Exact measures of income in a hyperbolic economy

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ABSTRACT. For a closed economy with human-made capital, non-renewable resource depletion and (possibly) exogenous, hyperbolic technical progress as explicit-form inputs to a production function, there is a feasible development path that is 'as if' optimal with respect to hyperbolic utility discounting. On this path, typically, welfare-equivalent income > wealth-equivalent income > Sefton-Weale income > net national product, with possibly dramatic differences among these measures; and sustainable income can be greater than, equal to, or less than NNP. For low enough discounting, growing consumption is optimal even when technical progress is zero. A particular discount rate makes all income measures and consumption constant and (except net national product) equal; and zero technical progress then gives the Solow (1974) maximin as a special case. The optimal path is time-consistent because of the way the utility discount rate is chosen to depend on the economy's stocks, and hence on absolute time.

1. Introduction

This paper gives exact formulae for five different definitions of income on the optimal development path of a theoretical economy with explicit functional forms. The economy is closed and deterministic, with constant population and a representative agent. There are three inputs to a Cobb–Douglas production function: the stock of human-made capital; the depletion of a finite, non-renewable resource; and (possibly) time in the form of an exogenously, linearly growing stock of technical knowledge. The 'optimal' path is the efficient path chosen by the economy as if it were maximizing the present value of utility over an infinite time horizon, using a hyperbolic utility discount factor (so the discount rate is not constant, a feature we call 'non-constant discounting').

Given the theoretical analysis of many different income measures in Asheim (2000), on which this paper builds, and the explicit functional form

of our 'hyperbolic economy', the purposes of studying this economy (and thereby what we will add to Asheim's results) need careful explanation. One is to show a property which some may find ethically attractive. This is that if discounting is hyperbolic and weak enough, forever-rising rather than constant consumption can be the optimal (present value maximizing) development path of an economy with human-made capital and a non-renewable resource, even with no technical progress. Hyperbolic discounting can thus avoid Solow's (1974: 41) concern that a maximum constant consumption path may 'perpetuate poverty' or give 'foolishly conservative injunctions', without causing the intergenerational inequity that may result from constant discounting, found in Dasgupta and Heal (1974) and others.

Another purpose is to give a clear example of why it may be hard to reach consensus on a single, exact definition of income. This will come from showing that four of the five income measures (including Sefton–Weale income, a measure not in Asheim 2000) have strictly different sizes in the hyperbolic economy, and that the differences can be dramatic for plausible parameter values. Next, considering the hyperbolic economy alongside other recent work on non-constant discounting, such as Henderson and Bateman (1995), Laibson (1997), and Weitzman (2001), yields additional insights into time-consistency. For example, our hyperbolic discount factor is a function of absolute time, which supports the chosen path in a time-consistent way. In Laibson, hyperbolic discounting is a function of relative time, so naive decision-making leads to time-inconsistent planning where people seek to constrain their own future choices. Finally, the hyperbolic economy adds to the range of algebraically exact economies which can be used to develop or check new theories about economies with both capital and non-renewable resources, and perhaps to reveal the often-limited generality of existing theories. This range otherwise seems to comprise only Solow's constant consumption path, the asymptotically steady growth path in Stiglitz (1974), and Pezzey and Withagen's (1998) solution of a 'single-peaked' economy.

Section 2 defines the hyperbolic economy, lists and interprets its results, and discusses whether its optimal path is time-consistent and well-motivated. All calculations use straightforward, though tedious and so omitted, algebra (marked by 'it can be shown that . . . '), that starts from the necessary first-order conditions of the optimal control problem.[1] Section 3 concludes.

2. The hyperbolic economy

2.1. General assumptions and definitions of income

The economy is a special case of that described in the appendix of Asheim (1997). Population is constant; consumers are identical and have no age structure, with each generation represented by one agent at an instant in continuous time, which stretches from zero to infinity; and the economy is closed to trade. The variables below are non-negative quantities along any development path in the economy, using terminology mostly similar to

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[1] Details are available from the author's website, http://cres.anu.edu.au/~pezzey.

that in Asheim (2000, 2003). Less familiar terms, or terms with ambiguous meanings in the literature, are highlighted in italics.

$K(t)$ is the non-depreciating, human-made capital stock; $K(0) = K_0 > 0$

$S(t)$ is the non-renewable, natural resource stock; $S(0) = S_0 > 0$

$T(t)$ is the exogenous stock of technological knowledge; $T(0) = T_0 > 0$

$C(t)$ is consumption of a single produced good

$R(t) = -\dot{S}(t)$ is the resource depletion flow, with zero extraction costs

$F(K(t), R(t), T(t))$ is output; $F = F(K(t), R(t))$ if technology is constant

$U(C(t))$ is instantaneous utility

$\phi_0(t)$ is the utility discount factor

$\Phi_0(t) := \phi_0(t)U_C(C)$ is the consumption discount factor

$W(t) := \int_t^\infty [\phi_0(s)/\phi_0(t)]U[C(s)]ds, t \geq 0$ is the present value of utility, that the representative agent acts as if to maximize (by choosing paths for consumption C and resource depletion R) at all times, in implementing an (intertemporally Pareto-efficient) utility path supported by discount factor ϕ_0. The W-maximizing path will be called optimal, and existence and uniqueness are assumed. If it had been that $\phi_0(t) = e^{-t}$, constant, then it would make sense to interpret $W(t)$ as (intertemporal) 'welfare'. However, with the non-constant discounting here, increasing $W(t)$ need not mean that 'things are getting better' (for example, $\dot{W}(t) > 0$ on the Solow constant-consumption path); so we just call $W(t)$ present value.

$\mu^K(t), \mu^S(t), \mu^T(t)$ are respectively the co-state variables of $K(t)$, $S(t)$ and $T(t)$ resulting from maximizing W

$\Theta(t) := \int_t^\infty [\Phi_0(s)/\Phi_0(t)]C(s)ds, t \geq 0$ is (current) *wealth*

$\delta(t) := -\dot{\phi}_0(t)/\phi_0(t)$ is the current (utility) discount rate (non-constant)

$\delta_\infty(t) := \int_t^\infty \phi_0(s)\delta(s)ds/\int_t^\infty \phi_0(s)ds$ is the *time-averaged discount rate*

$r(t) := -\dot{\Phi}_0(t)/\Phi_0(t)$ is the current interest rate (non-constant)

$r_\infty(t) := \int_t^\infty \Phi_0(s)r(s)ds/\int_t^\infty \Phi_0(s)ds$ is the *time-averaged interest rate.*

Five definitions of income are then:

$A(t) := U^{-1}[\delta_\infty(t)W(t)]$ is *welfare-equivalent income* (Asheim, 2000)[2]

$Y^e(t) := r_\infty(t)\Theta(t)$ is *wealth-equivalent income* (Asheim, 2000)

$SW(t) := [\int_t^\infty r(s)\Phi_0(s)C(s)ds]/\Phi_0(t)$ is *Sefton–Weale income*, after Sefton and Weale (1996)

$Y(t) := C(t) + [\mu^K(t)\dot{K}(t) + \mu^S(t)\dot{S}(t)]/U_C(t)$ is net national product (NNP), specifically 'green' NNP since it includes $\mu^S(t)\dot{S}(t)/U_C$, the value of resource change.[3]

$Y^m[K(t), S(t)] := \max C$ s.t. $C(t') \geq C$ for all $t' \geq t$, i.e. *sustainable income* or the maximum sustainable consumption level. Y^m is calculable only when there is no technical progress, because an analytic solution is generally unavailable when there is progress.

We give reasons for choosing these income measures in section 2.5 below, after we have derived values of the five measures for the hyperbolic

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[2] Given our comments in defining $W(t)$, $A(t)$ should strictly be called 'present-value-equivalent income', but like Asheim we use this more concise term.

[3] If NNP also included $\mu^T(t)\dot{T}(t)/U_C$, the value of technology change, it would be Pezzey (2004)'s 'augmented' (green) NNP.

economy, both analytically and for a numerical example. We also discuss there whether any of these measures is preferable to the others as a 'better' measure of income.

2.2. Specific assumptions, and the optimal, time-consistent path, for the hyperbolic economy

The specific functional forms used in the hyperbolic economy are

Production: $F = K^{\alpha}R^{\beta}T^{\nu} = C + \dot{K}$; $T(t) := (1 + \theta_0 t)T_0$; $\theta_0 > 0, \nu \geq 0$ (1)

Instantaneous utility: $U(C) = C^{1-\alpha}/(1-\alpha)$, $0 < \alpha < 1$

The technical progress rate in gross production here, $[(d/dt)(T^{\nu})]/T^{\nu} = \nu\theta_0/(1 + \theta_0 t)$, is positive but declining over time. This can be viewed as a compromise between the usual assumptions of either zero progress, or a constant, positive rate of progress.

Further necessary parameter restrictions, and algebraic abbreviations, are:

$0 < \beta < \alpha < \alpha + \beta \leq 1$ ($\beta < \alpha$ is needed to enable a constant consumption path in Solow, 1974) (2)

$\rho > 1 + \alpha - \beta + \nu$ (> 1) (ρ is a parameter that appears below in hyperbolic discount factors ϕ_0 and ϕ_x; this restriction is needed to make W converge) (3)

$\xi := (\rho - \alpha - \nu)/(1 - \beta)$ (> 1 from (3)) (4)

$\sigma := (\alpha + \nu - \beta\rho)/(1 - \alpha)(1 - \beta)$ (5)

$\Rightarrow \xi + \sigma = \rho + \alpha\sigma = [\rho(1 - \alpha - \beta) + \alpha(\alpha + \nu)]/(1 - \alpha)(1 - \beta)$ (> 0)

$\theta_0 := \left[\alpha(\xi - 1)^{\beta}S_0^{\beta}T_0^{\nu}/(\xi + \sigma)K_0^{1-\alpha}\right]^{1/(1-\beta)}$ (> 0) (6)

The reason for the restrictive value of θ_0 in (6) will be discussed below.

It can be shown that the following paths are then feasible and efficient:

Consumption $C(t) = [(\rho - \alpha)\theta_0 K_0/\alpha](1 + \theta_0 t)^{\sigma}$ (7)

Capital $K(t) = K_0(1 + \theta_0)^{\sigma+1}$ (8)

Resource stock $S(t) = S_0(1 + \theta_0 t)^{-(\xi-1)}$ (9)

Resource flow $R(t) = (\xi - 1)\theta_0 S_0(1 + \theta_0 t)^{-\xi}$

Output $F(t) = [(\xi + \sigma)/(\rho - \alpha)]C(t)$

Current interest rate $r(t) = (\xi + \sigma)\theta_0/(1 + \theta_0 t)$ (10)[4]

Time-averaged interest rate $r_{\infty}(t) = (\xi + \sigma - 1)\theta_0/(1 + \theta_0 t)$

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[4] Combining this result with (1), note that the definition of our economy follows the choice of linear production and (implicitly) a non-linear utility function and declining interest rate made by Hartwick (1977), rather than the non-linear production, linear utility function, and constant interest rate chosen by Weitzman (1976).

It can further be shown that the above paths will be followed and *will be time-consistent*, despite non-constant discounting being 'known' to make an optimal path time-inconsistent (Strotz, 1955/6), if the economy acts as if to maximize present value $W(t)$ using a hyperbolic utility discount factor

$$\phi_0(t) = (1 + \theta_0 t)^{-\rho}, \text{ with } \theta_0 \text{ as in (6), and } \rho \text{ restricted as in (3).} \quad (11)$$

This claim holds only if reoptimization at any time $t = x > 0$ (the test of time-consistency, but one not uniquely defined in the literature) is performed so that the utility weighting between $t = x$, and some subsequent time $t = x + s$ for any $s > 0$, *is exactly the same as the weighting used before reoptimization*. That is, a util enjoyed at $t = x + s$ must still weigh

$$\phi_0(x + s)/\phi_0(x) = [1 + \theta_0(x + s)]^{-\rho}/(1 + \theta_0 x)^{-\rho} \quad (< 1) \quad (12)$$

times as much as a util enjoyed at $t = x$.

This definition of reoptimization keeps discounting 'rooted' at absolute time $t = 0$; and the discount factor (12) for the interval $[x, x + s]$ then depends on absolute time $x + s$. Time-inconsistency is avoided by abandoning Strotz's requirement that the discount factor $\phi_0(t_2)/\phi_0(t_1)$, used to compare utilities at any time t_2 and t_1, must depend only on relative times $t_2 - t_1$ and psychological parameters (see also Asheim, 2000: 31). If by contrast we had chosen a definition of reoptimization which forced discounting to be 'uprooted', and restarted from time $t = x$ with the same initial parameter θ_0, then the weighting between $t = x$ and $t = x + s$ would be

$$(1 + \theta_0 s)^{-\rho} \quad (\neq \phi_0(x + s)/\phi_0(x)) \quad (13)$$

This weight depends only on relative time s and pyschological parameters; so it meets Strotz's requirement, and would indeed cause time-inconsistency.

How can we achieve the utility weights required in (12) using a new discount factor, defined with the new time scale that starts from the reoptimization time? Routine algebraic manipulation of (12) can show that

$$[1 + \theta_0(x + s)]^{-\rho}/(1 + \theta_0 x)^{-\rho} = (1 + \theta_x s)^{-\rho} \quad (14)$$

where θ_x is defined like θ_0 in (6), but using current stocks of K, S, and T

$$\theta_x := \{\alpha(\xi - 1)^\beta [S(x)]^\beta [T(x)]^\nu/(\xi + \sigma)[K(x)]^{1-\alpha}\}^{1/(1-\beta)} \quad (15)[5]$$

So our reoptimization approach can be defined more neatly by redefining the timescale at and after the point of reoptimization to be $s := t - x$, and using the discount factor (which now starts from 1 at $s = 0$)

$$\phi_x(s) = (1 + \theta_x s)^{-\rho} \quad (16)$$

As further confirmation of why a switch from the old discount factor $\phi_0(t)$ to the new factor $\phi_x(s)$ at any post-reoptimization time s (that is $t = x + s$)

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[5] I am grateful to a referee for this insight.

is time-consistent, note from (14) that the switch leaves the instantaneous discount rate unchanged

$$-\dot{\phi}_x(s)/\phi_x(s) = \rho\theta_0/[1 + \theta_0(x+s)] = -\dot{\phi}_0(x+s)/\phi_0(x+s) = -\dot{\phi}_0(t)/\phi_0(t)$$

It remains to comment on expressions (6) for θ_0 (the initial rate of decline of the discount rate, $(d/dt)\ln[-\dot{\phi}_0(t)/\phi_0(t)]|_{t=0}$), and (15) for θ_x (the rate of decline at $t = x$). These clearly depend on absolute time via the current stocks (K, S, T), in contrast to ρ, the general strength of discounting, which is unchanged over time. Without this particular dependence of θ_0 and θ_x on absolute time, the 'as if' optimal economy would not be exactly on a (hyperbolically) steady state path from time zero (only asymptotic rates-of-growth could then be computed, as in Stiglitz, 1974), and also would not be time-consistent. The fact that θ_0 and θ_x are both power functions of the marginal product of capital (respectively $F_K(0)$ and $F_K(x)$ from (1)) stems from the requirement that $F_K(t)$ and $C(t)$ satisfy the Ramsey rule for an optimal consumption path, which here is $F_K = -\dot{\phi}_0/\phi_0 + \alpha\dot{C}/C$.

Last but not least, the underlying motivation of the economy, in following paths (7)–(10), is a non-trivial question, addressed in section 2.6.

2.3. The five measures of income for the hyperbolic economy

From results (7)–(10), it can further be shown that the five measures of income on the optimal path of the hyperbolic economy are at any time

For any rate of technical progress, $\nu \geq 0$:

Welfare-equivalent income $A(t) = [1 + (1-\alpha)\sigma/(\xi-1)]^{1/(1-\alpha)}C(t)$ (17)

Wealth-equivalent income $Y^e(t) = [1 + \sigma/(\xi-1)]C(t)$ (18)

Sefton–Weale income $SW(t) = (1 + \sigma/\xi)C(t)$ (19)

NNP $Y(t) = [1 - \nu/(\rho-\alpha)](1 + \sigma/\xi)C(t)$ (20)

For $\alpha > \beta$, and no technical progress, $\nu = 0$, only:
Sustainable income

$$Y^m(t) = [(\xi+\sigma)(\alpha-\beta)/(\xi-1)\alpha]^{\beta/(1-\beta)}(1 + \sigma/\xi)C(t) \qquad (21)$$

Four features of these results are worth noting:

(a) Provided optimal consumption is growing ($\sigma > 0$), since all other parameters are positive, including $(1-\alpha)$, $(\xi-1)$ and $(\rho-\alpha)$ thanks to restrictions (2)–(4), the first four income measures are in the strict size order $A > Y^e > SW > Y$, consistent with (but stronger than) the non-strict general orders given in Asheim (2000). Finding more general conditions for this strict order to hold remains for further work.

(b) The $-\nu/(\rho-\alpha)$ term in NNP, and its absence in welfare-equivalent, wealth-equivalent and Sefton–Weale incomes, clearly reflects a 'technical progress premium' in the last three measures. This premium is absent from the national product definition of income because Y omits the value $\mu^T \dot{T}/U_C$ of the stock of exogenous technical knowledge, but this value is necessarily included in all the present-value-equivalent definitions; see also Pezzey (2003). It remains to be

seen if Weitzman's (1997) formula for the technical progress premium in an economy with a constant interest rate can be generalized to the case of a non-constant interest rate here.

(c)   It can be shown that if $\alpha > \beta$ and technical progress is zero ($\nu = 0$)

$$\alpha/\beta \gtrless \rho \, (> 1) \Leftrightarrow Y^m \gtrless Y \tag{22}$$

so that sustainable income Y^m is only loosely related to NNP Y.

(d)   From (5) and (7), the special case of $\alpha + \nu - \beta\rho = 0$ means that $\sigma = 0$ and all measures of income are constant, at levels which can be shown to be

$$A(t) = Y^e(t) = SW(t) = Y^m(t) = C(t) = Y(t)/[1 - \nu/(\rho - \alpha)] = \bar{C}$$
$$\forall t \geq 0$$

where

$$\bar{C} := [\alpha(1 - \beta) + \nu]\left\{K_0^{\alpha-\beta}[\alpha(\alpha + \nu - \beta)S_0]^\beta T_0^\nu/(\alpha + \nu)\right\}^{1/(1-\beta)} \tag{23}$$

Then if $\nu > 0$ (technical progress), the economy can forever consume (C) more than it 'produces' (Y), because time is itself productive but the value of time (that is of technical progress) is omitted from Y. ($\bar{C}$ then appears to be the first known algebraic expression for sustainable income Y^m in the case of technical progress, albeit restricted to this special case where $\theta_0 = \alpha\beta\{[\alpha(\alpha + \nu - \beta)S_0]^\beta T_0^\nu/(\alpha + \nu)K_0^{1-\alpha}\}^{1/(1-\beta)}$, from (6). But if $\nu = 0$ (no technical progress), $\bar{C}$ simplifies to the Solow (1974) constant consumption path $C(t) = (1 - \beta)\{K_0^{\alpha-\beta}[(\alpha - \beta)S_0]^\beta\}$, and only then are all five income measures defined, constant, and equal to consumption.

   The following numerical example gives an idea of how big differences among the income measures can be. If $\rho = 2, \alpha = 0.6, \beta = 0.05, \nu = 0.4, K_0 = 1000, S_0 = 100, T_0 = 1$ and time is measured in years, then to 3 decimal places, $\xi = 1.053, \sigma = 2.368$, and $\theta_0 = 0.010$. The various instantaneous, annual rates in the economy at time $t = 0$ are then:

utility discount rate	$\rho\theta_0$	$= 0.019;$
technical progress rate	$\nu\theta_0$	$= 0.004;$
consumption growth rate	$\sigma\theta_0$	$= 0.023;$
current interest rate	$(\xi + \sigma)\theta_0$	$= 0.033;$

$$(24)$$

These *initial* rates are the same order of magnitude as the constant rates used by Weitzman (1997) and other authors, and so are not altogether implausible. Inserting the numbers into (17)–(20), and adding a calculation of sustainable income Y^m done by numerical simulation just for time zero,[6] then shows that the income measures vary dramatically in this example,

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[6] See section 3.5 of an earlier version of this paper (Pezzey, 2002) for details of the simulation. A point of interest found in all simulations was that, contrary to Solow's (1974, p41) speculation, capital K does not approach zero on a sustainable income path, but grows without bound (which also happens in the constant consumption case (d) just discussed, where an analytic solution exists).

being (to one decimal place):

welfare-equivalent income	$A(t) = 1573.6\,C(t)$	
wealth-equivalent income	$Y^e(t) = 46.0\,C(t)$	
Sefton–Weale income	$SW(t) = 3.3\,C(t)$	(25)
sustainable income	$Y^m(0) = 3.1\,C(0)$	
NNP	$Y(t) = 2.3\,C(t)$	

The fact that $\alpha/\beta > p$ and $Y^m(0) > Y(0)$ here suggests that result (c) above may also apply to the case of positive technical progress.

Although, any empirical significance of these results is hard to judge, since the rates in (24) all decline over time as $1/(1 + \theta_0 t)$, contrary to empirical experience in Western economies over the last two centuries or so. Perhaps more significant are results from an exact asymptotic solution of the Stiglitz (1974) exponential economy, where it can be shown[7] that for the parameter values $\rho = 0.025, \alpha = 0.6, \beta = 0.05$ and $v = 0.01$ (a fairly standard set of constant rates, except for the role of α in $U(C)$), the asymptotic income measures are $A = 3.2C$, $Y^e = SW = 2.5C$, and $Y = 1.5C$.

2.4. Sustained growth
Another feature that could have been listed in the previous subsection, but deserves more prominence, is that optimal consumption in the hyperbolic economy is forever growing if the discount rate is low enough

$$\rho < (\alpha + v)/\beta \Rightarrow \sigma > 0 \Rightarrow \dot{C} > 0 \quad \forall t \tag{26}$$

Moreover, such sustained growth can be optimal even if there is no technical progress (just choose $\rho < \alpha/\beta$ when $v = 0$). This reflects how a hyperbolic utility discount rate declines over time, in a way that can match the declining return to capital in this economy. By contrast, in the seminal example of a capital-resource economy with no technical progress in Dasgupta and Heal (1974), the discount rate is constant, and ultimately becomes greater than the declining return to capital. Hence optimal consumption asymptotically falls toward zero there, no matter how small the discount rate.

We now discuss two topics noted earlier: why the five income measures were chosen, and whether some measures are better than others; and the motivation of the economy's chosen path.

2.5. Why these income measures, and is any measure better than the others?
A number of commenters on this paper felt it is unsatisfactory to give five, quantitatively quite different measures of income, and yet no reason for selecting them, or for preferring one measure to another. The answers to

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[7] The formulae for the corresponding special case of Stiglitz's economy are $F = K^\alpha R^\beta T^v$ but $T = e^t$, $\phi_0 = e^{-\rho t}$, $\zeta := (\rho - v)/(1 - \beta)$, $\omega := (v - \beta\rho)/(1 - \alpha)(1 - \beta)$, $(\zeta + \omega)K_0^{1-\alpha} = \alpha\zeta^\beta S_0^\beta$ (the parameter restriction needed to start on an analytic path), $C = (\rho K_0/\alpha)e^{\omega t}$, $A = [1 + (1 - \alpha)\omega/\zeta]^{1/(1-\alpha)}C$, $Y^e = SW = (1 + \omega/\zeta)C$, $Y = (1 - v/\rho)(1 + \omega/\zeta)C$.

both questions lie in the many different possible purposes for measuring income, for example:

> ...charting business cycles, comparing prosperity among nations, observing industrial structure, measuring factor shares and so on.... real income may be interpreted as a family of concepts, each member of which is best for some particular purpose. (Usher, 1994: 124)

These many purposes spring partly from the fact that taking proper account of the future, which many income measures try to do and current consumption clearly does not, still leaves undefined what kind of future society wants, and exactly how to take account of it. We justify the measures chosen here by noting a purpose for each of them, but this variety of purposes also makes it hard to prefer just one measure above the rest.

So NNP, $Y(t)$, would be the best measure of economic activity, and in a non-equilibrium model might serve for charting business cycles, but it has limited theoretical properties, particularly when there is exogenous technical progress (Asheim, 2000: 42). Testing the sustainability of development is a crucial purpose of a measure of income, and is clearly served here by ensuring that sustainable income $Y^m(t)$ exceeds consumption.[8] Welfare-equivalent income $A(t)$ can be used theoretically to compare prosperity across nations with proportional discount factors at the same point in time (Asheim, 2000: 30), but it is not measurable. Wealth-equivalent income $Y^e(t)$ is measurable and gives a lower bound to welfare-equivalent income (Asheim, 2000: 37), but our numerical results at (25) show that this bound can be very crude.[9] Sefton–Weale income $SW(t)$ is the present value of the product of the interest rate and consumption along the optimal consumption path (and has interesting properties for resource-trading economies which are not relevant for the closed economy here, see Sefton and Weale 1996).

The best-known writer on income (Hicks, 1946: ch. 14) emphasized both sustainable income (Y^m) and wealth-equivalent income (Y^e) as valid concepts. However, Hicks used a partial equilibrium framework (a person facing given prices, rather than a closed, general equilibrium economy facing endogenous prices, as above) where these two income definitions are indistinguishable. So the phrase 'Hicksian income' (used by Hamilton and Clemens, 1999; Nordhaus, 2000, and many other recent writers) is almost always contentious or ambiguous in the context of economic growth and development (see for example Vincent, 2000: footnote 2), and has been deliberately avoided here.

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[8] One may also ask whether some measure of income-exceeding consumption means that welfare is increasing, but we cannot answer this question, since as already noted, our present value $W(t)$ is not a global welfare index when discounting is non-constant.

[9] The degree to which A exceeds Y^m in (25) suggests that A can considerably exceed consumption, even while consumption is unsustainable. Such a case is shown for a Dasgupta–Heal economy with constant discounting and no technical progress by figure 4.1 in Pezzey and Toman (2002). Note, though, that in most of this figure, $Y > Y^e$ (the reverse order to what is found in this paper) when $A > C >$ sustainable income.

2.6. What is the economy's motivation?

An important question raised earlier, by the fact that the utility discount factor that supports the economy's chosen development path is hyperbolic ($\phi_0(t) = (1 + \theta_0 t)^{-\rho}$), is: What basic principle motivates the economy to follow the calculated optimal path?

There is no perfect answer. No elegant axiomatic foundation, of the kind that Koopmans (1960) established for exponential discounting, yet exists for hyperbolic discounting, and one may never exist. A partial answer lies in the discount factor ϕ_x in (15) which we would use in potentially reoptimizing the path at an arbitrary time x. The way in which ϕ_x varies with x, but only indirectly, via the stocks $K(x)$ of capital, $S(x)$ of resource, and $T(x)$ of technological knowledge, shows that the economy chooses (and wishes to choose) consumption and investments as stable functions of only these stocks. This also answers the question of time-consistency directly, since it means that the economy's choice of consumption and investments is a Markov program, and therefore time-consistent. However, it does not answer the further question of why the economy should make this choice, and why the hyperbolic form is preferred for ϕ_x. But neither is there any final resolution to the choice between constant discounting and constant consumption, even though this choice is clear axiomatically. Meanwhile, the above results on income measurement still seem useful.

3. Conclusions

Exact solutions have been presented for a feasible, intertemporally efficient path of a 'hyperbolic' theoretical economy with human-made capital, a non-renewable resource, and (possibly) exogenously growing technical knowledge as inputs to production, and specific functional forms. This economy illustrates some significant points in recent literature on income and sustainability accounting, adds some new points, and may prove useful as a testbed for future theoretical enquiry. We call the solutions optimal because they would be chosen by the economy behaving as if to maximize present value using hyperbolic utility discounting. We then have the property, which may be ethically attractive to some schools of thought, that optimal consumption grows forever if the overall strength of discounting is low enough, even when there is no exogenous technical progress. This avoids the well-known problem that the constant consumption path may 'perpetuate poverty' or be 'foolishly conservative' (Solow, 1974: 41); and the Solow path is a special case of the hyperbolic economy here, with a particular discount rate and zero technical progress.

Also in the hyperbolic economy, five measures of income levels – welfare-equivalent income, wealth-equivalent income, Sefton–Weale income, (green) net national product (NNP), and sustainable income, which all have interesting properties – are all distinct theoretically, and none of them can unambiguously be called 'Hicksian income'. The first four measures are typically in descending size order, and have quite different values in a plausible numerical example. This emphasizes that different measures of income serve different purposes, though testing sustainability could arguably be a natural purpose for the economy's context of measuring an income level in just one country. The optimal path is time-consistent,

given that the rate of decline of the utility discount rate is chosen to depend on the capital, resource and technology stocks, as well as parameters in the production function and the overall strength of discounting. Further research on any empirical significance of the above results would seem worthwhile.

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