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BICYCLIC AND BASS CYCLIC UNITS IN GROUP RINGS

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ABSTRACT. The subgroup generated by the Bass cyclic and bicyclic units is of infinite index in the group of units of the integral group ring ZG when G is either D or D_{16}^{+}.

Let G be a finite group, $U(\mathbf{Z}G)$ the group of units of the integral group ring $\mathbf{Z}G$ and $U_1(\mathbf{Z}G)$ the units of augmentation 1. If G is a finite nilpotent group, then Ritter and Sehgal [3] have shown that, under some restrictions, the Bass cyclic and bicyclic units generate a subgroup of finite index in $U(\mathbf{Z}G)$. The restrictions are on the Sylow-2 subgroups, and for 2-groups the situation is still not clear. Specifically, Ritter and Sehgal [3, p. 618] state that the question is open for the groups $D = \langle a,b,c \mid a^2 = b^2 = c^4 = 1, ac = ca, bc = cb, ba = c^2ab\rangle$ and $D_{16}^{+} = \langle a,b \mid a^8 = b^2 = 1, ba = a^5b\rangle$.

The purpose of this note is to show that for both D and D_{16}^{+}, the subgroup generated by the bicyclic and Bass cyclic units is of infinite index in $U(\mathbf{Z}G)$.

Our notation follows that in [4].

For $a \in G$, we denote by $\hat{a}$ the sum $1+a+a^2+\cdots+a^{\mathrm{ord}(a)-1}$. Recall that a bicyclic unit in $\mathbf{Z}G$ is a unit of the form $1+(1-a)b\hat{a}$ where $a,b \in G$; and a Bass cyclic unit is a unit of the form $(1+a+\cdots+a^{i-1})^m + \frac{1-i^m}{\mathrm{ord}(a)}\hat{a}$, where $a \in G$, $1 < i < \mathrm{ord}(a)$, $\big(i, \mathrm{ord}(a)\big) = 1$, $m = \varphi\big(\mathrm{ord}(a)\big)$, φ the Euler φ-function.

Let $\Gamma(2)$ denote the principal congruence subgroup modulo 2 of the Picard group. That is, $\Gamma(2)$ is obtained by factoring out $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}$ from the group of determinant 1 matrices of the form $\begin{pmatrix}1+2a & 2b\\ 2c & 1+2d\end{pmatrix}$ where a, b, c, d are Gaussian integers.

To begin, we recall the description of $U(\mathbf{Z}D)$ and $U(\mathbf{Z}D_{16}^{+})$ given by Jespers and Leal in Corollaries 4.5 and 4.7 of [2]. Note that Proposition 1 appears somewhat different from Corollary 4.5 as we have found it convenient to conjugate by $\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$. Also, Proposition 2 corrects some errors which appeared in the statement of Corollary 4.7 in [2].

PROPOSITION 1 ([2]). *In $U_1(\mathbf{Z}D)$, D has a torsion-free normal complement $V = \{u = 1+(1-c^2)\alpha \mid \alpha \in \Delta_{\mathbf{Z}}(D),\ u \text{ a unit}\}$. V is isomorphic to the subgroup of $\Gamma(2)$ consisting of those matrices $\begin{pmatrix}1+2a & 2b\\ 2c & 1+2d\end{pmatrix}$ for which $b+c$ is divisible by 2. One such*

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isomorphism maps

$$1 + (1 - c^2)\big(\alpha_0 + \alpha_1 c + (\beta_0 + \beta_1 c)a + (\gamma_0 + \gamma_1 c)b + (\delta_0 + \delta_1 c)ab\big)$$

to the matrix

$$\begin{pmatrix} 1 + 2(\alpha_0 - \delta_1) + 2(\alpha_1 + \delta_0)i & 2(\gamma_0 - \beta_1) + 2(\beta_0 + \gamma_1)i \\ 2(\gamma_0 + \beta_1) + 2(\gamma_1 - \beta_0)i & 1 + 2(\alpha_0 + \delta_1) + 2(\alpha_1 - \delta_0)i \end{pmatrix}.$$

PROPOSITION 2 ([2]). *In $U_1(\mathbf{Z}D_{16}^+)$, D_{16}^+ has a torsion-free normal complement $V = \{u = 1 + (1 - a^4)\alpha \mid \alpha \in \Delta_{\mathbf{Z}}(D_{16}^+),\ u \text{ a unit}\}$. V is isomorphic to the subgroup of $\Gamma(2)$ consisting of those matrices $\begin{pmatrix} 1+2a & 2b \\ 2c & 1+2d \end{pmatrix}$ for which $bi + c$ is divisible by 2. One such isomorphism maps*

$$1 + (1 - a^4)\big(\alpha_0 + \alpha_1 a^2 + (\beta_0 + \beta_1 a^2)a + (\gamma_0 + \gamma_1 a^2)b + (\delta_0 + \delta_1 a^2)ab\big)$$

to the matrix

$$\begin{pmatrix} 1 + 2(\alpha_0 + \gamma_0) + 2(\alpha_1 + \gamma_1)i & 2(\delta_1 - \beta_1) + 2(\beta_0 - \delta_0)i \\ 2(\beta_0 + \delta_0) + 2(\beta_1 + \delta_1)i & 1 + 2(\alpha_0 - \gamma_0) + 2(\alpha_1 - \gamma_1)i \end{pmatrix}.$$

It is shown in [1] that $\Gamma(2)$ is a subgroup of index 48 in $\mathrm{PSL}(2, \mathbf{Z}[i])$. Earlier, Waldinger [5] showed that the following 8 matrices also generate a subgroup of index 48 in $\mathrm{PSL}(2, \mathbf{Z}[i])$.

$$a_\ell = \begin{pmatrix} -1+2i & -2 \\ -2 & -1-2i \end{pmatrix} \qquad b_\ell = \begin{pmatrix} 3 & 2i \\ 2i & -1 \end{pmatrix}$$

$$\alpha_\ell = \begin{pmatrix} 3-2i & 2 \\ 4i & -1+2i \end{pmatrix} \qquad \beta_\ell = \begin{pmatrix} 1+2i & 2i \\ -4 & -3-2i \end{pmatrix}$$

$$a_r = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix} \qquad b_r = \begin{pmatrix} 1 & -2i \\ 0 & 1 \end{pmatrix}$$

$$\alpha_r = \begin{pmatrix} 3 & 2 \\ -2 & -1 \end{pmatrix} \qquad \beta_r = \begin{pmatrix} -1-2i & -2i \\ 2i & -1+2i \end{pmatrix}$$

Since all of the above matrices are in $\Gamma(2)$, we conclude that Waldinger's subgroup is, in fact, $\Gamma(2)$.

Waldinger also showed that the relations in $\Gamma(2)$ are $a_\ell b_\ell = b_\ell a_\ell$, $a_r b_r = b_r a_r$, $\alpha_\ell \beta_\ell = \beta_\ell \alpha_\ell$, $\alpha_r \beta_r = \beta_r \alpha_r$, $a_\ell \alpha_\ell = a_r \alpha_r$, $b_\ell \beta_\ell = b_r \beta_r$, $a_\ell b_\ell \alpha_\ell \beta_\ell = a_r b_r \alpha_r \beta_r$.

We will be interested in $\Gamma(2)/K$ where K is the normal closure in $\Gamma(2)$ of $\langle a_\ell, b_\ell, a_r, \alpha_r \rangle$. Since $\alpha_\ell = a_\ell^{-1} a_r \alpha_r$ and $\beta_\ell = b_\ell^{-1} b_r \beta_r$, $\Gamma(2)/K$ is generated by $\bar{b}_r$ and $\bar{\beta}_r$. The relations do not put any further restrictions on $\Gamma(2)/K$, so we conclude that $\Gamma(2)/K$ is a free group of rank two generated by $\bar{b}_r$ and $\bar{\beta}_r$.

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THEOREM 3. *The bicyclic and Bass cyclic units generate a subgroup of infinite index in $U(\mathbf{Z}D)$.*

PROOF. $U(\mathbf{Z}D)$ has no non-trivial Bass cyclic units, while, up to inverses, there are 12 bicyclic units as follows:

$$X_1 = 1 + (1-a)b\hat{a} = 1 + (1-c^2)(b-ab)$$
$$X_2 = 1 + (1-a)cb\hat{a} = 1 + (1-c^2)(cb-cab)$$
$$X_3 = 1 + (1-b)a\hat{b} = 1 + (1-c^2)(a+ab)$$
$$X_4 = 1 + (1-b)ca\hat{b} = 1 + (1-c^2)(ca+cab)$$
$$X_5 = 1 + (1-cab)a\widehat{cab} = 1 + (1-c^2)(a+cb)$$
$$X_6 = 1 + (1-cab)b\widehat{cab} = 1 + (1-c^2)(b-ca)$$
$$X_7 = 1 + (1-c^2a)b\widehat{c^2a} = 1 + (1-c^2)(b+ab)$$
$$X_8 = 1 + (1-c^2a)cb\widehat{c^2a} = 1 + (1-c^2)(cb+cab)$$
$$X_9 = 1 + (1-c^2b)a\widehat{c^2b} = 1 + (1-c^2)(a-ab)$$
$$X_{10} = 1 + (1-c^2b)ca\widehat{c^2b} = 1 + (1-c^2)(ca-cab)$$
$$X_{11} = 1 + (1-c^3ab)a\widehat{c^3ab} = 1 + (1-c^2)(a-cb)$$
$$X_{12} = 1 + (1-c^3ab)b\widehat{c^3ab} = 1 + (1-c^2)(b+ca)$$

Using Proposition 1, we obtain matrix representations for these bicyclic units.

$$X_1 = \begin{pmatrix} 1-2i & 2 \\ 2 & 1+2i \end{pmatrix} \qquad X_2 = \begin{pmatrix} 3 & 2i \\ 2i & -1 \end{pmatrix}$$
$$X_3 = \begin{pmatrix} 1+2i & 2i \\ -2i & 1-2i \end{pmatrix} \qquad X_4 = \begin{pmatrix} -1 & -2 \\ 2 & 3 \end{pmatrix}$$
$$X_5 = \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix} \qquad X_6 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$
$$X_7 = \begin{pmatrix} 1+2i & 2 \\ 2 & 1-2i \end{pmatrix} \qquad X_8 = \begin{pmatrix} -1 & 2i \\ 2i & 3 \end{pmatrix}$$
$$X_9 = \begin{pmatrix} 1-2i & 2i \\ -2i & 1+2i \end{pmatrix} \qquad X_{10} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$
$$X_{11} = \begin{pmatrix} 1 & 0 \\ -4i & 1 \end{pmatrix} \qquad X_{12} = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$$

In terms of the generators of $\Gamma(2)$, these bicyclic units can be expressed as follows.

$$X_1 = a_\ell,\ X_2 = b_\ell,\ X_3 = \beta_r,\ X_4 = \alpha_r^{-1},$$
$$X_5 = b_r^{-2},\ X_6 = a_r^2,\ X_7 = b_r^{-1}a_\ell b_r,\ X_8 = b_r^{-1}b_\ell b_r,$$
$$X_9 = a_r\beta_r a_r^{-1},\ X_{10} = a_r\alpha_r^{-1}a_r^{-1},\ X_{11} = (b_r^{-1}b_\ell)^2,\ X_{12} = (a_r\alpha_r)^2.$$

Let H be the subgroup of $\Gamma(2)$ generated by the bicyclic units and consider HK/K in $\Gamma(2)/K$. All generators of H except for X_3, X_5, X_9 and X_{11} are in K, so we see easily that HK/K is generated by $\bar{\beta}_r$ and $\bar{b}_r^2$. Thus HK/K is a proper free rank 2 subgroup of $\Gamma(2)/K$ and therefore is of infinite index in $\Gamma(2)/K$, and H is of infinite index in $\Gamma(2)$. We conclude from Proposition 1 that H is of infinite index in $U(\mathbf{Z}D)$. ■

THEOREM 4. *The bicyclic and Bass cyclic units generate a subgroup of infinite index in* $U(\mathbf{Z}D_{16}^{+})$.

PROOF. $U(\mathbf{Z}D_{16}^{+})$ has, up to inverses, 4 bicyclic units as follows:

$$X_1 = 1 + (1-b)a(1+b) = 1 + (1-a^4)(a+ab)$$
$$X_2 = 1 + (1-b)a^3(1+b) = 1 + (1-a^4)(a^3+a^3b)$$
$$X_3 = 1 + (1-a^4b)a(1+a^4b) = 1 + (1-a^4)(a-ab)$$
$$X_4 = 1 + (1-a^4b)a^3(1+a^4b) = 1 + (1-a^4)(a^3-a^3b)$$

Using Proposition 2, the matrix representations of these bicyclic units are

$$X_1 = \begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix} \qquad X_2 = \begin{pmatrix} 1 & 0 \\ 4i & 1 \end{pmatrix}$$
$$X_3 = \begin{pmatrix} 1 & 4i \\ 0 & 1 \end{pmatrix} \qquad X_4 = \begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$$

In terms of the generators of $\Gamma(2)$, these bicyclic units can be expressed as follows:

$$X_1 = (a_r\alpha_r)^2,\ X_2 = (b_\ell^{-1}b_r)^2,\ X_3 = b_r^{-2},\ X_4 = a_r^{-2}.$$

Let H be the subgroup of $\Gamma(2)$ generated by the bicyclic and Bass cyclic units and consider HK/K in $\Gamma(2)/K$. Note that X_1 and X_4 are in K, while X_2 and X_3 both generate the subgroup $\langle \bar{b}_r^2 \rangle$ modulo K. Since every Bass cyclic unit $\mathbf{Z}D_{16}^{+}$ is a power of the Bass cyclic unit $(1+a+a^2)^4 - 10\hat{a}$, HK/K is a proper subgroup of $\Gamma(2)/K$ requiring less than 3 generators. We conclude that HK/K is of infinite index in $\Gamma(2)/K$ and therefore, by Proposition 2, that H is of infinite index in $U(\mathbf{Z}D_{16}^{+})$. ∎

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