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Homogenization of composite electrets

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We study the two-scale homogenization of the diffraction interfacial condition for the diffusion equation relevant to a composite medium which has a periodic structure. The results are applied to the electric field potential within a dielectric composite body when there is a difference in dielectric permittivity between the composite components in the presence of interfacial static charges. The principal result is that the interfacial charge distribution is equivalent to an apparent bulk charge which can be calculated starting from the composite geometry. We perform the corrector analysis and establish that the corrector terms strongly depend on the interfacial charge.

Key words: Electret composite, interfacial and bulk charges, two-scale convergence, correctors

1 Introduction

The goal of the present study is to apply the two-scale homogenization tool to a new mathematical problem arising in composite electrets. Electrets are dielectric materials capable of storing oriented dipoles or an electric surplus charge for a long period of time without an external electric field. There are several ways to produce space-charge electrets. For example, electret polymer foams store charges on the inner surfaces of the voids after having been subjected to a corona discharge [14]. Another example of the electret composite material is the polymer-ferroelectric ceramic composite which is produced under the action of an electric discharge plasma and temperature. It is shown that this process results in strong oxidation of polymer chains. Such an oxidation is accompanied by an enhancement of interfacial interactions and an increase in the concentration of charge localization centres in the quasi-band gap of the polymer phase which upon polarization leads to an increase in interfacial charges [10]. Composite electrets find many applications including electrostatic filters, electret microphones, radiation dosimeters, etc. [8].

We consider a dielectric composite with static charges concentrated at the interface. Clearly, a body of such a composite, an electret, exhibits an electric field inside the body and close to it. Even though one knows in full details the charge interfacial distribution, it is almost impossible to calculate the resulting electric field due to the complex geometry of the composite structure. Under the assumption that the composite has a periodic structure, we apply the two-scale homogenization technique to prove that the interfacial charge distribution is equivalent to an "apparent" constant bulk charge. With such a bulk charge to hand, it is simpler to identify the resulting electric field by solving the homogenized charge conservation law for a homogeneous material.

We prove that the apparent bulk charge strongly depends not only on the interfacial charge value but on the geometry of the composite interface inside the representative cell of periodicity as well. To this end, we introduce a surface tortuosity coefficient and study its role in the homogenization process. We establish that it is due to high tortuosity that the apparent bulk charge can be strong even though the interfacial charge is weak.

We discover that besides the length l which characterizes the size of the representative cell, there are two more intrinsic lengths l_u and l_σ which depend on the interfacial and bulk charges. Hence, the homogenization strongly depends on the order relation between the lengths l, l_u, and l_σ. It may occur that $l \ll l_u \ll l_\sigma$ or $l \ll l_\sigma \ll l_u$, etc. We analyse some order relations which are of interest in geophysics. We clarify that the lengths l_u and l_σ characterize the electric field attenuation related to the interfacial and bulk charges. The role of intrinsic lengths in the homogenization of the Maxwell equations was highlighted in Amirat and Shelukhin [1]. Note that such a situation, when the homogenization depends on the values of coefficients, is rather general and arises for instance in poroelastic media [13].

Space-charge ionic electrets defy the conventional assumption in chemistry that the bulk matter is electrically neutral and that ionic materials must have an equal number of cationic and anionic charges [24]. Under the hypothesis of local neutrality of the composite, we perform the homogenization and prove that such a composite does not enjoy the electret property.

It is the essence of the homogenization method that the original heterogeneous problem is replaced approximately by an averaged one. We improve such approximation by constructing the so-called corrector terms which take into account the local fluctuations in each periodicity cell. We prove that the corrector terms strongly depend on the interfacial charge density.

Existence of interfacial charge implies that the normal component of the vector of electrical induction has a jump across the interface, with this jump being equal to the surface charge density. In terms of electric potential, such a jump condition is equivalent to a diffraction condition for the potential which solves a diffusion equation. Mathematically, we address the homogenization of the diffraction condition. Though such a condition is a classic one in the theory of elliptic equations, it has never been studied from the homogenization point of view. Note that the homogenization of elliptic equations with various boundary conditions (Dirichlet, Neumann, mixed conditions, Robin condition, Signorini condition, etc. . .) have been considered in many papers, see for instance [5–7, 9, 19] and the references therein. Problems of stationary diffusion in composites with an interface condition involving a jump of the temperature are considered in Auriault and Ene, [3], Lipton [12] and Monsurro [16].

It is proved in Amirat and Shelukhin [2] and Shelukhin *et al.* [23] that interfacial charges may appear due to ion transport in a fluid dielectric medium. The nature of this charge is the Maxwell–Wagner polarization which appears due to a difference in dielectric permittivities between the composite components. Contrary to the static electret interfacial charge, such a charge disappears as soon as the flow stops or an external electric field is removed.

2 Problem formulation

We consider a composite body consisting of two components. For convenience, we call them solid and fluid components. Let a solid domain Ω_s lie in a domain Ω, with $\Omega_f = \Omega\backslash\Omega_s$ being the fluid domain. In what follows, we use the Gaussian system of units. Let us consider the charge conservation law

$$\operatorname{div}(\varepsilon\mathbf{E}) = 4\pi q, \quad \mathbf{E} \equiv -\nabla u, \tag{2.1}$$

where $\mathbf{E}$ is the electric field, u is the electric potential, ε is the dielectric permittivity, and $q(x)$ is the bulk charge density. Here,

$$\varepsilon = \begin{cases} \varepsilon_f, & x \in \Omega_f, \\ \varepsilon_s, & x \in \Omega_s. \end{cases}$$

With $q_\sigma(x)$ standing for the surface charge density at the boundary Γ that separates the solid and fluid domains, we have the boundary conditions

$$[u] = 0, \quad [\varepsilon\mathbf{E}\cdot\mathbf{n}] = 4\pi q_\sigma. \tag{2.2}$$

Here, $\mathbf{n}$ is the unit normal vector to Γ, pointing from Ω_s to Ω_f, and the brackets $[v]$ stand for the jump of a function $v(x)$ across Γ. More precisely, denoting by v_f and v_s the values of v on either side of Γ, respectively, in the domains Ω_f and Ω_s, we set $[v] = v_f - v_s$. We set the following boundary condition

$$u|_{\partial\Omega} = 0. \tag{2.3}$$

A function u is called a strong solution of problem (2.1)–(2.3) if

$$u \in C^1\left(\overline{\Omega_i}\right) \cap C^2(\Omega_i), \quad i = s, f,$$

u solves equation (2.1) in the domains Ω_s and Ω_f, and it satisfies conditions (2.2) and (2.3).

The weak formulation of problem (2.1)–(2.3) is the following. We look for a function $u \in H_0^1(\Omega)$ such that.

$$\int_\Omega \varepsilon\nabla u\cdot\nabla\varphi - 4\pi q\varphi\, dx - \int_\Gamma 4\pi q_\sigma\varphi\, ds = 0, \quad \forall\varphi \in H_0^1(\Omega). \tag{2.4}$$

One can easily verify that any strong solution u of problem (2.1)–(2.3) belongs to the Sobolev space $H_0^1(\Omega)$ and satisfies equality (2.4).

3 One-dimensional problem

Let us consider a sequence of fluid and solid layers separated by points x_i $(0 \leqslant i \leqslant 2N)$:

$$0 = x_0 < x_1 < x_2 < \cdots < x_{2N} = L,$$

where N is a large integer. The intervals (x_{2i}, x_{2i+1}) and (x_{2i+1}, x_{2i+2}) represent the fluid and the solid parts respectively. With ϕ standing for the porosity $(0 \leqslant \phi \leqslant 1)$ and $l = L/N$, we have

$$x_{2i+1} - x_{2i} = \phi l, \quad x_{2i+2} - x_{2i+1} = (1-\phi)l.$$

We define

$$\varepsilon(y) = \begin{cases} \varepsilon_f & \text{for} \quad 0 < y < \phi, \\ \varepsilon_s & \text{for} \quad \phi < y < 1, \end{cases} \qquad 1_f(y) = \begin{cases} 1 & \text{for} \quad 0 < y < \phi, \\ 0 & \text{for} \quad \phi < y < 1, \end{cases}$$

and extend these functions periodically on $\mathbb{R}$. Let us denote

$$\varepsilon^\delta(x) = \varepsilon\left(\frac{x}{l}\right), \quad 1_f^\delta(x) = 1_f\left(\frac{x}{l}\right), \quad \delta = \frac{l}{L}.$$

The electric potential satisfies the boundary value problem posed in $\Omega = (0, L)$:

$$\begin{cases} -\varepsilon^\delta(x) u_{xx} = 4\pi q(x), \quad x_i < x < x_{i+1}, \\ [u]|_{x_i} = 0, \quad [\varepsilon^\delta(x) u_x]|_{x_i} = (-1)^i 4\pi q_\sigma, \quad 1 \leqslant i \leqslant 2N-1, \\ u(0) = u(L) = 0. \end{cases} \tag{3.1}$$

Here, q_σ is a given surface density constant.

Let us pass to dimensionless variables. Assume that the variables with the bar sign are reference values, then the dimensionless variables, with the prime sign, are

$$x = Lx', \quad u = \bar{u}u', \quad \varepsilon = \bar{\varepsilon}\varepsilon', \quad q = \bar{q}q', \quad q_\sigma = \bar{q}_\sigma q'_\sigma.$$

Observe that the parameters

$$l_u = \frac{\bar{u}\bar{\varepsilon}}{\bar{q}_\sigma}, \quad l_\sigma = \frac{\bar{q}_\sigma}{\bar{q}}, \tag{3.2}$$

have the dimension of length. We discuss their meaning later in Appendix. Thus, the above problem is characterized by four length scales l, L, l_u, and l_σ. We introduce the dimensionless parameters

$$a_1 = \frac{\bar{\varepsilon}\bar{u}}{\bar{q}_\sigma L} \equiv \frac{l_u}{L}, \quad a_2 = \frac{L\bar{q}}{\bar{q}_\sigma} \equiv \frac{L}{l_\sigma}.$$

In dimensionless variables, problem (3.1) becomes

$$\begin{cases} -a_1 \varepsilon'(x') u'_{x'x'} = 4\pi a_2 q'(x'), \quad x'_i < x' < x'_{i+1}, \\ [u']|_{x'_i} = 0, \quad a_1 [\varepsilon' u'_{x'}]|_{x'_i} = (-1)^i 4\pi q'_\sigma, \quad 1 \leqslant i \leqslant 2N-1, \\ u'|_{\partial\Omega} = 0, \quad \Omega' = \{x' : 0 < x' < 1\}. \end{cases}$$

For simplicity, we omit the prime sign in what follows. One can verify easily that the weak formulation of the above problem is the following:

$$\begin{cases} u^\delta \in H_0^1(\Omega), \\ \int\limits_\Omega a_1\varepsilon^\delta(x)u_x^\delta\varphi_x - 4\pi a_2 q\varphi\,dx + 4\pi q_\sigma \sum\limits_1^{2N-1}(-1)^i\varphi(x_i) = 0, \quad \forall\varphi \in H_0^1(\Omega). \end{cases}$$

Observe that in our notation the functions $\varepsilon^\delta(x)$ and $1_f^\delta(x)$ are periodic with period δ.

Simple calculations reveal that

$$\sum_{1=1}^{2N-1}(-1)^i\varphi(x_i) = -\int_0^{x_1}\varphi_x\,dx - \int_{x_2}^{x_3}\varphi_x\,dx - \cdots = -\int\limits_\Omega 1_f^\delta(x)\varphi_x\,dx.$$

Hence, we arrive at the following weak formulation:

$$\begin{cases} u^\delta \in H_0^1(\Omega), \\ \int_\Omega a_1\varepsilon^\delta(x)u_x^\delta\varphi_x - 4\pi q_\sigma 1_f^\delta(x)\varphi_x - 4\pi a_2 q\varphi\,dx = 0, \quad \forall\varphi \in H_0^1(\Omega). \end{cases} \tag{3.3}$$

Assuming that δ is a small number, we perform an asymptotic analysis, as $\delta \to 0$, under the hypothesis that

$$\frac{l}{L} = \delta, \quad \frac{l}{l_u} = \frac{\delta^{m_1}}{\bar{a}_1}, \quad \frac{l}{l_\sigma} = \delta^{m_2}\bar{a}_2. \tag{3.4}$$

So,

$$a_1 = \delta^{1-m_1}\bar{a}_1, \quad a_2 = \delta^{m_2-1}\bar{a}_2.$$

Here, we study the case $m_1 = m_2 = 1$. Other values of m_1 and m_2 are addressed when we consider three-dimensional boundary value problems in the next sections. Assuming that $q \in L^2(\Omega)$ and setting $\varphi = u$ in (3.3), one obtains that u^δ satisfies the estimate

$$\int_\Omega |u^\delta|^2 + |u_x^\delta|^2\,dx \leqslant c, \tag{3.5}$$

uniformly in δ. Therefore, there is a subsequence of u^δ (still denoted u^δ) and a function $u \in H_0^1(\Omega)$ such that u^δ converges to u weakly in $H_0^1(\Omega)$.

To characterize the function u, we use the notion of two-scale convergence introduced by G. Nguetseng [17]. Let Y denote the unit cell of periodicity, $Y = \{y : 0 < y < 1\}$. A sequence v^δ of functions in $L^2(\Omega)$ is said to weakly two-scale convergent to a function $v(x, y)$, $v \in L^2(\Omega \times Y)$, as $\delta \to 0$, if

$$\lim_{\delta\to 0}\int\limits_\Omega v^\delta(x)\varphi\left(x, \frac{x}{\delta}\right)\,dx = \frac{1}{|Y|}\int\limits_\Omega\int\limits_Y v(x,y)\varphi(x,y)\,dxdy, \;\; \forall\varphi \in C(\Omega; \mathcal{C}_{per}^\infty(Y)).$$

Shortly, we write it as $v^\delta \stackrel{2s}{\rightharpoonup} v(x,y)$. We emphasize that for each $x \in \Omega$, the test function $\varphi(x,\cdot)$ is Y-periodic in the variable y and belongs to $\mathcal{C}^\infty(Y)$. It is a crucial property of the two-scale convergence that for any sequence of functions $v^\delta(x)$ bounded in $L^2(\Omega)$ there is a subsequence (still denoted $v^\delta(x)$) and a function $v(x,y)$, $v \in L^2(\Omega \times Y)$, such that $v^\delta \overset{2s}{\rightharpoonup} v(x,y)$ [17]. We also have the following result [17]. If v^δ is bounded in $H^1(\Omega)$, there is a subsequence (still denoted v^δ) and functions $v \in L^2(\Omega)$, $v^1 \in L^2(\Omega; H^1_{per}(Y))$ such that both v^δ and v^δ_x two-scale converge weakly to $v(x)$ and $v_x(x) + v^1_y(x,y)$, respectively.

According to (3.5), there are functions $u(x) \in H^1_0(\Omega)$ and $u^1(x,y) \in L^2(\Omega; H^1_{per}(Y))$ such that

$$u^\delta \overset{2s}{\rightharpoonup} u, \quad u^\delta_x \overset{2s}{\rightharpoonup} u_x + u^1_y.$$

Setting

$$\varphi(x) = \varphi^0(x) + \delta\varphi^1\left(x, \frac{x}{\delta}\right), \quad \varphi^0 \in \mathcal{D}(\Omega), \ \ \varphi^1 \in \mathcal{D}\left(\Omega; \mathcal{C}^\infty_{per}(Y)\right),$$

in (3.3) and passing to the limit, as $\delta \to 0$, we conclude that

$$\int_\Omega \int_Y \left\{\bar{a}_1\varepsilon(y)\left[u_x(x) + u^1_y(x,y)\right] - 4\pi q_\sigma 1_f(y)\right\}\left(\varphi^0_x(x) + \varphi^1_y(x,y)\right)\,dxdy \\ - \int_\Omega \int_Y 4\pi\bar{a}_2 q\,\varphi^0(x)\,dxdy = 0. \tag{3.6}$$

Choosing $\varphi^0 = 0$, $\varphi^1(x,y) = \psi(x)\theta(y)$, $\psi \in \mathcal{D}(\Omega)$, and $\theta \in \mathcal{C}^\infty_{per}(Y)$, we obtain

$$\int_Y \theta_y(y)\left\{\bar{a}_1\varepsilon(y)\left[u_x(x) + u^1_y(x,y)\right] - 4\pi q_\sigma 1_f(y)\right\}\,dy = 0.$$

We look for u^1 in the form

$$u^1(x,y) = u_x(x)w^1(y) - 4\pi q_\sigma w^0(y). \tag{3.7}$$

The functions w^1 and w^0 can be identified from the following cell problems:

$$\begin{cases} w^1 \in H^1_{per}(Y), \\ \dfrac{d}{dy}\left\{\varepsilon(y)\left[1 + w^1_y(y)\right]\right\} = 0, \quad \displaystyle\int_Y w^1\,dy = 0, \end{cases} \tag{3.8}$$

and

$$\begin{cases} w^0 \in H^1_{per}(Y), \\ \dfrac{d}{dy}\left(\bar{a}_1\varepsilon(y)w^0_y(y) + 1_f(y)\right) = 0, \quad \displaystyle\int_Y w^0\,dy = 0. \end{cases} \tag{3.9}$$

Next, choosing $\varphi^1 = 0$ in (3.6), we obtain

$$\int_\Omega \int_Y \bar{a}_1\varepsilon(y)\left[u_x(x) + u^1_y(x,y)\right]\varphi^0_x(x) - 4\pi\bar{a}_2 q\varphi^0(x)\,dxdy = 0.$$

Using (3.7)–(3.9) we obtain the macro-equation

$$-\bar{a}_1\varepsilon^h u_{xx} = 4\pi\bar{a}_2 q \quad \text{in } \Omega, \quad u(0) = u(1) = 0, \tag{3.10}$$

with

$$\varepsilon^h = \varepsilon(y)(1 + w^1_y) = \text{constant} = \frac{1}{\phi/\varepsilon_f + (1-\phi)/\varepsilon_s}. \tag{3.11}$$

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As an example, we consider the following data

$$\varepsilon_f = \varepsilon_s = \varepsilon, \quad \phi = \frac{1}{2}, \quad q = 0.$$

In this case, the solution u^δ of (3.3) is given by the formulas

$$u^\delta = \begin{cases} -\dfrac{4\pi q_\sigma}{2\varepsilon}(x - x_{2i}), & x_{2i} < x < x_{2i+1}, \\ \dfrac{4\pi q_\sigma}{2\varepsilon}(x - x_{2i+2}), & x_{2i+1} < x < x_{2i+2}, \end{cases} \qquad u^\delta_x = \begin{cases} -\dfrac{4\pi q_\sigma}{2\varepsilon}, & x_{2i} < x < x_{2i+1}, \\ \dfrac{4\pi q_\sigma}{2\varepsilon}, & x_{2i+1} < x < x_{2i+2}. \end{cases}$$

One can verify easily that

$$\max_{0\leqslant x\leqslant 1} u^\delta(x) = u^\delta(x_{2i}) = 0, \quad \min_{0\leqslant x\leqslant 1} u^\delta(x) = -\frac{4\pi q_\sigma \delta}{2\varepsilon}.$$

It follows that $u^\delta \to 0$ strongly in $L^2(\Omega)$ and weakly in $H^1_0(\Omega)$. However, the convergence of u^δ does not occur in $H^1_0(\Omega)$ strongly since

$$\int_\Omega |u^\delta_x|^2\, dx = \frac{4\pi^2 q_\sigma^2}{\varepsilon^2}.$$

Observe that equation (3.10) does not depend on the surface charge density q_σ. However, q_σ can manifest itself through a corrector.

To derive a corrector for the function u^δ we argue by the formal expansion series approach [4, 21] and look for $u^\delta(x)$ in the form

$$u^\delta(x) = \sum_{k=0}^{\infty} \delta^k u^k(x, y), \tag{3.12}$$

where $y = x/\delta$ and the functions $u^k(x, y)$ are 1-periodic in the variable y. It is assumed that $u^0 \equiv u$ is defined by (3.10), (3.11), and the function $u^1(x, y)$ is defined by (3.7)–(3.9).

Given a function $u^k(x, y)$, we introduce the derivative operator

$$Du^k(x, y) = u^k_x(x, y) + \delta^{-1} u^k_y(x, y).$$

Clearly,

$$\frac{d}{dx} u^k\left(x, \frac{x}{\delta}\right) = Du^k(x, y)|_{y=x/\delta}.$$

Writing

$$F^k(x, y) = \bar{a}_1 \varepsilon(y) Du^k(x, y),$$

we find that

$$\sum_{k=0}^{\infty} \delta^k \left(F^k(x, y) - \bar{a}_1 \varepsilon(y) Du^k(x, y)\right) = 0. \tag{3.13}$$

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In terms of expansion series, one can write equation (3.3) as

$$\sum_{k=0}^{\infty} \delta^k DF^k(x,y) + 4\pi q(x)\bar{a}_2 - 4\pi q_\sigma(y) D1_f(y) = 0, \tag{3.14}$$

where $y = x/\delta$. We write (3.13) in the form $\sum_0^\infty \delta^k(\cdots)_k = 0$, with the functions $(\cdots)_k(x,y)$ not depending on δ, and arrive at the equalities $(\cdots)_k = 0$ for any k. Setting $k = 0$ and $k = 1$, we find that

$$F^0(x,y) = \bar{a}_1\varepsilon(y)\left(u_x(x)\left(1 + w_y^1(y)\right) - 4\pi q_\sigma w_y^0(y)\right),$$
$$F^1(x,y) = \bar{a}_1\varepsilon(y)\left(u_{xx}(x)w^1(y) + u_y^2(x,y)\right).$$

In finding the functions $u^k(x,y)$, $k = 2, 3, \ldots$, we assume that equality (3.14) holds for any x and y. Writing this equality in the form $\sum_{k=-1}^{\infty} \delta^k(\cdots)_k = 0$, with the functions $(\cdots)_k(x,y)$ not depending on δ, we arrive at the equalities $(\cdots)_k = 0$ for any $k = -1, 0, 1, \ldots$. Setting $k = -1$ and $k = 0$ and using the representation formulas for F^0 and F^1, we find that

$$\frac{\partial}{\partial y}\left\{\bar{a}_1\varepsilon(y)\left(u_x(x)\left(1 + w_y^1(y)\right) - 4\pi q_\sigma w_y^0(y)\right)\right\} = \frac{d}{dy}\left(4\pi q_\sigma 1_f(y)\right), \tag{3.15}$$

$$u_{xx}(x)\bar{a}_1\varepsilon(y)\left(1 + w_y^1(y)\right) + \frac{\partial}{\partial y}\left\{\bar{a}_1\varepsilon(y)\left(u_{xx}w_1(y) + u_y^2(x,y)\right)\right\} = -4\pi\bar{a}_2 q. \tag{3.16}$$

We look for $u^2(x,y)$ by the method of separation of variables, assuming that there is a function $w^2(y)$ such that $u^2(x,y) = u_{xx}(x)\,w^2(y)$. Inserting this representation formula into (3.16) and taking (3.10), (3.11) into account, we obtain that $w^2(y)$ should be a 1-periodic solution of the equation

$$\frac{d}{dy}\left\{\varepsilon(y)\left(w^1(y) + \frac{d}{dy}w^2(y)\right)\right\} = 0. \tag{3.17}$$

Clearly, this equation has a unique solution $w^2 \in H^1_{per}(Y)$ satisfying $\int_0^1 w^2 dy = 0$. We easily deduce from (3.17) that

$$w^2(y) = -\int_0^y w^1(\xi)\,d\xi, \quad y \in Y. \tag{3.18}$$

Let us now introduce the function

$$u^{c,\delta}(x) = u^c\left(x, \frac{x}{\delta}\right) + \delta u_b^\delta(x), \tag{3.19}$$

where

$$u^c(x,y) = u(x) + \delta\left(u_x(x)w^1(y) - 4\pi q_\sigma w^0(y)\right) + \delta^2 u_{xx}(x)w^2(y), \quad x \in \Omega, y \in Y, \tag{3.20}$$

and $u_b^\delta(x)$ is defined as a solution of the problem

$$\frac{d}{dx}\left(\varepsilon^\delta(x)\frac{du_b^\delta}{dx}\right) = 0, \quad u_b^\delta(0) = h_0,\ u_b^\delta(1) = h_1, \tag{3.21}$$

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with

$$h_0 = -\left(u_x(0)w^1(0) - 4\pi q_\sigma w^0(0)\right), \quad h_1 = -\left(u_x(1)w^1(1) - 4\pi q_\sigma w^0(1)\right). \tag{3.22}$$

The function $u_b^\delta(x)$ is a boundary layer of the first-order. One can verify that

$$(u_b^\delta(x) - h_0)\int_\Omega \frac{1}{\varepsilon^\delta(x)}\,dx = (h_1 - h_0)\int_0^x \frac{1}{\varepsilon^\delta(z)}\,dz.$$

We establish the following result.

Theorem 1 *Let u^δ and u be the solutions of problem* (3.3) *and problem* (3.10), (3.11), *respectively. Then, u^δ converges to u in $H^1(\Omega)$ weak. Moreover, if $q \in H^1(\Omega)$, the function $u^{c,\delta}$ given by* (3.19)–(3.22) *is a second-order corrector satisfying*

$$\|u^\delta - u^{c,\delta}\|_{H^1(\Omega)} \leqslant c\delta^2, \tag{3.23}$$

where c is a constant independent of δ.

Proof The weak convergence of u^δ towards u results from the notion of two-scale convergence. Let us prove (3.23). A straightforward calculation gives

$$\begin{aligned}
-\frac{d}{dx}\left(\bar{a}_1\varepsilon^\delta(x)\frac{du^{c,\delta}}{dx}\right) &= -\frac{1}{\delta}\left\{\frac{\partial}{\partial y}\left[\bar{a}_1\varepsilon(y)\left(u_x(x)\left(1 + w_y^1(y)\right) - 4\pi q_\sigma w_y^0(y)\right)\right]\right\}|_{y=\frac{x}{\delta}} \\
&\quad - u_{xx}(x)\,\bar{a}_1\varepsilon(y)\left(1 + w_y^1(y)\right)|_{y=\frac{x}{\delta}} \\
&\quad - u_{xx}(x)\left\{\frac{\partial}{\partial y}\left[\bar{a}_1\varepsilon(y)\left(w_1(y) + w_y^2(y)\right)\right]\right\}|_{y=\frac{x}{\delta}} \\
&\quad - \delta\, u_{xxx}(x)\,\bar{a}_1\varepsilon(y)\left(w_1(y) + w_y^2(y)\right)|_{y=\frac{x}{\delta}} \\
&\quad - \delta^2\frac{d}{dx}\left(\bar{a}_1 u_{xxx}(x)\,\varepsilon\left(\frac{x}{\delta}\right)w^2\left(\frac{x}{\delta}\right)\right) \\
&\equiv \sum_{i=1}^{i=5} A_i,
\end{aligned} \tag{3.24}$$

where the equality holds in the distributional sense. According to (3.15) we have

$$A_1 = -\frac{d}{dx}\left(4\pi q_\sigma 1_f\left(\frac{x}{\delta}\right)\right),$$

and according to (3.16) and (3.18) we have

$$A_2 + A_3 = 4\pi\bar{a}_2 q, \quad A_4 = 0.$$

We also have, using (3.10),

$$A_5 = \delta^2\frac{d}{dx}\left(4\pi\frac{\bar{a}_2}{\varepsilon^h}q_x(x)\,\varepsilon\left(\frac{x}{\delta}\right)w^2\left(\frac{x}{\delta}\right)\right).$$

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It follows that

$$-\frac{d}{dx}\left(\bar{a}_1\varepsilon^\delta(x)\frac{du^{c,\delta}}{dx}\right) = -\frac{d}{dx}\left(4\pi q_\sigma 1_f\left(\frac{x}{\delta}\right)\right) + 4\pi\bar{a}_2 q + \delta^2\frac{d}{dx}\left(4\pi\frac{\bar{a}_2}{\varepsilon^h}q_x(x)\,\varepsilon\left(\frac{x}{\delta}\right)w^2\left(\frac{x}{\delta}\right)\right).$$

Then, the difference $v^\delta = u^\delta - u^{c,\delta}$ solves the variational equation

$$\int_\Omega \bar{a}_1\varepsilon^\delta(x)v^\delta_x(x)\varphi_x(x)\,dx = -\delta^2\int_\Omega\left(4\pi\frac{\bar{a}_2}{\varepsilon^h}q_x(x)\,\varepsilon\left(\frac{x}{\delta}\right)w^2\left(\frac{x}{\delta}\right)\right)\varphi_x(x)\,dx, \tag{3.25}$$

$\forall\varphi\in H^1_0(\Omega)$. We have

$$v^\delta(0) = -\delta^2u^2_{xx}(0)w^2(0), \quad v^\delta(1) = -\delta^2u^2_{xx}(1)w^2(1).$$

Since

$$|v^\delta(1)-v^\delta(0)| \leqslant C\delta^2, \tag{3.26}$$

introducing the function

$$v^\delta_0(x) = x\left(v^\delta(1)-v^\delta(0)\right)+v^\delta(0),$$

we insert the function $\varphi_0 = v^\delta - v^\delta_0 \in H^1_0(\Omega)$ into (3.25) to obtain

$$\int_\Omega \bar{a}_1\varepsilon^\delta(x)|v^\delta_x|^2\,dx = -\delta^2\int_\Omega 4\pi\frac{\bar{a}_2}{\varepsilon^h}q_x(x)\,\varepsilon\left(\frac{x}{\delta}\right)w^2\left(\frac{x}{\delta}\right)\left[v^\delta_x(x)-v^\delta_{0x}(x)\right]\,dx + \int_\Omega \bar{a}_1\varepsilon^\delta(x)v^\delta_x v^\delta_{0x}\,dx.$$

Using the Cauchy–Schwarz inequality and (3.26) we obtain (3.23). □

Remark 1 *If q belongs only to $L^2(\Omega)$ one can easily show that*

$$\left\|u^\delta(x)-u(x)-\delta u^1\left(x,\frac{x}{\delta}\right)-\delta u^\delta_b(x)\right\|_{H^1(\Omega)} \leqslant c\delta,$$

where c is a constant independent of δ.

Remark 2 *In electret theory, it is the limit function $E = \lim_{\delta\to0}E^\delta$, $E^\delta \equiv -\partial u^\delta/\partial x$, which is of interest. Though the macro-equation (3.10) does not depend on the surface charge, this charge, by Theorem 1, should be taken into account as far as the electric field is concerned. Indeed,*

$$\left\|u^\delta(x)-u(x)-\delta u^1\left(x,\frac{x}{\delta}\right)-\delta u^\delta_b(x)\right\|_{H^1(\Omega)} = \left\|u^\delta(x)-u(x)-\delta u^1\left(x,\frac{x}{\delta}\right)-\delta u^\delta_b(x)\right\|_{L^2(\Omega)} + \left\|E^\delta(x)+\frac{d}{dx}\left(u(x)+\delta u^1\left(x,\frac{x}{\delta}\right)+\delta u^\delta_b(x)\right)\right\|_{L^2(\Omega)}.$$

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Hence,

$$\left\| E^{\delta}(x) - \frac{d}{dx}\left(-u(x) - \delta u^1\left(x, \frac{x}{\delta}\right) - \delta u_b^{\delta}(x)\right)\right\|_{L^2(\Omega)} \leqslant c\delta,$$

where c is a constant independent of δ.

4 Setting of the multi-dimensional problem

We return to the multi-dimensional problem (2.4) with periodic data $\varepsilon(x)$ and $q_\sigma(x)$ given in the representative cell

$$0 < x_i < l, \quad i = 1, 2, \ldots n,$$

$q_\sigma(x)$ being a function defined on the solid-fluid interfaces. We assume that $n = 2$ or $n = 3$. Let L stand for a characteristic size of the domain Ω. We rewrite problem (2.4) in dimensionless variables

$$x_j' = \frac{x_j}{L}, \quad u' = \frac{u}{\bar{u}}, \quad \varepsilon' = \frac{\varepsilon}{\bar{\varepsilon}}, \quad q' = \frac{q}{\bar{q}}, \quad q_\sigma' = \frac{q_\sigma}{\bar{q}_\sigma}.$$

Recall that $l/L = \delta$ and the parameters l_u and l_σ defined in (3.2) have the dimension length. We assume that these lengths can be compared with l by hypothesis (3.4). Under these assumptions, the function $u'(x')$, defined in the domain $\Omega' = \frac{1}{L}\Omega$, belongs to $H_0^1(\Omega')$ and solves the variational equation

$$\int_{\Omega'} \delta^{1-m_1} \bar{a}_1 \varepsilon' \nabla' u' \cdot \nabla' \varphi - 4\pi \delta^{m_2 - 1} \bar{a}_2 q' \varphi \, dx' = \int_{\Gamma'^{\delta}} 4\pi q_\sigma' \varphi \, ds', \quad \forall \varphi \in H_0^1(\Omega'). \tag{4.1}$$

Let us describe the fluid and solid domains in more details. We denote by Y the unit cube of $\mathbb{R}^n$, $Y = \{y = (y_1, \ldots, y_n) : 0 < y_i < 1\}$. We assume that Y is decomposed as

$$Y = Y_s \cup Y_f \cup \Gamma', \quad \Gamma' \equiv \partial Y_s \cap \partial Y_f,$$

where Y_s (the solid part) and Y_f (the fluid part) are open subsets of Y and Γ' is a smooth interface separating Y_s and Y_f. We assume that $|Y_s| > 0$ and $|Y_f| > 0$. Given $k \in \mathbb{Z}^n$, we define

$$Y^k = \{y \in \mathbb{R}^n : y - k \in Y\}.$$

Similarly, we define

$$\Gamma'^k = \Gamma' + k, \quad Y_i^k = Y_i + k \ \ (i = s, f).$$

As for the domain Ω, we assume, for simplicity, that its dimensionless replica Ω' is the unit cube of $\mathbb{R}^n$,

$$\Omega' = \{x \in \mathbb{R}^n : 0 < x_i < 1\},$$

and $\delta = 1/N$, $N \to \infty$.

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We define

$$\delta Y = \{y = (y_1, \ldots, y_n) : 0 < y_i < \delta\},$$

and similarly, we define δY_s, δY_f, δY_s^k, δY_f^k, and $\delta \Gamma'^k$. Let us write

$$K^\delta = \{k \in \mathbb{Z}^n : \delta Y_f^k \cap \Omega' \neq \emptyset\}, \quad \Gamma'^\delta = \Omega' \cap \left(\bigcup_{k \in K^\delta} \delta \Gamma'^k \right).$$

Finally, we set

$$\Omega_s'^\delta = \Omega' \cap \left(\bigcup_{k \in K^\delta} \delta Y_s^k \right), \quad \Omega_f'^\delta = \Omega' \backslash \overline{\Omega_s'^\delta}.$$

We introduce the notation

$$\varepsilon'^\delta(x') = \varepsilon' \left(\frac{x'}{\delta} \right), \quad q_\sigma'^\delta(x') = q_\sigma' \left(\frac{x'}{\delta} \right).$$

Let $u'^\delta \in H_0^1(\Omega')$ be a solution of the problem

$$\int_{\Omega'} \delta^{1-m_1} \bar{a}_1 \varepsilon'^\delta(x') \nabla' u'^\delta \cdot \nabla' \varphi - 4\pi \delta^{m_2 - 1} \bar{a}_2 q' \varphi \, dx' = 4\pi \langle \delta_{\Gamma'^\delta}, q_\sigma'^\delta \varphi \rangle, \tag{4.2}$$

$\forall \varphi \in H_0^1(\Omega')$, where

$$\langle \delta_{\Gamma'^\delta}, q_\sigma'^\delta \varphi \rangle = \int_{\Gamma'^\delta} q_\sigma'^\delta(x') \varphi(x') \, ds_{x'}.$$

We are interested in the asymptotic behaviour of the solution u'^δ of (4.2), as $\delta \to 0$. In what follows, we omit the prime index.

5 Reformulation of the multi-dimensional problem

In order to study the asymptotic behaviour of the solution u^δ of (4.2), as $\delta \to 0$, we reformulate the surface integral term $\int_{\Gamma^\delta} q_\sigma^\delta(x) \varphi(x) \, ds_x$.

First, we assume that q_σ is a given function defined on $\partial Y_s \cap \partial Y_f$ such that

$$q_\sigma \in L^2(\partial Y_s \cap \partial Y_f).$$

We extend q_σ by periodicity to $\cup_{k \in \mathbb{Z}^n} \left(\partial Y_s^k \cap \partial Y_f^k \right)$ where ∂Y_i^k denotes the boundary of Y_i^k, $i = s, f$. Then, we write

$$q_\sigma^\delta(x) = q_\sigma \left(\frac{x}{\delta} \right), \quad x \in \Gamma^\delta.$$

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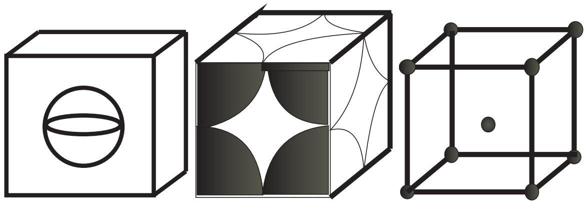

FIGURE 1. Cell solid domain: a - ball, b - S_8 domain with $r = 0.5$, c - S_9 domain (only sphere centres).

We also introduce the notation

$$\tilde{J}_\sigma = \int\limits_{\partial Y_s \cap \partial Y_f} |q_\sigma(y)|\, ds_y, \quad i_\sigma(y) = \frac{q_\sigma(y)}{\tilde{J}_\sigma},$$

$$s_\sigma = |\partial Y_s \cap \partial Y_f|^{-1} \int\limits_{\partial Y_s \cap \partial Y_f} i_\sigma(y)\, ds_y, \quad \tau = \frac{|\partial Y_s \cap \partial Y_f|}{|Y_s|},$$

and τ is the surface tortuosity coefficient.

Condition S. We assume that Y_s satisfies the following restriction. There is a periodic function $v(y)$ defined in the domain Y_s such that it solves the boundary-value problem

$$\begin{cases} \Delta_y v = \tau s_\sigma \quad \text{in } Y_s, \quad \displaystyle\int\limits_{Y_s} v\, dy = 0, \\ \nabla v \cdot \mathbf{n} = i_\sigma \quad \text{on } \partial Y_s, \end{cases} \tag{5.1}$$

where $\mathbf{n}$ denotes the unit outward normal vector to ∂Y_s.

Let us give some examples of the solid domain obeying condition S. Clearly, problem (5.1) has a unique solution $v \in H^1(Y_s)$ when the solid domain satisfies the following S_1-condition:

$$\overline{Y_s} \subset Y.$$

An example is given in Figure 1(a). Next we consider the solid domain as in Figure 1(b). Here, Y_s consists of eight components related to cell vertices and all the components are identical up to rotation; each component can be represented as an intersection $B \cap Y$, where B is a domain centred at the vertex and which has n planes of symmetry $y_i = const$. We call such a domain the S_8-domain. The most simple case is that B is a ball of a radius r such that $0 < r < r_p$, $r_p = \sqrt{2}/2$. Observe that the pore space loses connectivity if $r \geqslant r_p$ [22].

To prove solvability of the boundary value problem (5.1) in the case S_8, one can argue as shown in Figure 2. The solid domain consists of the domains 1–4. (For simplicity of presentation, these domains do not intersect with each other). First, we solve the problem for the domain B that consists of the domains 1, 2′, 3′, and 4′. Then, we define

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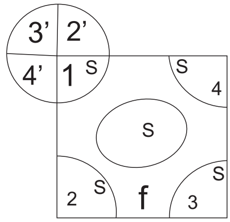

FIGURE 2. The solid domain Y_s of type S_2 with symmetrical vortex components.

$v(y)|_k = v(y)|_{k'}$, $k \neq 1$. In the case S_8, the porosity can be calculated by the formula ([20])

$$\Phi(r) = 1 - \frac{\pi}{6} - \frac{\pi}{2}\left(\frac{r}{2} - 1\right) + \frac{\pi}{4}\left(\frac{r}{2} - 1\right)^2 + \frac{\pi}{3}\left(\frac{r}{2} - 1\right)^3,$$

The S_8-solid domain retains essential features of many granular porous systems: (1) the pore spaces and grains form interconnecting channels, (2) the grains are of comparable size, and (3) the grains are joined at contacts that extend over a finite area.

To permit higher tortuosity, one can consider the S_9- domain which is the S_8- domain with one more solid ball in the centre of the cube (Figure 1(c)). The balls are of the same radius r, $0 < r < r_p$, $r_p = 3/\sqrt{32}$. Observe that the fluid component becomes isolated provided $r \geqslant r_p$. When $r = \sqrt{3}/4$ the centre sphere touches the vertex spheres.

In what follows, we set $v = 0$ in Y_f and then extend this new function periodically on $\mathbb{R}^n$. In what follows, we keep the same notation v for the extended function.

Lemma 1 *Let the solid domain Y_s satisfy condition S. Then, for any $\varphi \in H_0^1(\Omega)$, we have*

$$\langle \delta_{\Gamma^\delta}, q_\sigma^\delta \varphi \rangle = \tilde{J}_\sigma \int_\Omega 1_s(y) \left(\frac{\tau s_\sigma}{\delta} \varphi + \nabla\varphi \cdot \nabla_y v(y) \right) \Big|_{y=x/\delta} dx. \tag{5.2}$$

Proof Defining $v^\delta(x) = v(x/\delta)$, we have, for any multi-index j,

$$\Delta v^\delta(x)|_{\Omega_{sk}^\delta} = \frac{1}{\delta^2} \Delta_y v(y)|_{y=x/\delta} = \frac{\tau s_\sigma}{\delta^2},$$

where we have set $\Omega_{sk}^\delta = \delta Y_s^k$.

First, we consider the case S_1. Given $\varphi \in H_0^1(\Omega)$ we have

$$\int_{\Omega_{sk}^\delta} \tau s_\sigma \varphi \, dx = \delta^2 \int_{\Omega_{sk}^\delta} \varphi \Delta v^\delta \, dx = \delta^2 \int_{\Omega_{sk}^\delta} \operatorname{div}\left(\varphi \nabla v^\delta\right) - \nabla\varphi \cdot \nabla v^\delta \, dx$$
$$= \delta^2 \int_{\partial\Omega_{sk}^\delta} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) \, ds_x - \delta^2 \int_{\Omega_{sk}^\delta} \nabla\varphi \cdot \nabla v^\delta \, dx$$

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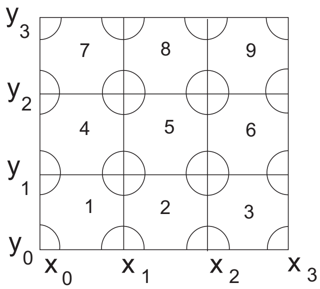

FIGURE 3. An example of periodical structure satisfying Condition S in $\mathbb{R}^2$. The solid phase is inside the circles.

$$= \delta \int_{\partial\Omega^\delta_{sk}} \varphi \left(\mathbf{n}(y) \cdot \nabla_y v(y)\right) |_{y=x/\delta}\, ds_x - \delta \int_{\Omega^\delta_{sk}} \nabla\varphi \cdot \nabla_y v(y)|_{y=x/\delta}\, dx$$
$$= \delta \int_{\partial\Omega^\delta_{sk}} \varphi i_\sigma\, ds_x - \delta \int_{\Omega^\delta_{sk}} \nabla\varphi \cdot \nabla_y v(y)|_{y=x/\delta}\, dx.$$

By summing over k we arrive at (5.2).

Let us consider the case S_8. Arguing like in the case S_1, we write

$$\int_{\Omega^\delta_{sk}} \tau s_\sigma \varphi\, dx = \delta^2 \int_{\Omega^\delta_{sk}} \varphi \Delta v^\delta\, dx = \delta^2 \int_{\Omega^\delta_{sk}} \operatorname{div}\left(\varphi \nabla v^\delta\right) - \nabla\varphi \cdot \nabla v^\delta\, dx$$
$$= \delta^2 \int_{\partial\Omega^\delta_{sk} \cap \partial\Omega^\delta_{fk}} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x - \delta^2 \int_{\Omega^\delta_{sk}} \nabla\varphi \cdot \nabla v^\delta\, dx + \delta^2 \int_{\partial\Omega^\delta_{sk} \cap \partial Y^\delta_k} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x.$$

We claim that

$$\sum_k \int_{\partial\Omega^\delta_{sk} \cap \partial Y^\delta_k} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x = 0. \tag{5.3}$$

To give an idea of the proof, we consider Figure 3 which illustrates the case $\mathbb{R}^2$ and $\delta = 1/3$. In this case, the total number of δcells is equal to 9 and the cells are numbered as shown in Figure 3. Let γ_{ij} stand for the boundary between the cell i and the cell j, with $\mathbf{n}$ pointing from the cell i to the cell j. Let γ^s_{ij} be the solid part of γ_{ij}. Since $\varphi \in H^1_0(\Omega)$, we have

$$\int_{\partial\Omega^\delta_{s1} \cap \partial Y^\delta_1} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x + \int_{\partial\Omega^\delta_{s2} \cap \partial Y^\delta_2} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x$$
$$= \int_{\gamma^s_{12} \cup \gamma^s_{14}} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x + \int_{\gamma^s_{21} \cup \gamma^s_{23} \cup \gamma^s_{25}} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x$$
$$= \int_{\gamma^s_{14}} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x + \int_{\gamma^s_{23} \cup \gamma^s_{25}} \varphi \left(\nabla v^\delta \cdot \mathbf{n}\right) ds_x.$$

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Here, we used the fact that $\int_{\gamma_{12}^s} = -\int_{\gamma_{21}^s}$. Now it is clear that the claim (5.3) is true and (5.2) follows. The case S_9 can be treated similarly. □

By Lemma 1, problem (4.2) becomes

$$\int_\Omega \delta^{1-m_1}\bar{a}_1\varepsilon_\delta(x)\nabla u^\delta\cdot\nabla\varphi - 4\pi\delta^{m_2-1}\bar{a}_2 q\varphi\,dx$$
$$= 4\pi\tilde{J}_\sigma\int_\Omega 1_s(y)\left(\frac{\tau s_\sigma}{\delta}\varphi + \nabla\varphi\cdot\nabla_y v(y)\right)\Big|_{y=x/\delta}dx, \quad \forall\varphi\in H_0^1(\Omega). \tag{5.4}$$

We assume that $q\in L^2(\Omega)$. Then, problem (5.4) has a unique solution $u^\delta\in H_0^1(\Omega)$. Depending on m_1 and m_2, we study different cases.

6 Both the bulk charge and the surface charge are strong

Let us consider the case when

$$l_\sigma \sim l \quad\text{and}\quad l_u\sim\frac{l}{\delta^2}, \quad i.e. \quad m_1 = 2 \quad\text{and}\quad m_2 = 0.$$

Under such assumptions problem (5.4) becomes

$$\int_\Omega \bar{a}_1\varepsilon_\delta(x)\nabla u^\delta\cdot\nabla\varphi - 4\pi\bar{a}_2 q\varphi\,dx$$
$$= 4\pi\tilde{J}_\sigma\int_\Omega 1_s(y)\left(\tau s_\sigma\varphi + \delta\nabla\varphi\cdot\nabla_y v(y)\right)\Big|_{y=x/\delta}dx, \quad \forall\varphi\in H_0^1(\Omega). \tag{6.1}$$

Problem (6.1) has a unique solution satisfying the estimate

$$\int_\Omega |u^\delta|^2 + |\nabla u^\delta|^2\,dx \leqslant c,$$

uniformly in δ. Clearly there is a subsequence, still denoted by u^δ, and there are functions $u(x)\in H_0^1(\Omega)$ and $u^1(x,y)\in L^2(\Omega;H^1_{per}(Y))$ such that

$$u^\delta \stackrel{2s}{\rightharpoonup} u, \quad \nabla u^\delta(x) \stackrel{2s}{\rightharpoonup} \nabla u(x) + \nabla_y u^1(x,y).$$

Taking in (6.1)

$$\varphi(x) = \varphi^0(x) + \delta\varphi^1\left(x,\frac{x}{\delta}\right), \quad \varphi^0\in\mathcal{D}(\Omega), \;\; \varphi^1\in\mathcal{D}\left(\Omega;C^\infty_{per}(Y)\right),$$

and passing to the limit, as $\delta\to 0$, we obtain

$$\int_\Omega\int_Y \bar{a}_1\varepsilon(y)\left(\nabla u + \nabla_y u^1\right)\cdot\left(\nabla\varphi^0 + \nabla_y\varphi^1\right) - 4\pi\bar{a}_2 q\varphi^0 = 4\pi\tilde{J}_\sigma\int_\Omega\int_Y \tau s_\sigma 1_s(y)\varphi^0\,dxdy. \tag{6.2}$$

Choosing $\varphi^0 = 0$, $\varphi^1(x, y) = \psi(x)\theta(y)$, $\psi \in \mathcal{D}(\Omega)$, and $\theta \in \mathcal{C}^\infty_{per}(Y)$, we obtain

$$\int_Y \varepsilon(y) \left(\nabla u + \nabla_y u^1\right) \cdot \nabla_y \theta \, dy = 0, \quad \text{a.e. in } \Omega.$$

We look for u^1 in the form

$$u^1(x, y) = w^k(y) \frac{\partial u}{\partial x_k}(x), \quad w^k \in H^1_{per}(Y), \tag{6.3}$$

and find that the function $w^k(y)$ $(k = 1, \ldots, n)$ solves the micro-equation

$$\frac{\partial}{\partial y_i} \left(\varepsilon(y) \left(\delta_{ik} + \frac{\partial w^k}{\partial y_i} \right) \right) = 0 \quad \text{in } Y. \tag{6.4}$$

Clearly, problem (6.4) has a unique solution $w^k \in H^1_{per}(Y)$ satisfying the condition $\int_Y w^k \, dy = 0$.

Taking $\varphi^1 = 0$ in (6.2) we find that

$$\int_\Omega \int_Y \bar{a}_1 \varepsilon(y) \left(\nabla u(x) + \nabla_y u^1\right) \cdot \nabla \varphi^0 - 4\pi \bar{a}_2 q \varphi^0 - 4\pi \tilde{J}_\sigma \tau s_\sigma 1_s(y) \varphi^0 \, dx dy = 0.$$

Next, using (6.3), we obtain that the function u solves the macro-equation

$$-\frac{\partial}{\partial x_i} \left(\bar{a}_1 \varepsilon^h_{ij} \frac{\partial u}{\partial x_j} \right) - 4\pi \bar{a}_2 q = 4\pi \tilde{J}_\sigma \tau s_\sigma (1 - \phi), \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0, \tag{6.5}$$

where

$$\phi = \int_Y 1_f(y) \, dy, \quad \varepsilon^h_{ik} = \int_Y \varepsilon(y) \left(\delta_{ik} + \frac{\partial w^k}{\partial y_i} \right) dy, \tag{6.6}$$

with ϕ being the porosity.

Let u^δ_b be a boundary layer function of first order defined as a solution to the Dirichlet problem

$$\operatorname{div} \left(\varepsilon_\delta(x) \nabla u^\delta_b \right) = 0 \quad \text{in } \Omega, \quad u^\delta_b(x) = -u^1 \left(x, \frac{x}{\delta} \right) \quad \text{on } \partial\Omega. \tag{6.7}$$

We have the following result.

Theorem 2 *Let u^δ and u be the unique solutions of problem* (6.1) *and problem* (6.5), (6.6), *respectively. Then u^δ converges to u in $H^1(\Omega)$ weakly. Moreover, if $u \in W^{2,\infty}(\Omega)$, we have*

$$\left\| u^\delta(x) - u(x) - \delta u^1 \left(x, \frac{x}{\delta} \right) - \delta u^\delta_b(x) \right\|_{H^1(\Omega)} \leqslant c\delta, \tag{6.8}$$

where $u^1(x, y)$ is defined by (6.3), (6.4), *$u^\delta_b(x)$ is defined by* (6.7), *and c is a constant independent of δ.*

Proof The weak convergence of u^δ towards u results from the notion of two-scale convergence. Let us prove (6.12). Given the functions $u(x)$ and $u^1(x,y)$, we define the function $u^2(x,y)$ as a solution to the problem

$$-\mathrm{div}_x\ \left(\bar{a}_1\varepsilon(y)\left(\nabla_x u+\nabla_y u^1\right)\right)-\mathrm{div}_y\ \left(\bar{a}_1\varepsilon(y)\left(\nabla_x u^1+\nabla_y u^2\right)\right)=4\pi\bar{a}_2 q+4\pi\tilde{J}_\sigma\tau s_\sigma 1_s(y), \tag{6.9}$$

assuming that $u^2(x,y)$ is Y-periodic with respect to y and satisfies the condition $\int_Y u^2(x,y)\,dy=0$.

Arguing like in Bensoussan *et al.* [4] and Moskow and Vogelius [15], one can verify that

$$u^2(x,y)=w^{kj}(y)\frac{\partial^2 u}{\partial x_k\partial x_j}(x)-4\pi\tilde{J}_\sigma w^0(y),$$

where $w^{kj}\in H^1_{per}(Y)$ is a periodic solution to the problem

$$\begin{aligned}-\mathrm{div}_y\ \left(\varepsilon(y)\,\nabla_y w^{kj}\right)=&\varepsilon(y)\left(\delta_{kj}+\frac{\partial w^k}{\partial y_j}\right)+\frac{\partial}{\partial y_j}\left(\varepsilon(y)w^k(y)\right)\\ &-\int_Y\varepsilon(y)\left(\delta_{kj}+\frac{\partial w^k}{\partial y_j}\right)dy,\qquad \int_Y w^{kj}\,dy=0,\end{aligned}$$

and $w^0(y)\in H^1_{per}(Y)$ is a periodic solution to the problem

$$-\mathrm{div}_y\ \left(\varepsilon(y)\,\nabla_y w^0\right)=\tau s_\sigma[(1-\phi)-1_s(y)],\quad \int_Y w^0\,dy=0.$$

We introduce the function

$$u^{c,\delta}(x)=u^c\left(x,\frac{x}{\delta}\right)+\delta u^\delta_b(x),$$

where

$$u^c(x,y)=u(x)+\delta u^1(x,y)+\delta^2u^2(x,y),\quad x\in\Omega,\ y\in Y.$$

Taking into account the definition of the functions $u^1(x,y)$ and $u^2(x,y)$, we easily verify that

$$-\mathrm{div}\ \left(\bar{a}_1\varepsilon_\delta(x)\,\nabla u^{c,\delta}\right)=r^\delta,$$

with

$$\begin{aligned}r^\delta(x)=4\pi\bar{a}_2q+4\pi\tilde{J}_\sigma\tau s_\sigma 1_s^\delta-\delta\left(\mathrm{div}_x\left\{\bar{a}_1\varepsilon(y)\left[\nabla_x u^1(x,y)+\nabla_y u^2(x,y)\right]\right\}\right)\Big|_{y=\frac{x}{\delta}}\\ -\delta\left\{\mathrm{div}_y\left[\bar{a}_1\varepsilon(y)\nabla_x u^2(x,y)\right]\right\}\Big|_{y=\frac{x}{\delta}}-\delta^2\left\{\mathrm{div}_x\left[\bar{a}_1\varepsilon(y)\nabla_x u^2(x,y)\right]\right\}\Big|_{y=\frac{x}{\delta}}.\end{aligned}$$

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We deduce that

$$
\begin{aligned}
-\mathrm{div}\left[\bar{a}_1\varepsilon_\delta(x)\nabla(u+\delta u^1+\delta u_b^\delta)\right] &= 4\pi\bar{a}_2 q(x)+4\pi\tilde{J}_\sigma\tau s_\sigma 1_s^\delta(x) \\
&\quad -\delta\left\{\mathrm{div}_x\left[\bar{a}_1\varepsilon(y)\nabla_x u^1(x,y)\right]\right\}\Big|_{y=\frac{x}{\delta}} \\
&\quad +\delta \mathrm{div}\left(\bar{a}_1\varepsilon_\delta(x)\nabla_y u^2\Big|_{y=\frac{x}{\delta}}\right) \\
&\quad -\delta\left\{\mathrm{div}_x\left(\bar{a}_1\varepsilon(y)\nabla_y u^2\right)\right\}\Big|_{y=\frac{x}{\delta}}.
\end{aligned}
\tag{6.10}
$$

Given a function $\varphi\in H_0^1(\Omega)$, we obtain from (6.10) that

$$
\begin{aligned}
\int_\Omega \bar{a}_1\varepsilon_\delta(x)\nabla(u+\delta u^1+\delta u_b^\delta)\cdot\nabla\varphi\,dx &= \int_\Omega\left(4\pi\bar{a}_2 q+4\pi\tilde{J}_\sigma\tau s_\sigma 1_s^\delta\right)\varphi\,dx \\
&\quad -\delta\int_\Omega\varphi\left[\mathrm{div}_x\left(\bar{a}_1\varepsilon(y)\nabla_x u^1\right)\right]\Big|_{y=\frac{x}{\delta}}dx \\
&\quad -\delta\int_\Omega\nabla\varphi\left(\bar{a}_1\varepsilon(y)\nabla_y u^2\right)\Big|_{y=\frac{x}{\delta}}dx \\
&\quad -\delta\int_\Omega\varphi\left[\mathrm{div}_x\left(\bar{a}_1\varepsilon(y)\nabla_y u^2\right)\right]\Big|_{y=\frac{x}{\delta}}dx.
\end{aligned}
$$

Setting $v^\delta=u^\delta-(u+\delta u^1+\delta u_b^\delta)$, we have that $v^\delta\in H_0^1(\Omega)$ and

$$
\begin{aligned}
\int_\Omega\bar{a}_1\varepsilon_\delta(x)\nabla v^\delta\cdot\nabla\varphi\,dx &= \int_\Omega\delta 4\pi\tilde{J}_\sigma 1_s^\delta\nabla\varphi\cdot\nabla_y v(y)|_{y=\frac{x}{\delta}}\,dx \\
&\quad +\delta\int_\Omega\varphi\left[\mathrm{div}_x\left(\bar{a}_1\varepsilon(y)\nabla_x u^1\right)\right]|_{y=\frac{x}{\delta}}\,dx \\
&\quad +\delta\int_\Omega\nabla\varphi\left(\bar{a}_1\varepsilon(y)\nabla_y u^2\right)|_{y=\frac{x}{\delta}}\,dx \\
&\quad +\delta\int_\Omega\varphi\left[\mathrm{div}_x\left(\bar{a}_1\varepsilon(y)\nabla_y u^2\right)\right]|_{y=\frac{x}{\delta}}\,dx.
\end{aligned}
\tag{6.11}
$$

In what follows we denote by c a generic constant that does not depend on δ. Using the Poincaré inequality $\|\varphi\|_2\leqslant c\|\nabla\varphi\|_2$, $\varphi\in H_0^1(\Omega)$, and the regularity of u^1 and u^2, we deduce from (6.11) that

$$
\int_\Omega\varepsilon_\delta(x)\nabla v^\delta\cdot\nabla\varphi\,dx\leqslant c\delta\left(\int_\Omega|\nabla\varphi|^2\,dx\right)^{\frac{1}{2}}.
$$

Hence, the inequality (6.8) follows. □

Corollary 1 *Let the assumptions be as in Theorem 2. We have*

$$
\left\|u^\delta(x)-u(x)-\delta u^1\left(x,\frac{x}{\delta}\right)\right\|_{H^1(\Omega)}\leqslant c\sqrt{\delta},
\tag{6.12}
$$

where c is a constant independent of δ.

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Proof The estimate in $H^1(\Omega)$ of the boundary layer u_b^δ, defined by (6.7), is a classical result, see refs. [11] and [18] for instance. It is proven that $\|u_b^\delta\|_{H^1(\Omega)} \leqslant c/\sqrt{\delta}$. The latter estimate, together with (6.8), gives estimate (6.12). □

Remark 3 *As far as the electric field is concerned, it follows from Corollary 1 that*

$$\left\| E^\delta(x) + \nabla \left(u(x) + \delta u^1 \left(x, \frac{x}{\delta} \right) \right) \right\|_{L^2(\Omega)} \leqslant c\sqrt{\delta},$$

where c denotes a constant independent of δ. Thus, to find an approximate electric field, it suffices to solve the micro-problems (6.4) *and equation* (6.5) *with constant coefficients.*

Remark 4 *Since* $\left\|u^1\left(x, \frac{x}{\delta}\right)\right\|_{L^2(\Omega)} \leqslant c$, *and* $\left\|u_b^\delta\right\|_{L^2(\Omega)} \leqslant c$, *see refs. [11] and [18] for instance, we deduce from* (6.8) *the following estimate*

$$\left\|u^\delta - u\right\|_{L^2(\Omega)} \leqslant c\delta,$$

where c denotes a constant independent of δ.

7 The surface charge with different strength

In what follows, we do not consider all the possible values of the powers m_1 and m_2 paying attention only to the most interesting cases. First, we assume that the surface charge is weak in the sense that

$$l_\sigma \sim \delta l \quad \text{and} \quad l_u \sim \frac{l}{\delta^3}, \quad i.e. \quad m_1 = 3 \quad \text{and} \quad m_2 = -1.$$

Under such assumptions problem (5.4) becomes

$$\int_\Omega \bar{a}_1 \varepsilon_\delta(x) \nabla u^\delta \cdot \nabla\varphi - 4\pi\bar{a}_2 q\varphi \, dx$$
$$= \delta^2 4\pi\tilde{J}_\sigma \int_\Omega 1_s(y) \left(\tau s_\sigma \varphi + \delta\nabla\varphi \cdot \nabla_y v(y)\right)\Big|_{y=x/\delta} dx, \quad \forall\varphi \in H_0^1(\Omega).$$

By the above arguments, we find that the weak limit u of u^δ satisfies the macro-equation

$$-\frac{\partial}{\partial x_i}\left(\bar{a}_1 \varepsilon_{ij}^h \frac{\partial u}{\partial x_j}\right) - 4\pi\bar{a}_2 q = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

where the constant matrix ε_{ij}^h is defined by (6.4) and (6.6).

Next, we consider the case when the surface charge is weak but the surface tortuosity coefficient is great:

$$l_\sigma \sim \delta l, \quad l_u \sim \frac{l}{\delta^3}, \quad \tau \sim \frac{1}{\delta}, \quad i.e. \quad m_1 = 3, \quad m_2 = -1, \quad \tau = \frac{\bar{\tau}}{\delta}.$$

Clearly, the weak limit u of u^δ satisfies the macro-equation

$$-\frac{\partial}{\partial x_i}\left(\bar{a}_1 \varepsilon_{ij}^h \frac{\partial u}{\partial x_j}\right) - 4\pi\bar{a}_2 q = 4\pi\tilde{J}_\sigma \bar{\tau} s_\sigma (1 - \phi) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0,$$

Now, we consider the case when the bulk charge is weak and the surface charge is strong in the following sense:

$$l_\sigma \sim \frac{l}{\delta} \quad \text{and} \quad l_u \sim \frac{l}{\delta^2}, \quad \textit{i.e.} \quad m_1 = 2 \quad \text{and} \quad m_2 = 1.$$

Then, the macro-equation becomes

$$-\frac{\partial}{\partial x_i}\left(\bar{a}_1 \varepsilon_{ij}^h \frac{\partial u}{\partial x_j}\right) = 4\pi \tilde{J}_\sigma \tau s_\sigma (1-\phi) \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

8 Neutral dielectric composites

Let us consider a composite material with a periodic structure under the assumption that both the interfacial charge density and the bulk charge density are periodic functions and the total electric charge of the periodicity cell is equal to zero. In dimensional variables, it implies that

$$q = q^\delta(x) \equiv q_0\left(\frac{x}{\delta L}\right), \quad q_0 \in L^2(Y),$$

and

$$\int\limits_{\delta Y^k} q^\delta(x)\, dx + \int\limits_{\delta \Gamma^k} q_\sigma^\delta(x)\, ds_x = 0, \quad \forall k \in K^\delta.$$

In dimensionless variables, this condition becomes

$$\int\limits_{Y} q_0(y)\, dy + \frac{l_\sigma}{l} \int\limits_{\Gamma} q_\sigma(y)\, ds_y = 0,$$

or equivalently,

$$\int\limits_{Y} q_0(y)\, dy + \frac{\tilde{J}_\sigma \bar{\tau} s_\sigma |Y_s|}{\delta^{m_2+m_3} \bar{a}_2} = 0,$$

where $\tau = \bar{\tau}\delta^{-m_3}$, $m_3 \geqslant 0$. Let us pass to the limit, as $\delta \to 0$, in equation (5.4) which is equivalent to

$$\begin{aligned}
&\int_\Omega \delta^{2-m_1+m_3} \bar{a}_1 \varepsilon^\delta(x) \nabla u^\delta \cdot \nabla\varphi - 4\pi \delta^{m_2+m_3} \bar{a}_2 q^\delta \varphi\, dx \\
&\quad = 4\pi \tilde{J}_\sigma \int_\Omega 1_s(y) \left(\bar{\tau} s_\sigma \varphi + \delta^{m_3+1} \nabla\varphi \cdot \nabla_y v(y)\right)\Big|_{y=x/\delta} dx, \quad \forall \varphi \in H_0^1(\Omega). \qquad (8.1)
\end{aligned}$$

Equation (8.1) holds for arbitrary δ provided $m_2 + m_3 = 0$. Assuming that $m_1 + m_2 = 2$ and acting like in the case of equation (6.5), we derive that u^δ converges weakly in $H^1(\Omega)$ to u and

$$\frac{\partial}{\partial x_i}\left(\bar{a}_1 \varepsilon_{ij}^h \frac{\partial u}{\partial x_j}\right) = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Hence, $u = 0$ and the neutral composite does not have the electret effect.

9 Conclusions

The above analysis shows that the macro-equation strongly depends on the surface charge value q_σ. Such a charge manifests itself through an additional constant bulk charge. As for the homogenized anisotropic dielectric permittivity matrix, it does not depend on the surface charge and its value is defined by the geometry of the solid-fluid interface. By the corrector technique, an algorithm is proposed for construction of an effective electric field; such an algorithm is based on solving cell micro-equations and a macro-equation with constant coefficients. One more conclusion is that the neutral composites do not have the electrets property.

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Appendix A

Let us discuss the meaning of the length $l_u = \bar{u}\bar{\varepsilon}/\bar{q}_\sigma$. To this end, we consider an electric field in a homogeneous dielectric medium around a charged sphere of dimensional radius L with a given constant surface charge density. In dimensionless variables, the potential depends on the radial variable only and solves the boundary-value problem

$$r > 1: \qquad \Delta u = 0, \quad -a_1 \varepsilon \frac{\partial u}{\partial r}\Big|_{r=1} = 4\pi q_\sigma, \quad a_1 = \frac{l_u}{L}.$$

One can verify that the dimensionless electric field $E(r) \equiv -\partial u/\partial r$ is given by the formula

$$E = \frac{1}{r^2} \cdot \frac{4\pi q_\sigma}{\varepsilon l_u/L}.$$

Thus, l_u is a distance of attenuation of the electric field in the sense that

$$E\big|_{r=l_u/L} = \frac{4\pi q_\sigma}{\varepsilon} \cdot \left(\frac{l_u}{L}\right)^{-3}.$$

As for the length l_σ, it works only in the presence of the bulk charge q. As above, we consider an electric field in a homogeneous dielectric medium around a charged sphere but in the presence of a bulk charge which decreases according to the law $q = q_0/r^2$. In this case, the electric field is given by the formula

$$E = \frac{4\pi q_0}{\varepsilon} \cdot \frac{1}{(l_\sigma/L)(l_u/L)} \left(\frac{r-1}{r^2}\right) + \frac{1}{r^2} \cdot \frac{4\pi q_\sigma}{\varepsilon l_u/L}.$$

Clearly, l_σ characterizes the rate of attenuation of the electric field associated with the bulk charge distribution.