^\eta$ satisfying (65). Since L_N is continuous for each N, we can find δ_1 such that $\mathrm{dist}(B^\alpha, B) + |\omega_\alpha - \omega| < \delta_1$ implies

$$|L_{\overline{N}}(B, \omega) - L_{\overline{N}}(B^\alpha, \omega_\alpha)| < \kappa. \tag{75}$$

https://doi.org/10.1017/S0143385709000704 Published online by Cambridge University Press

Pick δ_2 so that $|\omega_\alpha - \omega| < \delta_2$ implies $|a/q - \omega_\alpha| < 1/q^2$. Let $\delta_0 = \min(C(\kappa)^{-q^\eta} e^{-c_4 \kappa q}, \delta_1, \delta_2)$. Pick $(B^\alpha, \omega_\alpha)$ with $\text{dist}(B^\alpha, B) + |\omega - \omega_\alpha| < \delta_0$, $L(B^\alpha, \omega_\alpha) > 2\kappa$. Then by (75), $L_{\overline{N}}(B, \omega) > \kappa$. Thus, we can apply Lemma 5' to $B^1 = B,\ B^2 = B^\alpha,\ \omega_1 = \omega,\ \omega_2 = \omega_\alpha$. Let N_0 be as given in Lemma 5'. Thus, we have (66) and the argument used to arrive at (74) applies and by (74) we have

$$2\kappa < |L(B^\alpha, \omega_\alpha) - L(B, \omega)| < 4e^{-c_4 \kappa q} \tag{76}$$

a contradiction for large q. Hence, $L(B, \omega)$ is continuous. Also $L'(B, \omega)$ is then continuous by (6). □

Acknowledgement. The authors were supported in part by NSF grants DMS-0200974, DMS-0601081 and Grant No. 2002068 from the United States–Israel Binational Science Foundation (BSF), Jerusalem, Israel.

REFERENCES

[AJ] A. Avila and S. Jitomirskaya. The ten Martini problem. *Ann. of Math.* (2) **170**(1) (2009), 303–342.

[Bo] J. Bochi. Discontinuity of the Lyapunov exponent for non-hyperbolic cocycles. *Preprint*.

[Bo1] J. Bochi. Genericity of zero Lyapunov exponents. *Ergod. Th. & Dynam. Sys.* **22** (2002), 1667–1696.

[BV] J. Bochi and M. Viana. The Lyapunov exponents of generic volume preserving and symplectic maps. *Ann. of Math.* (2) **161**(3) (2005), 1423–1485.

[B1] J. Bourgain. *Green's Function Estimates for Lattice Schrödinger Operators and Applications (Annals of Mathematics Studies, 158)*. Princeton University Press, Princeton, NJ, 2005, p. 173.

[B2] J. Bourgain. Positivity and continuity of the Lyapunov exponent for shifts on $\mathbb{T}^d$ with arbitrary frequency vector and real analytic potential. *J. Anal. Math.* **96** (2005), 313–355.

[BG] J. Bourgain and M. Goldstein. On nonperturbative localization with quasi-periodic potential. *Ann. of Math.* (2) **152** (2000), 835–879.

[BGS] J. Bourgain, M. Goldstein and W. Schlag. Anderson localization for Schrödinger operators on $\mathbb{Z}$ with potentials given by the skew-shift. *Comm. Math. Phys.* **220**(3) (2001), 583–621.

[BJ] J. Bourgain and S. Jitomirskaya. Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential. *J. Stat. Phys.* **108**(5/6) (2002), 1028–1218.

[DL] D. Damanik and D. Lenz. A condition of Boshernitzan and uniform convergence in the multiplicative ergodic theorem. *Duke Math. J.* **133**(1) (2006), 95–123.

[F] A. Furman. On the multiplicative ergodic theorem for uniquely ergodic systems. *Ann. Inst. H. Poincaré* **33** (1997), 797–815.

[GJ] F. Germinet and S. Jitomirskaya. Strong dynamical localization for the almost Mathieu model. *Rev. Math. Phys.* **13** (2001), 755–765.

[GS] M. Goldshtein and W. Schlag. Hölder continuity of the integrated density of states for quasi-periodic Schrödinger equations and averages of shifts of subharmonic functions. *Ann. of Math.* (2) **154** (2001), 155–203.

[J] S. Jitomirskaya. Metal-insulator transition for the almost Mathieu operator. *Ann. of Math.* (2) **150** (1999), 1159–1175.

[JKS] S. Jitomirskaya, D. A. Koslover and M. Schulteis. Localization for a family of one-dimensional quasiperiodic operators of magnetic origin. *Ann. Inst. H. Poincaré* **6** (2005), 103–121.

[K] S. Klein. Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function. *J. Funct. Anal.* **218** (2005), 255–292.

[Kn] O. Knill. The upper Lyapunov exponent of SL(2, $\mathbb{R}$) cocycles: discontinuity and the problem of positivity. *Lyapunov Exponents (Oberwolfach, 1990) (Lecture Notes in Mathematics, 1486)*. Springer, Berlin, 1991, pp. 86–97.

[L] Y. Last. Spectral theory of Sturm–Louiville operators on infinite intervals: a review of recent developments. *Sturm–Louiville Theory*. Birkhäuser, Basel, 2005, pp. 99–120.

[Lo] S. Lojasiewicz. Sur le probléme de la division. *Studia Math.* **18** (1959), 87–136.

[S] B. Simon. *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory (AMS Colloquium Publications)*. American Mathematical Society, Providence, RI, 2005.

[T] D. Thouless. Bandwidth for a quasiperiodic tight-binding model. *Phys. Rev.* **28** (1983), 4272–4276.

[Th] J.-P. Thouvenot. An example of discontinuity in the computation of the Lyapunov exponents. *Proc. Steklov Inst. Math.* **216** (1997), 366–369.

[Y] L.-S. Young. Some open sets of non-uniformly hyperbolic cocycles. *Ergod. Th. & Dynam. Sys.* **13** (1993), 409–415.