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INTEGRAL GROUP RINGS OF SOME p-GROUPS

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1. Introduction. The group of units, $\mathcal{U}ZG$, of the integral group ring of a finite non-abelian group G is difficult to determine. For the symmetric group of order 6 and the dihedral group of order 8 this was done by Hughes-Pearson [**3**] and Polcino Milies [**5**] respectively. Allen and Hobby [**1**] have computed $\mathcal{U}\mathbf{Z}A_4$, where A_4 is the alternating group on 4 letters. Recently, Passman-Smith [**6**] gave a nice characterization of $\mathcal{U}\mathbf{Z}D_{2p}$ where D_{2p} is the dihedral group of order $2p$ and p is an odd prime. In an earlier paper [**2**] Galovich–Reiner–Ullom computed $\mathcal{U}\mathbf{Z}G$ when G is a metacyclic group of order pq with p a prime and q a divisor of $(p - 1)$. In this note, using the fibre product decomposition as in [**2**], we give a description of the units of the integral group rings of the two noncommutative groups of order p^3, p an odd prime. In fact, for these groups we describe the components of $\mathbf{Z}G$ in the Wedderburn decomposition of $\mathbf{Q}G$. The unit description is perhaps a little unsatisfying due to the difficulty in computing the units of commutative integral group rings. This difficulty does not arise if one considers the p-adic group ring $\mathbf{Z}_pG$, $|G| = p^3$, $\mathbf{Z}_p =$ the p-adic integers. Also, if $|G| = 27$, the commutative group involved is of exponent 3 and its integral group ring has only trivial units; and we can describe $\mathcal{U}\mathbf{Z}G$ as a group of 3×3 matrices over $\mathbf{Z}[\omega]$, $\omega^3 = 1$.

One of the groups of order p^3 has a normal cyclic subgroup of order p^2. We consider in Section 2 a group of order p^n having a normal cyclic group of index p and specialize to the case $n = 3$ in Section 3. The methods of this note can also handle extraspecial p-groups of order p^{2d+1} (see [**4**], p. 353) giving rise to matrices of size $p^d \times p^d$.

We are indebted to Ian Musson and the referee for improvements in this paper.

2. A group of order p^n. We consider the following group of order p^n:

$$H = \langle a, b \mid a^{p^{n-1}} = 1 = b^p, b^{-1}ab = a^{p^{n-2}+1} \rangle.$$

We need an easy fibre product diagram of rings. Let I and J be two

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ideals of a ring R such that $I \cap J = 0$. Then

$$\begin{array}{ccc} R & \longrightarrow & R/J \\ \downarrow & - & \downarrow \sim \\ R/I & \longrightarrow & R/(I+J) \end{array}$$

is a fibre product, in the sense that

$$R \simeq \{(\alpha, \beta) | \alpha \in R/I, \beta \in R/J, \bar{\alpha} = \tilde{\beta}\}.$$

This induces the fibre product of unit groups:

$$\begin{array}{ccc} \mathcal{U}(R) & \longrightarrow & \mathcal{U}(R/J) \\ \downarrow & & \downarrow \\ \mathcal{U}(R/I) & \longrightarrow & \mathcal{U}(R/I+J) \end{array}$$

This is to be applied to the group ring $\mathbf{Z}X$ with $J = \Delta(X, N)$ as the kernel of the natural homomorphism $\mathbf{Z}X \to \mathbf{Z}X/N$ with $N \lhd X$ and $I = \hat{N}\mathbf{Z}G$ where $\hat{N} = \sum_{x \in N} x$. We shall write $\hat{x}$ for $\langle \hat{x} \rangle$.

We shall need to number the entries of certain matrices by their pseudo-diagonals. Let us describe the n diagonals of the $n \times n$ matrix $A = [a_{ij}]$ as follows

0th diagonal: $a_{1,1}, a_{2,2}, \ldots, a_{n,n}$

1st diagonal: $a_{1,2}, a_{2,3}, \ldots, a_{n-1,n}, a_{n,1}$

2nd diagonal: $a_{1,3}, a_{2,4}, \ldots, a_{n-2,n}, a_{n-1,1}, a_{n,2}$

$\vdots \qquad\qquad \vdots$

$(n-1)$th diagonal: $a_{1,n}, a_{2,1}, \ldots, a_{n-1,n-2}, a_{n,n-1}$.

We shall have to number some matrices as $[x_{ij}]$ where x_{ij} is in the ith diagonal at the jth spot in the above numbering, $0 \leqq i, j \leqq n - 1$. Let ω be a primitive pth root of unity throughout this note.

PROPOSITION 1. *Suppose* $x_0, \ldots, x_{p-1} \in \mathbf{Z}[\xi]$ *are given with* $\xi^{p^{l-1}} = \omega$. *Then there exist* $t_i \in \mathbf{Z}[\xi]$ *satisfying*

$$\sum_{i=0}^{p-1} t_i \omega^{ji} = x_j, 0 \leqq j \leqq p - 1.$$

if and only if

$$\sum_{i=0}^{p-1} x_i \omega^{ki} \in p\mathbf{Z}[\xi] \quad \textit{for all } 0 \leqq k \leqq p - 1.$$

Proof. The given system of equations is

$$W \begin{bmatrix} t_0 \\ \cdot \\ \cdot \\ \cdot \\ t_{p-1} \end{bmatrix} = \begin{bmatrix} x_0 \\ \cdot \\ \cdot \\ \cdot \\ x_{p-1} \end{bmatrix} \quad \text{where } W = [\omega^{ij}], 0 \leqq i, j < p.$$

Since W is a character matrix, it follows by the orthogonality relations that

$$W^{-1} = \frac{1}{p}[\omega^{-ij}].$$

The system is equivalent to

$$\begin{bmatrix} t_0 \\ t_1 \\ \cdot \\ \cdot \\ \cdot \\ t_{p-1} \end{bmatrix} = W^{-1} \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{p-1} \end{bmatrix}.$$

Thus there is a solution $t_0, \ldots, p - 1$ if and only if

$$\frac{1}{p}\sum_{i=0}^{p-1} \omega^{-ik} x_i \in \mathbf{Z}[\xi]$$

for all $0 \leqq k \leqq p - 1$. This is equivalent to

$$\sum_{i=0}^{p-1} \omega^{ik} x_i \in p\mathbf{Z}[\xi], 0 \leqq k \leqq p - 1.$$

PROPOSITION 2. *Let A and B be $p \times p$ matrices over $\mathbf{Q}(\xi)$, $\xi^{p^{l-1}} = \omega$, given by*

$$B = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & \omega^{p-1} \end{bmatrix}, A = \begin{bmatrix} 0 & 1 & 0 & 0 \ldots 0 \\ 0 & 0 & 1 & 0 \ldots 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & 0 & 0 & \ldots 0 \end{bmatrix}.$$

The $\mathbf{Z}[\xi]$-span of the matrices $\{B^i A^j, 0 \leqq i, j \leqq p - 1\}$ consists of all

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$p \times p$ matrices over $\mathbf{Z}[\xi]$ of the form

$$M = \begin{bmatrix} x_{0,0} & x_{1,0} & \dots & x_{p-1,0} \\ x_{p-1,1} & x_{0,1} & \dots & x_{p-2,1} \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ x_{1,p-1} & x_{2,p-1} & \dots & x_{0,p-1} \end{bmatrix}$$

such that for each j and k, $0 \leqq j, k < p$,

$$(*) \quad \sum_{i=0}^{p-1} x_{ji}\omega^{ki} \in p\mathbf{Z}[\xi].$$

Proof. For a fixed j, the matrices B^iA^j, $0 \leqq i \leqq p - 1$, have non-zero entries only in the jth diagonal. The $\mathbf{Z}[\xi]$-vector $(x_0, x_1, \dots, x_{p-1})$ is a diagonal in the span of $\{B^iA^j\}$ if and only if there exist $t_i \in \mathbf{Z}[\xi]$ such that

$$\sum_{0}^{p-1} t_iB^i = \begin{bmatrix} x_0 & & & & \\ & x_1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & & x_{p-1} \end{bmatrix}.$$

This means that

$$\sum_i t_i\omega^{ji} = x_j, 0 \leqq j \leqq p - 1.$$

Applying the last proposition to each diagonal we get our result.

The next proposition is well known.

PROPOSITION 3. *Let $o_1 \subseteq o_2$ be $\mathbf{Z}$-orders in a rational algebra. If an element $\alpha \in o_1$ has an inverse in o_2 then it is a unit of o_1 already.*

Proof. We have for the indices of additive groups

$$(o_2 : \alpha o_1) = (\alpha o_2 : \alpha o_1) \leqq (o_2 : o_1),$$

which implies $\alpha o_1 = o_1$ and the result follows.

Now, we study our group of order p^n,

$$H = \langle a, b | a^{p^{n-1}} = 1 = b^p, b^{-1}ab = a^{p^{n-2}+1} \rangle.$$

Writing $a^{-1}b^{-1}ab = a^{p^{n-2}} = c$ we have $H' = \langle c \rangle$ of order p. Thus

$$\bar{H} = H/\langle c \rangle = \langle \bar{a} \rangle \times \langle \bar{b} \rangle.$$

Let λ be a primitive p^{n-2}th root of unity. Then

$$\mathbf{Q}H = \mathbf{Q}\bar{H} \oplus \mathbf{Q}(\lambda)_{p \times p}.$$

In fact,

$$\mathbf{Q}\bar{H} \simeq \mathbf{Q}H/\Delta(H, \langle c \rangle) \simeq \mathbf{Q}H\hat{c}, \mathbf{Q}(\lambda)_{p\times p} \simeq \mathbf{Q}H/\hat{c}\mathbf{Q}H.$$

Clearly,

$$\begin{array}{l} ZH \to ZH/\Delta(H, \langle c \rangle) \oplus \mathbf{Z}[\lambda]_{pxp} \\ \qquad\quad \wr\! \| \\ \qquad\quad \mathbf{Z}\bar{H} \end{array}$$

with the projection onto the first component. We shall compute the projection in the second component. It is easily checked that

$$\hat{c}\mathbf{Z}H + (1 - c)\mathbf{Z}H = p\mathbf{Z}H + (1 - c)\mathbf{Z}H$$

and

$$\hat{c}\mathbf{Z}H \cap (1 - c)\mathbf{Z}H = 0.$$

Thus we have the fibre product

$$\begin{array}{ccc} \mathbf{Z}H & \xrightarrow[\theta_3]{\bmod \langle c \rangle} & \mathbf{Z}\bar{H} \\ {\scriptstyle \bmod \hat{c}}\downarrow{\scriptstyle \theta_4} & & {\scriptstyle \theta_2}\downarrow{\scriptstyle \bmod p} \\ T = \mathbf{Z}H/\hat{c}\mathbf{Z}H & \xrightarrow{\theta_1} & \mathbf{Z}\bar{H}/p\mathbf{Z}\bar{H} \end{array}$$

with all maps natural. The $p \times p$ matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots 0 \\ 0 & 0 & 1 & 0 \dots 0 \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ 0 & 0 & & \dots 1 \\ \lambda & 0 & & \dots 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & & & & & \\ & \omega & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \omega^{p-1} \end{bmatrix}, \lambda^{p^{n-3}} = \omega$$

satisfy

$$A^p = \lambda I, B^p = I, B^{-1}AB = A^{p^{n-2}+1}.$$

That the matrices $\{B^i A^j\}$, $0 \leqq i, j < p$, are linearly independent over $\mathbf{Z}[\lambda]$ can be seen as follows.

Suppose that

$$\sum_{i,j} z_{ij} B^i A^j = 0.$$

Since A^j has non zero entries only in the jth diagonal we have

$$\sum_i z_{ij} B^i A^j = 0 \quad \text{for each } j.$$

It follows from the nonsingularity of A that

$$\sum_i z_{ij} B^i = 0.$$

This easily implies that $z_{ij} = 0$ for all i, j.

Let Sp be the $\mathbf{Z}[\lambda]$-span of the matrices $\{B^i A^j | 0 \leqq i, j < p\}$. We claim that T is isomorphic to Sp. Consider the map

$$\phi : \mathbf{Z}H \rightarrow Sp, \phi(a) = A, \phi(b) = B.$$

Since $\hat{c} = 1 + c + \ldots + c^{p-1}$ is mapped to $(1 + \omega + \ldots + \omega^{p-1})I = 0$, we have an induced map

$$\phi_0 : T \rightarrow Sp.$$

Since $\phi(a^{pk} a^i b^j) = \lambda^k A^i B^j$, ϕ_0 is onto Sp. Also, ϕ_0 is one to one as after tensoring with $\mathbf{Q}$ we see that both T and Sp have $\mathbf{Q}$-dimension $(p^n - p^{n-1})$.

Now we give a valuation theoretical description of Sp.

PROPOSITION 4. *The matrix $Z \in \mathbf{Z}[\lambda]_{p\times p} \in Sp$ if and only if the matrix $X = Z'$ satisfies*

$$\sum_{i=0}^{p-1} x_{ji} \omega^{ki} \in p\mathbf{Z}[\lambda], \quad \text{for all } 0 \leqq j, k < p$$

where Z' is obtained from Z by dividing all entries below the main diagonal by λ.

Proof. Observe that A^i has entries $1, 1, \ldots, 1, \underbrace{\lambda, \ldots, \lambda}_{i}$ in the ith diagonal and zeros elsewhere. Thus in order to compute Sp we need only calculate the span $\{B^j A^i, 0 \leqq j \leqq p\}$ separately for every i. Thus we need to find all $\mathbf{Z}[\lambda]$-vectors $(z_{i0}, \ldots, z_{ip-1})$ such that

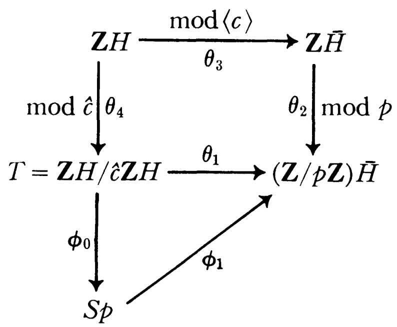
$$\sum_{=0}^{p-1} t_j B^j \begin{bmatrix} 1 & & & & & & \\ & \cdot & & & & & \\ & & \cdot & & & & \\ & & & 1 & & & \\ & & & & \lambda & & \\ & & & & & \cdot & \\ & & & & & & \cdot \\ & & & & & & & \lambda \end{bmatrix} = \begin{bmatrix} z_{i0} & & & & & \\ & z_{i1} & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & z_{ip-1} \end{bmatrix},$$

which is equivalent to

$$\sum_{j=0}^{p-1} t_j B^j = \begin{bmatrix} z_{i0} & & & & & & & \\ & \cdot & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & & & \\ & & & & z_{i,p-i-1} & & & \\ & & & & & z_{i,p-i}\lambda^{-1} & & \\ & & & & & & \cdot & \\ & & & & & & & \cdot \\ & & & & & & & & \cdot \\ & & & & & & & & & z_{i,p-1}\lambda^{-1} \end{bmatrix}.$$

The result now follows by Proposition 2.

We have the diagram



which is commutative by setting $\phi_1 = \theta_1\phi_0^{-1}$. Let us describe the map ϕ_1. Given $M \in Sp$ we wish to write M as $\sum \alpha_{ij}B^iA^j$ with $\alpha_{ij} \in \mathbf{Z}[\lambda]$, $0 \leqq i, j < p$. Let M' be obtained from M by dividing all entries below the main diagonal by λ. Then the jth diagonal $x_{j0}, \ldots, x_{j,p-1}$ of M' is the same as the main diagonal of $\sum_i \alpha_{ij}B^i$. We have

$$W \begin{bmatrix} \alpha_{0j} \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{p-1,j} \end{bmatrix} = \begin{bmatrix} x_{j0} \\ \cdot \\ \cdot \\ \cdot \\ x_{j,p-1} \end{bmatrix}, \quad W = \begin{bmatrix} 1 & 1 & \ldots 1 \\ 1 & \omega & \ldots \omega^{p-1} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ 1 & \omega^{p-1} & \ldots \omega^{(p-1)^2} \end{bmatrix}.$$

Thus

$$\alpha_{ij} = \frac{1}{p}\sum_k \omega^{-ik} x_{jk}.$$

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Writing

$$\alpha_{ij} = \sum d_{ijk}\lambda^k,\ d_{ijk} \in \mathbf{Z}[\omega],\ 0 \leqq k < p^{n-3}$$

we have

$$M = \sum d_{ijk}\lambda^k B^i A^j = \sum_{i,j} \left(\sum_k d_{ijk} A^{pk}\right) B^i A^j,$$

$$0 \leqq i, j < p,\ 0 \leqq k < p^{n-3}.$$

The commutative diagram implies that

$$\phi_1(M) = \sum \bar{d}_{ijk} \bar{b}^i \bar{a}^{j+pk}$$

where $\bar{d}_{ijk}$ is obtained from d_{ijk} by substituting $\omega = 1$ and going mod p.

In view of Proposition 3, we have proved

THEOREM 1. (a) $\mathbf{Z}H \simeq \{(\alpha, M) \in \mathbf{Z}\bar{H} \times \mathbf{Z}[\lambda]_{p\times p} | M'$ *satisfies* (*) *and* $\theta_2(\alpha) = \phi_1(M)\}$.

(b) $\mathcal{U}\mathbf{Z}H \simeq \{(\alpha, M) \in \mathcal{U}\mathbf{Z}\bar{H} \times \mathbf{Z}[\lambda]_{p\times p} | M$ *is a unit of* $\mathbf{Z}[\lambda]_{p\times p}$, M' *satisfies* (*) *and* $\phi_2(\alpha) = \phi_1(M)\}$. *Here*

(i) *M' is obtained from M by dividing all entries below the main diagonal by λ, λ is a primitive p^{n-2}th root of unity;*

(ii) *The condition* (*) *is*

$$\sum_{i=0}^{p-1} x_{ji}\omega^{ki} \in p\mathbf{Z}[\lambda],\ 0 \leqq j, k < p,\ \omega = \lambda^{p^{n-3}}$$

where $\{x_{ij}\}$ are the pseudo diagonals of M';

(iii) *$\theta_2 : \mathbf{Z}\bar{H} \to (\mathbf{Z}/p\mathbf{Z})\bar{H}$ is the natural map* mod *p;*

(iv) $\phi_1(M) = \sum_{i,j,k} \bar{d}_{ijk}\bar{b}^i\bar{a}^{j+pk}$ *where*

$$\alpha_{ij} = \frac{1}{p}\sum_l \omega^{-il} x_{jl} \in \mathbf{Z}[\lambda]$$

is written as $\sum_k d_{ijk}\lambda^k$, $d_{ijk} \in \mathbf{Z}[\omega]$, $0 \leqq k < p^{n-3}$ *and $\bar{d}_{ijk}$ is obtained from d_{ijk} by substituting $\omega = 1$ and going* mod p.

If we replace $\mathbf{Z}$ by the ring of p-adic integers $\mathbf{Z}_p$ then all the work above goes through. But in this case one knows explicitly that $\mathcal{U}\mathbf{Z}_p\bar{H}$ consists of all elements of nonzero augmentation. Therefore, a corresponding result for $\mathbf{Z}_pH$ is obtained.

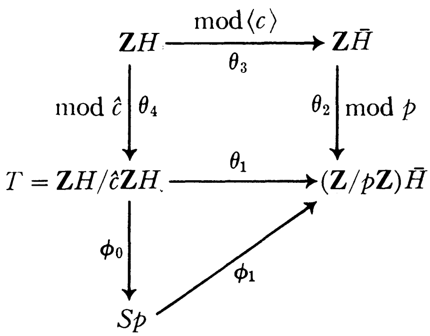
3. Groups of order p^3. If p is an odd prime, the two noncommutative groups of order p^3 are

$$H = \langle a, b |\ a^{p^2} = 1 = b^p,\ b^{-1}ab = a^{p+1}\rangle$$

and

$$G = \langle a, b|\ (a, b) = a^{-1}b^{-1}ab = c, ca = ac, cb = bc, a^p = 1 = b^p = c\rangle.$$

We reserve the letters G and H for these groups throughout this section. The first one is a special case of the group discussed in the last section, obtained by taking $n = 3$. We have $\omega = \lambda$, a primitive pth root of unity, $c = a^p$ and the fibre product diagram is as follows:



Theorem 1 specializes to

THEOREM 2. (a) $\mathbf{Z}H \simeq \{(\alpha, M) \in \mathbf{Z}\bar{H} \times \mathbf{Z}[\omega]_{p\times p}|M'$ *satisfies* (*) *and* $\theta_2(\alpha) = \phi_1(M)\}$.

(b) $\mathscr{U}\mathbf{Z}H \simeq \{(\alpha, M) \in \mathscr{U}\mathbf{Z}\bar{H} \times \mathbf{Z}[\omega]_{p\times p}|M$ *is a unit of* $\mathbf{Z}[\omega]_{p\times p}$, M' *satisfies* (*) *and* $\theta_2(\alpha) = \phi_1(M)\}$. *Here,*

(i) *M' is obtained from M by dividing all entries below the main diagonal by ω.*

(ii) *The condition* (*) *is*

$$\sum_{i=0}^{p-1} x_{ji}\omega^{ki} \in p\mathbf{Z}[\omega], 0 \leqq j, k < p, \omega^p = 1$$

where $\{x_{ij}\}$ are the pseudo diagonals of M'.

(iii) $\theta_2 : \mathbf{Z}\bar{H} \to (\mathbf{Z}/p\mathbf{Z})\bar{H}$ *is the natural map* mod p.

(iv) $\phi_1(M) = \sum_{i,j}\bar{\alpha}_{ij}\bar{b}^i\bar{a}^j$ *where*

$$\alpha_{ij} = \frac{1}{p}\sum_l \bar{\omega}^{il}x_{jl} \in \mathbf{Z}[\omega]$$

and $\bar{\alpha}_{ij}$ is obtained from α_{ij} by putting $\omega = 1$ and going mod p.

Now, we consider our second group of order p^3. The factor commutator group, $\bar{G} = G/\langle c\rangle$ is elementary abelian of order p^2, $\bar{G} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle$. We have the decomposition

$$\mathbf{Q}G \simeq \mathbf{Q}\bar{G} \oplus \mathbf{Q}(\omega)_{p\times p},$$

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where $\mathbf{Q}(\omega)_{p\times p}$ is the ring of all $p \times p$ matrices over $Q(\omega)$. In fact

$$\mathbf{Q}\bar{G} \simeq \mathbf{Q}G/\Delta(G, \langle c\rangle) \simeq \mathbf{Q}G\hat{c},$$
$$\mathbf{Q}(\omega)_{p\times p} \simeq \mathbf{Q}G/\hat{c}\mathbf{Q}G.$$

Clearly,

$$\begin{array}{l} \mathbf{Z}G \to \mathbf{Z}G/\Delta(G, \langle c\rangle) \oplus \mathbf{Z}[\omega]_{pxp} \\ \qquad\qquad\quad \wr\!\mid \\ \qquad\qquad\quad \mathbf{Z}\bar{G} \end{array}$$

with the projection onto the first component. We shall compute the projection in the second component. Consider the fibre product diagram

$$\begin{array}{ccc} \mathbf{Z}G & \xrightarrow[\theta_3]{\mod \langle c\rangle} & \mathbf{Z}\bar{G} \\ \mod \hat{c} \Big\downarrow \theta_4 & & \theta_2 \Big\downarrow \mod p \\ \mathbf{Z}G/\hat{c}\mathbf{Z}G & \xrightarrow{\theta_1} & (\mathbf{Z}/p\mathbf{Z})\bar{G} \end{array}$$

where θ_2, θ_3 and θ_4 are the natural projections and θ_1 is the map

$$\theta_1(\sum zc^i a^j b^k) = \sum \bar{z}\bar{a}^j \bar{b}^k, \quad z \in \mathbf{Z}.$$

It is worthwhile noting that $\mathbf{Z}G/\hat{c}\mathbf{Z}G$ is isomorphic to the twisted group ring $\mathbf{Z}[\omega] \circ \bar{G}$ with $\omega\bar{b}\bar{a} = \bar{a}\bar{b}$. The map θ_1 after this identification is given by

$$\theta_1(\sum \alpha\bar{a}^i\bar{b}^j) = \sum \bar{\alpha}\bar{a}^i\bar{b}^j, \quad \alpha \in \mathbf{Z}[\omega]$$

where $\bar{\alpha}$ is obtained from α by substituting $\omega = 1$.

Let us define a map ϕ_0 from $\mathbf{Z}[\omega] \circ \bar{G}$ to $\mathbf{Z}[\omega]_{p\times p}$ by

$$\bar{b} \to B = \begin{bmatrix} 1 & & & & \\ & \omega & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \omega^{p-1} \end{bmatrix}, \bar{a} \to A = \begin{bmatrix} 0 & 1 & 0 & & \dots 0 \\ 0 & 0 & 1 & 0 & \dots 0 \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & 0 & 0 & & \dots 0 \end{bmatrix}.$$

Then $\omega BA = AB$, $A^p = I = B^p$. Moreover, if $\sum_{i,j} \alpha_{ij}B^iA^j = 0$, $\alpha_{ij} \in \mathbf{Z}[\omega]$ and $0 \leqq i, j < p$, then $\alpha_{ij} = 0$ for all i, j. This can be seen as follows:

$$\sum_i \sum_j \alpha_{ij}B^iA^j = 0 \Rightarrow \sum_i \alpha_{ij}B^iA^j = 0$$

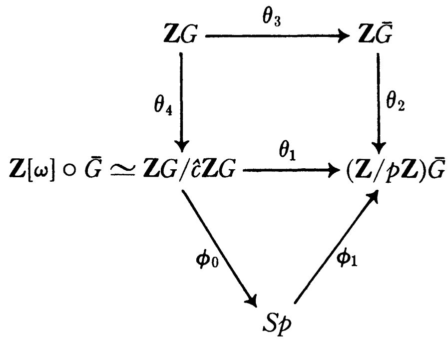
as A^j has nonzero entries only in the jth diagonal. Further, due to the nonsingularity of A it follows that $\sum_i \alpha_{ij}B^i = 0$, and this implies that $\alpha_{ij} = 0$ for all i, j. We have proved that

$$\mathbf{Z}G/\hat{c}\mathbf{Z}G \simeq \mathbf{Z}[\omega] \circ \bar{G} \simeq \text{span } \langle B^iA^j, 0 \leqq i, j < p\rangle = Sp,$$

the $\mathbf{Z}[\omega]$-span of the matrices $\{B^iA^j\}$. It follows by Proposition 2 that

$$Sp = \left\{ M \in \mathbf{Z}[\omega]_{p\times p} \mid M \text{ satisfies} \sum_{i=0}^{p-1} x_{ji}\omega^{ki} \in p\mathbf{Z}[\omega] \text{ for all } 0 \leqq k,j < p \right\}.$$

Let us understand the induced map $\phi_1 = \theta_1\phi_0^{-1}$:



Given $M \in Sp$, we wish to find $\phi_0^{-1}(M) \in \mathbf{Z}[\omega] \circ \bar{G}$. Let $\{x_{j,0}, \ldots, x_{j,p-1}\}$ be the jth diagonal of M. We wish to find $a_{ij} \in \mathbf{Z}[\omega]$ such that $\sum_{i,j} a_{ij}B^iA^j = M$. It is necessary to find a_{ij} such that

$$\sum_i a_{ij}B^i = \begin{bmatrix} x_{j,0} & & & \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ & & & & x_{j,p-1} \end{bmatrix}, 0 \leqq j \leqq p-1.$$

This is equivalent to

$$W \begin{bmatrix} a_{0,j} \\ \cdot \\ \cdot \\ \cdot \\ a_{p-1,j} \end{bmatrix} = \begin{bmatrix} x_{j,0} \\ x_{j,1} \\ \cdot \\ \cdot \\ \cdot \\ x_{j,p-1} \end{bmatrix}$$

where

$$W = \begin{bmatrix} 1 & 1 & & 1 \\ 1 & \omega & \ldots & \omega^{p-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 1 & \omega^{p-1} & \ldots & \omega^{(p-1)^2} \end{bmatrix}.$$

Hence, we see that

$$a_{ij} = \frac{1}{p}\sum_{k=0}^{p-1} \bar{\omega}^{ki} x_{jk}.$$

We have

$$\phi_0^{-1}(M) = \sum_{i,j} a_{ij}\bar{b}^i\bar{a}^j \quad \text{and} \quad \phi_1(M) = \sum_{i,j} \bar{a}_{ij}\bar{b}^i\bar{a}^j$$

where $\bar{a}_{ij}$ is obtained from a_{ij} by substituting $\omega = 1$. We have proved the first part of the next theorem. The second part follows from Proposition 3 in view of the fact that Sp is an order in $\mathbf{Z}[\omega]_{p\times p}$.

THEOREM 3.

(a) $\mathbf{Z}G \simeq \{(\alpha, M) \in \mathbf{Z}\bar{G} \times \mathbf{Z}[\omega]_{p\times p} |\ M$ *satisfies* $(*)$, $\theta_2(\alpha) = \phi_1(M)\}$.

(b) $\mathcal{U}\mathbf{Z}G \simeq \{(\alpha, M) \in \mathcal{U}\mathbf{Z}\bar{G} \times \mathbf{Z}[\omega]_{p\times p} |\ M$ *is a unit of* $\mathbf{Z}[\omega]_{p\times p}$, M *satisfies* $(*)$, $\theta_2(\alpha) = \phi_1(M)\}$.

Here,

(i) $\theta_2 : \mathbf{Z}\bar{G} \to (\mathbf{Z}/p\mathbf{Z})\bar{G}$ *is the natural map* mod p;

(ii) *The condition* $(*)$ *is*

$$\sum_{i=0}^{p-1} x_{ji}\omega^{ki} \in p\mathbf{Z}[\omega],\ 0 \leqq j, k < p,\ \omega^p = 1,$$

where $\{x_{ij}\}$ *are the pseudo diagonals of* M;

(iii) $\phi_1(M) = \sum_{i,j} \bar{a}_{ij}\bar{b}^i\bar{a}^j$ *where*

$$a_{ij} = \frac{1}{p}\sum_{k=0}^{p-1} \bar{\omega}^{ki} x_{jk} \in \mathbf{Z}[\omega]$$

and $\bar{a}_{ij}$ *is obtained from* a_{ij} *by putting* $\omega = 1$ *and going* mod p.

As in the case of Theorem 1 there is also a corresponding p-adic result.

4. Groups of order 27. Now, we specialize to the case $p = 3$; $\omega^3 = 1$. The groups are

$$G = \langle a, b | (a, b) = c, c \text{ central}, a^3 = b^3 = c^3 = 1\rangle, \quad \text{and}$$

$$H = \langle a, b | a^9 = 1 = b^3, b^{-1}ab = a^4\rangle.$$

Then $\bar{G} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle$, $\bar{H} = \langle \bar{a}\rangle \times \langle \bar{b}\rangle$ are both elementary abelian 3-groups. It is well known [7, p. 57] that $\mathcal{U}\mathbf{Z}\bar{G} = \pm\bar{G}$, $\mathcal{U}\mathbf{Z}\bar{H} = \pm\bar{H}$. We wish to give an explicit description for $\mathcal{U}\mathbf{Z}G$ and $\mathcal{U}\mathbf{Z}H$. The diagram for $\mathbf{Z}G$ is as follows:

$$B = \begin{bmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

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$$\begin{array}{ccc}
\mathbf{Z}G & \xrightarrow[\theta_3]{\bmod \langle c \rangle} & \mathbf{Z}\bar{G} \\
{\scriptstyle \bmod c}\big\downarrow{\scriptstyle \theta_4} & & {\scriptstyle \theta_2}\big\downarrow{\scriptstyle \bmod 3} \\
\mathbf{Z}[\omega] \circ \bar{G} \simeq \mathbf{Z}G/\hat{c}\mathbf{Z}G & \xrightarrow{\theta_1} & (\mathbf{Z}/3\mathbf{Z})\bar{G} \\
{\scriptstyle \phi_0}\big\downarrow & \nearrow{\scriptstyle \phi_1} & \\
Sp & &
\end{array}$$

Specializing the condition (*) to $p = 3$, we see that the matrix

$$M = \begin{bmatrix} x_{0,0} & x_{1,0} & x_{2,0} \\ x_{2,1} & x_{0,1} & x_{1,1} \\ x_{1,2} & x_{2,2} & x_{0,2} \end{bmatrix} \in \mathbf{Z}[\omega]_{3\times 3}$$

belongs to Sp if and only if it satisfies for each $0 \leqq i \leqq 2$ the conditions

$$x_{i0} + x_{i1} + x_{i2} \in 3\mathbf{Z}[\omega]$$
$$x_{i0} + x_{i1}\omega + x_{i2}\omega^2 \in 3\mathbf{Z}[\omega]$$
$$x_{i0} + x_{i1}\omega^2 + x_{i2}\omega \in 3\mathbf{Z}[\omega].$$

To find $\phi_1(M)$ we need $a_{ij} \in \mathbf{Z}[\omega]$ such that $M = \sum a_{ij}B^iA^j$. This gives

$$\begin{bmatrix} a_{0j} \\ a_{1j} \\ a_{2j} \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}\begin{bmatrix} x_{j0} \\ x_{j1} \\ x_{j2} \end{bmatrix}$$

and

$$\begin{aligned} a_{0j} &= \tfrac{1}{3}(x_{j0} + x_{j1} + x_{j2}) \\ (**) \quad a_{1j} &= \tfrac{1}{3}(x_{j0} + \omega^2 x_{j1} + \omega x_{j2}) \\ a_{2j} &= \tfrac{1}{3}(x_{j0} + \omega x_{j1} + \omega^2 x_{j2}). \end{aligned}$$

We have $\phi_1(M) = \sum \bar{a}_{ij}\bar{b}^i\bar{a}^j$. We know that the units of $\mathbf{Z}G$ are pairs $(\alpha, M), \alpha \in \mathcal{U}\mathbf{Z}\bar{G}$, $M \in Sp$ with $\phi_1(M) = \phi_2(\alpha)$. But, since $\mathcal{U}\mathbf{Z}\bar{G} = \pm\bar{G}$ we need matrices M such that

$$\phi_1(M) = \sum \bar{a}_{ij}\bar{b}^i\bar{a}^j = \pm\bar{a}^l\bar{b}^m = \theta_2(\pm a^l b^m)$$

for some l, m. This means that if $\pi = \omega - 1$,

(1) For two values of i and all j, $a_{ij} \equiv 0 \pmod{\pi}$;

(2) For the third value of i either

$$a_{i0} = \pm 1, a_{i1} \equiv a_{i2} \equiv 0 \pmod{\pi} \quad \text{or}$$
$$a_{i1} = \pm 1, a_{i0} \equiv a_{i2} \equiv 0 \pmod{\pi} \quad \text{or}$$
$$a_{i2} = \pm 1, a_{i0} \equiv a_{i1} \equiv 0 \pmod{\pi}.$$

We have proved

THEOREM 4. *$\mathscr{U}\mathbf{Z}G \simeq \{M \in \mathscr{U}\mathbf{Z}[\omega]_{3\times 3} | M$ satisfies (1) and (2) where a_{ij} are given by (**)$\}$.*

It is clear that the matrices $Y \in \mathscr{U}\mathbf{Z}[\omega]_{3\times 3}$ which are congruent to I mod π^3 are contained in $\mathscr{U}\mathbf{Z}G$ and hence $\mathscr{U}\mathbf{Z}G$ is a congruence subgroup in $SL(3, \mathbf{Z}[\omega])$.

Now, we describe $\mathscr{U}\mathbf{Z}H$. Recall that if we have a matrix

$$X = Z' = \begin{bmatrix} x_{0,0} & x_{1,0} & x_{2,0} \\ x_{2,1} & x_{0,1} & x_{1,1} \\ x_{1,2} & x_{2,2} & x_{0,2} \end{bmatrix}$$

satisfying (*) then the corresponding matrix in Sp is

$$Z = \begin{bmatrix} x_{0,0} & x_{1,0} & x_{2,0} \\ \omega x_{2,1} & x_{0,1} & x_{1,1} \\ \omega x_{1,2} & \omega x_{2,2} & x_{0,2} \end{bmatrix}.$$

If we write $Z = \sum a_{ij} B^i A^j$ then it can be checked that $\phi_1(Z) = \pm h$, $h \in H$ if and only if the matrix X satisfies (1) and (2). We have

THEOREM 5. *$\mathscr{U}\mathbf{Z}H \simeq \{Z \in \mathscr{U}\mathbf{Z}[\omega]_{3\times 3} | Z'$ satisfies (1) and (2) where a_{ij} are given by (**)$\}$.*

It is easily seen that $\mathscr{U}\mathbf{Z}H$ is a congruence subgroup in $SL(3, \mathbf{Z}[\omega])$.

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