PARABOLIC DIFFERENTIATION

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1. Introduction.

1.1. The purpose of this paper is the study of parabolically differentiable points of arcs in the real affine plane. In Section 2, two different definitions of convergence of a family of parabolas are given and it is observed (Theorem 1) that these are equivalent. In Section 3, tangent parabolas at a point p of an arc A are discussed and it is proved (Theorem 2) that all the non-degenerate non-tangent parabolas of A through p intersect A at p or that all of them support. In Section 4, osculating parabolas are introduced and the condition that an arc be twice parabolically differentiable at a point p is stated. Theorems 3, 4, 5, and 7 are concerned with properties of the osculating parabolas and it is proved (Theorem 6) that the non-osculating tangent parabolas of an arc A at p all support A at p. In Section 5, the third requirement for parabolic differentiability is given and the superosculating parabola is introduced. In Theorem 8, it is proved that if the third differentiability condition is satisfied at an interior point p, then the non-superosculating, non-degenerate osculating parabolas all support A at p or all intersect according as A has or has not a cusp at p.

The discussion is related to that in **(1)** on conformal differentiation but there are several differences. For example, while circular differentiation involves only two differentiability conditions, parabolic differentiation involves three. Three distinct points in the conformal plane determine a unique circle but if there is one parabola through four distinct points in the affine plane, there is, in general, a second one. The conformal plane is compact while the affine plane is not. The interior and the exterior of a circle are topologically equivalent in the conformal plane but there is an essential difference between the inside and the outside of a non-degenerate parabola in the affine plane. Finally, there is only one degenerate circle in the conformal plane, namely a point, but there are four types of degenerate parabolas which must be considered in the affine plane.

1.2. Notation. Some properties of parabolas. The letters p, t, $Q, \ldots$ usually denote points in the real affine plane, with the small italics indicating points of arcs. Gothic letters $\mathfrak{L}, \ldots$ denote lines. A parabola (which may be degenerate) will be denoted by π. In particular, a double ray (i.e., a ray counted

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twice), a single line, a pair of parallel lines, and a double line (i.e., a line counted twice) will also be included in the term parabola. A non-degenerate parabola π has a well-defined interior π_* and an exterior π^*, while a ray counted twice has an exterior but no interior. The interior of an oriented line counted once may be defined to lie to its left. The interior of a pair of parallel lines is the region between the parallels.

1.3. In affine co-ordinates the equation $y = cx^2$ represents a pencil of parabolas which touch the x-axis at the origin. As c becomes large through positive values the parabolas tend to the double ray $x^2 = 0$, $y \geqslant 0$. As c tends to 0 the parabolas tend to the line $y = 0$.

The equation $x^2 = ay + 1$ represents a family of parabolas through the points $(\pm 1, 0)$. As a tends to zero the parabolas tend to the pair of parallels parallel to $x = 0$.

The equation $x^2 = a^2y + a$ represents a family of parabolas which tend to the double line $x^2 = 0$ as a tends to zero through positive values. As a tends to zero through negative values, however, these parabolas do not have a limit.

1.4. Parabolas constitute a four-parameter family of conics; but four points in the real affine plane do not always determine a parabola.

Let C be the convex hull of the four points. If C is a strictly convex quadrangle then there are exactly two parabolas through the points. These are, in general, non-degenerate but when C is a trapezoid but not a parallelogram, one of them is non-degenerate and the other is a pair of parallel lines, and when C is a parallelogram they are both pairs of parallel lines.

If C is a non-degenerate triangle with all four points on its boundary there is exactly one parabola, viz., the line through the three collinear points and its parallel through the fourth point.

If C is a triangle containing one of the four points in its interior there is no parabola through them.

Finally, if C is a line segment then there are infinitely many degenerate parabolas through C, namely, any double ray through C, the line through C, the double line through C, and the line through C together with any other parallel line.

1.5. Associated with each non-degenerate parabola π is the pencil of parallel lines each of which meets π exactly once. We shall call this the *diametral pencil* of π. The line of this pencil through a point P is called the *diameter* of π through P.

The linear family of parabolas, each of which touches a given non-degenerate parabola π at the same point and has the same diametral pencil of lines will be called a *diametral pencil of parabolas.*

2. Convergence.

2.1. A *neighbourhood* of a point P is the interior of an ellipse which contains P in its interior.

A sequence of points $\{P_n\}$ is defined to be *convergent* to a point P if every neighbourhood of P contains P_n for all but a finite number of n.

A point P is defined to be an *accumulation point* [a *limit point*] of a sequence of parabolas $\{\pi_n\}$ if every neighbourhood of P contains points of π_n for infinitely many n [for all but a finite number of n].

A sequence of parabolas $\{\pi_n\}$ is defined to be (*pointwise*) *convergent* if it has at least one accumulation point and every accumulation point is a limit point of $\{\pi_n\}$.

We state the following lemmas without proof.

LEMMA 1. *The set consisting of the limit points of a convergent sequence of non-degenerate parabolas is a non-degenerate parabola or a line or a ray or a pair of parallel lines.*

LEMMA 2. *Every infinite sequence $\{\pi_n\}$ of non-degenerate parabolas which meet a bounded region has a convergent subsequence.*

2.2. Let π be a non-degenerate parabola and let P, Q, R, S, U be any five points of π. Thus the quadrangle $PQRS$ is strictly convex. Let P_n, Q_n, R_n, S_n, U_n converge to P, Q, R, S, U respectively. Thus the quadrangle $P_nQ_nR_nS_n$ is also convex if n is sufficiently large. Hence there are two parabolas π_n and π_n' through P_n, Q_n, R_n, S_n. Let π_0 be any limit parabola of $\{\pi_n\}$ or $\{\pi_n'\}$. Thus π_0 passes through P, Q, R, S. Hence π_0 is equal either to π or to the second parabola through P, Q, R, S. If, in addition, $U_n \in \pi_n$, then $U \in \pi$ and hence $\pi_0 = \pi$.

The above discussion remains valid if π is a pair of parallel lines with P and Q on one of them and R and S on the other.

LEMMA 3. *If P, Q, R, S is a strictly convex quadrangle and P_n, Q_n, R_n, S_n converge to P, Q, R, S respectively, then the two parabolas through P_n, Q_n, R_n, S_n converge to the two parabolas through P, Q, R, S.*

2.3. One can verify that if $\{\pi_n\}$ converges to a line π, then π_n meets any line $\mathfrak{L}$ which is not parallel to π in at least one point which converges to $\pi \cap \mathfrak{L}$. If there exists exactly one point of $\pi_n \cap \mathfrak{L}$ which converges to $\pi \cap \mathfrak{L}$, we say that $\{\pi_n\}$ *converges to the single line* π but if there are two points of $\pi_n \cap \mathfrak{L}$ which converge to $\pi \cap \mathfrak{L}$ we say that $\{\pi_n\}$ *converges to the double line* π.

It can be shown that the above property is independent of the choice of $\mathfrak{L} \nparallel \pi$.

2.4. In order to extend our concept of convergence we introduce *neighbourhoods of parabolas*.

A *neighbourhood of a non-degenerate parabola* π is the region which lies outside an ellipse in π_* and inside a hyperbola branch which contains π in its interior.

A *neighbourhood of a double ray* π is the inside of a hyperbola branch which contains π in its interior.

A *neighbourhood of a pair of parallel lines* is the common exterior of an ellipse which lies between the lines and a hyperbola which does not meet them.

A *neighbourhood of a double line* π is the exterior of a hyperbola which does not meet π.

A *neighbourhood of a single line* π is the common exterior of two ellipses which are separated by π.

2.5. A sequence of parabolas $\{\pi_n\}$ is called *globally convergent* to a parabola π if every neighbourhood of π contains π_n for all but a finite number of n and the following additional condition holds.

1. If π is non-degenerate or a pair of parallel lines, π_n contains the ellipse of the neighbourhood of π in its interior.
2. If π is a double ray, the vertex of π is a limit point of $\{\pi_n\}$.
3. If π is a double line, π_n intersects every line which is not parallel to π twice.
4. If π is a single line, π_n contains exactly one of the ellipses of the neighbourhood of π in its interior.

2.6. We need the following observations. Suppose that $\{\pi_n\}$ is globally convergent to π. Let n be sufficiently large.

1. If π is non-degenerate, then π_n is non-degenerate.
2. If π is a double ray, then π_n is non-degenerate or a double ray.
3. If π is a pair of parallel lines, then π_n is either non-degenerate or a pair of parallel lines.
4. If π is a double line, then π_n is not a single line.
5. If π is a single line, then π_n is neither a double ray nor a double line.

2.7. Suppose that every neighbourhood of π contains π_n for all but a finite number of n. It can readily be proved that if π is non-degenerate and π_n has a limit point on π or if π is a pair of parallel lines and $\{\pi_n\}$ has a limit point on each of them, then $\{\pi_n\}$ is globally convergent to π.

In this paper we shall be concerned mainly with sequences $\{\pi_n\}$ of non-degenerate parabolas each of which passes through a fixed point p. π will usually be non-degenerate or a double ray with the vertex p or a pair of parallel lines, one of which contains p while the other contains a limit point of $\{\pi_n\}$. Thus the initial assumption is sufficient to ensure the global convergence of $\{\pi_n\}$ to π.

2.8. One can prove the following theorem by considering the various cases separately.

THEOREM 1. *A sequence of parabolas* $\{\pi_n\}$ *is globally convergent to a parabola* π *if and only if* $\{\pi_n\}$ *is pointwise convergent to* π.

We include a proof of the case of Theorem 1 where π is non-degenerate.

(I) Let $\{\pi_n\}$ be globally convergent to π. Let $P \in \pi$. Let N be any neighbourhood of P. Let (ϵ, η) be a neighbourhood of π such that ϵ and η both meet N. By Section 2.5, $\epsilon \subset \pi_{n*}$ for all large n. Hence π_n meets N.

(II) Let $\{\pi_n\}$ be pointwise convergent to π. Let (ϵ, η) be any neighbourhood of π. We wish to show that $\pi_n \subset \epsilon^* \cap \eta_*$ and $\epsilon \subset \pi_{n*}$ for all large n.

1. If π_n meets ϵ for all large n, then these points will have an accumulation point, say, $Q \subset \pi \cap \epsilon$. Since this is impossible, π_n cannot meet ϵ when n is large. Thus $\pi_n \subset \epsilon^*$.

2. Let P, R, and R' be points of π such that PR and PR' are parallel to the asymptotes of η. If π_n meets η at Q_n for all large n, then the $\{Q_n\}$ cannot have a limit point on $\pi \cap \eta$. Hence the line P_nQ_n must converge to the line PR or PR'. However, the line P_nQ_n converges to the diameter of π through P. Since this is impossible, we conclude that $\pi_n \cap \eta = \emptyset$, for all large n.

Since π_n contains $P_n \subset \eta_*$, $\pi_n \subset \eta_*$.

3. Let $P, R \in \pi$ be such that PR meets ϵ_*. Let M and N be small neighbourhoods of P and R. If n is large, π_n contains points P_n and R_n in M and N and the line segment P_nR_n will meet ϵ. Hence ϵ contains points $\subset \pi_{n*}$. Since $\pi_n \cap \epsilon = \emptyset$, it follows that $\epsilon \subset \pi_{n*}$.

From now on a sequence $\{\pi_n\}$ which is either globally or pointwise convergent to π (and thus both globally and pointwise convergent to π) will be called *convergent* to π.

2.9. It may occur to the reader that the rather elaborate neighbourhood system which has been introduced in Section 2 could be replaced by the topology defined by regarding the parabolas $(ax + by)^2 + cx + dy + e = 0$ as points of a projective 4-space: $(a, b, c, d, e) \neq (0, 0, 0, 0, 1)$. This correspondence does not take care of the double rays, however, and the induced topology would not be identical with ours. In particular, as $a \to 0$ the points $(1, 0, 0, a, 0)$ in the projective 4-space converge to the unique point $(1, 0, 0, 0, 0)$, which we have to associate with the double line $x^2 = 0$. On the other hand, if $a \to 0$, $a > 0$, the parabolas $x^2 + ay = 0$ converge to the double ray $x^2 = 0$, $y \leqslant 0$, but if $a \to 0$, $a < 0$, they converge to the opposite double ray, $x^2 = 0$, $y \geqslant 0$.

The last example in Section 1.3 also shows that the two topologies are different.

2.10. Let π be a non-degenerate parabola. Let η be a hyperbola branch containing π in its interior and let ϵ be an ellipse in π_*. Any continuous arc joining a point on ϵ to a point on η will meet π.

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2.11. Let P be a point of π. A small neighbourhood N of P is decomposed by π into two regions unless P is an end-point of π. Let π' be any parabola through P. If $\pi' \cap N - P$ meets both regions [exactly one region], P is called a point of intersection [support] of π and π'. This language can be justified by observing that this relation is actually symmetric in π and π'.

The proof of the following lemmas is based on the properties of global convergence.

LEMMA 4. *Let $\{\pi_n\}$ converge to a non-degenerate parabola π and let $\{\pi_n'\}$ converge to π' where π' is non-degenerate, a pair of parallels, or a single line. If π and π' intersect in a point Q, then π_n and π_n' intersect in a point Q_n close to Q when n is large.*

LEMMA 5. *Let $\{\pi_n\}$ converge to a non-degenerate parabola π and let $\{\pi_n'\}$ converge to a double ray π'. Suppose that $P \in \pi \cap \pi'$ and P is the vertex of π'.*

If $\pi' \subset \pi_* \cup P$, then π_n and π_n' either do not meet outside a small neighbourhood N of P or else they meet twice outside N.

If π' meets π at a point $Q \neq P$, then π_n and π_n' meet in two distinct points near Q (unless π_n' is a double ray).

LEMMA 6. *Let $\{\pi_n\}$ converge to a pair of parallels π and let $\{\pi_n'\}$ converge to a double ray π'. Suppose that $P \in \pi \cap \pi'$ and P is the vertex of π'. If π' meets π at Q, where P and Q do not lie on the same parallel of π, then π_n and π_n' meet at two distinct points near Q (unless π_n' is a double ray).*

LEMMA 7. *Let $\{\pi_n\}$ converge to π and let $\{\pi_n'\}$ converge to π' where π and π' are both non-degenerate. Suppose that $P \in \pi \cap \pi'$. If $\pi' \subset \pi_* \cup P$, then π_n and π_n' either do not meet outside a small neighbourhood N of P or else they meet twice outside N.*

We illustrate the type of proof used in Lemmas 4–7 by considering the case of Lemma 4 where π and π' are non-degenerate.

Let N be a small neighbourhood of Q which contains no other point of $\pi \cap \pi'$. Let (ϵ, η) and (ϵ', η') be neighbourhoods of π and π' respectively such that ϵ' and η' meet ϵ and η inside N to form a curvilinear quadrilateral in N. For all large n, $\pi_n \in (\epsilon, \eta)$ and $\pi_n' \in (\epsilon', \eta')$. Since π_n meets one pair of opposite arcs of this quadrilateral, π_n meets π_n' in N; cf. Section 2.10.

2.12. LEMMA 8. *Suppose that a sequence of parabolas $\{\pi_n\}$ converges to a parabola π which is not a double line or a double ray and P_n and Q_n converge to P; $P_n \in \pi_n$, $Q_n \in \pi_n$, $P_n \neq Q_n$ (thus $P \in \pi$). Then P_nQ_n converges to the tangent of π at P.* (We define the tangent to a single line or one of the lines of a pair of parallels to be the line itself.)

2.13. Let τ be the (two-parameter) family of non-degenerate parabolas which touch a line $\mathfrak{T}$ at p. Thus $\tau = \tau(p, \mathfrak{T})$. We compactify τ by the addition of all its limit parabolas, and denote the compactified family by $\bar{\tau}$. Suppose that a degenerate parabola π is the limit of a sequence $\{\pi_n\}$ of parabolas of τ. Thus π_n lies in one of the closed half-planes determined by $\mathfrak{T}$. We may assume that π_n lies in the same closed half-plane for all large n. Thus the degenerate limit parabola π lies in this closed region and contains p. Hence π is the single line $\mathfrak{T}$, a double ray on $\mathfrak{T}$ through p, a double ray with the vertex p, the double line on $\mathfrak{T}$, or a pair of parallels one of which is $\mathfrak{T}$. We note that no double line which does not lie on $\mathfrak{T}$ belongs to $\bar{\tau}$.

Suppose that the degenerate parabola π is the limit of degenerate parabolas π_n of $\bar{\tau}$ listed above. Each π_n has been constructed as the limit of parabolas of τ. Thus π will also be the limit of parabolas of τ. This verifies that every degenerate limit parabola π of $\bar{\tau}$ can be obtained by the above step and the family $\bar{\tau}$ described above is already compact.

2.14. If the point $Q \notin \mathfrak{T}$, then any diametral pencil in τ will contain exactly one parabola through Q.

2.15. LEMMA 9. *Any two parabolas of τ which have three distinct points in common outside p coincide.*

2.16. The one-parameter subfamily ϕ. Let π_0 be any non-degenerate parabola through p and let $\mathfrak{T}$ denote its tangent at p. Then π_0 determines a one-parameter family $\phi = \phi(\pi_0)$ which consists of π_0 itself and those non-degenerate parabolas which touch $\mathfrak{T}$ at p and meet π_0 exactly once outside p. If $\pi_1 \in \phi(\pi_0)$, then $\phi(\pi_1) = \phi(\pi_0)$. Any two members of ϕ intersect at p. To every straight line $\mathfrak{L} \neq \mathfrak{T}$ through p there corresponds exactly one parabola of ϕ with the diameter $\mathfrak{L}$ through p.

All the parabolas of ϕ lie in the same half-plane. Through each point R of that half-plane there pass exactly two parabolas π_1 and π_2 of ϕ. Let π be any third parabola of ϕ. If $R \in \pi_*$ [$R \in \pi^*$] the diameters through p of π_1 and π_2 are separated [are not separated] by $\mathfrak{T}$ and the diameter $\mathfrak{D}$ of π through p. In the latter case these diameters pass through the region bounded by $\mathfrak{D}$ and $\mathfrak{T}$ which contains R.

If $\mathfrak{T}$ is oriented such that $\pi \subset \mathfrak{T}_* \cup p$, then π decomposes $\mathfrak{T}_*$ into three disjoint regions, one in π_* and the other two in π^*. Any other parabola of ϕ passes through π_* and exactly one of the regions in $\pi^* \cap \mathfrak{T}_*$.

3. Tangent parabolas of arcs.

3.1. Arcs. An *arc* A is defined as the continuous image in the affine plane of a real parameter interval. Thus, if a sequence of points of the parameter interval converges to a point p, the corresponding sequence of image points is defined to be convergent to the image of p. The same letters, $p, t, \ldots$ denote the points of the parameter interval and their images on A. The parameters s and p are supposed to be distinct and s will always be "sufficiently close" to

p. The *end-points* (*interior points*) of A are the respective images of the end-points (interior points) of the parameter interval.

A *neighbourhood* of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods. The images of distinct points of the parameter interval are to be considered to be different points of A even though they may coincide in the plane; nevertheless, the notation $Q \neq R$ will indicate that the points Q and R do not coincide.

3.2. Let p be a given point on an arc A. The arc A is called *differentiable* at p if the following condition is satisfied.

CONDITION I. *If the parameter s is sufficiently close to the parameter p, $s \neq p$, the line ps is uniquely determined. It converges if s tends to p.*

The limit straight line $\mathfrak{T}$ is the ordinary tangent of A at p. From now on, we assume that A satisfies Condition I at p.

We denote the family of non-degenerate parabolas which touch $\mathfrak{T}$ at p by τ and its compactification by $\bar{\tau}$, as in Section 2.13.

3.3. Suppose that the points p, Q, R are mutually distinct. Let s be a point of A. It can happen that there are parabolas through s, p, Q, R for all s sufficiently close to p. Then these parabolas have limits as s tends to p. Any such limit parabola π will pass through Q and R and, with an exception to be noted later on, will touch $\mathfrak{T}$ at p. Thus, neglecting this exception, π will belong to the compactified family $\bar{\tau}$ of all the parabolas which touch $\mathfrak{T}$ at p. We then call π a *tangent parabola of A at p.*

The next four sections show that any parabola of τ and some of the degenerate ones of $\bar{\tau}$ are tangent parabolas of A at p. It will be convenient to designate also the remaining parabolas of $\bar{\tau}$ as tangent parabolas.

3.4. If p, Q, R are not collinear and Q and R lie on the same side of $\mathfrak{T}$ the quadrangle $spQR$ will be convex when s is close to p. Hence there will be two parabolas π_1 and π_2 through these four points. When s tends to p, any limit parabola π will pass through p, Q, R and, by Lemma 8, will touch the limit $\mathfrak{T}$ of the line ps. Hence $\pi \in \bar{\tau}$ and π is one of the two parabolas of $\bar{\tau}$ which pass through Q and R. Both of these will be non-degenerate unless QR is parallel to $\mathfrak{T}$, in which case one of them will be a pair of parallel lines.

We denote the parabolas of $\bar{\tau}$ through Q and R by $\pi_i\,(\bar{\tau}; Q, R)$; $i = 1, 2$.

We may assume that the diameter through p of π_1 [π_2] passes through the pair of open regions defined by the lines pQ and pR which contains s [does not contain s]. This readily implies that the diameter through p of $\lim \pi_1$ [$\lim \pi_2$] passes through the same pair of open regions as s tends to p. Thus

$$\lim \pi_1 \neq \lim \pi_2.$$

3.5. If p, Q, R are not collinear and Q and R are separated by $\mathfrak{T}$, there is no parabola $psQR$ when s is close to p and there are no tangent parabolas of A at p through Q and R.

3.6. If Q lies on $\mathfrak{T}$ and R does not, there are two parabolas through $psQR$ when s is close to p provided that the line ps does not meet the open segment QR. If s lies on $\mathfrak{T}$, there is only one parabola, namely, $\mathfrak{T}$, and its parallel through R.

When s tends to p, any limit parabola π will pass through Q and R and will touch the limit of the line ps at p. Hence π is the unique parabola of $\bar{\tau}$ consisting of $\mathfrak{T}$ and its parallel through R.

3.7. Finally, let p, Q, R be distinct points on a line $\mathfrak{L}$. If $s \notin \mathfrak{T}$ in the case where $\mathfrak{L} = \mathfrak{T}$, there is only one parabola through $psQR$, viz., the line pQ and its parallel through s. It converges to the double line pQ when s tends to p. By the definition of $\bar{\tau}$ these double lines are not included in $\bar{\tau}$, except when $\mathfrak{L} = \mathfrak{T}$. If $\mathfrak{L} = \mathfrak{T}$ and there is a sequence of points $s \in A \cap \mathfrak{T}$ converging to p, then the parabola through $spQR$ is not uniquely defined.

If Q and R are collinear with p and lie on opposite sides of $\mathfrak{T}$ the tangent parabolas of A at p through Q and R will not be defined.

If Q and R lie on the same side of p, suppose that $\{R_n\}$ converges to R, $R_n \notin pQ$. Thus, both of the tangent parabolas of A at p through Q and R_n converge to the double ray through Q with the vertex p. It is convenient, in this case, to define this double ray to be the tangent parabola of A at p through Q and R but the reader should bear in mind that it cannot be obtained as the limit of parabolas through p, s, Q, R as s tends to p.

Thus, if p, Q, R are any mutually distinct points which lie in the same closed half-plane bounded by $\mathfrak{T}$, then the pair of tangent parabolas of A at p through Q_n and R_n always tend to the tangent parabolas of A at p through Q and R as Q_n tends to Q and R_n tends to R.

3.8. Non-tangent parabolas. Suppose that p is an interior point of an arc A. Then p is called a *point of support* [*intersection*] with respect to a parabola π if a sufficiently small neighbourhood of p on A is decomposed by p into two one-sided neighbourhoods which lie in the same region [in different regions] bounded by π. The parabola π is then called a supporting [intersecting] parabola of A at p. In particular, π supports A at p if $p \notin \pi$. It can happen that every neighbourhood of p on A has points $\neq p$ in common with π. Then π neither supports nor intersects A at p.

3.9. Let p be an interior point of an arc A. Suppose again that A satisfies Condition I at p.

LEMMA 10. *The lines $\neq \mathfrak{T}$ through p either all support A at p or they all intersect.*

It is well known that if A is differentiable at p, then A satisfies Lemma 10. For the reader's convenience, we include a proof.

Proof. If $\mathfrak{L}$ neither supports nor intersects A at p, then Condition I implies that $\mathfrak{L} = \mathfrak{T}$. Let $\mathfrak{L}$ and $\mathfrak{M}$ be two distinct lines through p. Suppose, for example, that $\mathfrak{L}$ intersects A at p and $\mathfrak{M}$ supports. We may assume that $A \subset \mathfrak{M}^* \cup p$. If $s \subset A \cap \mathfrak{L}_*$, the line sp lies in $(\mathfrak{L}_* \cap \mathfrak{M}^*) \cup (\mathfrak{L}^* \cap \mathfrak{M}_*) \cup p$. Letting $s \to p$ through $A \cap \mathfrak{L}_*$, we conclude that $\mathfrak{T}$ lies in the closure of this set. Symmetrically, letting $s \to p$ through $A \cap \mathfrak{L}^*$, we see that $\mathfrak{T}$ lies in the closure of $(\mathfrak{L}_* \cap \mathfrak{M}_*) \cup (\mathfrak{L}^* \cap \mathfrak{M}^*) \cup p$. Hence $\mathfrak{T} = \mathfrak{L}$ or $\mathfrak{M}$.

LEMMA 11. *If a non-tangent line $\mathfrak{L}$ through p intersects [supports] A at p, then every non-degenerate parabola π which touches $\mathfrak{L}$ at p also interects [supports] A at p.*

Proof. Let $\pi \subset \mathfrak{L}_* \cup p$, say. Let Q and R be any two points on π on opposite sides of $\mathfrak{T}$. Thus the lines pQ and pR separate $\mathfrak{L}$ and $\mathfrak{T}$. If N is a sufficiently small neighbourhood of p on A and $s \in N - p$, then the line ps meets the open segment QR and s lies inside the triangle pQR or in $\mathfrak{L}^*$. Hence s does not lie in the closure of either of the two regions $\mathfrak{L}_* \cap \pi^*$.

If $N - p$ lies in $\pi_* \cap \mathfrak{L}_*$ or in $\pi^* \cap \mathfrak{L}^*$, then π and $\mathfrak{L}$ both support A at p; otherwise π and $\mathfrak{L}$ both intersect A at p.

Lemmas 10 and 11 imply the following result.

THEOREM 2. *The non-degenerate non-tangent parabolas of A through p all intersect A at p or all of them support.*

3.10. Let $\{\pi(s)\}$ and $\{\pi'(s)\}$ be two sequences of parabolas of τ through s which converge to π and π' respectively as s tends to p. Here, we let s range through a certain sequence of points.

LEMMA 12. *If π has a point in $\mathfrak{T}_*$, then $s \subset \mathfrak{T}_*$.*

Proof. $\pi \subset \mathfrak{T}_* \cup \mathfrak{T}$, $\pi \not\subset \mathfrak{T}$ implies $\pi(s) \subset \mathfrak{T}_* \cup p$ and $s \subset \mathfrak{T}_*$.

COROLLARY. *If $\pi \subset \mathfrak{T}_* \cup \mathfrak{T}$ and π and π' do not lie on $\mathfrak{T}$, then $\pi' \subset \mathfrak{T}_* \cup \mathfrak{T}$.*

LEMMA 13. *π and π' do not intersect at two points outside p.*

Proof. Suppose that π and π' intersect at Q and R; $Q \neq R$, $Q \neq p$, $R \neq p$. By Lemma 4, $\pi(s)$ and $\pi'(s)$ will intersect at two points close to Q and R. Since $\pi(s)$ and $\pi'(s)$ belong to τ and also meet at s close to p, they coincide, by Lemma 9.

LEMMA 14. *If $\pi \in \tau$* and *$\pi' \in \tau$, then $\pi' \in \phi(\pi)$. In particular, π and π' do not support at p.*

Proof. (i) If π and π' intersect at a point $Q \neq p$, they cannot support or intersect at another point $\neq p, Q$. Hence π and π' intersect at p. By Section 2.16, $\pi' \in \phi(\pi)$.

(ii) Suppose that π and π' support at p. On account of Lemma 13 and Case (i), we assume that π and π' do not intersect outside p. By the Corollary of Lemma 12, we may assume that $\pi' \subset \pi_* \cup p$. Thus, either $\pi(s)$ and $\pi'(s)$ meet twice outside a small neighbourhood N of p, or else they do not meet outside N. In the former case, they coincide, by Lemma 9. In the latter case, they have the same diameter through p, and since they have the point s in common, they coincide. In either case, π and π' coincide.

LEMMA 15. *The following cases are impossible*:

(i) $\pi \in \tau$ *and* π' *is a pair of parallel lines, one of which is* $\mathfrak{T}$.

(ii) $\pi \in \tau$ *or* π *is a pair of parallel lines, one of which is* $\mathfrak{T}$, *and* π' *is a double ray in* $\mathfrak{T}_* \cup p$ *with the vertex* p.

Proof. By the corollary of Lemma 12 we may assume that $\pi \cup \pi' \subset \mathfrak{T}_* \cup \mathfrak{T}$. Case (i) is then excluded by Lemma 13.

Case (ii). Suppose that π and π' intersect at a point Q. By Lemma 5, $\pi(s)$ and $\pi'(s)$ intersect at two points near Q. By Lemma 9, $\pi(s)$ and $\pi'(s)$ coincide.

Finally, suppose that $\pi \in \tau$ and $\pi' \subset \pi_* \cup p$. By Lemma 5, $\pi(s)$ and $\pi'(s)$ either belong to the same diametral pencil, or they meet at two points outside a small neighbourhood N of p which includes s. By Lemma 9, $\pi(s)$ and $\pi'(s)$ coincide in either case and hence π and π' also coincide.

LEMMA 16. *If* π *is a double ray in* $\mathfrak{T}_* \cup p$ *or* $\pi \in \tau$, *then* π' *is not the single line* $\mathfrak{T}$.

Proof. Suppose that π' is the single line $\mathfrak{T}$. Since $\pi \neq \pi'$, we conclude that $\pi(s) \neq \pi'(s)$. Since $\pi(s)$ and $\pi'(s)$ have the point $s \neq p$ in common, $\pi(s)$ and $\pi'(s)$ cannot belong to the same diametral pencil of τ. Hence $\pi(s)$ and $\pi'(s)$ intersect twice outside a small neighbourhood N of p which includes s. By Lemma 9, $\pi(s)$ and $\pi'(s)$ coincide.

3.11. Let ϕ be a subfamily of τ of the type described in Section 2.16. Let p be an end-point of A, $A - p \subset \mathfrak{T}_*$. Let $\pi_i\,(\phi; s)$ denote the two parabolas of ϕ through s; $i = 1, 2$.

LEMMA 17. *The* $\pi_i\,(\phi; s)$ *can be numbered in such a way that*

$$\lim_{s \to p} \pi_2(\phi; s)$$

exists and is equal to a double ray on $\mathfrak{T}$ *with the vertex* p.

Proof. Let $\mathfrak{L}$ be any line through p; $\mathfrak{L} \neq \mathfrak{T}$. We may assume that A lies in one of the quadrants bounded by $\mathfrak{L}$ and $\mathfrak{T}$, say $A \subset (\mathfrak{L}_* \cap \mathfrak{T}_*) \cup p$.

Let $s \in A$; $s \neq p$. By Section 2.16, the line $\mathfrak{D} = ps$ is the diameter through p of one of the parabolas of ϕ. Since s lies in the interior of that parabola, 2.16 implies that the diameters of $\pi_1\,(\phi; s)$ and $\pi_2\,(\phi; s)$ are separated by $\mathfrak{D}$ and $\mathfrak{T}$. In particular, the diameter of, say, $\pi_2\,(\phi; s)$ must have points in the sector bounded by $\mathfrak{D}$ and $\mathfrak{T}$ in $\mathfrak{T}_* \cap \mathfrak{L}_*$.

Let $s \to p$. Then $\mathfrak{D} \to \mathfrak{T}$ and the diameter of $\pi_2(\phi; s)$ must also converge to $\mathfrak{T}$, i.e., $\pi_2(\phi; s)$ converges to the double ray on $p \cup (\mathfrak{T} \cap \mathfrak{L}_*)$ with the vertex p.

4. Osculating parabolas.

4.1. Since the tangent parabolas of A at p through Q and s are not defined when Q and s are separated by $\mathfrak{T}$ we shall restrict our attention from now on to arcs which are differentiable at p but which do not cross $\mathfrak{T}$. The point p may be either an interior point of A or an end-point.

CONDITION II. *Let A be differentiable at p and let all the points of $A - p$ close to p lie in one of the half-planes bounded by $\mathfrak{T}$, say in $\mathfrak{T}_* \cup \mathfrak{T}$. If $Q \in \mathfrak{T}_*$, then the two tangent parabolas of A at p through Q and s converge when s tends to p.*

The limit *osculating parabolas* of A at p through Q will be denoted by $\pi_i(\sigma; Q)$; $i = 1, 2$. The family of osculating parabolas of A at p will be denoted by σ. Thus σ is a subset of $\bar{\tau}$ and lies in $\mathfrak{T}_* \cup \mathfrak{T}$. If A satisfies Condition II at p, then A is called *twice parabolically differentiable at p.*

We observe that if A satisfies Condition II and there is a sequence of points $s \in A \cap \mathfrak{T}$ converging to p, then $\pi(\sigma; Q)$ consists of $\mathfrak{T}$ and the line through Q parallel to $\mathfrak{T}$; cf. Section 3.6.

4.2. Let $Q \in \mathfrak{T}$, $Q \neq p$.

If $s \notin \mathfrak{T}$ there is a unique degenerate parabola of $\bar{\tau}$ through Q and s, consisting of $\mathfrak{T}$ and the line through s parallel to $\mathfrak{T}$; cf. Section 3.6. As s tends to p this parabola converges to the double line on $\mathfrak{T}$.

If there is a sequence of points $s \in A \cap \mathfrak{T}$ converging to p, then the parabola of $\bar{\tau}$ through s and Q is not uniquely defined; cf. Section 3.7.

4.3. While it is assumed that Condition II holds for each point $Q \in \mathfrak{T}_*$, it is sufficient to assume that it holds for a single point in $\mathfrak{T}_*$.

THEOREM 3. *If Condition II holds for one point $Q \in \mathfrak{T}_*$, then it holds for every point $R \in \mathfrak{T}_*$.*

Proof. Let $Q \in \mathfrak{T}_*$, $R \in \mathfrak{T}_*$. Put $\pi(s) = \pi(\tau; s, Q)$, $\pi'(s) = \pi(\tau; s, R)$. Suppose that $\pi = \lim \pi(s)$ exists and let π' be any accumulation parabola of $\pi'(s)$ as s tends to p. We restrict s to a sequence of parameters converging to p, such that $\pi' = \lim \pi'(s)$.

Neither π nor π' is the single line $\mathfrak{T}$, the double line on $\mathfrak{T}$, or a double ray on $\mathfrak{T}$ through p, since none of these is the limit of a sequence of tangent parabolas through a fixed point in $\mathfrak{T}_*$. Thus π [π'] is non-degenerate, or a double ray in $\mathfrak{T}_* \cup p$ with the vertex p, or a pair of parallel lines, one of which is $\mathfrak{T}$.

By Lemma 15, if π is non-degenerate [a double ray with the vertex p; a pair of parallel lines one of which is $\mathfrak{T}$], then π' is also non-degenerate [a double ray with the vertex p; a pair of parallel lines one of which is $\mathfrak{T}$].

Thus, if π is a double ray with the vertex p, then π' is the unique double ray through R with the vertex p.

If π is a pair of parallel lines one of which is $\mathfrak{T}$, then π' is the unique parabola consisting of $\mathfrak{T}$ and the line through R parallel to $\mathfrak{T}$.

If $\pi \in \tau$, then $\pi' \in \tau$, and by Lemma 14, $\pi' \in \phi(\pi)$. Thus π' is one of the two parabolas of $\phi(\pi)$ through R; cf. Section 2.16.

4.4. THEOREM 4. *If A is twice differentiable at p, the set σ of the osculating parabolas of A at p is one of the following three subsets of $\bar{\tau}$:*

1. *σ is one of the one-parameter families ϕ of parabolas of τ which were described in Section* 2.16.

2. *σ consists of all the double rays of $\bar{\tau}$ with the common vertex p which lie in $\mathfrak{T}_* \cup p$.*

3. *σ consists of all the pairs of parallel lines of $\bar{\tau}$ which lie in $\mathfrak{T}_* \cup \mathfrak{T}$.*

Proof. Let $\pi \in \sigma$, $\pi' \in \sigma$. As in the proof of Theorem 3, we can first show that π [π'] belongs to one of the three classes 1, 2, and 3, of Theorem 4.

By Lemma 15, if π belongs to class 1 [class 2; class 3], then π' also belongs to class 1 [class 2; class 3]. Thus, all the parabolas of σ belong to the same one of the classes 1, 2, and 3.

We finally verify that every member of the respective class actually lies in σ. Let π belong to the class which contains σ and let $Q \in \pi$, $Q \in \mathfrak{T}_*$. If σ belongs to class 1 [class 2; class 3], then π coincides with one of the two non-degenerate osculating parabolas [the unique double ray; the unique pair of parallel lines] of σ through Q.

4.5. The compactified family $\bar{\sigma}$. We compactify the family σ of Type 1 or Type 2 by adding to σ the two double rays on $\mathfrak{T}$ with the vertex p.

In the case of Type 3, σ is compactified by the addition of the double line coincident with $\mathfrak{T}$.

4.6. Let δ denote the diametral pencil of τ whose diameters are parallel to a line $\mathfrak{L}$. Let $\pi(\delta; s)$ be the unique member of δ through s. Suppose that A satisfies Condition II at p.

THEOREM 5. *If the osculating parabolas of A at p are non-degenerate or double rays [pairs of parallel lines], then as $s \to p$, the parabola $\pi(\delta; s)$ of δ through s converges to the unique osculating parabola in δ [the single line $\mathfrak{T}$].*

Proof. Choose $\pi \in \sigma$. Let π' be any accumulation parabola of the $\pi(\delta; s)$. We restrict s to a sequence of parameters converging to p such that

$$\pi' = \lim \pi(\delta, s).$$

Let σ be of Type. 1. By Lemma 15, π' is not a double ray and by Lemma 16, π' is not the single line $\mathfrak{T}$. Hence $\pi' \in \tau$. By Lemma 14, $\pi' \in \phi(\pi)$. Hence $\pi' \in \sigma$.

If σ is of Type 2, Lemma 15 implies that $\pi' \notin \tau$ and Lemma 16 implies that π' is not the single line $\mathfrak{T}$. Hence π' is the double ray of δ in $\mathfrak{T}_* \cup p$.

If σ is of Type 3, Lemma 15 implies that $\pi' \notin \tau$ and π' is not a double ray in $\mathfrak{T}_* \cup p$. Hence π' is the single line $\mathfrak{T}$.

4.7. THEOREM 6. *The non-osculating tangent parabolas of A at an interior point p all support A at p.*

Proof. Let $\pi \in \bar{\tau}$. We may assume that π is non-degenerate. If π neither intersects nor supports A at p, then there exists a sequence of points $s \in \pi \cap A$, $s \neq p$, $s \to p$. Thus π may be regarded as a limit parabola of a sequence of tangent parabolas through s and a point $Q \in \pi$. Thus π is an osculating parabola.

Suppose that π intersects A at p. Let $\mathfrak{L}$ be a diameter of π and let $\pi(s)$ be the parabola of τ through s with $\mathfrak{L}$ as a diameter. Let s tend to p. We may assume that $s \notin \pi$. If $s \in \pi_* \cap A$, then $\pi(s) \subset \pi_* \cup p$ and

$$\lim \pi(s) \subset \pi_* \cup \pi.$$

Symmetrically, if $s \in \pi^* \cap A$, then $\lim \pi(s) \subset \pi^* \cup \pi$. Hence $\lim \pi(s) = \pi$ and, by Theorem 5, π is an osculating parabola.

4.8. Let π' be the limit of the double ray through s with the vertex p as s tends to p and let π'' be the symmetric double ray on $\mathfrak{T}$.

In this section we assume that A *is twice differentiable at its end-point* p.

If the osculating parabolas of A at p are double rays [pairs of parallel lines], then the osculating parabola through a point $s \in A$, $s \neq p$, tends to the double ray π' on $\mathfrak{T}$ with the vertex p [the double line coincident with $\mathfrak{T}$] as s tends to p. If p *is of Type* 1, Lemma 17 *implies, in particular, that* $\pi_2(\sigma; s)$ *also tends to* π'.

THEOREM 7. *If $\phi \neq \sigma$, then the parabolas $\pi_i(\phi; s)$ of ϕ through $s \in A$, $i = 1, 2$, converge to a double ray on $\mathfrak{T}$ with the vertex p as s tends to p. In particular*:

(i) $\pi_2(\phi; s) \to \pi'$.

(ii) $\pi_1(\phi; s) \to \pi''$ *if p is of Type* 2.

(iii) $\pi_1(\phi; s) \to \pi'$ *if p is of Type* 3.

(iv) *If p is of Type* 1 *and π_ϕ and π_σ are any two parabolas of ϕ and σ respectively, with the same diameter through p, then $\pi_1(\phi; s)$ tends to π' or to π'' according as $\pi_\phi \subset \pi_{\sigma*} \cup p$ or $\pi_\sigma \subset \pi_{\phi*} \cup p$.* (This result is independent of the choice of π_ϕ and π_σ.)

Proof. By Lemma 17, $\pi_2(\phi; s) \to \pi'$. Let π be any non-degenerate limit parabola of the $\pi_1(\phi; s)$ as s tends to p. Since $\bar{\phi}$ is a closed subset of $\bar{\tau}$, $\pi \in \bar{\phi}$. Since A satisfies Condition II at p, Lemma 15 implies that p is of Type 1. By Lemma 14, π is an osculating parabola of A at p. By Section 2.16, every parabola of ϕ is also an osculating parabola of A at p, i.e., $\phi = \sigma$. Thus, if $\phi \neq \sigma$, then $\lim \pi_1(\phi; s)$ is either π' or π''.

If π_ϕ is any non-degenerate parabola of ϕ, then, by Section 2.16, π separates or does not separate $\pi_1(\phi; s)$ and $\pi_2(\phi; s)$ in ϕ according as s lies in $\pi_{\phi*}$ or in $\pi_\phi{}^*$. Theorem 5 then implies the results (ii), (iii), and (iv).

5. Parabolically differentiable points.

5.1. Let A be twice parabolically differentiable at p. From now on we assume that the osculating parabolas of A at p are of Type 1. According to Lemma 17 we can number the $\pi_i(\sigma, s)$ such that

$$\lim_{s \to p+0} \pi_2(\sigma; s) \quad \text{and} \quad \lim_{s \to p-0} \pi_2(\sigma; s)$$

exist and are double rays on $\mathfrak{T}$ with the vertex p.

CONDITION III.

$$\pi(p) = \lim_{s \to p} \pi_1(\sigma; s)$$

exists.

By its definition, $\pi(p)$ will belong to the family $\bar{\sigma}$ of Type 1 described in Theorem 4. We call $\pi(p)$ the *superosculating parabola* of A at p.

Since $\pi(p) \in \bar{\sigma}$, it is either a non-degenerative osculating parabola or a double ray on $\mathfrak{T}$ with vertex p.

The example $y = x^2 + x^3 \sin(1/x)$ shows that Condition II does not imply Condition III.

Throughout Section 5 we shall assume that Condition III holds.

5.2. Suppose that p is an end-point of A. Let $\mathfrak{D}(\pi)$ denote the diameter of π through p. Thus $\mathfrak{D}\{\pi(p)\}$ is the diameter of the superosculating parabola of A at p.

LEMMA 18. *Let $\pi_0 \in \sigma$ and let $A - p \subset \mathfrak{D}(\pi_0)_*$. The ray of $\mathfrak{D}\{\pi(p)\}$ in $\pi(p)_* \cup \pi(p)$ lies in the closure of $\mathfrak{D}(\pi_0)^* \cap \mathfrak{T}_*$ or $\mathfrak{D}(\pi_0)_* \cap \mathfrak{T}_*$ according as $A - p$ lies in π_{0*} or in $\pi_0{}^*$.*

Proof. By Section 2.16, $\mathfrak{D}\{\pi_1(\sigma; s)\}$ and $\mathfrak{D}\{\pi_2(\sigma; s)\}$ are separated or are not separated by $\mathfrak{D}(\pi_0)$ and $\mathfrak{T}$ according as $s \in \pi_0{}^*$ or $s \in \pi_{0*}$. By Theorem 7, $\mathfrak{D}\{\pi_2(\sigma; s)\}$ tends to $\mathfrak{T}$ through the sector $\mathfrak{D}(\pi_0)_* \cap \mathfrak{T}_*$ as s tends to p. Hence $\mathfrak{D}\{\pi_1(\sigma; s)\}$ tends to its limit $\mathfrak{D}\{\pi(p)\}$ through $\mathfrak{D}(\pi_0)^* \cap \mathfrak{T}_*$ or through $\mathfrak{D}(\pi_0)_* \cap \mathfrak{T}_*$ according as $A - p$ lies in π_{0*} or in $\pi_0{}^*$.

COROLLARY. *If*

$$\pi(p) = \lim_{s \to p} \pi_2(\sigma; s),$$

then $A - p$ *lies in* $\pi_0{}^*$.

5.3. THEOREM 8. *If* p *is an interior point of* A, *then the parabolas of* $\sigma - \pi(p)$ *all support* A *at* p *or all intersect according as* A *has or has not a cusp at* p.

Proof. Let M be a small neighbourhood of p on A. Thus p decomposes M into two disjoint arcs N and N' and $M = N \cup p \cup N'$. Let π be a non-degenerate osculating parabola of A at p.

(i) If π has points $\neq p$ in common with every neighbourhood of p on A, then there exists a sequence of points $s \in A \cap \pi$, $s \neq p$, $s \to p$. Thus π is an osculating parabola of A at p through s for each s. From Condition III, π will be the superosculating parabola of A at p.

(ii) Suppose that some line $\neq \mathfrak{T}$ through p intersects A at p. By Lemma 10, every line $\neq \mathfrak{T}$ through p, in particular $\mathfrak{D}(\pi)$, will intersect A at p. We wish to show that π also intersects A at p unless $\pi = \pi(p)$.

Let $N \subset \mathfrak{D}_*(\pi) \cup \pi_*$ $[N' \subset \mathfrak{D}^*(\pi) \cap \pi_*]$. Then, by Lemma 18, the ray of $\mathfrak{D}\{\pi(p)\}$ in $\pi(p)_* \cup \pi(p)$ lies in the closure of $\mathfrak{D}^*(\pi) \cap \mathfrak{T}_*$ $[\mathfrak{D}_*(\pi) \cap \mathfrak{T}_*]$. Thus $\mathfrak{D}\{\pi(p)\} = \mathfrak{D}(\pi)$, i.e., $\pi = \pi(p)$. The case

$$N \subset \mathfrak{D}_*(\pi) \cap \pi^*, \qquad N' \subset \mathfrak{D}^*(\pi) \cap \pi^*$$

is dealt with similarly.

(iii) If a line $\neq \mathfrak{T}$ through p supports A at p, a similar argument shows that π also supports A at p if $\pi \neq \pi(p)$.

(iv) Theorem 8 follows from the Corollary of Lemma 18 in the case where $\pi(p)$ is a double ray.

5.4. We give some examples of the various types of differentiable end-points. In each case $s \geqslant 0$. The point in question will be the origin, given by $s = 0$; m, n, and r are positive integers.

Type 1. $\quad x = s^m, \quad y = s^{2m} + as^{2m+n}$.

Type 1(a) $\quad [\pi(p)$ non-degenerate$]$: $m \leqslant n$.

Type 1(b) $\quad \left[\pi(p) = \lim_{s \to p} \pi_2(\sigma; s)\right]$: $m > n$, $a < 0$.

Type 1(c) $\quad \left[\pi(p) = \text{double ray symmetric to } \lim_{s \to p} \pi_2(\sigma; s)\right]$: $m > n$, $a > 0$.

Type 2. $\quad x = s^m, \quad y = s^{m+r}$, $0 < r < m$.

Type 3. $\quad x = s^m, \quad y = s^{m+r}$, $r > m$.

In all these examples, the tangent parabolas have the same equations

$$(y - \lambda x)^2 = \mu y.$$

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The osculating parabolas have the equations

$$(y - \lambda x)^2 = \lambda^2 y \qquad \text{(Type 1)},$$
$$(y - \lambda x)^2 = 0, \qquad y \geqslant 0 \qquad \text{(Type 2)},$$
$$y(y - k) = 0, \qquad k \geqslant 0 \qquad \text{(Type 3)}.$$

In Type 1(a), the equation of the non-degenerate superosculating parabola is $y = x^2$ if $m < n$ and $(ay + 2x)^2 = 4y$ if $m = n$.

Reference

1. N. D. Lane and P. Scherk. *Differentiable points in the conformal plane*, Can. J. Math., 5 (1953), 512–518.

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