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Coherent quantum hollow beam creation in a plasma wakefield accelerator

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Abstract. A theoretical investigation of the propagation of a relativistic electron (or positron) particle beam in an overdense magnetoactive plasma is carried out within a fluid plasma model, taking into account the individual quantum properties of beam particles. It is demonstrated that the collective character of the particle beam manifests mostly through the self-consistent macroscopic plasma wakefield created by the charge and the current densities of the beam. The transverse dynamics of the beam–plasma system is governed by the Schrödinger equation for a single-particle wavefunction derived under the Hartree mean field approximation, coupled with a Poisson-like equation for the wake potential. These two coupled equations are subsequently reduced to a nonlinear, non-local Schrödinger equation and solved in a strongly non-local regime. An approximate Glauber solution is found analytically in the form of a Hermite–Gauss ring soliton. Such non-stationary ('breathing' and 'wiggling') coherent structure may be parametrically unstable and the corresponding growth rates are estimated analytically.

1. Introduction

Recently, we developed a self-consistent description of the transverse quantum dynamics of a cold relativistic electron (positron) beam in a cold magnetized plasma (Fedele et al. 2012). We assumed that the plasma is immersed in a strong constant and uniform external magnetic field $\mathbf{B}_0 = B_0\hat{z}$. The ions are immobile, forming a uniform background of density n_0. The electron/positron beam, before entering the plasma, is traveling along the z-axis (called the *longitudinal direction*) with the velocity $\beta c\hat{z}$ ($\beta \approx 1$). We study an *overdense regime*, in which the beam's unperturbed density n_b is much smaller than the plasma density, $n_b \approx N/\pi\sigma_\perp^2\sigma_z$, where N, σ_z, and $\sigma_\perp$ are the number of particles, beam length, and beam spot size respectively. The beam is sufficiently long so that its longitudinal dynamics is ignored, i.e. all physical quantities depend on the variable $\xi = z - \beta ct$ and the transverse coordinates $\mathbf{r}_\perp = (x, y)$. We consider a *long beam limit*, in which the beam is much longer than the plasma wavelength, $\partial^2/\partial\xi^2 \ll \omega_{pe}^2/c^2$, and a *paraxial* beam motion, $dx/d\xi \sim dy/d\xi \ll 1$, i.e. we consider electron rays whose slopes (to $\hat{z}$) are very small, which implies that the transverse motion is non-relativistic. For each individual particle, at each ξ, we account for the uncertainty relation between its transverse position and the transverse momentum, $\langle x^2\rangle^{1/2}\langle p_x^2\rangle^{1/2} \geqslant \hbar/2$ and $\langle y^2\rangle^{1/2}\langle p_y^2\rangle^{1/2} \geqslant \hbar/2$ respectively. For simplicity, we take $\langle x\rangle = \langle y\rangle = \langle p_x\rangle = \langle p_y\rangle = 0$. In the quantum picture, the beam is represented as an ensemble of electron rays whose slopes are affected by the uncertainty due to quantum diffraction (quantum paraxial diffraction). The collective quantum nature of the beam particles, which manifests when the interparticle distance δ is comparable or smaller than the thermal de Broglie wavelength λ_T, viz., $\delta \lesssim \lambda_T$, is assumed to be negligible. Since the particle motion is relativistic in the z-direction, we have $\lambda_T = h/m_0\gamma_0 v_T$, where $v_T = \sqrt{k_B T/m_0\gamma_0}$ is the thermal velocity (k_B and T being the Boltzmann constant and the transverse beam temperature respectively). Such condition readily leads to $T \lesssim (m_0c^2/\gamma_0 k_B)(n_b\lambda_c^3)^{2/3} \propto n_b^{2/3}/\gamma_0$, where λ_c is the electron Compton wavelength. For a typical electron/positron beam, with the density $n_b \sim 10^{13}-10^{15}$ cm^{-3}, the energy $\gamma_0 \sim 10^2-10^4$, and moderate temperature that permits the (classical) cold beam approximation, $10^{-2} \ll T \ll 10^4$ K, we can ignore the overlapping of wavefunctions of the particles and take into account only the individual quantum properties (the uncertainty principle and the spin).

Under these conditions, the transverse beam dynamics is described by correcting the paraxial geometric optics of electrons with their paraxial wave optics, given in terms of a single-particle wavefunction. In Jagannathan et al. (1989) and Khan and Jagannathan (1995), this was done in the presence of external electromagnetic fields, but without the self-consistent collective effects due to electromagnetic interaction. Remarkably, the concomitance of the individual quantum nature of the particles and the collective effects enable the beam to manifest quantum behavior at macroscopic level, which was also observed in the Sokolov–Ternov effect (Sokolov and Ternov 1963), quantum excitation (Huang et al. 1995; Huang and Ruth 1998), quantum free electron lasers (Preparata 1988; Bonifacio et al. 2005), etc. Within the Hartree's mean field approximation, this physical concomitance has been recently considered to describe the macroscopic manifestation of the individual quantum particle nature via classical collective effects due to the plasma wakefield (PWF) excitation, leading to the formation of coherent structures (Chen et al. 1985; Rosenzweig et al. 1988). Recently, the nonlinear and collective quantum evolution of a relativistic charged particle beam has been studied in detail (Fedele et al. 2012) for both strictly local and moderately non-local plasma wakefield response.

2. Basic equations

The fluid model for the beam–plasma system was introduced with the Lorentz–Maxwell system of equations where the charges and currents were provided by the plasma and the relativistic beam. Under the conditions described above, the Lorentz–Maxwell system can be linearized and reduced to (see Fedele et al. 2012):

$$\left(\nabla_\perp^2 - \frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}\right) U_w = \frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}\frac{\rho_b}{n_0\gamma_0}, \qquad (2.1)$$

where $U_w = U_w(\mathbf{r}_\perp, \xi) = -q(A_{1z} - \phi_1)/m_0\gamma_0 c^2$ is the dimensionless wake potential, $A_{1z}(\mathbf{r}_\perp, \xi)$ and $\phi_1(\mathbf{r}_\perp, \xi)$ are the perturbations of the longitudinal component of the vector potential and the scalar potential respectively, and $\rho_b = \rho_b(\mathbf{r}_\perp, \xi)$ is the beam number density. Here $q = -e$ for electrons or $q = e$ for positrons, $\omega_{pe} = (4\pi n_0 e^2/m_0)^{1/2}$ is the electron plasma frequency, $\omega_{ce} = -eB_0/m_0 c$ is the electron–cyclotron frequency and $\omega_{UH} = (\omega_{pe}^2 + \omega_{ce}^2)^{1/2}$ is the upper hybrid frequency.

In the paraxial approximation and for a stationary solution in the co-moving frame, the full classical single-particle relativistic Hamiltonian H can be expanded in the non-relativistic transverse motion, with the unperturbed relativistic longitudinal motion as the leading order motion. Then the *effective* single-particle Hamiltonian $\mathscr{H} = \Delta H/H_0 = (H - H_0)/H_0$ (where $H_0 = m_0\gamma_0 c^2$ is the unperturbed energy of single particle) can be cast in the form

$$\mathscr{H}(\mathbf{r}_\perp, \mathscr{P}_\perp, \xi) = \frac{1}{2}\mathscr{P}_\perp^2 + \frac{1}{2}k_c\hat{z}\cdot(\mathbf{r}_\perp \times \mathscr{P}_\perp) + U_w(\mathbf{r}_\perp, \xi) + \frac{Kr^2}{2}, \qquad (2.2)$$

where $\mathscr{P}_\perp = \mathbf{p}_\perp/m_0\gamma_0 c$ is the dimensionless perpendicular momentum, $k_c = -qB_0/m_0\gamma_0 c^2$, $K \equiv (k_c/2)^2 = \omega_{ce}^2/4\gamma_0^2 c^2$, and $r = |\mathbf{r}_\perp|$. Note that the interaction of the beam particles with the plasma is realized as a *mean field* effect through the wake potential U_w, which is generated by the charges and currents of both the plasma and the beam particles. Introducing in (2.2), the correspondence rules $\mathscr{H} \to \hat{\mathscr{H}} = i\epsilon\partial/\partial\xi$ and $\mathscr{P}_\perp \to \hat{\mathscr{P}}_\perp = -i\epsilon\nabla_\perp$, where $\epsilon \equiv \hbar/m_0\gamma_0 c = \lambda_c/\gamma_0$, for a paraxial beam in which the individual wavefunctions do not overlap, we obtain the following two-dimensional spinorial Schrödinger equation:

$$i\epsilon\frac{\partial\vec{\Psi}}{\partial\xi} = -\frac{\epsilon^2}{2}\nabla_\perp^2\vec{\Psi} - \frac{i\epsilon k_c}{2}\hat{z}\cdot(\mathbf{r}_\perp \times \nabla_\perp)\vec{\Psi} + U_w\vec{\Psi} + \frac{Kr^2}{2}\vec{\Psi} + \epsilon k_c\hat{s}_z\cdot\vec{\Psi}, \qquad (2.3)$$

where $\vec{\Psi}$ is the spinor, whose component is (Ψ_s), with s ranging in the set $\{-1/2, 1/2\}$ of eigenvalues of $\hat{s}_z$ that correspond to the 'spin down' and 'spin up' states respectively. In general, U_w is a functional of the wave function Ψ_s, where $s = \pm 1/2$, and therefore (2.3) is nonlinear. It can be decomposed into two scalar equations for Ψ_s, viz.

$$i\epsilon\frac{\partial\Psi_s}{\partial\xi} = -\frac{\epsilon^2}{2}\nabla_\perp^2\Psi_s - \frac{i\epsilon k_c}{2}\hat{z}\cdot(\mathbf{r}_\perp \times \nabla_\perp)\Psi_s + U_w\Psi_s + \frac{Kr^2}{2}\Psi_s + \hat{s}_z\epsilon k_c\Psi_s. \qquad (2.4)$$

The beam is composed of two kinds of electrons (with 'down' and 'up' spins), and its density is given by $\rho_b(\mathbf{r}_\perp, \xi) = (N/2\sigma_z)|\vec{\Psi}|^2 = (N/2\sigma_z)(|\Psi_{-1/2}|^2 + |\Psi_{1/2}|^2)$, where we have assumed that statistically the beam population is equipartitioned into 'down' and 'up' spin states and that each Ψ_s is normalized. Thus, (2.1) becomes

$$\left(\nabla_\perp^2 - \frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}\right) U_w = \frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}\frac{N}{2n_0\gamma_0\sigma_z} \times \left(|\Psi_{-1/2}|^2 + |\Psi_{1/2}|^2\right). \qquad (2.5)$$

Obviously, the wake potential is a functional of the form $U_w = U_w[|\Psi_{-1/2}|^2 + |\Psi_{1/2}|^2]$.

Equations (2.4) and (2.5) provide the scalar quantum description of the beam–plasma interaction. In the cylindrical coordinates (r, φ, ξ) and for the solutions of the form $\Psi_s(r, \xi, \varphi) = \psi_m(r, \xi)\exp\{i[m\varphi - (k_c/2)(m + 2s)\xi]\}$, where m is an integer, called the *vortex charge*, $U_w(|\psi_m|^2)$ becomes cylindrically symmetric and the Zakharov-like system (2.4) and (2.5) can be written in the following

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spin-independent form:

$$i\epsilon\frac{\partial\psi_m}{\partial\xi}=-\frac{\epsilon^2}{2}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_m}{\partial r}\right)+U_w\psi_m+\left(\frac{Kr^2}{2}+\frac{m^2\epsilon^2}{2r^2}\right)\psi_m, \tag{2.6}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_w}{\partial r}\right)-\frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}U_w=\frac{\omega_{pe}^2}{\omega_{UH}^2}\frac{\omega_{pe}^2}{c^2}\frac{N}{n_0\gamma_0\sigma_z}|\psi_m|^2. \tag{2.7}$$

From the above we note that the beam density structure is governed by nonlinearity, the paraxial quantum diffraction, and the trapping effect provided by $\mathbf{B}_0$. It carries the signature of both the orbital angular momentum and the quantum nature of individual particles that are exhibited at the macroscopic level. Standard 2D solitons (without vorticity, i.e. zero vortex charge) and ring solitons (with vorticity, i.e. non-zero vortex charge) have been found (Fedele et al. 2012) for strictly local [i.e. $\nabla_\perp^2\ll k_H^2$, where $k_H=\omega_{pe}^2/(c\,\omega_{UH})$] and moderately non-local ($\nabla_\perp^2\sim k_H^2$) regimes, while the stability criteria have been discussed for weak, moderate, and strong focusing conditions.

3. Transverse beam dynamics in a strong locality regime

In the strongly focused regime,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U_w}{\partial r}\right)\gg\frac{\omega_{pe}^2}{\omega_{uh}^2}\frac{\omega_{pe}^2}{c^2}U_w, \tag{3.1}$$

and Poisson's equation (2.7) is readily integrated as

$$U_w=\frac{\omega_{pe}^2}{\omega_{uh}^2}\frac{\omega_{pe}^2}{c^2}\frac{N}{n_0\gamma_0\sigma_z}\left[\log\left(\frac{r}{\bar r}\right)\int_0^r r'\,dr'\,|\psi_m(r')|^2+\int_r^\infty r'\,dr'\,\log\left(\frac{r'}{\bar r}\right)|\psi_m(r')|^2\right], \tag{3.2}$$

where $\bar r$ is a constant of integration. Obviously, in (3.2), the relation between the wake potential $U_w(r)$ and the nonlinear source term $|\psi_m(r')|^2$ is non-local. For this reason, in the literature, the corresponding Green's function is often referred to as the *non-local response function*. In the above, the limits of integration are adopted such that the wake potential U_w vanishes at $r=0$, while the unavoidable singularity at $r\to\infty$ is minimized, i.e. reduced to a weak logarithmic singularity that does not affect the solution of the Schrödinger equation (2.6), because it is 'not accessible' for the wavefunction ψ_m in the presence of the external potential Kr^2. Namely, for large values of r the latter grows much faster than $\log r$, and the related singularity of U_w produces no effect. For a wavefunction ψ_m with a finite width, $\bar r$ is adopted to be the position of its 'center of mass', viz.

$$\int_0^{\bar r} r'\,dr'\,|\psi_m(r')|^2=\int_{\bar r}^\infty r'\,dr'\,|\psi_m(r')|^2=\frac{1}{2}\int_0^\infty r'\,dr'\,|\psi_m(r')|^2. \tag{3.3}$$

The Schrödinger equation (2.6) is simplified by introducing the transformation $\psi_m(r,\xi)=\chi(r,\xi)/\sqrt{r}$ when it reduces to

$$i\epsilon\frac{\partial\chi}{\partial\xi}=-\frac{\epsilon^2}{2}\frac{\partial^2\chi}{\partial r^2}+V\chi,\qquad\text{with}$$

$$V=U_w+\frac{1}{2}Kr^2+\frac{\epsilon^2}{2r^2}\left(m^2-\frac{1}{4}\right). \tag{3.4}$$

The effective potential V is a nonlinear functional of ψ_m that can be expanded as

$$V\approx V_0(\xi)+V_1(\xi)(r-r_0)+\frac{1}{2}V_2(\xi)(r-r_0)^2, \tag{3.5}$$

where $V_0(\xi)=V(r_0,\xi)$, $V_1(\xi)=\partial V(r_0,\xi)/\partial r_0$, and $V_2(\xi)=\partial^2V(r_0,\xi)/\partial r_0^2$. Adopting $r_0(\xi)$ to be in the minimum of the effective potential, (3.4) and (3.5) yield

$$V_1=\frac{\mathscr{I}}{r_0}\left(\frac{1}{2}-\Delta_1\right)+Kr_0-\frac{\epsilon^2}{r_0^3}\left(m^2-\frac{1}{4}\right)=0,$$

$$\Delta_1=\frac{\int_{r_0}^{\bar r}dr'\,|\chi(r')|^2}{\int_0^\infty dr'\,|\chi(r')|^2}, \tag{3.6}$$

$$V_2=-\frac{\mathscr{I}}{r_0^2}\left(r_0\frac{d\Delta_1}{dr_0}+\frac{1}{2}-\Delta_1\right)+K+\frac{3\epsilon^2}{r_0^4}\left(m^2-\frac{1}{4}\right), \tag{3.7}$$

and $\mathscr{I}=(N\omega_{pe}^4/n_0\gamma_0\sigma_zc^2\omega_{uh}^2)\int_0^\infty dr'\,|\chi(r')|^2$. To determine the position of the minimum of the effective potential V, we conveniently rewrite (3.6) as

$$V_1^{(0)}(r_0)\equiv\frac{\mathscr{I}}{2r_0}+Kr_0-\frac{\epsilon^2}{r_0^3}\left(m^2-\frac{1}{4}\right)=\frac{\mathscr{I}}{r_0}\Delta_1. \tag{3.8}$$

When the 'center of mass', defined by (3.3), is sufficiently close to the bottom of the potential well, $\bar r\approx r_0(\Rightarrow\Delta_1\ll Kr_0^2/\mathscr{I})$, the above equation is readily solved in the form $r_0=r_0^{(0)}+\delta r_0$ ($\delta r_0\ll r_0$), where

$$r_0^{(0)}=\frac{1}{2K^{\frac{1}{2}}}\left\{\left[\mathscr{I}^2+16K\epsilon^2\left(m^2-1/4\right)\right]^{\frac{1}{2}}-\mathscr{I}\right\}^{\frac{1}{2}},$$

$$\delta r_0=\left[\frac{dV_1^{(0)}(r_0^{(0)})}{dr_0^{(0)}}\right]^{-1}\frac{\mathscr{I}}{r_0^{(0)}}\Delta_1. \tag{3.9}$$

However, since Δ_1 depends on the position of the centroid of the wave function, $\bar r(\xi)$, to be determined self-consistently from (2.6), this problem remains essentially nonlinear even in a strongly nonlocal limit.

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4. Hollow ring soliton as a Hermite–Gaussian coherent state

Our (3.4) with (3.5) describes the motion of an isolated quantum particle in the external potential V, the latter being both time-dependent and propagating in space. Such equation for a linear oscillator possesses solutions in the form of wave packets, whose centroids follow the motion of a classical particle. These coherent packets were described by Schrödinger (1926), but eventually got forgotten. The coherent packets were rediscovered nearly 40 years later by Glauber (1963) and now, in the literature, they are known as the 'coherent' or 'Glauber states' (see e.g. Schulten 2000, pp. 90–95). A Glauber state is characterized by two distinct types of motions: the 'wiggling' associated with the classical propagation of its centroid, and the quantum 'breathing' associated with the oscillations of the width of the wave packet. Generalizing the procedure of de Nicola et al. (1995) and de Martino et al. (1997) to the case of a time-dependent parabolic potential in the presence of an external force, the generalized Glauber solution of (3.4) and (3.5) can be written as a superposition of Hermite–Gaussian modes, viz.

$$\chi = \sum_k \alpha_k \, \chi_k \left(r, \xi\right), \tag{4.1}$$

where α_k are arbitrary constants and the wiggling/breathing Hermite–Gaussian modes $\chi_k(r,\xi)$, given by

$$\begin{aligned} \chi_k \left(r, \xi\right) = & \left[\pi \, 2^{2k+1} (k!)^2 \sigma\left(\xi\right)^2\right]^{-\frac{1}{4}} \, H_k \left[\frac{r - \overline{r}\left(\xi\right)}{\sqrt{2}\,\sigma\left(\xi\right)}\right] \\ & \times \exp\left\{ i\phi_k\left(\xi\right) + \frac{i}{\epsilon}\, r\, \overline{r}'\left(\xi\right) \right. \\ & \left. + \left[\frac{i}{2\epsilon}\frac{\sigma'\left(\xi\right)}{\sigma\left(\xi\right)} - \frac{1}{4\,\sigma\left(\xi\right)^2}\right]\left[r - \overline{r}\left(\xi\right)\right]^2 \right\}, \end{aligned} \tag{4.2}$$

are orthogonal and constitute a complete set. Here the prime denotes the first derivative and H_k is the Hermite polynomial of order k and the Hermite–Gaussian functions χ_k are normalized as

$$\int_{-\infty}^{\infty} \left|\chi_k\left(r, \xi\right)\right|^2 \, dr = 1. \tag{4.3}$$

The cylindrical coordinate r is positive definite and the wavefunction $\chi(r,\xi)$, formally, has no physical meaning when $r < 0$. We consider a strongly non-local regime, i.e. a 'thin ring' with $\sigma \ll \overline{r}$, and the contribution of the region $r \in (-\infty, 0]$ to the integral in (4.3) is negligible (it contains only a distant tail of the Gaussian in $\chi_k(r,\xi)$). As for a normalized wavefunction (4.1) we have $\sum_k |\alpha_k|^2 = 1$, and we can express the intensity $\mathscr{I}$ as

$$\mathscr{I} = \frac{\omega_{pe}^2}{\omega_{uh}^2}\frac{\omega_{pe}^2}{c^2}\frac{N}{n_0 \gamma_0 \sigma_z}. \tag{4.4}$$

The radially independent part of the phase of the kth mode, $\phi_k(\xi)$, is given by

$$\begin{aligned} \phi_k\left(\xi\right) = \phi_{k,0} - \int_0^{\xi} d\xi' & \left\{ \frac{\epsilon\left(2k+1\right)}{4\,\sigma^2\left(\xi'\right)} + \frac{V_2\left(\xi'\right)}{2\epsilon} \right. \\ & \left. \times \left[r_0\left(\xi'\right)^2 - \overline{r}\left(\xi'\right)^2\right] + \frac{\overline{r}\left(\xi'\right)^2}{2\epsilon} + \frac{V_0\left(\xi'\right)}{\epsilon} \right\}, \end{aligned} \tag{4.5}$$

where $\phi_{k,0}$ are arbitrary constants. The position of the 'center of mass', $\overline{r}(\xi)$, satisfies the equation of motion of a classical particle,

$$\overline{r}''\left(\xi\right) = -V_2\left(\xi\right)\left[\overline{r}\left(\xi\right) - r_0\left(\xi\right)\right], \tag{4.6}$$

which is readily integrated as

$$\begin{aligned} \overline{r}\left(\xi\right) = & \, r_0^{(0)} + \delta\overline{r}_0 \, \frac{\sigma\left(\xi\right)}{\sigma\left(0\right)} \, \cos\left[\int_0^{\xi} \frac{\epsilon \, d\xi'}{2\,\sigma(\xi')^2}\right] \\ & + \frac{2\sigma\left(\xi\right)}{\epsilon}\int_0^{\xi} d\xi' \;\sigma\left(\xi'\right)\, V_2\left(\xi'\right)\, \delta r_0\left(\xi'\right) \\ & \times \, \sin\left[\int_{\xi'}^{\xi} \frac{\epsilon\, d\xi''}{2\sigma\left(\xi''\right)^2}\right], \end{aligned} \tag{4.7}$$

where $\delta\overline{r}_0$ is an arbitrary amplitude of the radial excursion (or wiggling) of the wave packet. To be consistent with the strong localization regime in which our equations are valid, the latter needs to scale as $\delta\overline{r}_0 < \sigma$ or $\delta\overline{r}_0 \sim \sigma$. As the temporal dependence of the position of the centroid of the wave packet $\overline{r}$ is *known in advance*, the description of coherent structures in the relativistic particle beam via the Hermite–Gaussian Glauber modes in a strongly non-local regime remains *essentially linear*. Thus, one overcomes the conceptual problem of the nonlinearity of moving structures as discussed in the preceding section. The characteristic length $\sigma(\xi)$ satisfies the Ermakov–Pinney equation

$$\sigma''\left(\xi\right) + V_2(\xi)\,\sigma\left(\xi\right) - \frac{\epsilon^2}{4\,\sigma\left(\xi\right)^3} = 0, \tag{4.8}$$

which can be integrated with the use of the estimate for $V_2(\xi)$ that is obtained from (3.7) using (4.2), viz.

$$\begin{aligned} V_2\left(\xi\right) = & \, K + \frac{3\epsilon^2}{r_0^{(0)4}}\left(m^2 - \frac{1}{4}\right) + \frac{\mathscr{I}}{2 r_0^{(0)2}}\left[\frac{r_0^{(0)}}{\sigma\left(\xi\right)\,\lambda_2} - 1\right] \\ & + \mathscr{O}\left[\frac{\sigma\left(\xi\right)}{r_0^{(0)}}\right], \end{aligned} \tag{4.9}$$

where, for the $k = 0$ mode whose 'center of mass' is located close to the bottom of the potential well, we have $\lambda_2 \gtrsim \sqrt{\pi}$. For higher modes, $k = 1, 2, 3, \ldots$, the wavefunction χ_k, and consequently the effective potential V, features several minima and maxima close to each other, with a separation $\sim \sigma$, and the simple expansion (3.5) appears to be inappropriate. A qualitatively viable solution is obtained if we use instead an effective wake potential $\overline{U}_w$, calculated as a 'moving average' across the wavefunction, that possesses only one minimum. Then using the leading order part of (4.9) in σ/r_0, viz.

$$V_2(\xi) \approx \frac{\mathscr{I}}{2r_0^{(0)}\sigma(\xi)\lambda_2}, \tag{4.10}$$

the Ermakov–Pinney equation (4.8) takes the form

$$\sigma''(\xi) + \frac{\mathscr{I}}{2r_0^{(0)}\lambda_2} - \frac{\epsilon^2}{4\,\sigma(\xi)^3} = 0, \tag{4.11}$$

whose simplest solution is the stationary state $\sigma = s =$ constant, where

$$s = \left(\epsilon^2 r_0^{(0)}\lambda_2/2\mathscr{I}\right)^{\frac{1}{3}}. \tag{4.12}$$

For an infinitesimally small deviation from the stationary state (4.12), setting $\sigma = s + d\sigma$ and solving with the accuracy of the first order in $d\sigma$, (4.11) yields a solution that is oscillating with a 'breathing frequency' $\Omega_0^{(B)}$, viz.

$$\sigma = s + \delta s\,\cos\left(\Omega_0^{(B)}\xi\right), \quad \text{where} \quad \Omega_0^{(B)} = \frac{\sqrt{3}\,\epsilon}{2s^2}, \tag{4.13}$$

and δs is an arbitrary constant satisfying $\delta s \ll s$. Such oscillations of the characteristic length σ give rise to the characteristic 'breathing' (i.e. periodic swellings and contractions) of the wavefunction.

In the case of finite deviations from the stationary state (4.12), the 'breathing frequency' $\Omega^{(B)}$ is determined by the integration of the Ermakov–Pinney equation (4.11) in quadratures, which yields

$$\left(\frac{d\sigma}{d\xi}\right)^2 = \sigma_0^2\eta_0^2 - \frac{\mathscr{I}}{\lambda\,r_0^{(0)}}(\sigma - \sigma_0) - \frac{\epsilon^2}{4}\left(\frac{1}{\sigma^2} - \frac{1}{\sigma_0^2}\right), \tag{4.14}$$

where the initial values $\sigma_0 = \sigma(\xi = 0)$ and $\eta_0 = \eta(\xi = 0)$ are arbitrary constants. Without loss of generality, we may adopt $\eta_0 = 0$ (which corresponds to a particular choice of the initial 'time' $\xi = 0$) and integrate (4.14) as

$$\xi = \left(\frac{\lambda\,r_0^{(0)}}{\mathscr{I}}\right)^{\frac{1}{2}} \int_{\sigma_0}^{\sigma} \frac{|\sigma'\,d\sigma'|}{\sqrt{(\sigma_0 - \sigma')(\sigma' - \sigma_1)(\sigma' - \sigma_2)}}, \tag{4.15}$$

where the roots $\sigma_{1,2}$ are given by

$$\sigma_{1,2} = \left(s^3/4\sigma_0^2\right)\left(1 \pm \sqrt{1 + 8\,\sigma_0^3/s^3}\right). \tag{4.16}$$

Without further loss of generality, we may also take $\sigma_0 \geqslant s$, when (4.16) gives the ordering of the roots $\sigma_0 \geqslant s \geqslant \sigma_1 \geqslant 0 \geqslant \sigma_2$, and the square root in (4.15) is a real quantity when $\sigma_0 \geqslant \sigma \geqslant \sigma_1$. Conversely, if we adopt the initial value σ_0 in the domain $s > \sigma_0 \geqslant 0$ (or $\sigma_0 < 0$), we obtain the same result as above when we exchange σ_0 and σ_1 (or σ_0 and σ_2 respectively). In the special case $\sigma_0 = s$, there is a double root $\sigma_0 = \sigma_1 = s$, with $\sigma_2 = -s/2$, corresponding to a stationary solution $\sigma(\xi) = s =$ constant.

Introducing the change of variables $t' = \sqrt{(\sigma_0 - \sigma')/(\sigma_0 - \sigma_1)}$ and the parameter $\mu = \sqrt{(\sigma_0 - \sigma_1)/(\sigma_0 - \sigma_2)}$, (4.15) is rewritten as

$$\xi = \left(\frac{\lambda\,r_0^{(0)}}{\mathscr{I}}\right)^{\frac{1}{2}} \frac{2}{\sqrt{(1-\mu^2)(\sigma_1 - \sigma_2)}} \int_0^t dt' \times \left(\sigma_1\frac{\sqrt{1-\mu^2 t'^2}}{\sqrt{1-t'^2}} - \mu^2\sigma_2\frac{\sqrt{1-t'^2}}{\sqrt{1-\mu^2 t'^2}}\right), \tag{4.17}$$

where the upper boundary is given by $t = \sqrt{(\sigma_0 - \sigma)/(\sigma_0 - \sigma_1)}$. The integrals in (4.17) are readily evaluated as

$$\int_0^t dt'\,\frac{\sqrt{1-\mu^2 t'^2}}{\sqrt{1-t'^2}} = E\left[\sin^{-1}(t)\middle|\,\mu^2\right] \quad \text{and}$$

$$\int_0^t dt'\,\frac{\sqrt{1-t'^2}}{\sqrt{1-\mu^2 t'^2}} = E\left[\sin^{-1}(\mu\,t)\middle|\,\frac{1}{\mu^2}\right], \tag{4.18}$$

where $E(\phi|n) = \int_0^\phi d\vartheta\sqrt{1 - n\sin^2\vartheta}$ is the elliptic integral of the second kind. The evolution equation (4.15) describes a periodic motion whose half-period $T_{1/2}$ is identified as the 'time' ξ required for propagation from $\sigma' = \sigma_0$ to $\sigma' = \sigma_1$ (i.e. from $t' = 0$ to $t' = 1$ in (4.17) and (4.18)). The dependence of the corresponding breathing frequency $\Omega^{(B)} = \pi/T_{1/2}$ on the initial value σ_0 can be easily found by the numerical integration of (4.17), revealing a nearly linear dependence and the possibility of a parametric instability when the ratio $\Omega^{(B)}/\omega_0$ has an integer value. The most likely candidate is $\Omega^{(B)}/\omega_0 = 2$, obtained when $\sigma_0/s = 1.0757$.

The solution presented in this section has the form of a hollow beam, which can be identified as a breathing/wiggling *ring soliton*, displayed in Fig. 1. Our procedure is not applicable if the beam is located exactly at the axis, $\bar{r} = 0$, when the function $\log r'$ in (3.2) is rapidly varying across the beam and the applied separation of spatial scales is not possible.

5. Parametric instability of a ring soliton

Using the leading order expression (4.10) for the coefficient of restoring force and (3.6), the position of the center of mass of the wave packet takes the form

$$\bar{r}(\xi) = r_0^{(0)} + \delta\bar{r}_0\,\frac{\sigma(\xi)}{\sigma(0)}\,\cos\left[\int_0^\xi \frac{\epsilon\,d\xi'}{2\,\sigma(\xi')^2}\right] + \frac{\mathscr{I}\,\sigma(\xi)}{\epsilon\,r_0^{(0)}\lambda_2}\int_0^\xi d\xi'\;\delta r_0(\xi')\;\sin\left[\int_{\xi'}^{\xi}\frac{\epsilon\,d\xi''}{2\sigma(\xi'')^2}\right]. \tag{5.1}$$

We can see from above that the center of mass may propagate to a large distance if the quantities $\delta r_0(\xi')$ and $\sin[\int_{\xi'}^{\xi} d\xi''\,\epsilon/2\sigma(\xi'')^2]$ in (5.1) are in resonance with each other. Then the motion of the external potential produces an additional force on the wave packet during each cycle, which gives rise to a secular growth of its

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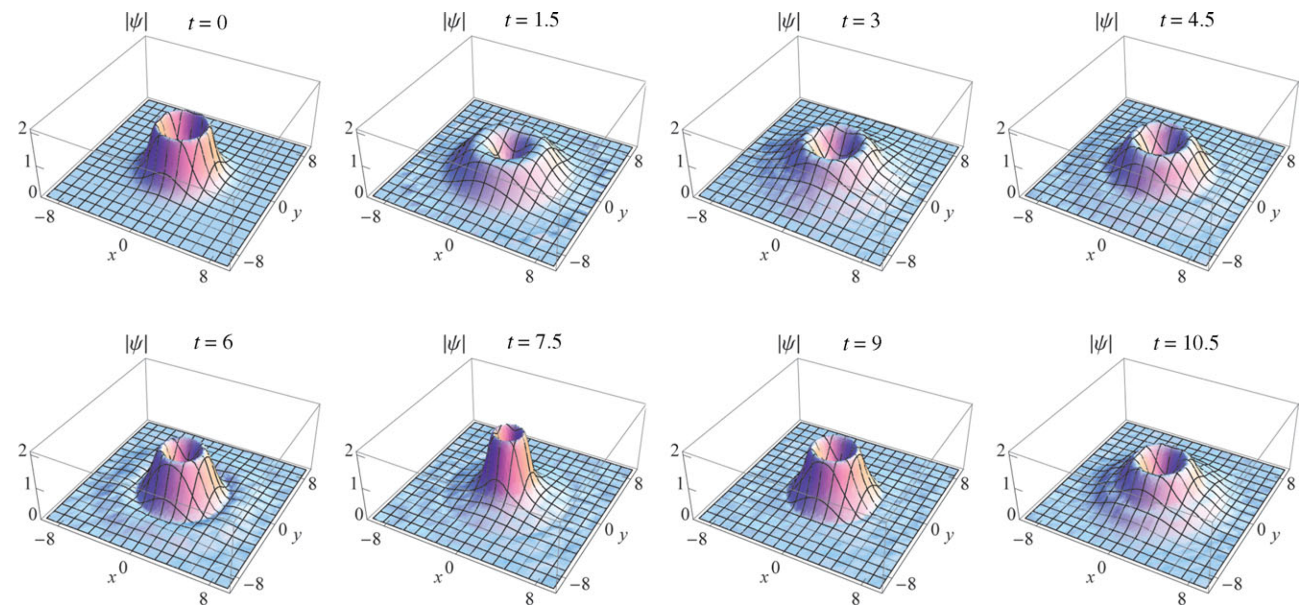

Figure 1. (Colour online) The evolution of a typical Glauber mode, found as the numerical solution of the (3.4) for a linear quantum oscillator $2i\,\partial\chi/\partial\xi = \partial^2\chi/\partial r^2 - r^2\chi$. The initial condition has the form of a Gaussian, displaced from the bottom of the potential well, $\chi(0,r) = \exp[-(r-5)^2/2]$.

position $\bar{r}$. Such behavior is recognized as the parametric instability of the ring soliton (4.2).

The characteristic frequency of the motion of the effective potential can be estimated as follows. For the Hermite–Gaussian states, the quantity Δ_1 (3.6) is evaluated as

$$\Delta_1 = -\frac{1}{\sqrt{\pi}\,2^k k!}\int_0^{\frac{r_0(\xi)-\bar{r}(\xi)}{\sqrt{2}\sigma(\xi)}} d\rho\,\exp\left(-\rho^2\right)\,H_k\left(\rho\right)^2, \quad (5.2)$$

and from (3.9) we see that the location of the potential minimum, δr_0, depends on the time-like variable ξ only through the combination $[r_0(\xi)-\bar{r}(\xi)]/\sigma(\xi)$, viz.

$$\delta r_0\left(\xi\right) = \frac{\mathscr{I}}{r_0^{(0)}}\left[\frac{dV_1^{(0)}(r_0^{(0)})}{dr_0^{(0)}}\right]^{-1}\Delta_1\left[\frac{r_0\left(\xi\right)-\bar{r}\left(\xi\right)}{\sqrt{2}\,\sigma\left(\xi\right)}\right]. \quad (5.3)$$

Since Δ_1 is a monotonous function, (5.3) indicates that $\delta r_0(\xi)$ has the same periodicity as the argument of the function Δ_1. We note that the numerator $r_0(\xi) - \bar{r}(\xi)$ oscillates with the characteristic wiggling frequency $\Omega^{(W)} \sim \epsilon/2\sigma$ (see (4.7)), while the denominator is an elliptic function (see (4.17) and (4.18)), which can be written as $\sigma(\xi) = \langle\sigma\rangle + \tilde{\sigma}(\xi)$, where $\langle\sigma\rangle$ = constant and $\tilde{\sigma}(\xi)$ oscillates with the breathing frequency $\Omega^{(B)} \gtrsim \sqrt{3}\,\epsilon/2\sigma$, see (4.13). Thus, the 'temporal' evolution of $\delta r_0(\xi)$ is, to the leading order, distinguished by the wiggling (with the characteristic frequency $\Omega^{(W)}$) and the beating between the breathing and the wiggling (with the characteristic frequency $\Omega^{(B)} \mp \Omega^{(W)}$). Both these processes can be resonant of Green's function $\sim \sin[\int_{\xi'}^{\xi} d\xi''\,\epsilon/2\sigma(\xi'')^2]$ inside the integral in (5.1) and can give rise to a secular growth of excursions of the wave packet, i.e. to the parametric instability of coherent state. The growth rate Γ of such parametric instability is estimated by balancing the perturbation of the left-hand side of (3.4) with the nonlinear drive due to the motion of the potential V, viz. $\Gamma\chi \sim (dr_0/d\xi)(\partial\chi/\partial r)$, together with (3.9), the strong localization condition, and the orderings $dV_1^{(0)}/dr_0 \sim 4K$ and $r_0^{(0)} \sim (m^2\epsilon^2/K)^{\frac{1}{4}}$ give

$$\frac{\Gamma}{\Omega^{(W)}} \sim \frac{1}{4\sqrt{2\pi}}\,\frac{\mathscr{I}}{\epsilon K^{\frac{1}{2}}}\,\frac{\delta\bar{r}_0}{\sigma}. \quad (5.4)$$

It should be noted that expansion (3.5) is not valid when the amplitude of excursions exceeds the width of the wave packet, $\delta\bar{r}_0 \geqslant \sigma$. The simple estimate (4.10) for the coefficient V_2, used in the integration of the Pinney–Ermakov equation, also becomes inaccurate. Furthermore, in the regime $\delta\bar{r}_0 \gtrsim \sigma$ the growthrate Γ of the parametric instability becomes comparable with the characteristic frequency of the Hermite–Gauss mode $d\phi_k/d\xi \sim \epsilon/4\sigma^2$. In such a case, a full set of equations is to be used to describe the evolution of a relativistic particle beam, which is a more demanding task.

6. Conclusions

We have self-consistently studied the strongly non-local plasma wakefield response driven by a relativistic cold electron (or positron) beam in a cold, overdense, strongly magnetized plasma, accounting for the individual quantum nature of beam particles. The density, energy, and temperature of the system were assumed to be such that the overlapping of particle wave functions was negligible. Within the Hartree's mean field approximation, it has been shown that the system manifests the individual quantum nature at the macroscopic level leading to the formation of coherent structures. For a sufficiently long beam, the self-consistent non-local plasma wakefield response is governed by a Poisson-like equation for the wake potential and the spinorial Schrödinger equation.

In the cylindrical symmetry, the spinorial equation has been reduced to a scalar Schrödinger equation for a wavefunction whose squared modulus is proportional to the beam density profile. These governing equations have a strong non-local character when the beam spot size is much smaller than the characteristic magnetized plasma wavelength, i.e. for $\nabla_\perp^2 \gg \omega_{pe}^4/(c^2\,\omega_{UH}^2)$.

Employing the procedure proposed by Krolikowski et al. (2004) and extensively used in nonlinear optics, we have obtained an analytic, fully nonlinear solution in the form of a hollow beam. These ring solitons are similar to the self-similar structures known as *accessible solitons* in nonlinear optics (Briedis et al. 2005; He et al. 2008; Belić and Zhong 2009; Zhang and Yi 2009). However, due to the presence of the singularity of the non-local response function in the center of a relativistic particle beam, the strongly non-local analogue of the fundamental 2D optical solitons has not been found in our case. We have demonstrated that the ring solitions in particle beams, besides stable stationary states, may also feature the 'breathing' (the oscillations of their width) and the 'wiggling' (oscillations around the equilibrium position) coherent states. These motions may resonantly couple with the natural frequency of the ring soliton, yielding the parametric instability and the destruction of the coherence of particle bunch. The conditions for such resonant coupling and the resulting growth rates have been estimated.

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