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REVIEW

Three papers on the reverse mathematics of Jullien's Indecomposability Theorem

Antonio Montalbán, *Indecomposable linear orderings and hyperarithmetic analysis*. ***Journal of Mathematical Logic***, vol. 6 (2006), no. 1, pp. 89–120.

Itay Neeman, *The strength of Jullien's indecomposability theorem*. ***Journal of Mathematical Logic***, vol. 8 (2008), no. 1, pp. 93–119.

———, *Necessary use of Σ^1_1 induction in a reversal*. ***Journal of Symbolic Logic***, vol. 76 (2011), no. 2, pp. 561–574.

Officially, reverse mathematics is the investigation of the axiomatic strength of theorems of second-order arithmetic—given some theorem provable in the theory Z_2 of second-order arithmetic, reverse mathematics asks for a fragment of Z_2 both sufficient to prove the theorem and necessary in the sense that the theorem (together with some weak "base theory") implies the whole fragment of Z_2.

In practice, much of reverse mathematics is concerned with relative computability. If we consider only ω-models (models where the first-order part is the actual natural numbers) then a model of the standard base theory $\mathbf{RCA_0}$ is just a collection of sets of numbers closed under Turing reductions—that is, a *Turing ideal*. In particular, a standard method for separating two theorems—showing that $\mathbf{RCA_0} + \mathbf{S}$ does not imply $\mathbf{T}$—is to construct a Turing ideal satisfying $\mathbf{S}$ but not $\mathbf{T}$.

In rare cases, however, $\mathbf{RCA_0}$ is not a sufficiently strong base theory for fully comparing two theorems. It may be that every Turing ideal satisfying $\mathbf{S}$ does satisfy $\mathbf{T}$, but $\mathbf{S}$ and $\mathbf{T}$ can still be separated in a nonstandard model. That is, although $\mathbf{RCA_0} + \mathbf{S} \nvdash \mathbf{T}$, it may be that there is some additional induction axiom ϕ so that $\mathbf{RCA_0} + \phi + \mathbf{S} \vdash \mathbf{T}$.

The papers under review here concern theories of *hyperarithmetic analysis*: theories $\mathbf{T}$ extending $\mathbf{RCA_0}$ with the property that all ω-models of $\mathbf{T}$ are closed under hyperarithmetic reductions, and the principal is true in all ω-models of the form $HYP(Y)$, the set of X hyperarithmetically reducible to Y. (Recall that X is hyperarithmetically reducible to Y if X is Turing reducible to the α-th jump $Y^{(\alpha)}$ where α is some countable ordinal which has a Y-computable presentation.)

The notion of a hyperarithmetic reduction is more subtle than a Turing reduction, due to the existence of *pseudo-well-orderings*. A model of second-order arithmetic might contain an ill-founded linear ordering $(L, \leq_L)$, but not contain any infinite descending sequence through this ordering, so that the ordering is well-founded in the sense of the model, but not actually well-founded. The ordinal α in the definition of a hyperarithmetic reduction ranges over (presentations of) *actual* well-orderings. For example, although closure under hyperarithmetic reductions is a perfectly reasonable statement in the language of second order arithmetic, it is *not* a theory of hyperarithmetic analysis: it is the theory $\mathbf{ATR_0}$, one of the "big five" theories of reverse mathematics, and the hyperarithmetic sets do not give a model of $\mathbf{ATR_0}$. This is because there exist computable ill-founded orderings with no hyperarithmetic infinite descending sequence. For each of these, an ω-model of $\mathbf{ATR_0}$ must either include an infinite descending sequence or make sense of iterating the jump operation along the ordering; either way, such a model must include nonhyperarithmetic sets.

So, while $\mathbf{RCA_0}$ itself is essentially the only theory exactly characterizing sets closed under Turing reductions, there are many theories of hyperarithmetic analysis (and indeed, provably no weakest theory). Most examples of these theories come from logic—they include the axiom of Δ^1_1-comprehension, which states that Δ^1_1-definable sets exist, and various forms of the axiom of choice for Σ^1_1 formulas. Montalbán's paper identifies several new hyperarithmetic theories,

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but the most striking is a theorem of classical mathematics: Jullien's Indecomposability Theorem.

A *cut* in a linear ordering $(U, <_U)$ is a partition $U = L \cup R$ into disjoint sets so that if $a \in L$ and $b \in R$, $a <_U b$. U is *indecomposable* if whenever (L, R) is a cut of U, either U embeds into L or U embeds into R. Jullien's Indecomposability Theorem states then if $(U, <_U)$ is a countable indecomposable ordering which does not contain a copy of the rationals then either for every cut with $L \neq \emptyset$, U embeds into L, or for every cut with $R \neq \emptyset$, U embeds into R. In the traditional style of reverse mathematics, Jullien's indecomposability theory is given the abbreviation **INDEC**.

One way to view indecomposability is in terms of the *middle cut*: let L be those a such that U embeds into $U_{>a}$ and let R be those a such that U embeds into $U_{\leq a}$. If U does not contain a copy of the rationals, L and R must be disjoint, and when U is indecomposable, $L \cup R = U$. So the assumptions of the theorem make (L, R) a cut, and the indecomposability theorem says that either $L = \emptyset$ or $R = \emptyset$.

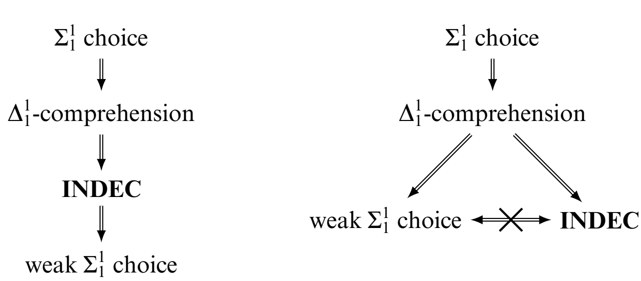
Montalbán shows that that the middle cut is Δ^1_1, and that if the middle cut is nontrivial (i.e. $L \neq \emptyset$ and $R \neq \emptyset$) and present in the model then U contains a copy of the rationals. It follows that Δ^1_1-comprehension implies **INDEC**, and so, like Δ^1_1-comprehension, **INDEC** is true in any model $HYP(Y)$. On the other hand, he shows that for each set Y and each Y-computable ordinal α, there exists an Y-computable linear ordering C such that $Y^{(\alpha)}$ is computable from the middle cut of C and the linear order has a property known as *weak indecomposability*. He shows that **INDEC** implies the middle cut of a weakly indecomposable linear order exists (even in the absence of Δ^1_1-comprehension). It follows that any ω-model of **INDEC** contains, for every set Y and every Y-computable ordinal α, the linear ordering C, and so the middle cut of C, and therefore the set $Y^{(\alpha)}$, so the ω-model is closed under hyperarithmetic reductions.

In the second paper reviewed here, Neeman refines this by placing ω-models of **INDEC** strictly between Δ^1_1-comprehension and another principle, weak Σ^1_1 choice. Recall that the Σ^1_1 axiom of choice says, for a Σ^1_1 sentence ϕ, that if for every n there is a set X so that $\phi(n, X)$ holds then there is a single set X so that for each n, $\phi(n, X_n)$ holds (where $X_n = \{i \mid (n, i) \in X\}$).

The weak axiom of choice is a weakening which applies only when for each n there is a *unique* set X so that $\phi(n, X)$. Van Wesep (1977) had shown that weak Σ^1_1 choice is strictly weaker than Δ^1_1-comprehension. By contrast, Steel (1978) had shown that the full Σ^1_1 axiom of choice is strictly stronger than Δ^1_1-comprehension.

Neeman shows that, together with Σ^1_1-induction, **INDEC** implies weak Σ^1_1 choice, and that none of these implications reverse: **INDEC** does not imply Δ^1_1-comprehension, weak Σ^1_1 choice does not imply **INDEC**, and in the third paper he shows that in the absence of Σ^1_1-induction, **INDEC** does not imply weak Σ^1_1 choice. This makes the relationship between **INDEC** and weak Σ^1_1 choice one of the rare examples where an additional induction axiom is needed to make two theorems comparable, and the only known example where the induction axiom is as strong as Σ^1_1-induction.

Over base theory $\mathbf{RCA_0} + \Sigma^1_1$ induction   Over base theory $\mathbf{RCA_0}$



All these nonimplications (as well as additional ones in Montalbán's paper) are shown by constructing collections of sets closed under hyperarithmetic reductions using Steel's *tagged tree forcing*. Despite the name, tagged tree forcing is a two stage process, only the

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first stage of which is actually forcing. In the first stage we fix a linear ordering $\prec$ and force (with finite conditions) over $L_{\omega_1^{CK}}$ to add a countable tree T of sequences of natural numbers, an enumeration $f : \mathbb{N} \to \mathcal{B}$ of countably many branches of T, and a *tag* h which is an order-preserving map from the part of T which is not an initial segment of some branch $f(i)$ to $\prec$. If $\prec$ is a well-ordering then $h(\sigma)$ is precisely an explicit ordinal bound on the height of the tree above σ. Following van Wesep, Neeman takes $\prec$ to be a pseudo-well-ordering with no hyperarithmetic infinite descending sequences; consequently, while there may be branches in T not in the range of f, the genericity of the forcing construction ensures that these branches are not hyperarithmetic in (T, f).

In the second stage, we choose a set K and take a model M_K consisting of all sets hyperarithmetic in T together with $\{f(i) \mid i \in F\}$ for some finite $F \subseteq K$. By varying the choice of K one obtains different models which satisfy different principles. These models always have a standard first-order part, and the second order part is a countable union of collections of the form $HYP(Y)$.

Van Wesep constructed a model of weak Σ^1_1 choice which does not satisfy Δ^1_1-comprehension by constructing an appropriate K; Neeman shows that this model does not satisfy **INDEC** either, proving that weak Σ^1_1 choice does not imply **INDEC**.

Neeman constructs a different model by creating a Cohen real H (forcing over $L_{\omega_1^{CK}}[T, f, h]$) and choosing K so that H will be Δ^1_1 over M_K. M_K certainly does not satisfy Δ^1_1-comprehension, since it does not contain H, and he shows that it does satisfy **INDEC**. (Since the first-order part is standard, it necessarily satisfies Σ^1_1-induction, and so also weak Σ^1_1 choice.)

Showing that **INDEC** requires Σ^1_1-induction to imply weak Σ^1_1 choice means constructing a model with a nonstandard first-order part; this is what Neeman does in the third paper reviewed here. Neeman first uses the Steel forcing over $L_{\omega_1^{CK}}$ to add (T, f, h) as before. Then he takes a nonstandard model elementarily equivalent to $L_{\omega_1^{CK}}[T, f, h]$; this nonstandard extension contains an ill-founded end-extension of ω and a nonstandard version of (T, f, h). He constructs a model M_K by allowing sets generated by the branches $\{f(i) \mid i \in F\}$ where F is an actually finite (in the sense of the standard model) subset of K. Note that the model M_K requires working outside the nonstandard model of $L_{\omega_1^{CK}}[T, f, h]$, in a standard model of set theory which contains a copy of this nonstandard model.

K is chosen so that in M_K, for each k there is a unique branch beginning with (k, i) for some i. However, for any nonstandard n, there does not exist a sequence of branches $\langle b_0, \dots, b_n \rangle$ where each b_k begins with some (k, i). Therefore, both weak Σ^1_1 choice and Σ^1_1-induction fail. On the other hand, by passing back and forth between properties of the original, standard (T, f, h) and the nonstandard version of the same, he shows that the model satisfies **INDEC** and Δ^1_1-induction.

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