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Abstract

In this paper we construct an action of the affine Hecke category (in its 'Soergel bimodules' incarnation) on the principal block of representations of a simply connected semisimple algebraic group over an algebraically closed field of characteristic bigger than the Coxeter number. This confirms a conjecture of G. Williamson and the second author, and provides a new proof of the tilting character formula in terms of antispherical p-Kazhdan–Lusztig polynomials.

Contents

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1. Introduction

1.1 Representation theory of reductive algebraic groups and the Hecke category

Let G be a connected reductive algebraic group over an algebraically closed field $\Bbbk$ of characteristic p (assumed to be strictly bigger than the Coxeter number of G), and let W_{aff} be the associated affine Weyl group. A choice of a Borel subgroup B in G determines a subset $S_{\mathrm{aff}} \subset W_{\mathrm{aff}}$ of 'simple reflections' such that $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system. It has long been expected (following ideas of Verma [Ver75], later expanded by Lusztig [Lus80] in particular) that the combinatorics of the category $\mathsf{Rep}(G)$ of finite-dimensional algebraic G-modules should be expressible in terms of the Kazhdan–Lusztig combinatorics of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. This expectation is now known to be exact if p is large (thanks to work of Kazhdan and Lusztig, Kashiwara and Tanisaki, and Andersen, Jantzen and Soergel, or a later version of Fiebig), but not for some smaller values of p (as shown by Williamson); see [RW18] for details and references.

In [RW18], G. Williamson and the second author of the present paper started advocating the idea that the combinatorics of $\mathsf{Rep}(G)$ should rather be expressed in terms of the

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p-Kazhdan–Lusztig combinatorics, introduced a few years before by G. Williamson (partly in collaboration; see [JMW14, JW17]) as some 'combinatorial shadow' of the Hecke category D_{BS} over $\Bbbk$ attached to $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$. (Here by 'Hecke category' we mean the diagrammatic category introduced by Elias and Williamson in [EW16]; this category is closely related to those of Soergel bimodules and parity complexes on flag varieties.) In that paper it was observed in particular that a concrete incarnation of this idea (a character formula for indecomposable tilting modules in the principal block, in terms of antispherical p-Kazhdan–Lusztig polynomials) was a consequence of the following conjecture, where for $s \in S_{\mathrm{aff}}$ we denote by B_s the object of D_{BS} naturally associated with s.

CONJECTURE 1.1 [RW18]. *There exists a ($\Bbbk$-linear) right action of the monoidal category D_{BS} on the principal block $\mathsf{Rep}_0(G)$ of $\mathsf{Rep}(G)$ such that for any $s \in S_{\mathrm{aff}}$ the object B_s acts via a functor isomorphic to the wall-crossing functor associated with s.*

The formulation of Conjecture 1.1 was motivated in particular by the philosophy of categorical actions of Lie algebras; it was proved in [RW18, Part II] (and independently by Elias and Losev [EL17]) in the special case when $G = \mathrm{GL}_n(\Bbbk)$ for some n, using the machinery of 2-Kac–Moody algebras [Rou08].

Later the 'combinatorial' consequence of Conjecture 1.1 (the tilting character formula) was proved for general G (in fact, in two very different ways; see [AMRW19, RW20]), but these proofs use other tools, and none of them implies the original categorical conjecture. The main result of the present paper is a proof of Conjecture 1.1 (which, as explained above, provides in particular a third general proof of the tilting character formula). In contrast to the other approaches to such questions, our proof does not involve constructible sheaves in any way; it uses coherent sheaves, but mostly over affine schemes, and hence can be considered essentially algebraic.

1.2 Localization for Harish-Chandra bimodules

The action from Conjecture 1.1 will be constructed from the natural action of Harish-Chandra $\mathcal{U}\mathfrak{g}$-bimodules (where $\mathcal{U}\mathfrak{g}$ is the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G), that is, G-equivariant finitely generated $\mathcal{U}\mathfrak{g}$-bimodules on which the diagonal action of $\mathcal{U}\mathfrak{g}$ is the differential of the G-action (see § 3.4 for details). Namely, the category $\mathsf{Rep}(G)$ can be naturally seen as a full subcategory of the category of G-equivariant $\mathcal{U}\mathfrak{g}$-modules via differentiation. Thus it admits an action of the monoidal category of Harish-Chandra bimodules; moreover, wall-crossing functors (and, more generally, translation functors) can be described as the action of certain specific completed Harish-Chandra bimodules.

More specifically, recall that (under suitable assumptions on p) the center of $\mathcal{U}\mathfrak{g}$ identifies with functions on the fiber product

$$\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$$

where the superscript (1) denotes Frobenius twist, $\mathfrak{t}$ is the Lie algebra of a maximal torus T contained in B, and W is the Weyl group of (G, T), acting on $\mathfrak{t}^*$ via the 'dot action' $\bullet$. The subalgebra $\mathscr{O}(\mathfrak{g}^{*(1)})$ is realized as the 'Frobenius center' and the subalgebra $\mathscr{O}(\mathfrak{t}^*/(W, \bullet))$ as the 'Harish-Chandra center'. For $\lambda, \mu \in X^*(T)$, in §§ 3.5–3.7 we will construct a certain monoidal category

$$\mathsf{HC}^{\hat{\lambda}, \hat{\mu}}$$

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of 'completed Harish-Chandra bimodules', which are certain G-equivariant finitely generated modules over

$$(\mathcal{U}\mathfrak{g} \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))} \mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))^{\hat{\lambda},\hat{\mu}},$$

where $\mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))^{\hat{\lambda},\hat{\mu}}$ is the completion of $\mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))$ with respect to the maximal ideal determined by (λ,μ). Once this definition is in place, to construct the desired action it therefore suffices to construct a monoidal functor from the Hecke category D_{BS} to the category $\mathsf{HC}^{\hat{0},\hat{0}}$ sending each B_s to an object isomorphic to the completed bimodule realizing the corresponding wall-crossing functor. This will be realized in Theorem 6.3.

The main tool we will use for this construction is a localization theory for Harish-Chandra bimodules. Even though in the end we are interested in G-modules, which when seen as $\mathcal{U}\mathfrak{g}$-modules have a *trivial* Frobenius central character, we will localize our bimodules on the *regular part* of $\mathscr{O}(\mathfrak{g}^{*(1)})$, and more precisely on a Kostant section $\mathbf{S}^* \subset \mathfrak{g}^{*(1)}$ to the (co)adjoint quotient. We will therefore set $\mathcal{U}_{\mathbf{S}}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathscr{O}(\mathbf{S}^*)$, and consider a certain group scheme over $\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)$ constructed out of the universal centralizer group scheme $\mathbb{J}^*_{\mathbf{S}}$ over $\mathbf{S}^* \cong \mathfrak{t}^{*(1)}/W$. For $\lambda,\mu \in X^*(T)$, we will define a certain monoidal category $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ of finitely generated equivariant modules over

$$\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} := (\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))} \mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))^{\hat{\lambda},\hat{\mu}};$$

see § 3.9 for the precise definition. By construction we have a natural monoidal functor

$$\mathsf{HC}^{\hat{\lambda},\hat{\mu}} \to \mathsf{HC}^{\hat{\lambda},\hat{\lambda}}_{\mathbf{S}},$$

and the main result of § 3 (Proposition 3.7) states that this functor is fully faithful on a certain subcategory $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}}$ of 'diagonally induced' bimodules which contains the objects realizing translation and wall-crossing functors.

It is a classical observation that $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is an Azumaya algebra over its center (see Proposition 4.1 for details and references); as a consequence, the category of (finitely generated) bimodules over this algebra such that the left and right actions of its center coincide is equivalent to the category of (finitely generated) modules over this center (see § 4.1 for details and references). This property is not directly applicable to our problem, since the two actions of this center on Harish-Chandra bimodules do *not* coincide in general; however, by using bimodules realizing translation to and from the 'most singular' Harish-Chandra character (namely, the opposite of the half-sum of the positive roots), we construct in § 4 an equivariant splitting bundle for $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ in the case where λ and μ belong to the lower closure of the fundamental alcove. As a consequence, for such λ,μ we obtain an equivalence of categories between $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ and the category of coherent representations of the pullback of $\mathbb{J}^*_{\mathbf{S}}$ to the spectrum of $\mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))^{\hat{\lambda},\hat{\mu}}$; see Corollary 4.8.

A general theory of localization for modules over $\mathcal{U}\mathfrak{g}$ has been developed by the first author with Mirković and Rumynin; see [BMR06, BMR08, BM13]. The localization that we require here is, however, slightly different, and the present paper does not formally rely on any substantial result from [BMR06, BMR08, BM13]. One difference is that we are interested not in modules but in *bimodules*, which are equivariant for the diagonal G-action. Some of the constructions in [BMR06, BMR08, BM13] (in particular, the non-canonicity of the choice of splitting bundle) make this theory difficult to use in an equivariant setting, and our construction is slightly different. Finally, as explained above, we only need to consider the *regular* part of the Frobenius

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center, which simplifies the situation a lot, and in particular allows us to work completely at the level of *abelian* categories, without having to consider the more involved *derived* categories.

1.3 The Hecke category and representations of the universal centralizer

The other crucial ingredient of our proof is a new incarnation of the Hecke category (for any Coxeter system $(\mathcal{W}, \mathcal{S})$) recently found by Abe [Abe21].

The Hecke category is a categorification of the Hecke algebra of $(\mathcal{W}, \mathcal{S})$, depending on a choice of extra data (comprising a representation V of $\mathcal{W}$), which admits several different incarnations. An early definition of this category in terms of Soergel bimodules [Soe07] applies to 'reflection faithful' representations of Coxeter systems, which include natural examples of representations over fields of characteristic 0 (e.g. geometric representations of finite Coxeter systems and representations appearing in the theory of Kac–Moody Lie algebras for crystallographic Coxeter systems), but does *not* include important examples over fields of positive characteristic (e.g. some natural representations of affine Weyl groups of reductive groups). Under this assumption Soergel bimodules can be defined as a full subcategory of the category of graded bimodules over the polynomial algebra $\mathscr{O}(V)$. More recently Elias and Williamson [EW16] have proposed a definition of the Hecke category in terms of generators and relations which applies (and behaves as one might expect) in a much greater generality, encompassing the representation of the affine Weyl group that we require. It is in terms of this construction that Conjecture 1.1 was stated. (For more on the Hecke category, see also [JW17, Wil18].)

The main drawback of the construction in [EW16], however, is that it is much less concrete than Soergel's original definition, and does not involve $\mathscr{O}(V)$-bimodules. This drawback is exactly compensated by Abe's work; under minor technical assumptions he proves in [Abe21] that the category of Elias and Williamson identifies with a category of 'enhanced Soergel bimodules', which are certain graded bimodules over $\mathscr{O}(V)$ endowed with a decomposition of its tensor product with $\mathrm{Frac}(\mathscr{O}(V))$ (on the right) parametrized by $\mathcal{W}$.

Based on Abe's work, in the case of the affine Weyl group acting on $X^*(T) \otimes_{\mathbb{Z}} \Bbbk$ through the natural action of W, we realize the Hecke category as a full subcategory in ($\mathbb{G}_{\mathrm{m}}$-equivariant) coherent representations of the pullback of $\mathbb{J}^*_{\mathbf{S}}$ to $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$; see Theorem 2.10. This construction allows us to define a monoidal functor from the Hecke category to the category of representations considered in § 1.2, and then (using our localization theorem) a monoidal functor

$$\mathsf{D}_{\mathrm{BS}} \to \mathsf{HC}^{\hat{0},\hat{0}}_{\mathbf{S}}. \tag{1.1}$$

(This construction applies more generally for the category $\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}_{\mathbf{S}}$ when λ belongs to the fundamental alcove; in this case natural étale maps allow us to identify the completions of the schemes $\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)$ and $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$ at the images of (λ,λ).)

Remark 1.2. Although the concrete incarnation of this idea that is relevant in the present paper is new, the fact that affine Soergel bimodules are closely related to representations of the universal centralizer was already known: it dates back (at least) to [Dod11]; see also [MR18] for an adaptation of these ideas to positive characteristic coefficients.

At this point, to conclude our proof it only remains to show that our functor (1.1) sends the objects of the Hecke category labeled by simple reflections to the images in $\mathsf{HC}^{\hat{0},\hat{0}}_{\mathbf{S}}$ of the bimodules realizing the wall-crossing functors. (In fact, this property will also ensure that the functor takes values in the essential image of our fully faithful functor $\mathsf{HC}^{\hat{0},\hat{0}}_{\mathrm{diag}} \to \mathsf{HC}^{\hat{0},\hat{0}}_{\mathbf{S}}$, which will imply that it 'lifts' to a functor from D_{BS} to $\mathsf{HC}^{\hat{0},\hat{0}}$.) In the case when the simple reflection belongs to the

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finite Weyl group W, this can be checked explicitly, using localization at a character involving a weight on the corresponding wall of the fundamental alcove; see Proposition 6.6. The general case is reduced to this one using a standard trick (used, for example, in [Ric10, BM13]), based on the observation that in the *extended* affine Weyl group each simple reflection is conjugate to a simple reflection which belongs to W. The concrete proof involves the study of an analogue of the affine braid group action from [BR12] in our present context; in this case the situation simplifies, however (once again because we work over the regular part of the Frobenius center), and this action in fact factors through an action of the extended affine Weyl group.

Remark 1.3. One of the motivations for Abe's work [Abe21] was an attempt to prove Conjecture 1.1. What Abe was actually able to construct is rather an action on the principal block of the category of G_1T-modules (where G_1 is the first Frobenius kernel of G), which is interesting but less applicable to character computations; see [Abe19].

1.4 Towards a coherent realization of the Hecke category

Thanks to work of Kazhdan and Lusztig [KL87] and Ginzburg [CG97], it is known that the Hecke algebra of $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ identifies with the Grothendieck group of the category of equivariant coherent sheaves on the Steinberg variety of triples St.[1] The fiber product $\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$ considered in § 1.3 identifies with the preimage of the Kostant section $\mathbf{S}^*$ in the Frobenius twist of St, and the construction of § 1.3 can be seen to provide a fully faithful monoidal functor from the Hecke category to the category of equivariant coherent sheaves on the *regular part* of St. In future work we will upgrade this construction to a fully faithful monoidal functor to the category of equivariant coherent sheaves on the whole Steinberg variety.[2] This construction will be part of our project (in part jointly with L. Rider) of constructing a 'modular' version of the equivalence constructed by the first author in [Bez16]; see [BRR20] for a first step towards this goal.

1.5 Contents

In § 2 we recall Abe's results, and use them to construct our monoidal functor from the Hecke category to the appropriate category of representations of the universal centralizer. In § 3 we introduce the categories of completed Harish-Chandra bimodules we will work with, and prove that restriction to a Kostant section is fully faithful on diagonally induced bimodules. In § 4 we develop our localization theory for Harish-Chandra bimodules. In § 5 we prove (for later use) some technical results using the relation between $\mathcal{U}\mathfrak{g}$ and differential operators on the flag variety of G. In § 6 we prove the main result of the paper, that is, we construct the Hecke action on the principal block and prove that objects associated with simple reflections act via wall-crossing functors. Finally, Appendix A contains an index of the main notation used in the paper.

2. The affine Hecke category and representations of the regular centralizer

In this section we explain that the affine Hecke category attached to a connected reductive algebraic group $\mathbf{G}$ can be described as a category of representations of (a pullback of) the universal centralizer attached to $\mathbf{G}$. Our main tool will be a description of the Hecke as a category of 'enhanced Soergel bimodules' recently obtained by Abe [Abe21]. In later sections the group $\mathbf{G}$

[1] Here, by *Steinberg variety of triples* we mean the fiber product of two copies of the Grothendieck resolution over the dual of the Lie algebra, and not the version involving the Springer resolution. This distinction is not important for the results of [KL87, CG97], but it is for our considerations here.

[2] We understand that Ivan Losev has found a different proof of this statement, also based on the results of the present paper.

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will be chosen as the Frobenius twist of the group G appearing in Conjecture 1.1; however, this construction applies in a slightly more general context, and might be of independent interest.

2.1 The affine Weyl group and the associated Hecke category

We let $\Bbbk$ be an algebraically closed field of characteristic p (possibly equal to 0), and $\mathbf{G}$ be a connected reductive algebraic group over $\Bbbk$. We fix a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ and a maximal torus $\mathbf{T} \subset \mathbf{B}$. The Lie algebras of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$ will be denoted $\mathbf{g}$, $\mathbf{b}$ and $\mathbf{t}$ respectively. We set $\mathbf{X} := X^*(\mathbf{T})$ (respectively, $\mathbf{X}^\vee := X_*(\mathbf{T})$), and denote by $\Phi \subset \mathbf{X}$ (respectively, $\Phi^\vee \subset \mathbf{X}^\vee$) the root system (respectively, coroot system) of $(\mathbf{G}, \mathbf{T})$. The canonical bijection $\Phi \xrightarrow{\sim} \Phi^\vee$ will be denoted $\alpha \mapsto \alpha^\vee$. The choice of $\mathbf{B}$ determines a subset $\Phi^+ \subset \Phi$ of positive roots, consisting of the $\mathbf{T}$-weights in $\mathbf{g}/\mathbf{b}$; the corresponding basis of Φ will be denoted Φ^{s}. In this section we will make the following assumptions.

(i) p is good for $\mathbf{G}$.
(ii) Neither $\mathbf{X}/\mathbb{Z}\Phi$ nor $\mathbf{X}^\vee/\mathbb{Z}\Phi^\vee$ has p-torsion.
(iii) There exists a $\mathbf{G}$-equivariant isomorphism $\mathbf{g} \xrightarrow{\sim} \mathbf{g}^*$.

For simplicity we will fix once and for all a $\mathbf{G}$-equivariant isomorphism $\kappa : \mathbf{g} \xrightarrow{\sim} \mathbf{g}^*$. By equivariance this also provides an identification of $\mathbf{t}$ and $\mathbf{t}^*$ (where $\mathbf{t}^*$ is identified with the subspace in $\mathbf{g}^*$ consisting of linear forms that vanish on all root subspaces).

Let $\mathbf{W} = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group of $(\mathbf{G}, \mathbf{T})$. The associated *affine Weyl group* is the semi-direct product

$$W_{\mathrm{aff}} := \mathbf{W} \ltimes \mathbb{Z}\Phi$$

where $\mathbb{Z}\Phi \subset \mathbf{X}$ is the lattice generated by the roots. For $\lambda \in \mathbb{Z}\Phi$ we will denote by t_λ the image of λ in W_{aff}. It is well known that W_{aff} is generated by the subset S_{aff} consisting of the reflections s_α with $\alpha \in \Phi^{\mathrm{s}}$, together with the products $t_\beta s_\beta$ where $\beta \in \Phi$ is such that $\beta^\vee$ is a maximal coroot. Moreover, the pair $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ is a Coxeter system; see [Jan03, §II.6.3]. We will sometimes need to 'enlarge' this group by considering translations by all elements of $\mathbf{X}$. Namely, the *extended affine Weyl group* is the semi-direct product

$$W_{\mathrm{ext}} := \mathbf{W} \ltimes \mathbf{X}.$$

Then W_{aff} is a normal subgroup in W_{ext}.

We will consider the balanced 'realization' of W_{aff} over $\Bbbk$ (in the sense of [EW16]) defined as follows.

– The underlying $\Bbbk$-vector space is $\mathbf{t}^*$.
– If $\alpha \in \Phi^{\mathrm{s}}$ and $s = s_\alpha$, then the 'root' $\alpha_s \in \mathbf{t}$ (respectively, 'coroot' $\alpha_s^\vee \in \mathbf{t}^*$) associated with s is the differential of $\alpha^\vee$ (respectively, of α).
– If $\beta \in \Phi^+$ is such that $\beta^\vee$ is a maximal coroot and $s = t_\beta s_\beta$ then the 'root' α_s (respectively, 'coroot' $\alpha_s^\vee$) associated with s is the differential of $-\beta^\vee$ (respectively, of $-\beta$).

This realization is an example of a Cartan realization in the sense of [AMRW17, § 10.1]. Our assumption (ii) implies that this realization satisfies the 'Demazure surjectivity' condition of [EW16]. There is an associated action of W_{aff} on $\mathbf{t}^*$, which is simply the natural action of $\mathbf{W}$, seen as an action of W_{aff} via the projection $W_{\mathrm{aff}} \to \mathbf{W}$.

We will denote by D_{BS} the diagrammatic Hecke category defined by Elias–Williamson [EW16] for the Coxeter system $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ and this choice of realization. (For a discussion of this definition, see also [AMRW17, Chap. 2].) The technical conditions necessary for this category to be defined (and well behaved) are somewhat subtle, and not all of them are made explicit

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in [EW16]; see [EW20, § 5] for a detailed discussion of this question. As explained in [EW20, § 5.1], a sufficient condition (in addition to the fact that the data define a realization satisfying Demazure surjectivity) that ensures that all the results of [EW16] are applicable is that for any pair of distinct simple reflections s, t such that st has finite order, the restriction of the action to the subgroup generated by s and t is faithful. It follows from [AMRW17, Lemma 8.1.1] that this condition is satisfied in our context, except possibly if $p = 2$ and $st = ts$. (The assumptions of [AMRW17, Lemma 8.1.1] are easily checked by hand for Cartan realizations in good characteristic.) However, in this case s and t act non-trivially by Demazure surjectivity, and st acts non-trivially since $\alpha_s^\vee$ and $\alpha_t^\vee$ are not collinear thanks to assumption (ii).

The category D_{BS} is a $\Bbbk$-linear (non-additive) monoidal category. By definition its objects are pairs $(\underline{w}, n)$ where $\underline{w}$ is a word in S_{aff} and $n \in \mathbb{Z}$; the product is given by concatenation of words and addition of integers, and for any words $\underline{w}, \underline{w}'$ the direct sum of morphism spaces

$$\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathsf{D}_{\mathrm{BS}}}((\underline{w}, 0), (\underline{w}', n))$$

is a graded bimodule over $R := \mathscr{O}(\mathbf{t}^*) = \mathrm{Sym}(\mathbf{t})$ (where the grading is such that elements in $\mathbf{t}$ have degree 2). Following usual conventions, the object $(\underline{w}, n)$ will rather be denoted $B_{\underline{w}}(n)$. Then there exists a natural 'grading shift' autoequivalence of D_{BS} such that $(B_{\underline{w}}(n))(1) = B_{\underline{w}}(n+1)$ for any $\underline{w}$ and any $n \in \mathbb{Z}$.

Remark 2.1. The Hecke category D_{BS} (as, more generally, Hecke categories attached to Cartan realizations of crystallographic Coxeter systems) admits an incarnation in terms of parity complexes on a flag variety; see [RW18, Part III]. Although important for some other purposes, this realization of the Hecke category will not play any role in the present paper. (The relation between Soergel bimodules and constructible sheaves on flag varieties was first obtained, in a characteristic-0 context, by Soergel [Soe01] in the case of finite crystallographic groups (using the earlier definition of Soergel bimodules in this case in [Soe92]) and by Härterich for Kac–Moody groups [Har99].)

2.2 Abe's incarnation of the Hecke category

For our present purposes we will need a description of D_{BS} in terms of R-bimodules due to Abe [Abe21], which is close to the definition of Soergel bimodules [Soe07], and which we now recall. Once again, in order to apply these results one needs some technical assumptions. A sufficient condition (in terms of vanishing of quantum binomial coefficients) for the results of [Abe21] to be applicable is given in [Abe20]. One can check by explicit computation that this condition is automatically satisfied for Cartan realizations.

We will denote by Q the fraction field of R. Following [Abe21], we denote by C' the category defined as follows. Objects are pairs consisting of a graded R-bimodule M together with a decomposition

$$M \otimes_R Q = \bigoplus_{w \in W_{\mathrm{aff}}} M_Q^w \tag{2.1}$$

as (R, Q)-bimodules such that:

- there exist only finitely many elements w such that $M_Q^w \neq 0$;
- for any $w \in W_{\mathrm{aff}}$, $r \in R$ and $m \in M_Q^w$ we have

$$m \cdot r = w(r) \cdot m. \tag{2.2}$$

Morphisms in C' are defined in the obvious way, as morphisms of graded bimodules compatible with the decompositions (2.1). We also denote by C the full subcategory of C' whose objects are

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those whose underlying graded R-bimodule M is finitely generated as an R-bimodule and flat as a right R-module. As explained in [Abe21, Lemma 2.6], the underlying R-bimodule of any object in C is in fact finitely generated as a left and as a right R-module; this property shows that the tensor product over R induces in a natural way a monoidal product on C. We also have a 'grading shift' autoequivalence of C, which only changes the grading of the underlying graded R-bimodule in such a way that $M(1)^i = M^{i+1}$.

For $s \in S_{\mathrm{aff}}$, we consider the s-invariants $R^s \subset R$, and the graded R-bimodule $B_s^{\mathrm{Bim}} := R \otimes_{R^s} R(1)$. This object has a natural 'lift' as an object in C, which will also be denoted by B_s^{Bim} (see [Abe21, § 2.4]).

The following result is a special case of [Abe21, Theorem 5.9]; see also [Abe20, Theorem 1.3].

THEOREM 2.2. *There exists a canonical fully faithful monoidal functor*

$$\mathsf{D}_{\mathrm{BS}} \to \mathsf{C}$$

sending B_s to B_s^{Bim} for any $s \in S_{\mathrm{aff}}$ and intertwining the grading shifts (1).

It will be convenient to consider also a slight extension of the category C, adapted to the group W_{ext}. Namely, the action of W_{aff} on $\mathbf{t}^*$ extends in a natural way to W_{ext} (using now the projection $W_{\mathrm{ext}} \to \mathbf{W}$). We will denote by $\mathsf{C}'_{\mathrm{ext}}$ the category whose objects are pairs consisting a graded R-bimodule M together with a decomposition

$$M \otimes_R Q = \bigoplus_{w \in W_{\mathrm{ext}}} M_Q^w$$

as (R, Q)-bimodules such that:

- there exist only finitely many elements w such that $M_Q^w \neq 0$;
- for any $w \in W_{\mathrm{ext}}$, $r \in R$ and $m \in M_Q^w$ we have $m \cdot r = w(r) \cdot m$,

and where morphisms are defined in the obvious way. We will also denote by $\mathsf{C}_{\mathrm{ext}}$ the full subcategory of $\mathsf{C}'_{\mathrm{ext}}$ whose objects are those whose underlying graded R-bimodule M is finitely generated as an R-bimodule and flat as a right R-module. It is clear that C' is a full subcategory in $\mathsf{C}'_{\mathrm{ext}}$, that C is a full subcategory in $\mathsf{C}_{\mathrm{ext}}$, and that the tensor product $\otimes_R$ defines a monoidal structure on $\mathsf{C}_{\mathrm{ext}}$.

Remark 2.3. Some adaptations of the 'Soergel calculus' of [EW16] to *extended* affine Weyl groups have been discussed in work of Mackaay and Thiel [MT17] and Elias [Eli18, § 3]. The analogue of Theorem 2.2 in this context is most likely true, but since it is not needed in this paper we will not investigate this question further.

In addition to the objects B_s^{Bim} considered above, the category $\mathsf{C}_{\mathrm{ext}}$ possesses 'standard' objects $(\Delta_x : x \in W_{\mathrm{ext}})$ defined as follows. For any $x \in W_{\mathrm{ext}}$, Δ_x is isomorphic to R as a graded vector space, and the R-bimodule structure is given by

$$r \cdot m \cdot r' = rmx(r')$$

for $r, r' \in R$ and $m \in \Delta_x$. The decomposition of $\Delta_x \otimes_R Q$ is defined so that this object is concentrated in degree x. For any $x, y \in W_{\mathrm{ext}}$ we have a canonical isomorphism

$$\Delta_x \otimes_R \Delta_y \xrightarrow{\sim} \Delta_{xy} \tag{2.3}$$

in $\mathsf{C}_{\mathrm{ext}}$, defined by $m \otimes m' \mapsto mx(m')$.

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Lemma 2.4. *Let $s,t \in S_{\mathrm{aff}}$ and $x \in W_{\mathrm{ext}}$ be such that $s = xtx^{-1}$. Then there exists a canonical isomorphism*

$$B_s^{\mathrm{Bim}} \cong \Delta_x \otimes_R B_t^{\mathrm{Bim}} \otimes_R \Delta_{x^{-1}}.$$

Proof. The isomorphism of R-bimodules

$$\Delta_x \otimes_R B_t^{\mathrm{Bim}} \otimes_R \Delta_{x^{-1}} \xrightarrow{\sim} B_s^{\mathrm{Bim}}$$

is defined by

$$r_1 \otimes (r_2 \otimes r_3) \otimes r_4 \mapsto (r_1 x(r_2)) \otimes (x(r_3)x(r_4)).$$

We leave it to the reader to check that this morphism is well defined, and indeed defines an isomorphism in $\mathsf{C}_{\mathrm{ext}}$. □

In § 6 we will also need the following standard claim, for which we refer to [EW16, § 3.4].

Lemma 2.5. *For any $s \in S_{\mathrm{aff}}$, there exist exact sequences of R-bimodules*

$$\Delta_s \hookrightarrow R \otimes_{R^s} R \twoheadrightarrow \Delta_e, \qquad \Delta_e \hookrightarrow R \otimes_{R^s} R \twoheadrightarrow \Delta_s.$$

2.3 Universal centralizer and Kostant section

We will denote by $\mathbf{g}_{\mathrm{reg}} \subset \mathbf{g}$ the open subset consisting of regular elements, that is, elements whose centralizer has dimension $\dim(T)$. The 'regular universal centralizer' is the affine group scheme

$$\mathbb{J}_{\mathrm{reg}} := \mathbf{g}_{\mathrm{reg}} \times_{\mathbf{g}_{\mathrm{reg}} \times \mathbf{g}_{\mathrm{reg}}} (\mathbf{G} \times \mathbf{g}_{\mathrm{reg}})$$

over $\mathbf{g}_{\mathrm{reg}}$, where the morphism $\mathbf{g}_{\mathrm{reg}} \to \mathbf{g}_{\mathrm{reg}} \times \mathbf{g}_{\mathrm{reg}}$ is the diagonal embedding, and the map $\mathbf{G} \times \mathbf{g}_{\mathrm{reg}} \to \mathbf{g}_{\mathrm{reg}}$ sends (g,x) to $(g \cdot x, x)$. For any $x \in \mathbf{g}_{\mathrm{reg}}$, the fiber of $\mathbb{J}_{\mathrm{reg}}$ over x is the scheme-theoretic centralizer of x for the adjoint $\mathbf{G}$-action. By construction $\mathbb{J}_{\mathrm{reg}}$ is a closed subgroup scheme in $\mathbf{G} \times \mathbf{g}_{\mathrm{reg}}$, and as explained in [Ric17, Corollary 3.3.6] it is smooth over $\mathbf{g}_{\mathrm{reg}}$. We will also denote by $\mathbf{g}^*_{\mathrm{reg}}$ the image of $\mathbf{g}_{\mathrm{reg}}$ under κ, and by $\mathbb{J}^*_{\mathrm{reg}}$ the smooth affine group scheme over $\mathbf{g}^*_{\mathrm{reg}}$ obtained by pushforward from $\mathbb{J}_{\mathrm{reg}}$. (It is easily seen that these objects do not depend on the choice of κ.)

There exists a canonical morphism

$$\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathrm{reg}} \to (\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{g}^*_{\mathrm{reg}}) \times \mathbf{T} \tag{2.4}$$

of group schemes over $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{g}^*_{\mathrm{reg}}$, whose construction we now explain. Let $\mathbf{n}$ be the Lie algebra of the unipotent radical $\mathbf{U}$ of $\mathbf{B}$. Recall that the *Grothendieck resolution* is the $\mathbf{G}$-equivariant vector bundle over $\mathbf{G}/\mathbf{B}$ given by

$$\tilde{\mathbf{g}} := \{(\xi, g\mathbf{B}) \in \mathbf{g}^* \times \mathbf{G}/\mathbf{B} \mid \xi_{|g \cdot \mathbf{n}} = 0\}.$$

We have natural maps

$$\pi : \tilde{\mathbf{g}} \to \mathbf{g}^*, \quad \vartheta : \tilde{\mathbf{g}} \to \mathbf{t}^*.$$

(The morphism π is induced by the first projection. The morphism ϑ sends a pair $(\xi, g\mathbf{B})$ to $\xi_{|g \cdot \mathbf{b}}$, seen as an element in $(g \cdot \mathbf{b}/g \cdot \mathbf{n})^* \cong (\mathbf{b}/\mathbf{n})^* \cong \mathbf{t}^*$, where the first isomorphism is induced by conjugation by the inverse of any representative for the coset $g\mathbf{B}$.) If we denote by $\tilde{\mathbf{g}}_{\mathrm{reg}}$ the preimage of $\mathbf{g}^*_{\mathrm{reg}}$ in $\tilde{\mathbf{g}}$, then these maps induce an isomorphism of schemes

$$\tilde{\mathbf{g}}_{\mathrm{reg}} \xrightarrow{\sim} \mathbf{g}^*_{\mathrm{reg}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*; \tag{2.5}$$

see [Ric17, Lemma 3.5.3]. Moreover, under this identification, by [Ric17, Proposition 3.5.6] the group scheme $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathrm{reg}}$ identifies with the universal stabilizer associated with the action

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of $\mathbf{G}$ on $\tilde{\mathbf{g}}_{\mathrm{reg}}$ (defined by the same procedure as for $\mathbb{J}_{\mathrm{reg}}$ above), which is such that the fiber over $(\xi, g\mathbf{B})$ is the scheme-theoretic stabilizer of ξ for the action of $g\mathbf{B}g^{-1}$. Now as above in the definition of ϑ, there exists for any $g \in \mathbf{G}$ a canonical isomorphism $g\mathbf{B}g^{-1}/g\mathbf{U}g^{-1} \cong \mathbf{T}$, which allows us to define the wished-for morphism (2.4).

Let $\mathbf{g}_{\mathrm{rs}} \subset \mathbf{g}$ denote the open subset of semisimple regular elements, and set $\mathbf{g}^*_{\mathrm{rs}} := \kappa(\mathbf{g}_{\mathrm{rs}})$. We will denote by $\mathbb{J}_{\mathrm{rs}}$ (respectively, $\mathbb{J}^*_{\mathrm{rs}}$) the restriction of $\mathbb{J}_{\mathrm{reg}}$ (respectively, $\mathbb{J}^*_{\mathrm{reg}}$) to $\mathbf{g}_{\mathrm{rs}}$ (respectively, $\mathbf{g}^*_{\mathrm{rs}}$). Recall that the adjoint quotient $\mathbf{g}/\mathbf{G}$ identifies canonically with $\mathbf{t}/\mathbf{W}$ (see [BC19, § 4.1]); as a consequence, under our assumptions the coadjoint quotient $\mathbf{g}^*/\mathbf{G}$ identifies canonically with $\mathbf{t}^*/\mathbf{W}$.

LEMMA 2.6. *The morphism* (2.4) *restricts to an isomorphism*

$$\mathbb{J}^*_{\mathrm{rs}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \xrightarrow{\sim} \mathbf{T} \times (\mathbf{g}^*_{\mathrm{rs}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

of group schemes over $\mathbf{g}^*_{\mathrm{rs}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$.

Proof. It is sufficient to prove the analogous statement for $\mathbf{g}$ in place of $\mathbf{g}^*$. If we denote by $\tilde{\mathbf{g}}_{\mathrm{rs}}$ the inverse image of $\mathbf{g}_{\mathrm{rs}}$ under π, then by [Jan04, Lemma 13.4] there exists a canonical isomorphism

$$\mathbf{G} \times^{\mathbf{T}} \mathbf{t}_{\mathrm{rs}} \xrightarrow{\sim} \tilde{\mathbf{g}}_{\mathrm{rs}},$$

where $\mathbf{t}_{\mathrm{rs}} := \mathbf{t} \cap \mathbf{g}_{\mathrm{rs}}$. From the comments above and the compatibility of universal stabilizers with open embeddings, $\mathbf{t} \times_{\mathbf{t}/\mathbf{W}} \mathbb{J}_{\mathrm{rs}}$ identifies with the universal stabilizer associated with the action of $\mathbf{G}$ on $\tilde{\mathbf{g}}_{\mathrm{rs}}$. Since the latter scheme identifies with $\mathbf{G} \times^{\mathbf{T}} \mathbf{t}_{\mathrm{rs}} = \mathbf{G}/\mathbf{T} \times \mathbf{t}_{\mathrm{rs}}$, the universal stabilizer identifies with $\mathbf{T} \times \tilde{\mathbf{g}}_{\mathrm{rs}}$, as desired. □

Let us choose a Kostant section to the adjoint quotient, that is, a closed subscheme $\mathbf{S} \subset \mathbf{g}$ contained in $\mathbf{g}_{\mathrm{reg}}$ and such that the composition $\mathbf{S} \hookrightarrow \mathbf{g} \to \mathbf{g}/\mathbf{G}$ (where the second map is the adjoint quotient morphism) is an isomorphism. (For a construction of such a section in the present generality, see [Ric17, § 3].) We will denote by $\mathbf{S}^*$ the image of $\mathbf{S}$ under κ, so that the composition $\mathbf{S}^* \hookrightarrow \mathbf{g}^* \to \mathbf{g}^*/\mathbf{G}$ is an isomorphism, and by $\mathbb{J}^*_{\mathbf{S}}$ the restriction of $\mathbb{J}^*_{\mathrm{reg}}$ to $\mathbf{S}^*$ (a closed subgroup scheme of $\mathbf{G} \times \mathbf{S}^*$, smooth over $\mathbf{S}^*$).

As explained in [MR18, § 4.4], for example, there exists a natural action of the multiplicative group $\mathbb{G}_{\mathrm{m}}$ on $\mathbf{S}^*$ such that the isomorphism $\mathbf{S}^* \xrightarrow{\sim} \mathbf{g}^*/\mathbf{G}$ is $\mathbb{G}_{\mathrm{m}}$-equivariant, where $t \in \Bbbk^\times$ acts on $\mathbf{g}^*$ by multiplication by t^{-2}, and on $\mathbf{g}^*/\mathbf{G}$ by the induced action. The isomorphism $\mathbf{t}^*/\mathbf{W} \xrightarrow{\sim} \mathbf{g}^*/\mathbf{G}$ is also $\mathbb{G}_{\mathrm{m}}$-equivariant, where the action on $\mathbf{t}^*/\mathbf{W}$ is induced by the action on $\mathbf{t}^*$ where $t \in \Bbbk^\times$ acts by multiplication by t^{-2}. As explained in [MR18, § 4.5, p. 2302] there exists a natural $\mathbb{G}_{\mathrm{m}}$-action on $\mathbb{J}^*_{\mathbf{S}}$ such that the structure morphism $\mathbb{J}^*_{\mathbf{S}} \to \mathbf{S}^*$, the multiplication map $\mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{S}^*} \mathbb{J}^*_{\mathbf{S}} \to \mathbb{J}^*_{\mathbf{S}}$ and the inversion morphism $\mathbb{J}^*_{\mathbf{S}} \to \mathbb{J}^*_{\mathbf{S}}$ are $\mathbb{G}_{\mathrm{m}}$-equivariant.

2.4 Representations of the universal centralizer and Abe's category

The actions of $\mathbb{G}_{\mathrm{m}}$ on $\mathbf{t}^*$ and $\mathbf{t}^*/\mathbf{W}$ considered in § 2.3 provide an action on the fiber product $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$. Let us now consider the category

$$\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

of $\mathbb{G}_{\mathrm{m}}$-equivariant coherent representations of the smooth affine group scheme

$$\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$$

over $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ (where the morphism $\mathbb{J}^*_{\mathbf{S}} \to \mathbf{t}^*/\mathbf{W}$ is obtained via the identification $\mathbf{S}^* \xrightarrow{\sim} \mathbf{t}^*/\mathbf{W}$), that is, $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$-modules equipped with a structure of $\mathbb{G}_{\mathrm{m}}$-equivariant coherent sheaf on $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$, such that the action map is $\mathbb{G}_{\mathrm{m}}$-equivariant. Since R is finite over

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$R^{\mathbf{W}}$, this category admits a natural convolution product $\star$, such that the $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$-module underlying the product $M \star N$ is the tensor product $M \otimes_R N$ (where $R = \mathscr{O}(\mathbf{t}^*)$ acts on M via the second projection $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \to \mathbf{t}^*$ and on N via the first projection $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \to \mathbf{t}^*$). In this way, $(\mathsf{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*), \star)$ is a monoidal category. We will denote by

$$\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$$

the full subcategory of $\mathsf{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ whose objects are the representations whose underlying coherent sheaves are flat with respect to the second projection $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \to \mathbf{t}^*$. It is not difficult to check that this subcategory is stable under $\star$, hence also admits a canonical monoidal category structure.

Proposition 2.7. *There exists a canonical fully faithful monoidal functor*

$$\big(\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*), \star\big) \to (\mathsf{C}_{\mathrm{ext}}, \otimes_R).$$

Proof. We start by constructing a functor

$$\mathsf{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*) \to \mathsf{C}'_{\mathrm{ext}}. \tag{2.6}$$

By definition, any object in $\mathsf{Rep}^{\mathbb{G}_m}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ is in particular a $\mathbb{G}_m$-equivariant coherent sheaf on $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$, hence can be seen as a graded R-bimodule. To equip this graded bimodule with the structure of an object in $\mathsf{C}'_{\mathrm{ext}}$, we must provide a decomposition of its tensor product with Q parametrized by W_{ext}. In fact, we will provide such a decomposition for its tensor product with $\mathscr{O}(\mathbf{t}^*_{\mathrm{rs}})$, where $\mathbf{t}^*_{\mathrm{rs}} := \mathbf{t}^* \cap \mathbf{g}^*_{\mathrm{rs}}$ (which is sufficient since Q is a further localization of $\mathscr{O}(\mathbf{t}^*_{\mathrm{rs}})$).

First, the open subset $\mathbf{t}^*_{\mathrm{rs}} \subset \mathbf{t}^*$ is the complement of the kernels of the differentials of the coroots. This open subset is stable under the action of $\mathbf{W}$, and the restriction of this action is free; see [Ric17, Lemma 2.3.3]. In particular, we have an open subset $\mathbf{t}^*_{\mathrm{rs}}/\mathbf{W} \subset \mathbf{t}^*/\mathbf{W}$, the morphism $\mathbf{t}^*_{\mathrm{rs}} \to \mathbf{t}^*_{\mathrm{rs}}/\mathbf{W}$ is étale, and the map $(w, x) \mapsto (x, w(x))$ induces an isomorphism of schemes

$$\mathbf{W} \times \mathbf{t}^*_{\mathrm{rs}} \xrightarrow{\sim} \mathbf{t}^*_{\mathrm{rs}} \times_{\mathbf{t}^*_{\mathrm{rs}}/\mathbf{W}} \mathbf{t}^*_{\mathrm{rs}};$$

see [SGA1, Exp. V, § 2]. As a consequence, for any coherent sheaf $\mathscr{F}$ on $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$, the tensor product

$$\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{F}) \otimes_R \mathscr{O}(\mathbf{t}^*_{\mathrm{rs}})$$

admits a canonical decomposition (as an $\mathscr{O}(\mathbf{t}^*_{\mathrm{rs}})$-bimodule) parametrized by $\mathbf{W}$, such that the action on the factor corresponding to $w \in \mathbf{W}$ factors through the quotient

$$\mathscr{O}(\mathbf{t}^*_{\mathrm{rs}} \times \mathbf{t}^*_{\mathrm{rs}}) \twoheadrightarrow \mathscr{O}(\mathrm{Gr}(w, \mathbf{t}^*_{\mathrm{rs}}))$$

(where on the right-hand side $\mathrm{Gr}(w, \mathbf{t}^*_{\mathrm{rs}})$ denotes the graph of w acting on $\mathbf{t}^*_{\mathrm{rs}}$), that is, satisfies the condition in (2.2).

Next, let us explain how this decomposition can be refined if $\mathscr{F}$ belongs to $\mathsf{Rep}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$. For this, we consider the restriction $\mathbf{M}_w$ of $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ to $\mathrm{Gr}(w, \mathbf{t}^*_{\mathrm{rs}})$. Identifying the latter subscheme with $\mathbf{t}^*_{\mathrm{rs}}$ via the first projection and using Lemma 2.6, we obtain a canonical isomorphism of group schemes

$$\mathbf{M}_w \xrightarrow{\sim} \mathbf{t}^*_{\mathrm{rs}} \times \mathbf{T}.$$

This means that the category of representations of $\mathbf{M}_w$ on coherent sheaves on $\mathrm{Gr}(w, \mathbf{t}^*_{\mathrm{rs}})$ is canonically equivalent to the category of $\mathbf{X}$-graded coherent sheaves on $\mathbf{t}^*_{\mathrm{rs}}$. Starting with an object $\mathscr{F}$ in $\mathsf{Rep}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$, we therefore obtain a decomposition of $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{F}) \otimes_R \mathscr{O}(\mathbf{t}^*_{\mathrm{rs}})$

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parametrized by W_{ext} by defining, for $\lambda \in \mathbf{X}$ and $w \in \mathbf{W}$, the summand associated with $t_\lambda w$ as the λ-graded part in the summand associated with w (which is a representation of $\mathbf{M}_w$). This finishes the description of the functor (2.6).

It is clear from construction that this functor sends objects in $\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ to objects in $\mathsf{C}_{\mathrm{ext}}$, which therefore provides the functor of the statement. This functor is also easily seen to be monoidal. Let us now explain why it is fully faithful. Consider $\mathscr{F}, \mathscr{G}$ in $\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$, and denote their images by M, N—so that the underlying graded bimodule of M (respectively, N) is $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{F})$ (respectively, $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{G})$). By construction, morphisms in $\mathsf{C}_{\mathrm{ext}}$ from M to N are morphisms of graded bimodules from $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{F})$ to $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{G})$ whose restriction to $\mathbf{t}^* \times \{\eta\}$ (where η is the generic point of $\mathbf{t}^*$) commutes with the action of the restriction of $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$. Now, since by assumption $\Gamma(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*, \mathscr{G})$ is flat as a right R-module and $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ is flat over $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$, such a morphism is automatically a morphism of $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$-comodules. This proves the desired full faithfulness. □

LEMMA 2.8. *For any $w \in W_{\mathrm{ext}}$, the object Δ_w belongs to the essential image of the functor of Proposition 2.7.*

Proof. The isomorphism (2.3) reduces the proof to the case where w belongs to either $\mathbf{W}$ or $\mathbf{X}$. The case $w \in \mathbf{W}$ is obvious: in this case Δ_w is the image of its underlying graded R-bimodule, endowed with the trivial structure as a representation. For the case $w \in \mathbf{X}$, in view of the construction of the functor in Proposition 2.7, the claim follows from the fact that the isomorphism of Lemma 2.6 is the restriction of the morphism (2.4). □

For $w \in W_{\mathrm{ext}}$, we will denote by $\Delta_w^{\mathbb{J}}$ the unique object in $\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ which is sent to Δ_w.

2.5 Representations of the universal centralizer and the Hecke category

By Proposition 2.7 the category $\mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ can be seen as a full monoidal subcategory in Abe's category $\mathsf{C}_{\mathrm{ext}}$, and by Theorem 2.2 the same is true for the Hecke category D_{BS}. We now investigate the relation between these two subcategories.

LEMMA 2.9. *The essential image of the functor of Theorem 2.2 is contained in the essential image of the functor of Proposition 2.7.*

Proof. By definition, the category D_{BS} is generated under convolution and grading shift by the objects $(B_s : s \in S_{\mathrm{aff}})$. Hence, to prove the lemma it suffices to prove each B_s^{Bim} belongs to the essential image of the functor of Proposition 2.7.

If $s = s_\alpha$ for some $\alpha \in \Phi^{\mathrm{s}}$, then B_s^{Bim} is the image of the appropriate shift of $\mathscr{O}(\mathbf{t}^* \times_{\mathbf{t}^*/\{e,s\}} \mathbf{t}^*)$, endowed with the trivial structure as a representation of $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$. If $s \in \mathbf{S}_{\mathrm{aff}}$ is not of this form, then there exist $x \in W_{\mathrm{ext}}$ and $t \in S_{\mathrm{aff}}$ such that $t = s_\alpha$ for some $\alpha \in \Phi^{\mathrm{s}}$ and $s = xtx^{-1}$. (In fact, such a statement is even true in the braid group associated with W_{ext}; see [Ric10, Lemma 6.1.2] or [BM13, Lemma 2.1.1] for the proof in the setting where $\mathbf{G}$ is semisimple and simply connected, from which one can deduce the general case using restriction to the derived subgroup.) By Lemma 2.4 we then have $B_s^{\mathrm{Bim}} \cong \Delta_x \otimes_R B_t^{\mathrm{Bim}} \otimes_R \Delta_{x^{-1}}$; since B_t^{Bim} is now known to belong to the essential image of our functor, and since Δ_x also satisfies this property by Lemma 2.8, this finishes the proof. □

From this lemma we deduce the following claim, which will be crucial for our constructions in § 6.

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Theorem 2.10. *There exists a canonical fully faithful monoidal functor*

$$\mathsf{D}_{\mathrm{BS}} \to \mathsf{Rep}_{\mathrm{fl}}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}_{\mathbf{S}}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*).$$

3. Some categories of equivariant $\mathcal{U}\mathfrak{g}$-bimodules

3.1 Weights

From now on we assume that $p > 0$. We consider a simply connected semisimple algebraic group G over $\Bbbk$, and its category

$$\mathsf{Rep}(G)$$

of finite-dimensional algebraic representations.

For a $\Bbbk$-scheme X we will denote by $X^{(1)}$ the associated Frobenius twist, defined as the fiber product $X^{(1)} := X \times_{\mathrm{Spec}(\Bbbk)} \mathrm{Spec}(\Bbbk)$, where the morphism $\mathrm{Spec}(\Bbbk) \to \mathrm{Spec}(\Bbbk)$ is associated with the map $x \mapsto x^p$. (The projection $X^{(1)} \to X$ is an isomorphism of $\mathbb{F}_p$-schemes, but not of $\Bbbk$-schemes.) We will assume that

$$p \text{ is very good for } G.$$

Then the group $\mathbf{G} := G^{(1)}$ satisfies the assumptions of § 2. We will denote by $\mathrm{Fr} : G \to \mathbf{G}$ the Frobenius morphism of G, and will use the same notation for its restriction to the various subgroups considered below.

The subgroups $\mathbf{B}$, $\mathbf{T}$, $\mathbf{U}$ of $\mathbf{G}$, when seen as subschemes in G, determine subgroups B, T, U whose Frobenius twists are $\mathbf{B}$, $\mathbf{T}$, $\mathbf{U}$, respectively. We will denote by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{n}$ the respective Lie algebras of G, B, T, U (so that $\mathbf{g} = \mathfrak{g}^{(1)}$, and similarly for B, U, T), and by W the Weyl group of (G, T). We set $\mathbb{X} := X^*(T)$, and denote by $\mathfrak{R} \subset \mathbb{X}$ the root system of (G, T). The choice of B determines a system of positive roots $\mathfrak{R}^+ \subset \mathfrak{R}$, chosen as the T-weights in $\mathfrak{g}/\mathfrak{b}$. We will denote by $\mathfrak{R}^{\mathrm{s}} \subset \mathfrak{R}$ the corresponding subset of simple roots, and by $\rho \in \mathbb{X}$ the half-sum of the positive roots. We also set $\mathbb{X}^\vee := X_*(T)$, and denote by $\mathfrak{R}^\vee \subset \mathbb{X}^\vee$ the coroot system. The canonical bijection $\mathfrak{R} \xrightarrow{\sim} \mathfrak{R}^\vee$ will be denoted as usual $\alpha \mapsto \alpha^\vee$.

The Frobenius morphism Fr induces an isomorphism

$$N_G(T)/T \xrightarrow{\sim} N_{\mathbf{G}}(\mathbf{T})/\mathbf{T},$$

which allows us to identify the Weyl group $\mathbf{W}$ of $\mathbf{G}$ with W. It is a standard fact that the morphism from $\mathbf{X} = X^*(T^{(1)})$ to $\mathbb{X}$ induced by $\mathrm{Fr} : T \to \mathbf{T}$ is injective, and that its image is $p \cdot \mathbb{X}$, which allows us to identify $\mathbf{X}$ with $p \cdot \mathbb{X}$. The identification $\mathbf{X} = p \cdot \mathbb{X}$ is W-equivariant, and the root system Φ of $(\mathbf{G}, \mathbf{T})$ is $\Phi = \{p \cdot \alpha : \alpha \in \mathfrak{R}\}$; similarly, we have $\Phi^+ = p\mathfrak{R}^+$ and $\Phi^{\mathrm{s}} = p\mathfrak{R}^{\mathrm{s}}$. In particular, the affine Weyl group W_{aff} of § 2.1 identifies with $W \ltimes (p\mathbb{Z}\mathfrak{R})$, and the extended affine Weyl group W_{ext} identifies with $W \ltimes (p \cdot \mathbb{X})$. Recall also our subset of Coxeter generators $S_{\mathrm{aff}} \subset W_{\mathrm{aff}}$. The subgroup $W \subset W_{\mathrm{aff}}$ is a parabolic subgroup; its longest element will be denoted (as usual) by w_0. We will consider the 'dot' action of W_{ext} (or its subgroup W_{aff}) on $\mathbb{X}$ defined by

$$(t_\mu w) \bullet \lambda = w(\lambda + \rho) - \rho + \mu$$

for $\mu \in p\mathbb{X}$, $w \in W$ and $\lambda \in \mathbb{X}$.

Given a character $\lambda \in \mathbb{X}$, we will denote by $\bar{\lambda} \in \mathfrak{t}^*$ the differential of λ. We set

$$\mathfrak{t}^*_{\mathbb{F}_p} := \{\bar{\lambda} : \lambda \in \mathbb{X}\} \subset \mathfrak{t}^*.$$

In this way, the map $\lambda \mapsto \bar{\lambda}$ induces an isomorphism of abelian groups

$$\mathbb{X}/p\mathbb{X} \xrightarrow{\sim} \mathfrak{t}^*_{\mathbb{F}_p}.$$

(In particular, $\mathfrak{t}^*_{\mathbb{F}_p}$ is finite.)

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The group W naturally acts on $\mathfrak{t}^*$. We also have a 'dot' action of W on $\mathfrak{t}^*$, defined by

$$w \bullet \xi := w(\xi + \bar{\rho}) - \bar{\rho}.$$

With this definition the map $\mathbb{X} \to \mathfrak{t}^*$ sending λ to $\bar{\lambda}$ is W_{ext}-equivariant, where W_{ext} acts on $\mathbb{X}$ via the dot action and on $\mathfrak{t}^*$ via the projection $W_{\mathrm{ext}} \to W$ and the dot action of W on $\mathfrak{t}^*$. This observation legitimates the use of the same notation for these actions. It also shows that the subset $\mathfrak{t}^*_{\mathbb{F}_p} \subset \mathfrak{t}^*$ is stable under the dot action of W. Below we will consider the quotient $\mathfrak{t}^*/(W, \bullet)$ of the dot action of W on $\mathfrak{t}^*$. For $\lambda \in \mathbb{X}$, we will denote by $\tilde{\lambda}$ the image of $\bar{\lambda}$ in $\mathfrak{t}^*/(W, \bullet)$.

As mentioned above, our assumption that p is very good for G implies in particular that the quotient $\mathbb{X}/\mathbb{Z}\mathfrak{R}$ has no p-torsion, or in other words that

$$\mathbb{Z}\mathfrak{R} \cap p\mathbb{X} = p\mathbb{Z}\mathfrak{R}. \tag{3.1}$$

This equality has the following consequences.

LEMMA 3.1. *Let $\lambda \in \mathbb{X}$.*

(i) *We have*

$$W_{\mathrm{aff}} \bullet \lambda = (W_{\mathrm{ext}} \bullet \lambda) \cap (\lambda + \mathbb{Z}\mathfrak{R}).$$

(ii) *The stabilizer of $\bar{\lambda}$ for the dot action of W on $\mathfrak{t}^*$ is the image under the natural surjection $W_{\mathrm{aff}} \to W$ of the stabilizer of λ for the dot action of W_{aff} on $\mathbb{X}$.*

Proof. (i) Since $W \bullet \lambda \subset \lambda + \mathbb{Z}\mathfrak{R}$, we have

$$(W_{\mathrm{ext}} \bullet \lambda) \cap (\lambda + \mathbb{Z}\mathfrak{R}) = (W \bullet \lambda + p\mathbb{X}) \cap (\lambda + \mathbb{Z}\mathfrak{R}) = W \bullet \lambda + (\mathbb{Z}\mathfrak{R} \cap p\mathbb{X}).$$

In view of (3.1), the right-hand side equals $W \bullet \lambda + p\mathbb{Z}\mathfrak{R} = W_{\mathrm{aff}} \bullet \lambda$, as desired.

(ii) For $w \in W$ we have

$$w \bullet \bar{\lambda} = \overline{w \bullet \lambda},$$

so that $w \bullet \bar{\lambda} = \bar{\lambda}$ if and only if $w \bullet \lambda \in \lambda + p\mathbb{X}$. Since $w \bullet \lambda \in \lambda + \mathbb{Z}\mathfrak{R}$, as above this condition is equivalent to $w \bullet \lambda \in \lambda + p\mathbb{Z}\mathfrak{R}$, that is, to the existence of $\mu \in p\mathbb{Z}\mathfrak{R}$ such that $t_\mu w \in W_{\mathrm{aff}}$ stabilizes λ. □

For any subset $I \subset \mathfrak{R}^{\mathrm{s}}$, we will denote by $W_I \subset W$ the subgroup generated by the reflections $\{s_\alpha : \alpha \in I\}$. Recall that an element of $\mathbb{X}$ is called *regular* if its stabilizer in W_{aff} (for the dot action) is trivial. As a consequence of Lemma 3.1, we obtain, in particular, the following claim.

LEMMA 3.2. *Let $\lambda \in \mathbb{X}$, and assume that the stabilizer of λ for the dot action of W_{aff} is W_I. Then the morphism*

$$\mathfrak{t}^*/(W_I, \bullet) \to \mathfrak{t}^*/(W, \bullet)$$

induced by the quotient morphism $\mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet)$ is étale at the image of $\bar{\lambda}$. In particular, if λ is regular then the quotient morphism $\mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet)$ is étale at $\bar{\lambda}$.

Proof. By Lemma 3.1(ii), the stabilizer of $\bar{\lambda}$ for the dot action of W on $\mathfrak{t}^*$ is W_I. Hence, the claim follows from the general criterion [SGA1, Exp. V, Proposition 2.2]. □

3.2 The center of the enveloping algebra

Consider the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of $\mathfrak{g}$. Its center $Z(\mathcal{U}\mathfrak{g})$ can be described as follows. We set

$$Z_{\mathrm{HC}} := (\mathcal{U}\mathfrak{g})^G.$$

(Here, the subscript 'HC' stands for Harish-Chandra.) Next, as the Lie algebra of an algebraic group over a field of characteristic p, $\mathfrak{g}$ admits a 'restricted pth power' operation $x \mapsto x^{[p]}$, which

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stabilizes the Lie algebra of any algebraic subgroup of G. We will denote by

$$Z_{\mathrm{Fr}}$$

the $\Bbbk$-subalgebra of $\mathcal{U}\mathfrak{g}$ generated by the elements of the form $x^p - x^{[p]}$ for $x \in \mathfrak{g}$. Then by [MR99, Theorem 2] multiplication induces an isomorphism

$$Z_{\mathrm{Fr}} \otimes_{Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}}} Z_{\mathrm{HC}} \xrightarrow{\sim} Z(\mathcal{U}\mathfrak{g}).$$

It is well known that $\mathcal{U}\mathfrak{g}$ is finite as a Z_{Fr}-algebra (hence a fortiori as a $Z(\mathcal{U}\mathfrak{g})$-algebra).

These central subalgebras can be described geometrically as follows. It is well known that the map $x \mapsto x^p - x^{[p]}$ induces a $\Bbbk$-algebra isomorphism

$$\mathscr{O}(\mathfrak{g}^{*(1)}) \xrightarrow{\sim} Z_{\mathrm{Fr}}. \tag{3.2}$$

We also have $Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}} = (Z_{\mathrm{Fr}})^G$, and the G-action on $\mathfrak{g}^{*(1)}$ factors through the Frobenius morphism Fr, so that we obtain an isomorphism

$$\mathscr{O}(\mathfrak{g}^{*(1)}/G^{(1)}) \xrightarrow{\sim} Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}}.$$

On the other hand, the 'Harish-Chandra isomorphism' provides a $\Bbbk$-algebra isomorphism

$$\mathscr{O}(\mathfrak{t}^*/(W, \bullet)) \xrightarrow{\sim} Z_{\mathrm{HC}}, \tag{3.3}$$

see [MR99, Theorem 1(2)].

The *Artin–Schreier morphism*

$$\mathrm{AS} : \mathfrak{t}^* \to \mathfrak{t}^{*(1)}$$

is the morphism associated with the algebra map $\mathscr{O}(\mathfrak{t}^{*(1)}) \to \mathscr{O}(\mathfrak{t}^*)$ defined by $h \mapsto h^p - h^{[p]}$ for $h \in \mathfrak{t}$. It is well known that AS is a Galois covering with Galois group $\mathfrak{t}^*_{\mathbb{F}_p}$ (acting on $\mathfrak{t}^*$ via addition). The morphism AS is W-equivariant, where W acts on $\mathfrak{t}^*$ via the dot action and on $\mathfrak{t}^{*(1)}$ via the natural action. It therefore induces a morphism

$$\mathfrak{t}^*/(W, \bullet) \to \mathfrak{t}^{*(1)}/W. \tag{3.4}$$

Recall the Chevalley isomorphism

$$\mathfrak{t}^{*(1)}/W \xrightarrow{\sim} \mathfrak{g}^{*(1)}/G^{(1)}$$

already encountered in § 2.3. Under this identification, the embedding $Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}} \hookrightarrow Z_{\mathrm{HC}}$ is induced by (3.4).

Combining all these descriptions, and setting

$$\mathfrak{C} := \mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

we therefore obtain a $\Bbbk$-algebra isomorphism

$$\mathscr{O}(\mathfrak{C}) \xrightarrow{\sim} Z(\mathcal{U}\mathfrak{g}),$$

see [MR99, Corollary 3].

Using this identification one can consider $\mathcal{U}\mathfrak{g}$ as an $\mathscr{O}(\mathfrak{C})$-algebra. The G-action on $\mathfrak{C}$ induced by the adjoint G-action on $\mathcal{U}\mathfrak{g}$ is the action obtained by pullback via the Frobenius morphism Fr of the $G^{(1)}$-action on $\mathfrak{C}$ induced by the coadjoint $G^{(1)}$-action on $\mathfrak{g}^{*(1)}$. Using this action, one can therefore see $\mathcal{U}\mathfrak{g}$ as a G-equivariant $\mathscr{O}(\mathfrak{C})$-algebra.

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3.3 Central reductions

In view of (3.2), the maximal ideals in Z_{Fr} are in a canonical bijection with elements in $\mathfrak{g}^{*(1)}$. Given $\eta \in \mathfrak{g}^{*(1)}$, we will denote by $\mathfrak{m}_\eta \subset Z_{\mathrm{Fr}}$ the corresponding maximal ideal, and set

$$\mathcal{U}_\eta\mathfrak{g} := \mathcal{U}\mathfrak{g}/\mathfrak{m}_\eta \cdot \mathcal{U}\mathfrak{g}.$$

Similarly, in view of (3.3) the maximal ideals in Z_{HC} are in a canonical bijection with closed points in $\mathfrak{t}^*/(W,\bullet)$, that is, with $(W,\bullet)$-orbits in $\mathfrak{t}^*$. Given a closed point $\xi \in \mathfrak{t}^*/(W,\bullet)$, we will denote by $\mathfrak{m}^\xi \subset Z_{\mathrm{HC}}$ the corresponding maximal ideal, and set

$$\mathcal{U}^\xi\mathfrak{g} := \mathcal{U}\mathfrak{g}/\mathfrak{m}^\xi \cdot \mathcal{U}\mathfrak{g}.$$

If η and ξ have the same image in $\mathfrak{t}^{*(1)}/W$, then $\mathfrak{m}_\eta \cdot Z(\mathcal{U}\mathfrak{g}) + \mathfrak{m}^\xi \cdot Z(\mathcal{U}\mathfrak{g})$ is a maximal ideal in $Z(\mathcal{U}\mathfrak{g})$, and we can also set

$$\mathcal{U}^\xi_\eta\mathfrak{g} := \mathcal{U}\mathfrak{g}/(\mathfrak{m}_\eta \cdot \mathcal{U}\mathfrak{g} + \mathfrak{m}^\xi \cdot \mathcal{U}\mathfrak{g}).$$

In the cases we will encounter more specifically below, the point ξ will often be the image $\tilde{\lambda}$ of the differential of a character $\lambda \in \mathbb{X}$. In this setting we will write $\mathfrak{m}^\lambda$, $\mathcal{U}^\lambda\mathfrak{g}$ and $\mathcal{U}^\lambda_\eta\mathfrak{g}$ instead of $\mathfrak{m}^{\tilde{\lambda}}$, $\mathcal{U}^{\tilde{\lambda}}\mathfrak{g}$ and $\mathcal{U}^{\tilde{\lambda}}_\eta\mathfrak{g}$. The image of any element of $\mathfrak{t}^*_{\mathbb{F}_p}$ under the Artin–Schreier map is 0; therefore, if we denote by

$$\mathcal{N}^* \subset \mathfrak{g}^*$$

the preimage under the coadjoint morphism $\mathfrak{g}^* \to \mathfrak{t}^*/W$ of the image of 0, then, given any $\lambda \in \mathbb{X}$, the elements $\eta \in \mathfrak{g}^{*(1)}$ whose image in $\mathfrak{t}^{*(1)}/W$ coincides with that of $\tilde{\lambda}$ are exactly those in $\mathcal{N}^{*(1)}$.

3.4 Harish-Chandra bimodules

We will denote by HC the category whose objects are the $\mathcal{U}\mathfrak{g}$-bimodules V endowed with an (algebraic) action of G which satisfy the following conditions.

(i) The action morphisms $\mathcal{U}\mathfrak{g} \otimes V \to V$ and $V \otimes \mathcal{U}\mathfrak{g} \to V$ are G-equivariant (for the diagonal actions on $\mathcal{U}\mathfrak{g} \otimes V$ and $V \otimes \mathcal{U}\mathfrak{g}$).
(ii) The $\mathfrak{g}$-action on V obtained by differentiating the G-action is given by $(x, v) \mapsto x \cdot v - v \cdot x$.
(iii) V is finitely generated both as a left and as a right $\mathcal{U}\mathfrak{g}$-module.

Morphisms in the category HC are morphisms of bimodules which also commute with the G-actions. Objects in this category are called Harish-Chandra bimodules. It is easily seen that the tensor product $\otimes_{\mathcal{U}\mathfrak{g}}$ of bimodules endows HC with the structure of a monoidal category, where the G-action on the tensor product is the diagonal action.

If M belongs to HC, the $\mathcal{U}\mathfrak{g}$-action obtained by differentiating the G-action must vanish on $Z_{\mathrm{Fr}} \cap (\mathfrak{g} \cdot \mathcal{U}\mathfrak{g})$. In view of condition (ii) above, this implies that the two actions of Z_{Fr} on M obtained by restriction of the left and right $\mathcal{U}\mathfrak{g}$-actions coincide; in other words, the action of $\mathcal{U}\mathfrak{g} \otimes_{\Bbbk} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ on M must factor through an action of $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$. However, the two actions of Z_{HC} on a Harish-Chandra bimodule might differ. Note that $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is in a natural way a finite algebra over the commutative ring

$$\mathcal{Z} := Z(\mathcal{U}\mathfrak{g}) \otimes_{Z_{\mathrm{Fr}}} Z(\mathcal{U}\mathfrak{g}) = Z_{\mathrm{HC}} \otimes_{Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}}} Z_{\mathrm{Fr}} \otimes_{Z_{\mathrm{Fr}} \cap Z_{\mathrm{HC}}} Z_{\mathrm{HC}} \cong \mathscr{O}(\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C}).$$

Note also that since $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is finitely generated both as a left and as a right $\mathcal{U}\mathfrak{g}$-module, condition (iii) above can be equivalently replaced by the condition that the object is finitely generated as a $\mathcal{U}\mathfrak{g}$-bimodule (or as a left $\mathcal{U}\mathfrak{g}$-module, or as a right $\mathcal{U}\mathfrak{g}$-module). It is easily seen

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that forgetting the right (respectively, left) action of $\mathcal{U}\mathfrak{g}$ defines an equivalence of categories

$$\mathsf{HC} \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g}) \quad (\text{respectively, } \mathsf{HC} \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g}^{\mathrm{op}})), \tag{3.5}$$

where $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g})$ is the category of G-equivariant finitely generated $\mathcal{U}\mathfrak{g}$-modules, and similarly for $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g}^{\mathrm{op}})$. For instance, for the first functor, one can reconstruct the right action of $\mathcal{U}\mathfrak{g}$ on a G-equivariant $\mathcal{U}\mathfrak{g}$-module M by setting $m \cdot x := x \cdot m - \varrho(x)(m)$ for $m \in M$ and $x \in \mathfrak{g}$, where ϱ denotes the differential of the G-action.

We will also denote by $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$ the category of G-equivariant finitely generated (left) modules over $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$. As above, since $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is finitely generated both as a left and as a right $\mathcal{U}\mathfrak{g}$-module, the tensor product $\otimes_{\mathcal{U}\mathfrak{g}}$ endows this category with a monoidal structure, and as explained above we have a fully faithful monoidal functor

$$\mathsf{HC} \to \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}). \tag{3.6}$$

If M belongs to $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$, then one obtains an extra $\mathcal{U}_0\mathfrak{g}$-action on M by setting $x \cdot m = (x \otimes 1 - 1 \otimes x)m$ for $x \in \mathfrak{g}$ and $m \in M$. Since $\mathcal{U}_0\mathfrak{g}$ identifies canonically with the distribution algebra $\mathrm{Dist}(G_1)$ of the kernel G_1 of Fr, M becomes in this way a $G \ltimes G_1$-equivariant $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$-module, where the action of $G \ltimes G_1$ on $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is obtained from the G-action by composition with the product morphism $G \ltimes G_1 \to G$. As for (3.5), forgetting the right (respectively, left) action of $\mathcal{U}\mathfrak{g}$ defines an equivalence of categories

$$\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}^{G \ltimes G_1}(\mathcal{U}\mathfrak{g}) \quad (\text{respectively, } \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}^{G \ltimes G_1}(\mathcal{U}\mathfrak{g}^{\mathrm{op}})).$$

From the point of view of these equivalences and those in (3.5), the essential image of (3.6) consists of equivariant modules on which the action of $G \ltimes G_1$ factors through the product morphism $G \ltimes G_1 \to G$.

One can construct interesting objects in HC from G-modules as follows. Given V in $\mathsf{Rep}(G)$, we consider the Harish-Chandra bimodule

$$V \otimes \mathcal{U}\mathfrak{g},$$

where the left $\mathcal{U}\mathfrak{g}$-action is diagonal (with respect to the action on V obtained by differentiation, and the action on $\mathcal{U}\mathfrak{g}$ by left multiplication), the right $\mathcal{U}\mathfrak{g}$-action is induced by right multiplication on $\mathcal{U}\mathfrak{g}$, and the G-action is diagonal (with respect to the given action on V and the adjoint action on $\mathcal{U}\mathfrak{g}$). In particular, for $x, y, z \in \mathcal{U}\mathfrak{g}$ and $v \in V$ we have

$$x \cdot (v \otimes z) \cdot y = (x_{(1)} \cdot v) \otimes (x_{(2)} z y),$$

where we use Sweedler's notation for the comultiplication in the Hopf algebra $\mathcal{U}\mathfrak{g}$. It is easily seen that the map $(x \otimes y) \otimes v \mapsto (x_{(1)} \cdot v) \otimes (x_{(2)} y)$ induces an isomorphism of G-equivariant $\mathcal{U}\mathfrak{g}$-bimodules

$$(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathcal{U}\mathfrak{g}} V \xrightarrow{\sim} V \otimes \mathcal{U}\mathfrak{g},$$

where the tensor product over $\mathcal{U}\mathfrak{g}$ on the left-hand side is taken with respect to the morphism $\mathcal{U}\mathfrak{g} \to \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ defined by $x \mapsto x_{(1)} \otimes S(x_{(2)})$, where S is the antipode. In particular, the modules $V \otimes \mathcal{U}\mathfrak{g}$ are 'induced from the diagonal'. For V, V' in $\mathsf{Rep}(G)$, we have a canonical isomorphism of Harish-Chandra bimodules

$$(V \otimes \mathcal{U}\mathfrak{g}) \otimes_{\mathcal{U}\mathfrak{g}} (V' \otimes \mathcal{U}\mathfrak{g}) \xrightarrow{\sim} (V \otimes V') \otimes \mathcal{U}\mathfrak{g}. \tag{3.7}$$

One can similarly consider, again for V in $\mathsf{Rep}(G)$, the Harish-Chandra bimodule

$$\mathcal{U}\mathfrak{g} \otimes V$$

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where now the actions of $\mathcal{U}\mathfrak{g}$ are defined by

$$x \cdot (z \otimes v) \cdot y = (xzy_{(1)}) \otimes (S(y_{(2)}) \cdot v)$$

for $x, y, z \in \mathcal{U}\mathfrak{g}$ and $v \in V$ (and the G-action is still diagonal). As above, we have an isomorphism of G-equivariant $\mathcal{U}\mathfrak{g}$-bimodules

$$(\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathcal{U}\mathfrak{g}} V \xrightarrow{\sim} \mathcal{U}\mathfrak{g} \otimes V,$$

now given by $(x \otimes y) \otimes v \mapsto (xy_{(1)}) \otimes (S(y_{(2)}) \cdot v)$ for $x, y \in \mathcal{U}\mathfrak{g}$ and $v \in V$. In particular, the objects $V \otimes \mathcal{U}\mathfrak{g}$ and $\mathcal{U}\mathfrak{g} \otimes V$ are isomorphic; explicitly, the isomorphism is given by

$$v \otimes x \mapsto x \otimes (S(x) \cdot v)$$

for $x \in \mathcal{U}\mathfrak{g}$ and $v \in V$.

3.5 Completed Harish-Chandra bimodules

We now need to adapt the considerations of § 3.4 to the setting of completed Harish-Chandra characters.

Let us set

$$\mathfrak{D} := \mathrm{Spec}(Z_{\mathrm{HC}} \otimes_{Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}} Z_{\mathrm{HC}}) \cong \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

so that $\mathcal{Z} = \mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D})$. For $\lambda, \mu \in \mathbb{X}$, we will also set

$$\mathcal{I}^{\lambda,\mu} := \mathfrak{m}^{\lambda} \otimes_{Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}} Z_{\mathrm{HC}} + Z_{\mathrm{HC}} \otimes_{Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}} \mathfrak{m}^{\mu} \cdot Z_{\mathrm{HC}},$$

and will denote by $\mathfrak{D}^{\hat{\lambda},\hat{\mu}}$ the spectrum of the completion of $\mathscr{O}(\mathfrak{D})$ with respect to the maximal ideal $\mathcal{I}^{\lambda,\mu}$. Finally, we set

$$\mathcal{U}^{\hat{\lambda},\hat{\mu}} := (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathcal{Z}} \mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \cong (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}).$$

(Note that $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$ is *not* the completion of $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ with respect to the ideal generated by $\mathcal{I}^{\lambda,\mu}$, since $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is not of finite type as an $\mathscr{O}(\mathfrak{D})$-module.)

The algebra $\mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$ is Noetherian (see [Sta20, Tag 05GH]), hence $\mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D}^{\hat{\lambda},\hat{\mu}})$ is Noetherian too, being finitely generated over $\mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$ (see [Sta20, Tag 00FN]). Finally, since $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$ is finitely generated as an $\mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D}^{\hat{\lambda},\hat{\mu}})$-module it is left and right Noetherian (as a non-commutative ring), and a $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$-module is finitely generated if and only if it is finitely generated as an $\mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D}^{\hat{\lambda},\hat{\mu}})$-module.

The G-action on $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ induces an algebraic G-module structure on $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$, and we can consider the category of G-equivariant finitely generated modules over this algebra; this (abelian) category will be denoted by $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$. The full subcategory whose objects are the modules M such that the differential of the G-action coincides with the action given by $x \cdot m = xm - mx$ for $x \in \mathfrak{g}$ and $m \in M$ will be denoted $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$; its objects will be called *completed Harish-Chandra bimodules*.

Given $\lambda, \mu \in \mathbb{X}$, we have a natural exact functor

$$\mathsf{C}^{\lambda,\mu} : \mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \to \mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}),$$

defined by

$$\mathsf{C}^{\lambda,\mu}(M) = \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \otimes_{\mathscr{O}(\mathfrak{D})} M,$$

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which restricts to a functor $\mathsf{HC} \to \mathsf{HC}^{\hat{\lambda},\hat{\mu}}$. (Exactness of this functor follows from the fact that $\mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$ is flat over $\mathscr{O}(\mathfrak{D})$; see [Sta20, Tag 00MB].) We will denote by

$$\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}}$$

the full additive subcategory of $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$ whose objects are direct summands of objects of the form $\mathsf{C}^{\lambda,\mu}(V \otimes \mathcal{U}\mathfrak{g})$ with V in $\mathsf{Rep}(G)$. (In view of the comments at the end of § 3.4, objects in this category will sometimes be referred to as completed diagonally induced Harish-Chandra bimodules.) In the case where $\lambda = \mu$, we will set

$$\mathcal{U}^{\hat{\lambda}} = \mathsf{C}^{\lambda,\lambda}(\Bbbk \otimes \mathcal{U}\mathfrak{g}),$$

where $\Bbbk$ is the trivial G-module.

For later use, we also introduce some completed bimodules which are closely related to the translation functors for G-modules (see § 6.3 below for details). Recall that a weight $\lambda \in \mathbb{X}$ is said to belong to the *fundamental alcove* (respectively, to the *closure of the fundamental alcove*) if it satisfies

$$0 < \langle \lambda + \rho, \alpha^\vee \rangle < p \quad (\text{respectively, } 0 \leq \langle \lambda + \rho, \alpha^\vee \rangle \leq p),$$

for any positive root α. With this notation, the set of weights which belong to the closure of the fundamental alcove is a fundamental domain for the $(W_{\mathrm{aff}}, \bullet)$-action on $\mathbb{X}$. Moreover, if $\lambda \in \mathbb{X}$ belongs to the closure of the fundamental alcove, then its stabilizer in W_{aff} is the parabolic subgroup generated by the elements $s \in S_{\mathrm{aff}}$ such that $s \bullet \lambda = \lambda$; see [Jan03, §II.6.3].

Remark 3.3. The weight lattice $\mathbb{X}$ contains weights which belong to the fundamental alcove if and only if $p \geq h$, where h is the Coxeter number of G; see [Jan03, § 6.2]. Even though this condition will be imposed only in § 6, some of our statements in § 5 involve weights in the fundamental alcove; these statements will simply be empty in the case where $p < h$.

Let $\mathbb{X}^+ \subset \mathbb{X}$ be the subset of dominant weights determined by $\mathfrak{R}^+$. For any $\nu \in \mathbb{X}^+$, we will denote by $\mathsf{L}(\nu)$ the simple G-module with highest weight ν, that is, the unique simple submodule in $\mathrm{Ind}_B^G(\nu)$. Given two weights $\lambda, \mu \in \mathbb{X}$ which belong to the closure of the fundamental alcove, we set

$$\mathsf{P}^{\lambda,\mu} := \mathsf{C}^{\lambda,\mu}(\mathsf{L}(\nu) \otimes \mathcal{U}\mathfrak{g}) \in \mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}},$$

where ν is the unique dominant W-translate of $\lambda - \mu$.

3.6 Comparison of completions

For notational simplicity, let us now fix a subset $\Lambda \subset \mathbb{X}$ such that the map $\lambda \mapsto \tilde{\lambda}$ restricts to a bijection $\Lambda \xrightarrow{\sim} \mathfrak{t}^*_{\mathbb{F}_p}/(W, \bullet)$. (In other words, Λ is a set of representatives for the $\bullet$-action of W_{ext} on $\mathbb{X}$.)

We will denote by $\mathcal{I} \subset \mathscr{O}(\mathfrak{t}^{*(1)}/W) = Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}$ the maximal ideal corresponding to the image of $0 \in \mathfrak{t}^{*(1)}$. Then, in the notation of § 3.2, $\mathcal{I} \cdot Z_{\mathrm{Fr}}$ is the ideal of definition of $\mathcal{N}^{*(1)} \subset \mathfrak{g}^{*(1)}$, and each ideal $\mathcal{I}^{\lambda,\mu}$ contains $\mathcal{I} \cdot \mathscr{O}(\mathfrak{D})$. We will denote by $\mathfrak{D}^\wedge$ the spectrum of the completion of $\mathscr{O}(\mathfrak{D})$ with respect to the ideal $\mathcal{I} \cdot \mathscr{O}(\mathfrak{D})$. Note that since $\mathscr{O}(\mathfrak{D})$ is finite as an $\mathscr{O}(\mathfrak{t}^{*(1)}/W)$-module (because the morphisms $\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ and $\mathfrak{t}^{*(1)} \to \mathfrak{t}^{*(1)}/W$ are finite), if we denote by $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$ the completion of $\mathscr{O}(\mathfrak{t}^{*(1)}/W)$ with respect to $\mathcal{I}$ we have a canonical isomorphism

$$\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{D}) \xrightarrow{\sim} \mathscr{O}(\mathfrak{D}^\wedge); \tag{3.8}$$

see [Sta20, Tag 00MA].

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LEMMA 3.4. *The natural morphism*

$$\mathscr{O}(\mathfrak{D}^{\wedge}) \to \prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$$

is a ring isomorphism.

Proof. The lemma will basically follow from the observation that the closed points in the fiber of the morphism (3.4) over the (closed) point corresponding to $\mathcal{I}$ are those corresponding to the ideals $\mathfrak{m}^{\lambda}$ with $\lambda \in \mathbb{X}$, which itself follows from the fact that the fiber of $\mathrm{AS} : \mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ over 0 is $\mathfrak{t}^*_{\mathbb{F}_p}$.

More precisely, the morphism considered in this statement is the product of the morphisms $\mathscr{O}(\mathfrak{D}^{\wedge}) \to \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$ induced by the natural morphisms $\mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D})) \twoheadrightarrow \mathscr{O}(\mathfrak{D})/(\mathcal{I}^{\lambda,\mu})^n$. This morphism is clearly a ring morphism; to prove that it is invertible we will construct its inverse.

Let us fix some $n \geq 1$, and consider the quotient $\mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D}))$. Here, as explained above, $\mathscr{O}(\mathfrak{D}) = Z_{\mathrm{HC}} \otimes_{Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}} Z_{\mathrm{HC}}$ is a finite $\mathscr{O}(\mathfrak{t}^{*(1)}/W)$-module; therefore, this algebra is finite-dimensional. Its maximal ideals are in bijection with the maximal ideals of $\mathscr{O}(\mathfrak{D})$ containing $\mathcal{I} \cdot \mathscr{O}(\mathfrak{D})$, hence with $\Lambda \times \Lambda$ through $(\lambda, \mu) \mapsto \mathcal{I}^{\lambda,\mu}/\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D})$. In view of the general theory of Artin rings (see, for example, [AM69, Chap. 8]), for any $\lambda, \mu \in \Lambda$ the quotient

$$\mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D}) + (\mathcal{I}^{\lambda,\mu})^m)$$

does not depend on m for $m \gg 0$, and the natural morphism from $\mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D}))$ to the product of these rings (over $\Lambda \times \Lambda$) is an isomorphism.

We are now ready to define the wished-for inverse morphism

$$\prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \to \mathscr{O}(\mathfrak{D}^{\wedge}).$$

For this it suffices to define, for any $n \geq 1$, a ring morphism

$$\prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \to \mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D})). \tag{3.9}$$

We fix m such that the natural morphism

$$\mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D})) \to \prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D}) + (\mathcal{I}^{\lambda,\mu})^m) \tag{3.10}$$

is an isomorphism. Then we have natural ring morphisms

$$\prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \to \prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D})/(\mathcal{I}^{\lambda,\mu})^m \to \prod_{\lambda,\mu\in\Lambda} \mathscr{O}(\mathfrak{D})/(\mathcal{I}^n \cdot \mathscr{O}(\mathfrak{D}) + (\mathcal{I}^{\lambda,\mu})^m).$$

Composing with the inverse of (3.10), we deduce the desired map (3.9).

It is easy (and left to the reader) to check that the two morphisms considered above are inverse to each other. □

Remark 3.5. If we denote by $Z^{\wedge}_{\mathrm{HC}}$ the completion of Z_{HC} with respect to the ideal $\mathcal{I} \cdot Z_{\mathrm{HC}}$, then we have as in (3.8) a canonical isomorphism $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} Z_{\mathrm{HC}} \xrightarrow{\sim} Z^{\wedge}_{\mathrm{HC}}$, and therefore a canonical isomorphism

$$\mathscr{O}(\mathfrak{D}^{\wedge}) \xrightarrow{\sim} Z^{\wedge}_{\mathrm{HC}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} Z^{\wedge}_{\mathrm{HC}}.$$

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Denoting by $Z_{\mathrm{HC}}^{\hat{\lambda}}$ the completion of Z_{HC} with respect to the ideal $\mathfrak{m}^{\lambda}$, for $\lambda \in \mathbb{X}$, the same considerations as for Lemma 3.4 show that we have a canonical isomorphism

$$Z_{\mathrm{HC}}^{\wedge} \xrightarrow{\sim} \prod_{\lambda \in \Lambda} Z_{\mathrm{HC}}^{\hat{\lambda}},$$

from which we obtain a decomposition

$$\mathscr{O}(\mathfrak{D}^{\wedge}) \xrightarrow{\sim} \prod_{\lambda,\mu \in \Lambda} Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} Z_{\mathrm{HC}}^{\hat{\mu}}.$$

This decomposition is in fact 'the same' as the decomposition of Lemma 3.4, in the sense that for any $\lambda, \mu \in \Lambda$ we have an identification

$$\mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} Z_{\mathrm{HC}}^{\hat{\mu}}. \tag{3.11}$$

We will also set

$$\mathcal{U}^{\wedge} := (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^{\wedge}) \cong (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge},$$

where the isomorphism uses (3.8). Then, as in § 3.5, $\mathcal{U}^{\wedge}$ is a left and right Noetherian ring, endowed with a structure of an algebraic G-module, and Lemma 3.4 implies that the natural morphism

$$\mathcal{U}^{\wedge} \to \prod_{\lambda,\mu \in \Lambda} \mathcal{U}^{\hat{\lambda},\hat{\mu}} \tag{3.12}$$

is an algebra isomorphism. Below we will consider various categories of $\mathcal{U}^{\wedge}$-modules; in fact we have

$$\begin{aligned}\mathcal{U}^{\wedge} \cong (\mathcal{U}\mathfrak{g} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}) \otimes_{Z_{\mathrm{Fr}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} \\ \times (\mathcal{U}\mathfrak{g} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge})^{\mathrm{op}};\end{aligned}$$

a $\mathcal{U}^{\wedge}$-module is therefore the same as a $(\mathcal{U}\mathfrak{g} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge})$-bimodule on which the left and right actions of $Z_{\mathrm{Fr}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}$ coincide.

We will denote by $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge})$ the abelian category of G-equivariant finitely generated $\mathcal{U}^{\wedge}$-modules. In view of (3.12), we have a canonical equivalence of categories

$$\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge}) \cong \bigoplus_{\lambda,\mu \in \Lambda} \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}). \tag{3.13}$$

We also have a canonical exact functor

$$\mathsf{C}^{\wedge} : \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \to \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge}) \tag{3.14}$$

defined by $\mathsf{C}^{\wedge}(M) = \mathscr{O}(\mathfrak{D}^{\wedge}) \otimes_{\mathscr{O}(\mathfrak{D})} M$. For the same reasons as above, for any M in $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$ we have a canonical isomorphism

$$\mathsf{C}^{\wedge}(M) \cong \bigoplus_{\lambda,\mu \in \Lambda} \mathsf{C}^{\lambda,\mu}(M).$$

An object M in $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge})$ will be called a *completed Harish-Chandra bimodule* if the differential of the G-action coincides with the action given by $x \cdot m = xm - mx$ for $x \in \mathfrak{g}$ and $m \in M$, and we will denote by $\mathsf{HC}^{\wedge}$ the full subcategory of $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge})$ consisting of such objects; this terminology

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is compatible with that of § 3.5 in the sense that (3.13) restricts to an equivalence of categories

$$\mathsf{HC}^\wedge \cong \bigoplus_{\lambda,\mu\in\Lambda} \mathsf{HC}^{\hat\lambda,\hat\mu}.$$

We will denote by $\mathsf{HC}^\wedge_{\mathrm{diag}}$ the full additive subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$ whose objects are the direct summands of objects of the form $\mathsf{C}^\wedge(V\otimes\mathcal{U}\mathfrak{g})$ with V in $\mathsf{Rep}(G)$. With this definition, (3.13) restricts further to an equivalence of categories

$$\mathsf{HC}^\wedge_{\mathrm{diag}} \cong \bigoplus_{\lambda,\mu\in\Lambda} \mathsf{HC}^{\hat\lambda,\hat\mu}_{\mathrm{diag}}.$$

3.7 Monoidal structure

We now want to define some analogue of the monoidal structure on $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}})$ for the categories $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\mu})$. More specifically, given λ,μ,ν in $\mathbb{X}$, we want to define a canonical bifunctor

$$(-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-) : \mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\mu})\times\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\mu,\hat\nu})\to\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\nu}) \tag{3.15}$$

right exact on both sides, which restricts to a bifunctor

$$\mathsf{HC}^{\hat\lambda,\hat\mu}\times\mathsf{HC}^{\hat\mu,\hat\nu}\to\mathsf{HC}^{\hat\lambda,\hat\nu},$$

these bifunctors satisfying natural unit and associativity axioms. Explicitly, we require that:

– if $\mu=\lambda$ we have a canonical isomorphism of functors

$$\mathcal{U}^{\hat\lambda}\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-)\cong\mathrm{id},$$

and if $\nu=\mu$ we have a canonical isomorphism

$$(-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,\mathcal{U}^{\hat\mu}\cong\mathrm{id};$$

– for four weights $\lambda,\mu,\nu,\eta\in\mathbb{X}$ we have an isomorphism

$$((-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-))\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-)\xrightarrow{\sim}(-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,((-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-))$$

of functors from

$$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\mu})\times\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\mu,\hat\nu})\times\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\nu,\hat\eta})$$

to $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\eta})$.

In particular, in the case where $\lambda=\mu=\nu$, this construction will equip $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\lambda})$ with the structure of a monoidal category.

For this we can assume that all the weights involved belong to the subset Λ chosen in § 3.6. It therefore suffices to construct a monoidal structure of the category $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$, with monoidal unit $\mathsf{C}^\wedge(\Bbbk\otimes\mathcal{U}\mathfrak{g})$; the bifunctor (3.15) will then be deduced by restriction to the factor $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\mu})\times\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\mu,\hat\nu})$ in the decomposition (3.13).

Recall that we have

$$\mathcal{U}^\wedge=(\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}})\otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)}\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge.$$

Given M,N in $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$, we set

$$M\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,N:=M\otimes_{\mathcal{U}\mathfrak{g}\otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)}\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge}N,$$

where the right action of $\mathcal{U}\mathfrak{g}\otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)}\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$ on M is induced by the action of the second copy of $\mathcal{U}\mathfrak{g}$, and the left action on N is induced by the action of the first copy of $\mathcal{U}\mathfrak{g}$. This tensor

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product admits compatible actions of $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ (induced by the action of the first copy of $\mathcal{U}\mathfrak{g}$ on M, and of the second copy of $\mathcal{U}\mathfrak{g}$ on N) and of $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$, hence of $\mathcal{U}^\wedge$. This module is, moreover, finitely generated, since it is finitely generated over $\mathscr{O}(\mathfrak{g}^{*(1)}) \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$, and it admits a canonical (diagonal) G-module structure. In this way we obtain a bifunctor

$$\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge) \times \mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge) \to \mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge),$$

which is easily seen to provide a monoidal structure with monoidal unit $\mathsf{C}^\wedge(\Bbbk \otimes \mathcal{U}\mathfrak{g})$. It is easily seen also that the functor (3.14) has a canonical monoidal structure; using (3.7), we deduce that the full subcategory $\mathsf{HC}^\wedge_{\mathrm{diag}}$ is a monoidal subcategory.

If $\lambda, \mu, \nu \in \mathbb{X}$ and if M belongs to $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$ and N belongs to $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\mu},\hat{\nu}})$, then seeing M and N as objects in $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$ via (3.13), the product $M \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} N$ belongs to the factor $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^{\hat{\lambda},\hat{\nu}})$, which provides the desired bifunctor (3.15). (In fact, the action of the left copy of Z_{HC} factors through an action of $Z_{\mathrm{HC}}^{\hat{\lambda}}$, and that of the right copy factors through an action of $Z_{\mathrm{HC}}^{\hat{\nu}}$, which justifies the claim in view of (3.11).) From the corresponding property for $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}^\wedge)$ we deduce that the subcategories $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$ and $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}}$ are stable (in the obvious sense) under this bifunctor. In this setting the functor $\mathsf{C}^{\hat{\lambda},\hat{\mu}}$ satisfies

$$\mathsf{C}^{\hat{\lambda},\hat{\mu}}(M \otimes_{\mathcal{U}\mathfrak{g}} N) \cong \bigoplus_{\nu \in \Lambda} \mathsf{C}^{\hat{\lambda},\hat{\nu}}(M) \,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\, \mathsf{C}^{\hat{\nu},\hat{\mu}}(N)$$

for any M, N in $\mathsf{Mod}_{\mathrm{fg}}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$.

3.8 Restriction to the Kostant section

Recall the constructions of § 2.3 applied to the group $\mathbf{G} = G^{(1)}$. In particular, we have a Kostant section $\mathbf{S}^* \subset \mathfrak{g}^* = \mathfrak{g}^{*(1)}$, and group schemes $\mathbb{J}^*_{\mathrm{reg}}$ over $\mathfrak{g}^{*(1)}_{\mathrm{reg}}$ and $\mathbb{J}^*_{\mathbf{S}}$ over $\mathbf{S}^*$.

We also set

$$\mathbb{I}^*_{\mathbf{S}} := (G \times \mathbf{S}^*) \times_{G^{(1)} \times \mathbf{S}^*} \mathbb{J}^*_{\mathbf{S}},$$

where the map $G \times \mathbf{S}^* \to G^{(1)} \times \mathbf{S}^*$ is the product of the Frobenius morphism of G and the identity of $\mathbf{S}^*$. Since G is smooth its Frobenius morphism is flat (see, for example, [BK05, § 1.1]), and therefore $\mathbb{I}^*_{\mathbf{S}}$ is a flat affine group scheme over $\mathbf{S}^*$. By construction $\mathbb{I}^*_{\mathbf{S}}$ contains $G_1 \times \mathbf{S}^*$ as a normal subgroup, and the quotient identifies with $\mathbb{J}^*_{\mathbf{S}}$. Note also that the morphism (2.4) induces (after restriction to $\mathbf{S}$ and composition with the projection $\mathbb{I}^*_{\mathbf{S}} \to \mathbb{J}^*_{\mathbf{S}}$) a group-scheme morphism

$$\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}^*_{\mathbf{S}} \to (\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbf{S}^*) \times T^{(1)}. \tag{3.16}$$

Finally, we set

$$\mathcal{U}_{\mathbf{S}}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathscr{O}(\mathbf{S}^*),$$

where $\mathscr{O}(\mathbf{S}^*)$ is seen as a Z_{Fr}-algebra via the identification (3.2). If we set

$$\mathfrak{C}_{\mathbf{S}} := \mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet),$$

then the projection $\mathfrak{C}_{\mathbf{S}} \to \mathfrak{t}^*/(W, \bullet)$ is an isomorphism, and $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is an $\mathscr{O}(\mathfrak{C}_{\mathbf{S}})$-algebra. Recall that the algebra $\mathcal{U}\mathfrak{g}$ can be seen as a G-equivariant $\mathscr{O}(\mathfrak{C})$-algebra (see § 3.2). Using the general construction recalled in [MR18, § 2.2], from this we deduce on $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ a natural module structure for the flat affine group scheme $\mathfrak{C}_{\mathbf{S}} \times_{\mathbf{S}^*} \mathbb{I}^*_{\mathbf{S}}$ over $\mathfrak{C}_{\mathbf{S}}$, such that the multiplication morphism is equivariant.

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3.9 (Completed) Harish-Chandra bimodules for $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$

We now want to define, given $\lambda, \mu \in \mathbb{X}$, categories analogous to $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$ and $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$ but for the algebra $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$ in place of $\mathcal{U}_{\mathfrak{g}}$. We start with the non-completed version.

Let us consider the category $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}})$ of finitely generated $\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}$-modules endowed with a compatible $\mathbb{I}^*_{\mathbf{S}}$-module structure. Since $\mathbb{I}^*_{\mathbf{S}}$ is flat over $\mathbf{S}^*$, this category is abelian. Here $\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} (\mathcal{U}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$ is an algebra over

$$\mathcal{Z}_{\mathbf{S}} := \mathscr{O}(\mathfrak{t}^*/(W,\bullet)) \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^*/(W,\bullet)),$$

which identifies with $\mathscr{O}(\mathfrak{D})$ via the composition of natural algebra morphisms

$$\mathscr{O}(\mathfrak{D}) \to \mathscr{O}(\mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{D}) = \mathcal{Z} \to \mathcal{Z}_{\mathbf{S}}.$$

As in § 3.4, the tensor product $\otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}$ defines a monoidal structure on this category, and, using the construction of [MR18, § 2.2] considered above, the functor $\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} (-)$ defines a monoidal functor

$$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}_{\mathfrak{g}} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}_{\mathfrak{g}}{}^{\mathrm{op}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}). \tag{3.17}$$

An object M in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}})$ will be called a *Harish-Chandra* $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$*-bimodule* if the restriction of the action of $\mathbb{I}^*_{\mathbf{S}}$ to $G_1 \times \mathbf{S}^*$, seen as an action of the algebra $\mathrm{Dist}(G_1) = \mathcal{U}_0\mathfrak{g}$, coincides with the action determined by the rule $x \cdot m = xm - mx$ for $x \in \mathfrak{g}$ and $m \in M$. We will denote by $\mathsf{HC}_{\mathbf{S}}$ the full subcategory of $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}})$ consisting of such objects; then the functor (3.17) restricts to a functor

$$\mathsf{HC} \to \mathsf{HC}_{\mathbf{S}}.$$

As in § 3.4, for any M in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}})$ the action of $\mathbb{I}^*_{\mathbf{S}}$ on M extends to an action of the semi-direct product

$$\mathbb{I}^*_{\mathbf{S}} \ltimes G_1 = \mathbb{I}^*_{\mathbf{S}} \ltimes_{\mathbf{S}} (G_1 \times \mathbf{S});$$

the Harish-Chandra $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$-bimodules are those objects on which this action factors through the multiplication morphism $\mathbb{I}^*_{\mathbf{S}} \ltimes G_1 \to \mathbb{I}^*_{\mathbf{S}}$.

Now we add completions to the picture. Given $\lambda, \mu \in \mathbb{X}$, we will denote by $\mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ the completion of $\mathcal{Z}_{\mathbf{S}}$ with respect to the maximal ideal $\mathcal{I}^{\lambda,\mu}_{\mathbf{S}} := \mathcal{I}^{\lambda,\mu} \cdot \mathcal{Z}_{\mathbf{S}}$, so that we have a canonical isomorphism $\mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} \mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$. We also set

$$\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} := \mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} \otimes_{\mathcal{Z}_{\mathbf{S}}} (\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}).$$

In this setting $\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}$ is finitely generated as a $\mathcal{Z}_{\mathbf{S}}$-module, so that $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ identifies with the completion of $\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}$ with respect to the ideal $\mathcal{I}^{\lambda,\mu} \cdot (\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}})$. If we denote by $\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ the pullback of $\mathbb{I}^*_{\mathbf{S}}$ under the natural morphism $\mathrm{Spec}(\mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}) \to \mathbf{S}^*$, then $\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ is a flat group scheme over $\mathrm{Spec}(\mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}})$, and the $\mathbb{I}^*_{\mathbf{S}}$-module structure on $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$ induces a natural $\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$-module structure on $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$.

The algebra $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$ is left and right Noetherian, and we will denote by $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}})$ the abelian category of $\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$-equivariant finitely generated $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$-modules. Note that we have

$$\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} = \mathcal{U}^{\hat{\lambda},\hat{\mu}} \otimes_{Z_{\mathrm{Fr}}} \mathscr{O}(\mathbf{S}^*);$$

in particular, the functor $\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} (-)$ defines a natural functor

$$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}). \tag{3.18}$$

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(Standard arguments show that this functor is exact, but this will not play any role below.) If $\lambda, \mu \in \mathbb{X}$ belong to the closure of the fundamental alcove, we will also set

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} := \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathsf{P}^{\lambda,\mu}.$$

One defines the notion of *completed Harish-Chandra* $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$*-bimodules* by imposing the same condition as for $\mathsf{HC}_{\mathbf{S}}$. The full subcategory of $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ consisting of such objects will be denoted by $\mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$; it is clear that the functor (3.18) restricts to a functor

$$\mathsf{HC}^{\hat{\lambda},\hat{\mu}} \to \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}.$$

Using considerations similar to those of § 3.7, one constructs, again for $\lambda, \mu, \nu \in \mathbb{X}$, a canonical bifunctor

$$(-) \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \times \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}) \tag{3.19}$$

which factors through a bifunctor

$$\mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \times \mathsf{HC}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}} \to \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}},$$

this construction being unital, associative and compatible in the natural way with the bifunctors $(-)\widehat{\otimes}_{\mathcal{U}_{\mathfrak{g}}}(-)$ via the functors (3.18). More explicitly, one remarks that if $\mathcal{Z}_{\mathbf{S}}^{\wedge}$ is the completion of $\mathcal{Z}_{\mathbf{S}}$ with respect to the ideal $\mathcal{I} \cdot \mathcal{Z}_{\mathbf{S}}$, then if we set

$$\mathcal{U}_{\mathbf{S}}^{\wedge} := \mathcal{Z}_{\mathbf{S}}^{\wedge} \otimes_{\mathcal{Z}_{\mathbf{S}}} (\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}) = \mathscr{O}(\mathbf{S}^{*(1)}) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathcal{U}^{\wedge},$$

as in (3.12) we have a canonical algebra isomorphism

$$\mathcal{U}_{\mathbf{S}}^{\wedge} \xrightarrow{\sim} \prod_{\lambda,\mu\in\Lambda} \mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}.$$

If we denote by $\mathbb{I}_{\mathbf{S}}^{\wedge}$ the pullback of $\mathbb{I}_{\mathbf{S}}^*$ to $\mathrm{Spec}(\mathcal{Z}_{\mathbf{S}}^{\wedge})$, then we can consider the abelian category $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge})$ of finitely generated $\mathbb{I}_{\mathbf{S}}^{\wedge}$-equivariant $\mathcal{U}_{\mathbf{S}}^{\wedge}$-modules, and the bifunctor

$$(-) \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \times \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \tag{3.20}$$

defined by

$$M \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} N = M \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} N$$

defines a monoidal structure on this category. Note that any finitely generated $\mathcal{U}_{\mathbf{S}}^{\wedge}$-module M is also finitely generated as a $\mathcal{Z}_{\mathbf{S}}^{\wedge}$-module, so that the natural morphism $M \to \varprojlim_n M/\mathcal{I}^n \cdot M$ is an isomorphism (see [Sta20, Tag 00MA]); the monoidal product considered above therefore satisfies

$$M \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} N \cong \varprojlim_{n\geq 1} (M/\mathcal{I}^n \cdot M) \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} (N/\mathcal{I}^n \cdot N)$$

as $\mathcal{U}_{\mathbf{S}}^{\wedge}$-modules, for any M, N in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge})$. The functor $\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} (-)$ also induces a functor

$$\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\wedge}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \tag{3.21}$$

which admits a canonical monoidal structure. The composition of this functor with the functor $\mathsf{C}^{\wedge}$ of (3.14) will be denoted by $\mathsf{C}_{\mathbf{S}}^{\wedge}$.

As in (3.13), we have

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \cong \bigoplus_{\lambda,\mu\in\Lambda} \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}), \tag{3.22}$$

and the bifunctor (3.19) is then obtained by restriction of (3.20) to the appropriate summands. In the case where $\lambda = \mu = \nu$, this bifunctor equips $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$ with a monoidal category structure,

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with unit object

$$\mathcal{U}_{\mathbf{S}}^{\hat{\lambda}} := \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathcal{U}^{\hat{\lambda}}.$$

It is clear that the functor (3.18) is compatible with the bifunctors $(-)\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}(-)$ and $(-)\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$ in the obvious way.

LEMMA 3.6. *For any $\lambda, \mu \in \mathbb{X}$ which belong to the closure of the fundamental alcove and any $\nu \in \mathbb{X}$, the functor*

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$$

is both left and right adjoint to the functor

$$\mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}).$$

A similar property holds for the functors $(-)\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}$ and $(-)\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}$.

Proof. We prove the case of convolution on the left; convolution on the right can be treated similarly. We remark that for any $V \in \mathsf{Rep}(G)$, the functor

$$\mathsf{C}_{\mathbf{S}}^{\wedge}(V \otimes \mathcal{U}\mathfrak{g}) \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge})$$

is both left and right adjoint to the functor

$$\mathsf{C}_{\mathbf{S}}^{\wedge}(V^* \otimes \mathcal{U}\mathfrak{g}) \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} (-) : \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}).$$

(In fact, these functors can be realized more concretely as tensor product with V and V^* respectively.) On the other hand, the inclusion functor

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge})$$

(see (3.22)) is both left and right adjoint to the corresponding projection functor

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}),$$

and similarly for μ in place of λ. The desired claim follows, since the functors $\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$ and $\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$ are isomorphic to compositions of functors of this form. (More specifically, if $\nu \in \mathbb{X}$ is the only dominant W-translate of $\mu - \lambda$, the functor $\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$ involves the module $\mathsf{L}(\nu)$, and the functor $\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$ involves the module $\mathsf{L}(-w_0(\nu))$; here we fix an isomorphism $\mathsf{L}(\nu)^* \cong \mathsf{L}(-w_0(\nu))$.) □

3.10 Restriction to the Kostant section for diagonally induced bimodules

In this subsection we aim to prove the following claim.

PROPOSITION 3.7. *For any $\lambda, \mu \in \mathbb{X}$, the functor (3.18) is fully faithful on the subcategory $\mathsf{HC}_{\mathrm{diag}}^{\hat{\lambda},\hat{\mu}}$.*

The proof of this proposition will use some standard properties stated in the following lemma. Here, k is a commutative ring, A is a left Noetherian k-algebra, and H is an affine k-group scheme. For any commutative k-algebra k' we set $H_{k'} := \mathrm{Spec}(k') \times_{\mathrm{Spec}(k)} H$. If A admits an action of H by algebra automorphisms, we denote by $\mathsf{Mod}^H(A)$ the category of H-equivariant A-modules. (This category is abelian if H is flat over k.)

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Lemma 3.8.

(i) *If M, N are A-modules with M finitely generated, for any flat k-module V we have a canonical isomorphism*

$$\mathrm{Hom}_A(M, N \otimes_k V) \xrightarrow{\sim} \mathrm{Hom}_A(M, N) \otimes_k V$$

where $N \otimes_k V$ is regarded as an A-module for the action on the first factor.

(ii) *If H is flat over k, and if M, N are H-equivariant A-modules, with M finitely generated as an A-module, then the k-module $\mathrm{Hom}_A(M, N)$ admits a canonical H-module structure, and we have a canonical isomorphism*

$$\mathrm{Hom}_{\mathsf{Mod}^H(A)}(M, N) \xrightarrow{\sim} (\mathrm{Hom}_A(M, N))^H.$$

(iii) *If k' is a flat commutative k-algebra, for any H-module we have a canonical isomorphism*

$$(k' \otimes_k M)^{H_{k'}} \xrightarrow{\sim} k' \otimes_k M^H.$$

Proof. (i) We consider a presentation $A^{\oplus n} \to A^{\oplus m} \to M \to 0$; we then have exact sequences

$$0 \to \mathrm{Hom}_A(M, N \otimes_k V) \to \mathrm{Hom}_A(A^{\oplus m}, N \otimes_k V) \to \mathrm{Hom}_A(A^{\oplus n}, N \otimes_k V)$$

and

$$0 \to \mathrm{Hom}_A(M, N) \otimes_k V \to \mathrm{Hom}_A(A^{\oplus m}, N) \otimes_k V \to \mathrm{Hom}_A(A^{\oplus n}, N) \otimes_k V,$$

where we use the flatness of V. It is clear that the second and third terms in these exact sequences identify, and we deduce an identification of the first terms.

(ii) We consider the morphism

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N \otimes_k \mathscr{O}(H))$$

which sends a morphism $\varphi : M \to N$ to the composition

$$M \xrightarrow{(\mathrm{id} \otimes S) \otimes \Delta_M} M \otimes_k \mathscr{O}(H) \xrightarrow{\varphi \otimes \mathrm{id}} N \otimes_k \mathscr{O}(H) \to N \otimes_k \mathscr{O}(H)$$

where the third morphism sends $n \otimes g$ to $n_{(1)} \otimes f_{(2)} g$ in Sweedler's notation. By (i) we have $\mathrm{Hom}_A(M, N \otimes_k \mathscr{O}(H)) \cong \mathrm{Hom}_A(M, N) \otimes_k \mathscr{O}(H)$, so that this morphism can be seen as a morphism $\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(M, N) \otimes_k \mathscr{O}(H)$, which can be checked to provide an $\mathscr{O}(H)$-comodule structure (i.e. an H-module structure) on $\mathrm{Hom}_A(M, N)$. The isomorphism

$$\mathrm{Hom}_{\mathsf{Mod}^H(A)}(M, N) \xrightarrow{\sim} (\mathrm{Hom}_A(M, N))^H$$

is then clear from definitions.

(iii) See [Jan03, § I.2.10, equation (3)]. □

With these tools we can give the proof of the proposition.

Proof of Proposition 3.7. To prove the proposition, it suffices to prove that the functor (3.21) is fully faithful on the subcategory $\mathsf{HC}^\wedge_{\mathrm{diag}}$, which will follow if we prove that it induces an isomorphism

$$\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)}(\mathsf{C}^\wedge(M), \mathsf{C}^\wedge(V \otimes \mathcal{U}\mathfrak{g})) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}^\wedge_{\mathbf{S}})}(\mathsf{C}^\wedge_{\mathbf{S}}(M), \mathsf{C}^\wedge_{\mathbf{S}}(V \otimes \mathcal{U}\mathfrak{g}))$$

for any M in $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$ and $V \in \mathsf{Rep}(G)$.

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For N in $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)}(\mathsf{C}^\wedge(M), \mathsf{C}^\wedge(N)) &\cong (\mathrm{Hom}_{\mathcal{U}^\wedge}(\mathsf{C}^\wedge(M), \mathsf{C}^\wedge(N)))^G \\ &\cong \left(\mathrm{Hom}_{\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}}}(M, \mathsf{C}^\wedge(N))\right)^G \\ &\cong \left(\mathrm{Hom}_{\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}}}(M, N) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^\wedge)\right)^G,\end{aligned}$$

where the first isomorphism uses Lemma 3.8(ii), and the third one uses Lemma 3.8(i). Using Lemma 3.8(iii), we deduce isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)}(\mathsf{C}^\wedge(M), \mathsf{C}^\wedge(N)) &\cong \left(\mathrm{Hom}_{\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}}}(M, N)\right)^G \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^\wedge) \\ &\cong \mathrm{Hom}_{\mathsf{Mod}^G(\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}})}(M, N) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^\wedge).\end{aligned}$$

Assuming now that $N = V \otimes \mathcal{U}\mathfrak{g}$, we claim that the functor $\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} (-)$ induces an isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}})}(M, V \otimes \mathcal{U}\mathfrak{g}) \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}\mathfrak{g}\otimes_{\mathscr{O}(\mathbf{S}^*)}\mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})}(\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} M, V \otimes \mathcal{U}_{\mathbf{S}}\mathfrak{g}). \qquad (3.23)\end{aligned}$$

In fact, the algebra $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$ is a G-equivariant finite $\mathscr{O}(\mathfrak{g}^{*(1)})$-algebra. Therefore, it identifies with the global sections of a G-equivariant coherent sheaf of $\mathscr{O}_{\mathfrak{g}^{*(1)}}$-algebras $\mathscr{U}$ on $\mathfrak{g}^{*(1)}$. Moreover, the restriction $\mathscr{U}_{\mathbf{S}}$ of $\mathscr{U}$ to $\mathbf{S}^*$ is an $\mathbb{I}^*_{\mathbf{S}}$-equivariant sheaf of $\mathscr{O}_{\mathbf{S}^*}$-algebras on $\mathbf{S}^*$, whose global sections are $\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} (\mathcal{U}_{\mathbf{S}}\mathfrak{g})^{\mathrm{op}}$. Consider the open embedding $j : \mathfrak{g}^{*(1)}_{\mathrm{reg}} \hookrightarrow \mathfrak{g}^{*(1)}$, and set $\mathscr{U}_{\mathrm{reg}} := j^*(\mathscr{U})$. Let us denote by $\mathsf{QCoh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}})$ the category of G-equivariant quasi-coherent sheaves on $\mathfrak{g}^{*(1)}_{\mathrm{reg}}$ equipped with a $\mathscr{U}_{\mathrm{reg}}$-module structure, compatible with the G-equivariant structure in the natural way, and by $\mathsf{Coh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}})$ the subcategory of coherent modules. Then we have a restriction functor

$$j^* : \mathsf{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \to \mathsf{QCoh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}})$$

which admits a right adjoint

$$j_* : \mathsf{QCoh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}}) \to \mathsf{Mod}^G(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$$

coinciding with the usual pushforward functor at the level of quasi-coherent sheaves on $\mathfrak{g}^{*(1)}_{\mathrm{reg}}$ and $\mathfrak{g}^{*(1)}$. Since the complement of $\mathfrak{g}^{*(1)}_{\mathrm{reg}}$ has codimension at least 2, the natural morphism $\mathscr{O}_{\mathfrak{g}^{*(1)}} \to j_*j^*\mathscr{O}_{\mathfrak{g}^{*(1)}}$ is an isomorphism. Since $\mathcal{U}\mathfrak{g}$ is free over Z_{Fr} it follows that the morphism $V \otimes \mathcal{U}\mathfrak{g} \to j_*j^*(V \otimes \mathcal{U}\mathfrak{g})$ is also an isomorphism, and then that the functor j^* induces an isomorphism

$$\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{Fr}}}\mathcal{U}\mathfrak{g}^{\mathrm{op}})}(M, V \otimes \mathcal{U}\mathfrak{g}) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{Coh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}})}(j^*M, j^*(V \otimes \mathcal{U}\mathfrak{g})).$$

It is a standard observation (see [Ric17, Proposition 3.3.11]) that restriction to $\mathbf{S}^*$ induces an equivalence of abelian categories

$$\mathsf{Coh}^{G^{(1)}}(\mathfrak{g}^{*(1)}_{\mathrm{reg}}) \xrightarrow{\sim} \mathsf{Rep}(\mathbb{J}^*_{\mathbf{S}}),$$

where the right-hand side denotes the category of representations of the affine group scheme $\mathbb{J}^*_{\mathbf{S}}$ (see § 3.8) on coherent $\mathscr{O}_{\mathbf{S}^*}$-modules. The same considerations provide an equivalence of categories

$$\mathsf{Coh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}) \xrightarrow{\sim} \mathsf{Rep}(\mathbb{I}^*_{\mathbf{S}}).$$

(Here we use the fact that the Frobenius morphism of G is flat and surjective, hence faithfully flat.) This equivalence is monoidal with respect to the natural tensor product on each

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side, and the image of the algebra $\mathscr{U}_{\mathrm{reg}}$ is $\mathscr{U}_{\mathbf{S}}$; therefore, it induces an equivalence of abelian categories

$$\mathsf{Coh}^G(\mathfrak{g}^{*(1)}_{\mathrm{reg}}, \mathscr{U}_{\mathrm{reg}}) \xrightarrow{\sim} \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}}).$$

From this we finally obtain that (3.23) is an isomorphism.

Combining (3.23) with the preceding isomorphisms, we obtain a canonical isomorphism

$$\begin{aligned}&\mathrm{Hom}_{\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)}(\mathsf{C}^\wedge(M), \mathsf{C}^\wedge(V \otimes \mathcal{U}\mathfrak{g}))\\ &\quad\cong \mathrm{Hom}_{\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})}(\mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} M, V \otimes \mathcal{U}_{\mathbf{S}}\mathfrak{g}) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^\wedge).\end{aligned}$$

Considerations similar to those of the beginning of the proof allow us to identify the right-hand side with

$$\mathrm{Hom}_{\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}^\wedge_{\mathbf{S}})}(\mathsf{C}^\wedge_{\mathbf{S}}(M), \mathsf{C}^\wedge_{\mathbf{S}}(V \otimes \mathcal{U}\mathfrak{g})),$$

which finishes the proof. □

4. Localization for Harish-Chandra bimodules

4.1 Azumaya algebras

We start by recalling the basic theory of Azumaya algebras.

Let R be a commutative ring. Recall that an R-module P is called *faithfully projective* if it is projective of finite type and if, moreover, the only R-module M such that $P \otimes_R M = 0$ is $M = 0$. By [KO74, Chap. I, Lemme 6.2] this condition is equivalent to requiring that P is projective of finite type and faithful (i.e. its annihilator in R is trivial). An R-module P is finitely generated and projective if and only if it is finitely presented and, moreover, the localization $P_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ for any $\mathfrak{p} \in \mathrm{Spec}(R)$; see [KO74, Chap. I, Lemme 5.2] or [Sta20, Tag 00NX]. In this setting, P is faithful if and only if the rank of $P_{\mathfrak{p}}$ is positive for any $\mathfrak{p}$; see [KO74, Chap. I, Lemme 6.1]. This notion is important in Morita theory since if P is a faithfully projective R-module, then we obtain quasi-inverse equivalences of categories

$$\mathsf{Mod}(R) \rightleftarrows \mathsf{Mod}(\mathrm{End}_R(P))$$

given by $M \mapsto P \otimes_R M$ and $N \mapsto \mathrm{Hom}_R(P, R) \otimes_{\mathrm{End}_R(P)} N$ where $\mathsf{Mod}(A)$ is the category of left A-modules for any ring A; see [KO74, Chap. I, Lemme 7.2]. In the case where R is Noetherian, the ring $\mathrm{End}_R(P)$ is left Noetherian (as a non-commutative ring), and these equivalences restrict to equivalences

$$\mathsf{Mod}_{\mathrm{fg}}(R) \rightleftarrows \mathsf{Mod}_{\mathrm{fg}}(\mathrm{End}_R(P)) \tag{4.1}$$

between subcategories of finitely generated modules. (Here, a left $\mathrm{End}_R(P)$-module is finitely generated if and only if it is finitely generated as an R-module.)

Let A be an R-algebra.[3] Recall (see [KO74, § III.5]) that A is called an *Azumaya R-algebra* if it satisfies one of the following equivalent conditions.

– A is faithfully projective as an R-module, and the morphism sending $a \otimes b$ to the map $x \mapsto axb$ is an isomorphism of R-algebras

$$A \otimes_R A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_R(A).$$

[3] Let us insist that by an R-algebra we mean a (not necessarily commutative) ring A endowed with a ring morphism from R to the *center* of A.

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– A is finite as an R-module, the ring morphism $R \to A$ is injective, and for any maximal ideal $\mathfrak{m} \subset R$ the finite-dimensional $R/\mathfrak{m}$-algebra $A/\mathfrak{m}A$ is a central simple algebra.

In particular, the first characterization and the facts recalled above show that in this case we have canonical equivalences of categories

$$\mathsf{Mod}(R) \rightleftarrows \mathsf{Mod}(A \otimes_R A^{\mathrm{op}}).$$

4.2 Azumaya property of $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$

The following property is standard (see [BG97, BG01]); we recall its proof for the reader's convenience.

PROPOSITION 4.1. *The $\mathscr{O}(\mathfrak{C}_{\mathbf{S}})$-algebra $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is Azumaya.*

Proof. We will use the second characterization of Azumaya algebras recalled in § 4.1. Since $\mathcal{U}\mathfrak{g}$ is finite over $Z(\mathcal{U}\mathfrak{g}) = \mathscr{O}(\mathfrak{C})$, $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is finite over $\mathscr{O}(\mathfrak{C}_{\mathbf{S}})$. To prove that the morphism $\mathscr{O}(\mathfrak{C}_{\mathbf{S}}) \to \mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is injective, we consider the composition

$$G \times \mathbf{S}^* \to G^{(1)} \times \mathbf{S}^* \to \mathfrak{g}^{*(1)}_{\mathrm{reg}} \to \mathfrak{g}^{*(1)}.$$

Here the first morphism is flat since G is smooth (see § 3.8), the second morphism is smooth by [Ric17, Lemma 3.3.1], and the third one is an open embedding; this composition is therefore flat. The algebras $Z(\mathcal{U}\mathfrak{g})$ and $\mathcal{U}\mathfrak{g}$ can be considered as G-equivariant coherent sheaves on $\mathfrak{g}^{*(1)}$, and the induced morphism $\mathscr{O}(\mathfrak{C}_{\mathbf{S}}) \to \mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is obtained from the embedding $Z(\mathcal{U}\mathfrak{g}) \to \mathcal{U}\mathfrak{g}$ by the pullback functor

$$\mathsf{Coh}^G(\mathfrak{g}^{*(1)}) \to \mathsf{Coh}^G(G \times \mathbf{S}^*)$$

associated with the flat morphism considered above, followed by the obvious equivalences

$$\mathsf{Coh}^G(G \times \mathbf{S}^*) \cong \mathsf{Coh}(\mathbf{S}^*) \cong \mathsf{Mod}_{\mathrm{fg}}(\mathscr{O}(\mathbf{S}^*));$$

it is therefore injective.

What is left to prove is that if $\mathfrak{m} \subset Z(\mathcal{U}\mathfrak{g})$ is a maximal ideal which belongs to $\mathfrak{C}_{\mathbf{S}} = \mathbf{S}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$, then $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$ is a central simple algebra. In fact, this property holds more generally if $\mathfrak{m}$ belongs to $\mathfrak{C}_{\mathrm{reg}} := \mathfrak{g}^{*(1)}_{\mathrm{reg}} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$. Indeed, let N be the maximal dimension of a simple $\mathcal{U}\mathfrak{g}$-module. By [BG97, Proposition 3.1], if $\mathfrak{m} \subset Z(\mathcal{U}\mathfrak{g})$ is a maximal ideal such that $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$ admits a simple module V of dimension N, then $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$ is a central simple algebra; more specifically, the algebra morphism $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g} \to \mathrm{End}_{\Bbbk}(V)$ is an isomorphism. Now by [PS99, Theorem 4.4] we have $N = p^{\#\mathfrak{R}^+}$. And by [PS99, Theorem 5.6], if $\mathfrak{m}$ belongs to $\mathfrak{C}_{\mathrm{reg}}$ then any simple $\mathcal{U}\mathfrak{g}/\mathfrak{m}\mathcal{U}\mathfrak{g}$-module has dimension divisible by $p^{\#\mathfrak{R}^+}$, hence equal to N. □

It follows in particular from Proposition 4.1 that $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$ is faithfully projective as an $\mathscr{O}(\mathfrak{C}_{\mathbf{S}})$-module.

Below we will use a slightly more concrete version of Proposition 4.1, as follows. First we need to recall the definition of baby Verma modules. Consider some element $\eta \in \mathfrak{g}^{*(1)}$, and some Borel subgroup $B' \subset G$ with unipotent radical U' such that η vanishes on $\mathrm{Lie}(U')^{(1)}$. (Such a Borel subgroup exists for any η; see [Jan98, Lemma 6.6].) Then η defines an element η' in $(\mathrm{Lie}(B')/\mathrm{Lie}(U'))^{*(1)}$. Let $\xi \in \mathfrak{t}^*$ be an element whose image under the map

$$\mathfrak{t}^* \cong (\mathrm{Lie}(B)/\mathrm{Lie}(U))^* \xrightarrow{\sim} (\mathrm{Lie}(B')/\mathrm{Lie}(U'))^* \to (\mathrm{Lie}(B')/\mathrm{Lie}(U'))^{*(1)}$$

is η', where the second map is induced by conjugation by an element $g \in G$ such that $gBg^{-1} = B'$ (it is well known that the isomorphism does not depend on the choice of g), and the second one

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is the Artin–Schreier map associated with the torus B'/U'. Then we can consider the associated *baby Verma module*

$$\mathsf{Z}_{\eta,B'}(\xi) := \mathcal{U}_\eta\mathfrak{g} \otimes_{\mathcal{U}_\eta \operatorname{Lie}(B')} \Bbbk_\xi,$$

where $\mathcal{U}_\eta \operatorname{Lie}(B')$ is the central reduction (with respect to the Frobenius center) of the enveloping algebra of $\operatorname{Lie}(B')$ at the image of η in $\operatorname{Lie}(B')^{*(1)}$, and $\Bbbk_\xi$ is its one-dimensional module defined by the image of ξ in $(\operatorname{Lie}(B')/\operatorname{Lie}(U'))^*$. This module has dimension $p^{\#\mathfrak{R}^+}$; furthermore, if we assume that $\eta \in \mathfrak{g}^{*(1)}_{\mathrm{reg}}$, then the considerations in the proof of Proposition 4.1 imply that this module is simple, and that the algebra morphism

$$\mathcal{U}^{\xi'}_\eta\mathfrak{g} \to \operatorname{End}_\Bbbk(\mathsf{Z}_{\eta,B'}(\xi)) \tag{4.2}$$

is an isomorphism, where we denote by ξ' the image of ξ in $\mathfrak{t}^*/(W,\bullet)$.

4.3 Splitting bundles for the algebras $\mathcal{U}_{\mathbf{S}}^{\hat{\boldsymbol{\lambda}},\hat{\boldsymbol{\mu}}}$

Our goal in this section is to construct some tools that will allow us to study the categories $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat\lambda,\hat\mu})$ and $\mathsf{HC}^{\hat\lambda,\hat\mu}$ via geometric methods. First we introduce the categories of modules that will be involved in these constructions. Our model will be the category $\mathsf{Coh}^G(\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C})$ of G-equivariant coherent sheaves on $\mathfrak{C} \times_{\mathfrak{g}^{*(1)}} \mathfrak{C}$, or in other words of G-equivariant finitely generated $\mathcal{Z}$-modules, which is a monoidal category for the operation sending a pair (M,N) to

$$M \otimes_{Z(\mathcal{U}\mathfrak{g})} N,$$

where in the tensor product $Z(\mathcal{U}\mathfrak{g})$ acts on M via the right action and on N via the left action. The $\mathcal{Z}$-action on $M \otimes_{Z(\mathcal{U}\mathfrak{g})} N$ comes from the left action of $Z(\mathcal{U}\mathfrak{g})$ on M and the right action on N. In practice, however, we will have to restrict to $\mathbf{S}^*$, and add generalized characters to this picture.

First, recall the group scheme $\mathbb{I}^\wedge_{\mathbf{S}}$ over $\operatorname{Spec}(\mathcal{Z}^\wedge_{\mathbf{S}})$ introduced in § 3.9. We consider the abelian category $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}})$ of representations of the flat affine group scheme $\mathbb{I}^\wedge_{\mathbf{S}}$ on finitely generated $\mathcal{Z}^\wedge_{\mathbf{S}}$-modules. Here a $\mathcal{Z}^\wedge_{\mathbf{S}}$-module is a $Z_{\mathrm{HC}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$-bimodule on which the left and right actions of $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$ coincide; the category of such modules therefore admits a canonical tensor product, which preserves finitely generated modules, and induces a monoidal product

$$(-)\,\hat\star_{\mathbf{S}}\,(-) : \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}}) \times \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}})$$

with unit object $Z_{\mathrm{HC}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$, with diagonal $\mathcal{Z}^\wedge_{\mathbf{S}}$-module structure and trivial action of $\mathbb{I}^\wedge_{\mathbf{S}}$. If we set

$$\mathbb{J}^\wedge_{\mathbf{S}} := \operatorname{Spec}(\mathcal{Z}^\wedge_{\mathbf{S}}) \times_{\mathbf{S}} \mathbb{J}^*_{\mathbf{S}},$$

then we can also consider the abelian category $\mathsf{Mod}^{\mathbb{J}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}})$ of representations of $\mathbb{J}^\wedge_{\mathbf{S}}$ on finitely generated $\mathcal{Z}^\wedge_{\mathbf{S}}$-modules; the quotient morphism $\mathbb{I}^\wedge_{\mathbf{S}} \to \mathbb{J}^\wedge_{\mathbf{S}}$ induces an exact and fully faithful functor

$$\mathsf{Mod}^{\mathbb{J}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^\wedge_{\mathbf{S}})$$

whose image is stabilized by the convolution product $\hat\star_{\mathbf{S}}$.

Next, for $\lambda,\mu \in \mathbb{X}$ we have the affine group scheme $\mathbb{I}^{\hat\lambda,\hat\mu}_{\mathbf{S}}$ over $\operatorname{Spec}(\mathcal{Z}^{\hat\lambda,\hat\mu}_{\mathbf{S}})$ also introduced in § 3.9, and we can consider the abelian category $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}^{\hat\lambda,\hat\mu}_{\mathbf{S}})$ of representations of this affine

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group scheme on finitely generated $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-modules. As in (3.22), we have a decomposition

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\wedge}) \cong \bigoplus_{\lambda,\mu\in\Lambda} \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}).$$

Given M in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ and N in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}})$, seen as objects in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\wedge})$, the object $M \,\hat{\star}_{\mathbf{S}}\, N$ belongs to $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$; in other words, the bifunctor $\hat{\star}_{\mathbf{S}}$ restricts to a bifunctor

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \times \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}) \tag{4.3}$$

for any $\lambda,\mu,\nu \in \mathbb{X}$. In particular, each category $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$ admits a monoidal structure; the unit object $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$ for this structure is $Z_{\mathrm{HC}}^{\hat{\lambda}}$, endowed with the natural structure of a $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$-module (induced by the product morphism $Z_{\mathrm{HC}} \otimes_{Z_{\mathrm{HC}}\cap Z_{\mathrm{Fr}}} Z_{\mathrm{HC}} \to Z_{\mathrm{HC}}$) and the trivial structure as a representation of $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$.

We can also play the same game starting with $\mathbb{J}_{\mathbf{S}}^{*}$ in place of $\mathbb{I}_{\mathbf{S}}^{*}$; we obtain in this way affine group schemes $\mathbb{J}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$, categories $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ and exact fully faithful functors

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$$

whose essential images are stabilized by the bifunctor (4.3) in the obvious sense.

Remark 4.2. In view of (3.11), for any λ,μ we have an algebra isomorphism

$$\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \xrightarrow{\sim} Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^{\wedge}} Z_{\mathrm{HC}}^{\hat{\mu}}.$$

From this point of view, the bifunctor (4.3) is induced by the tensor product $(-) \otimes_{Z_{\mathrm{HC}}^{\hat{\mu}}} (-)$.

Recall that a weight $\lambda \in \mathbb{X}$ is said to belong to the *lower closure of the fundamental alcove* if it satisfies

$$0 \leq \langle \lambda+\rho, \alpha^{\vee}\rangle < p$$

for any positive root α. Recall also the completed bimodules introduced in § 3.5. In particular, given $\lambda,\mu \in \mathbb{X}$ which belong to the lower closure of the fundamental alcove, we have the objects

$$\mathsf{P}^{\lambda,-\rho} = \mathsf{C}^{\lambda,-\rho}(\mathsf{L}(\lambda+\rho)\otimes\mathcal{U}\mathfrak{g}) \in \mathsf{HC}_{\mathrm{diag}}^{\hat{\lambda},\widehat{-\rho}},$$
$$\mathsf{P}^{-\rho,\mu} = \mathsf{C}^{-\rho,\mu}(\mathsf{L}(-w_0\mu+\rho)\otimes\mathcal{U}\mathfrak{g}) \in \mathsf{HC}_{\mathrm{diag}}^{\widehat{-\rho},\hat{\mu}}.$$

We set

$$\mathsf{M}^{\lambda,\mu} := \mathsf{P}^{\lambda,-\rho} \,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\, \mathsf{P}^{-\rho,\mu} \in \mathsf{HC}_{\mathrm{diag}}^{\hat{\lambda},\hat{\mu}}.$$

We also set $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} := \mathscr{O}(\mathbf{S}^{*}) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathsf{M}^{\lambda,\mu}$, so that

$$\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} = \mathsf{P}_{\mathbf{S}}^{\lambda,-\rho} \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}\, \mathsf{P}_{\mathbf{S}}^{-\rho,\mu}$$

where we use the notation of § 3.9.

The main technical result of this section is the following theorem. Its proof will be given in § 4.5, after some preliminaries treated in § 4.4.

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Theorem 4.3. *For any $\lambda, \mu \in \mathbb{X}$ in the lower closure of the fundamental alcove, the $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-module $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ is faithfully projective, and the natural algebra morphism*

$$\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \to \mathrm{End}_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}}(\mathsf{M}_{\mathbf{S}}^{\lambda,\mu})$$

is an isomorphism.

4.4 Study of some stalks

Recall that $\mathrm{Spec}(\mathcal{Z}_{\mathbf{S}})$ identifies naturally with

$$\mathfrak{D} = \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet);$$

see § 3.9. We also set

$$\tilde{\mathfrak{D}} := \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*.$$

Since the Artin–Schreier map $\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$ is a Galois covering with Galois group $\mathfrak{t}^*_{\mathbb{F}_p}$, we have a canonical isomorphism

$$\mathfrak{t}^*_{\mathbb{F}_p} \times \mathfrak{t}^* \xrightarrow{\sim} \tilde{\mathfrak{D}}$$

defined by $(\eta, \xi) \mapsto (\eta + \xi, \xi)$. For $\lambda \in \mathbb{X}$ we will denote by $\tilde{\mathfrak{D}}(\lambda)$ the image of $\{\overline{\lambda + \rho}\} \times \mathfrak{t}^*$ in $\tilde{\mathfrak{D}}$; if $\tilde{\Lambda} \subset \mathbb{X}$ is a subset of representatives for the quotient $\mathfrak{t}^*_{\mathbb{F}_p} = \mathbb{X}/p\mathbb{X}$, we then have

$$\tilde{\mathfrak{D}} = \bigsqcup_{\lambda \in \tilde{\Lambda}} \tilde{\mathfrak{D}}(\lambda).$$

Recall that if A is a finitely generated $\Bbbk$-algebra, then by [Sta20, Tag 02J6] $\mathrm{Spec}(A)$ is a Jacobson space in the sense of [Sta20, Tag 005T]; in other words, closed points are dense in any closed subset of $\mathrm{Spec}(A)$. Here we have a natural finite morphism

$$\tilde{\mathfrak{D}} \to \mathfrak{D}, \tag{4.4}$$

and it is easily seen that the image of this morphism contains all the closed points of $\mathfrak{D}$; this morphism is therefore surjective. For any $\lambda \in \mathbb{X}$ we denote by $\mathfrak{D}(\lambda)$ the scheme-theoretic image of $\tilde{\mathfrak{D}}(\lambda)$ in $\mathfrak{D}$; in other words, $\mathscr{O}(\mathfrak{D}(\lambda))$ is the image of the composition

$$\mathscr{O}(\mathfrak{D}) \to \mathscr{O}(\tilde{\mathfrak{D}}) \twoheadrightarrow \mathscr{O}(\tilde{\mathfrak{D}}(\lambda));$$

see [Sta20, Tag 056A]. The morphism (4.4) then factors through a finite morphism $\tilde{\mathfrak{D}}(\lambda) \to \mathfrak{D}(\lambda)$, which is surjective since its image is closed and dense; see [Sta20, Tag 01R8]. Since $\tilde{\mathfrak{D}}(\lambda)$ is integral, so is $\mathfrak{D}(\lambda)$. Moreover, we can check that

$$\mathfrak{D}(\lambda) = \mathfrak{D}(\mu) \quad \text{if and only if } \tilde{\lambda} = \tilde{\mu},$$

where the operation $\lambda \mapsto \tilde{\lambda}$ is as in § 3.1. If $\Lambda \subset \mathbb{X}$ is (as in § 3.6) a subset of representatives for $\mathfrak{t}^*_{\mathbb{F}_p}/(W,\bullet)$, we therefore have

$$\mathfrak{D} = \bigcup_{\lambda \in \Lambda} \mathfrak{D}(\lambda),$$

and this constitutes the decomposition of $\mathfrak{D}$ into its irreducible components.

Let us consider the open subset

$$\mathfrak{t}^*_\circ := \{\xi \in \mathfrak{t}^* \mid \forall w \in W,\ w \bullet \xi - \xi \notin \mathfrak{t}^*_{\mathbb{F}_p} \smallsetminus \{0\}\} \subset \mathfrak{t}^*.$$

Then $\mathfrak{t}^*_\circ$ is stable under the $(W,\bullet)$-action, and is in fact the pullback of an open subset of $\mathfrak{t}^*/(W,\bullet)$, which therefore identifies with the quotient $\mathfrak{t}^*_\circ/(W,\bullet)$.

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Recall the Grothendieck resolution $\tilde{\mathfrak{g}}$[4] (for the reductive group $\mathbf{G} = G^{(1)}$) and the morphism $\vartheta : \tilde{\mathfrak{g}} \to \mathfrak{t}^{*(1)}$; see § 2.3 and § 3.8. If we denote by $\tilde{\mathbf{S}}^*$ the (scheme-theoretic) preimage of $\mathbf{S}^*$ in $\tilde{\mathfrak{g}}$, then by [Ric17, Proposition 3.5.5] the morphism ϑ restricts to an isomorphism $\tilde{\mathbf{S}}^* \xrightarrow{\sim} \mathfrak{t}^{*(1)}$. In concrete terms, this means that, given $\zeta \in \mathfrak{t}^{*(1)}/W$ identified with an element in $\mathbf{S}^*$, the datum of a preimage of ζ in $\mathfrak{t}^{*(1)}$ is equivalent to the datum of a Borel subgroup $B' \subset G$ such that $\zeta_{|\mathrm{Lie}(U')^{(1)}} = 0$, where U' is the unipotent radical of B'.

PROPOSITION 4.4. *Let $\lambda \in \mathbb{X}$ be a weight which belongs to the lower closure of the fundamental alcove. Consider some element $\xi \in \mathfrak{t}^*_\circ$, and denote by $(\zeta_1, \zeta_2) \in \mathfrak{D}(\lambda)$ the image of $(\xi + \overline{\lambda+\rho}, \xi) \in \tilde{\mathfrak{D}}(\lambda)$ in $\mathfrak{D}$ and by $\eta \in \mathbf{S}^*$ the element corresponding to the images of ζ_1 and ζ_2 in $\mathfrak{t}^{*(1)}/W$. As explained above, the image of ξ in $\mathfrak{t}^{*(1)}$ determines a Borel subgroup $B' \subset G$ with unipotent radical U' such that $\eta_{|\mathrm{Lie}(U')^{(1)}} = 0$.*

If we denote by $i : \mathrm{Spec}(\Bbbk) \to \mathfrak{D}$ the morphism defined by (ζ_1, ζ_2), there exists an isomorphism of $\mathcal{U}_\eta^{\zeta_1}\mathfrak{g} \otimes (\mathcal{U}_\eta^{\zeta_2}\mathfrak{g})^{\mathrm{op}}$-modules

$$i^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}}\mathfrak{g}) \cong \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho}) \otimes \mathsf{Z}_{\eta,B'}(\xi)^*.$$

Proof. By definition we have

$$i^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}}\mathfrak{g}) \cong \Bbbk_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} \big(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_\eta^{\zeta_2}\mathfrak{g}\big).$$

By construction, the image of ξ in $\mathfrak{t}^{*(1)}$ corresponds to the element in $(\mathrm{Lie}(B')/\mathrm{Lie}(U'))^{*(1)}$ defined by η; by (4.2), we therefore have a canonical isomorphism

$$\mathcal{U}_\eta^{\zeta_2}\mathfrak{g} \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(\mathsf{Z}_{\eta,B'}(\xi)) \cong \mathsf{Z}_{\eta,B'}(\xi) \otimes \mathsf{Z}_{\eta,B'}(\xi)^*,$$

under which the action of $\mathcal{U}\mathfrak{g}$ induced by left multiplication on the left-hand side corresponds to the natural action on $\mathsf{Z}_{\eta,B'}(\xi)$. We deduce an isomorphism

$$i^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}}\mathfrak{g}) \cong \Bbbk_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} (\mathsf{L}(\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi)) \otimes \mathsf{Z}_{\eta,B'}(\xi)^*,$$

which shows that to conclude the proof it suffices to construct an isomorphism of $\mathcal{U}_\eta^{\zeta_1}$-modules

$$\Bbbk_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} (\mathsf{L}(\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi)) \cong \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho}). \tag{4.5}$$

As above, we have a canonical isomorphism

$$\mathcal{U}_\eta^{\zeta_1}\mathfrak{g} \xrightarrow{\sim} \mathrm{End}_{\Bbbk}(\mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho}));$$

therefore, any $\mathcal{U}_\eta^{\zeta_1}\mathfrak{g}$-module is isomorphic to a direct sum of copies of $\mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho})$. To analyze how many copies we have for the specific module in the left-hand side of (4.5), we observe that

$$\begin{aligned}
&\mathrm{Hom}_{\mathcal{U}_\eta^{\zeta_1}\mathfrak{g}}(\Bbbk_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} (\mathsf{L}(\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi)), \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho})) \\
&\quad = \mathrm{Hom}_{\mathcal{U}_\eta\mathfrak{g}}(\mathsf{L}(\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi), \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho})) \\
&\quad \cong \mathrm{Hom}_{\mathcal{U}_\eta\mathfrak{g}}(\mathsf{Z}_{\eta,B'}(\xi), \mathsf{L}(-w_0\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho})).
\end{aligned}$$

We now consider the $\mathcal{U}_\eta\mathfrak{g}$-module $\mathsf{L}(-w_0\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho})$, and more specifically the direct summand on which Z_{HC} acts with a generalized character corresponding to ζ_2. We have a canonical isomorphism of $\mathcal{U}_\eta\mathfrak{g}$-modules

$$\mathsf{L}(-w_0\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho}) \cong \mathcal{U}_\eta\mathfrak{g} \otimes_{\mathcal{U}_\eta \mathrm{Lie}(B')} (\mathsf{L}(-w_0\lambda+\rho)_{|B'} \otimes \Bbbk_{\xi+\overline{\lambda+\rho}}).$$

[4] Of course, $\tilde{\mathfrak{g}}$ is the Frobenius twist of the Grothendieck resolution attached to the group G.

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The B'-module $\mathsf{L}(-w_0\lambda+\rho)_{|B'}$ admits a filtration

$$0 \subset M_1 \subset \cdots \subset M_n = \mathsf{L}(-w_0\lambda+\rho)_{|B'}$$

where each M_i/M_{i-1} is one-dimensional; moreover, these modules are associated with the characters of $B'/U' \cong B/U \cong T$ corresponding to the T-weights of $\mathsf{L}(-w_0\lambda+\rho)$, counted with multiplicities. This filtration induces a filtration of $\mathsf{L}(-w_0\lambda+\rho)_{|B'} \otimes \Bbbk_{\xi+\overline{\lambda+\rho}}$, and then of $\mathsf{L}(-w_0\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho})$, whose subquotients are of the form $\mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho+\mu})$, where μ runs over the T-weights of $\mathsf{L}(-w_0\lambda+\rho)$, counted with multiplicities.

We claim that there exists exactly one subquotient in this filtration on which Z_{HC} acts via the character ζ_2, corresponding to the multiplicity-1 weight $-\lambda-\rho$ of $\mathsf{L}(-w_0\lambda+\rho)$. Indeed, assume that Z_{HC} acts with character ζ_2 on $\mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho+\mu})$. Then there exists $w \in W$ such that $\xi+\overline{\lambda+\rho+\mu} = w \bullet \xi$. Since ξ belongs to $\mathfrak{t}^*_\circ$, this condition implies that $\xi+\overline{\lambda+\rho+\mu} = \xi$, hence that $\lambda+\rho+\mu \in p\mathbb{X}$. On the other hand, μ is a weight of $\mathsf{L}(-w_0\lambda+\rho)$, hence it belongs to $-w_0\lambda+\rho+\mathbb{Z}\mathfrak{R} = -\lambda-\rho+\mathbb{Z}\mathfrak{R}$. In view of (3.1) these conditions imply that $\lambda+\rho+\mu \in p\mathbb{Z}\mathfrak{R}$, that is, that $\lambda+\mu \in -\rho+p\mathbb{Z}\mathfrak{R} = W_{\mathrm{aff}} \bullet (-\rho)$. By [Jan03, Lemma II.7.7] (applied to the pair of elements $(\lambda,-\rho)$), there must then exist $w \in W_{\mathrm{aff}}$ such that $w \bullet \lambda = \lambda$ and $\lambda+\mu = w \bullet (-\rho)$. Here, since λ belongs to the lower closure of the fundamental alcove, the first condition implies that $w \in W$ (see § 3.5); it follows that $w \bullet (-\rho) = -\rho$, hence that $\lambda+\mu = -\rho$, which finishes the proof of our claim.

This claim implies that the direct summand of $\mathsf{L}(-w_0\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho})$ corresponding to the generalized character of Z_{HC} given by ζ_2 is isomorphic to $\mathsf{Z}_{\eta,B'}(\xi)$; it follows that

$$\mathrm{Hom}_{\mathcal{U}_\eta^{\zeta_1}\mathfrak{g}}\big(\Bbbk_{\zeta_1} \otimes_{Z_{\mathrm{HC}}} (\mathsf{L}(\lambda+\rho) \otimes \mathsf{Z}_{\eta,B'}(\xi)), \mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho})\big)$$

is one-dimensional, which finally proves (4.5). □

The statement of Proposition 4.4 is not symmetric, in that the conditions we impose imply that ζ_2 necessarily belongs to $\mathfrak{t}^*_\circ/(W,\bullet)$, whereas ζ_1 might not. Below we will also need the other variant of this statement, in which the *first* component has to belong to $\mathfrak{t}^*_\circ/(W,\bullet)$. Its proof is analogous to that of Proposition 4.4. (More precisely, in this case the counterpart of (4.5) can be obtained directly, without recourse to the computation in the paragraph following this equation.)

Proposition 4.5. *Let $\mu \in \mathbb{X}$ be a weight which belongs to the lower closure of the fundamental alcove. Consider some element $\xi \in \mathfrak{t}^*_\circ$, and denote by $(\zeta_1,\zeta_2) \in \mathfrak{D}(-w_0\mu)$ the image of $(\xi, \xi+\overline{\mu+\rho}) \in \tilde{\mathfrak{D}}(-w_0\mu)$ in $\mathfrak{D}$ and by $\eta \in \mathbf{S}^*$ the element corresponding to the images of ζ_1 and ζ_2 in $\mathfrak{t}^{*(1)}/W$. As explained above Proposition 4.4, the image of ξ in $\mathfrak{t}^{*(1)}$ determines a Borel subgroup $B' \subset G$ with unipotent radical U' such that $\eta_{|\mathrm{Lie}(U')^{(1)}} = 0$.*

If we denote by $i : \mathrm{Spec}(\Bbbk) \to \mathfrak{D}$ the morphism defined by (ζ_1,ζ_2), there exists an isomorphism of $\mathcal{U}_\eta^{\zeta_1}\mathfrak{g} \otimes (\mathcal{U}_\eta^{\zeta_2}\mathfrak{g})^{\mathrm{op}}$-modules

$$i^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathfrak{S}}\mathfrak{g}) \cong \mathsf{Z}_{\eta,B'}(\xi) \otimes \mathsf{Z}_{\eta,B'}(\xi+\overline{\mu+\rho})^*.$$

4.5 Proof of Theorem 4.3

The proof of Theorem 4.3 will require two more preliminary lemmas.

Lemma 4.6. *Let X be a reduced scheme locally of finite type over $\Bbbk$, and let $\mathscr{F}$ be a coherent sheaf on X. Assume that there exists $d \geq 0$ such that for any morphism $i : \mathrm{Spec}(\Bbbk) \to X$ the pullback $i^*(\mathscr{F}) \in \mathsf{Coh}(\mathrm{Spec}(\Bbbk)) = \mathsf{Vect}_\Bbbk$ has dimension d. Then $\mathscr{F}$ is a locally free $\mathscr{O}_X$-module of rank d.*

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Proof. Of course we can assume that X is also affine and of finite type, that is, that $X = \mathrm{Spec}(A)$ for some finitely generated reduced $\Bbbk$-algebra A. With this notation, recall that closed points are dense in any closed subset of $\mathrm{Spec}(A)$; see § 4.4.

Let us denote by M the A-module corresponding to $\mathscr{F}$. In this setting the datum of a morphism $i : \mathrm{Spec}(\Bbbk) \to X$ is equivalent to the datum of a maximal ideal $\mathfrak{m} \subset A$, and we have $i^*(\mathscr{F}) = M/\mathfrak{m} \cdot M$. In view of [Sta20, Tag 0FWG], to show that M is locally free of rank d it suffices to prove that for any $\mathfrak{p} \in \mathrm{Spec}(A)$ we have

$$\dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}) = d.$$

Now by [Pes96, Theorem 7.33], the function

$$\mathfrak{p} \mapsto \dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p})$$

is upper semi-continuous. By assumption, this function is constant (equal to d) on the subset of $\mathrm{Spec}(A)$ consisting of maximal ideals, that is, of closed points. Hence, the open subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}) \leq d\}$$

contains all closed points, hence is the whole of $\mathrm{Spec}(A)$. On the other hand, the open subset

$$\{\mathfrak{p} \in \mathrm{Spec}(A) \mid \dim_{A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}) \leq d-1\}$$

does not contain any closed point, hence is empty. □

LEMMA 4.7. *The morphism*

$$\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet) \to \mathfrak{D}$$

induced by projection on the first and third factors is étale at any point of the form $(\tilde{\lambda}, \widetilde{-\rho}, \tilde{\mu})$ *with* $\lambda, \mu \in \mathbb{X}$.

Proof. To prove this claim it suffices to prove that the morphism $\mathfrak{t}^*/(W,\bullet) \to \mathfrak{t}^{*(1)}/W$ is étale at $\widetilde{-\rho}$. The dot action of W and the natural action of $\mathfrak{t}^*_{\mathbb{F}_p}$ on $\mathfrak{t}^*$ combine to provide an action of the semi-direct product $\mathfrak{t}^*_{\mathbb{F}_p} \rtimes W$ (where W acts on $\mathfrak{t}^*_{\mathbb{F}_p}$ through the natural, unshifted, action) defined by $(\bar{\lambda} w) \bullet \xi = w(\xi + \bar{\rho}) - \bar{\rho} + \bar{\lambda}$ for $\bar{\lambda} \in \mathfrak{t}^*_{\mathbb{F}_p}$ and $w \in W$. Moreover, the composition $\mathfrak{t}^* \to \mathfrak{t}^*/(W,\bullet) \to \mathfrak{t}^{*(1)}/W$ is the quotient morphism for this action. Since $\overline{-\rho}$ is stabilized by W, the claim then follows from [SGA1, Exp. V, Proposition 2.2]. □

For $\lambda \in \mathbb{X}$, whose image in $\mathfrak{t}^*_{\mathbb{F}_p}/(W,\bullet)$ is that of $\lambda' \in \Lambda$, we set

$$\mathfrak{D}_\circ(\lambda) := (\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*_\circ/(W,\bullet)) \smallsetminus \Big(\bigcup_{\mu \in \Lambda \smallsetminus \{\lambda'\}} \mathfrak{D}(\mu) \Big).$$

Then $\mathfrak{D}_\circ(\lambda)$ is an open subset of $\mathfrak{D}$, contained in $\mathfrak{D}(\lambda)$. We will denote by $j_\lambda : \mathfrak{D}_\circ(\lambda) \to \mathfrak{D}$ the embedding. Continuing with the same notation, we also set

$$\mathfrak{D}'_\circ(\lambda) := (\mathfrak{t}^*_\circ/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)) \smallsetminus \Big(\bigcup_{\mu \in \Lambda \smallsetminus \{\lambda'\}} \mathfrak{D}(\mu) \Big),$$

and we denote by $j'_\lambda : \mathfrak{D}'_\circ(\lambda) \to \mathfrak{D}$ the open embedding.

Proof of Theorem 4.3. Let $\lambda, \mu \in \mathbb{X}$ belong to the lower closure of the fundamental alcove. The vector $\overline{-\rho}$ belongs to $\mathfrak{t}^*_\circ$ since this point is stable under the dot action of W. On the other hand, if $\nu \in \mathbb{X}$ is such that $(\tilde{\lambda}, \widetilde{-\rho}) \in \mathfrak{D}(\nu)$, then there exists $\xi \in \mathfrak{t}^*$ such that the point $(\xi + \overline{\nu+\rho}, \xi) \in \tilde{\mathfrak{D}}$ has image $(\tilde{\lambda}, \widetilde{-\rho})$ in $\mathfrak{D}$; we then have $\xi \in W \bullet \overline{-\rho} = \{\overline{-\rho}\}$ and $\xi + \overline{\nu+\rho} \in W \bullet \bar{\lambda}$,

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so that $\tilde{\lambda} = \tilde{\nu}$. We have finally checked that $(\tilde{\lambda}, \widetilde{-\rho}) \in \mathfrak{D}_\circ(\lambda)$; similar considerations show that $(\widetilde{-\rho}, \tilde{\mu}) \in \mathfrak{D}'_\circ(-w_0\mu)$.

By construction, $\mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu)$ is an open subscheme in the fiber product

$$\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet).$$

Consider the morphism

$$f : \mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu) \to \mathfrak{t}^*/(W,\bullet)$$

induced by projection on the middle factor. The algebra $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$ is an $\mathscr{O}(\mathfrak{t}^*/(W,\bullet))$-algebra; it therefore defines a coherent sheaf of $\mathscr{O}_{\mathfrak{t}^*/(W,\bullet)}$-algebras $\mathscr{A}$ on $\mathfrak{t}^*/(W,\bullet)$. Consider also the projections

$$p : \mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu) \to \mathfrak{D}_\circ(\lambda),$$
$$q : \mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu) \to \mathfrak{D}'_\circ(-w_0\mu).$$

The sheaves $p^* j_\lambda^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}})$ and $q^*(j'_{-w_0\mu})^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}})$ are naturally sheaves of (right and left, respectively) modules for $f^*\mathscr{A}$, so that we can consider the tensor product

$$p^* j_\lambda^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}) \otimes_{f^*\mathscr{A}} q^*(j'_{-w_0\mu})^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}). \tag{4.6}$$

We claim that this sheaf is a locally free $\mathscr{O}_{\mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu)}$-module, of rank $p^{2\#\mathfrak{R}^+}$. In fact, by Lemma 4.6, to prove this it suffices to prove that for any closed point $(\zeta_1, \zeta_2, \zeta_3) \in \mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu)$, denoting by $i : \mathrm{Spec}(\Bbbk) \to \mathfrak{D}_\circ(\lambda) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{D}'_\circ(-w_0\mu)$ the corresponding morphism, the vector space

$$i^*(p^* j_\lambda^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathcal{S}\mathfrak{g}}) \otimes_{f^*\mathscr{A}} q^*(j'_{-w_0\mu})^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathcal{S}\mathfrak{g}})) \tag{4.7}$$

has dimension $p^{2\#\mathfrak{R}^+}$. If we denote by $i_1 : \mathrm{Spec}(\Bbbk) \to \mathfrak{D}$ and $i_2 : \mathrm{Spec}(\Bbbk) \to \mathfrak{D}$ the embeddings of the points (ζ_1, ζ_2) and (ζ_2, ζ_3) respectively, then this vector space can be written as

$$i_1^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}) \otimes_{\mathcal{U}_\eta^{\zeta_2}\mathfrak{g}} i_2^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}),$$

where $\eta \in \mathbf{S}^*$ is the image of the elements ζ_i. Let $\xi \in \mathfrak{t}^*$ be such that (ζ_1, ζ_2) is the image of $(\xi + \overline{\lambda+\rho}, \xi)$, and let $B' \subset G$ be the Borel subgroup with unipotent radical U' such that $\eta_{|\mathrm{Lie}(U')^{(1)}} = 0$ determined by the image of ξ in $\mathfrak{t}^{*(1)}$ (see the comments above Proposition 4.4). By Proposition 4.4 we have

$$i_1^*(\mathsf{L}(\lambda+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}) \cong \mathsf{Z}_{\eta,B'}(\xi + \overline{\lambda+\rho}) \otimes \mathsf{Z}_{\eta,B'}(\xi)^*.$$

Similarly, if $\xi' \in \mathfrak{t}^*$ is such that (ζ_2, ζ_3) is the image of $(\xi', \xi' + \overline{\mu+\rho})$, and if $B'' \subset G$ is the Borel subgroup with unipotent radical U'' such that $\eta_{|\mathrm{Lie}(U'')^{(1)}} = 0$ determined by the image of ξ' in $\mathfrak{t}^{*(1)}$, then by Proposition 4.5 we have

$$i_2^*(\mathsf{L}(-w_0\mu+\rho) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}) \cong \mathsf{Z}_{\eta,B''}(\xi') \otimes \mathsf{Z}_{\eta,B''}(\xi' + \overline{\mu+\rho})^*.$$

Here $\mathsf{Z}_{\eta,B'}(\xi)$ and $\mathsf{Z}_{\eta,B''}(\xi')$ are two simple modules over the matrix algebra $\mathcal{U}_\eta^{\zeta_2}\mathfrak{g}$, see § 4.2; they must therefore be isomorphic. Fixing an isomorphism $\varphi : \mathsf{Z}_{\eta,B'}(\xi) \xrightarrow{\sim} \mathsf{Z}_{\eta,B''}(\xi')$, we obtain a pairing

$$\mathsf{Z}_{\eta,B'}(\xi)^* \otimes \mathsf{Z}_{\eta,B''}(\xi') \to \Bbbk$$

defined by $f \otimes v \mapsto f(\varphi^{-1}(v))$, which induces an isomorphism

$$\mathsf{Z}_{\eta,B'}(\xi)^* \otimes_{\mathcal{U}_\eta^{\zeta_2}\mathfrak{g}} \mathsf{Z}_{\eta,B''}(\xi') \xrightarrow{\sim} \Bbbk.$$

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Combining these observations, we obtain that the vector space in (4.7) is isomorphic to

$$\mathsf{Z}_{\eta,B'}(\xi+\overline{\lambda+\rho})\otimes\mathsf{Z}_{\eta,B''}(\xi'+\overline{\mu+\rho})^*,$$

hence has dimension $p^{2\#\mathfrak{R}^+}$, as desired.

We now consider the morphism

$$\mathfrak{D}_\circ(\lambda)\times_{\mathfrak{t}^*/(W,\bullet)}\mathfrak{D}'_\circ(-w_0\mu)\to\mathfrak{D}$$

obtained from that of Lemma 4.7 by restriction to the open subset

$$\mathfrak{D}_\circ(\lambda)\times_{\mathfrak{t}^*/(W,\bullet)}\mathfrak{D}'_\circ(-w_0\mu)\subset\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet).$$

This lemma ensures that this morphism is étale at $(\tilde\lambda,\widetilde{-\rho},\tilde\mu)$; it therefore identifies the completion of $\mathfrak{D}_\circ(\lambda)\times_{\mathfrak{t}^*/(W,\bullet)}\mathfrak{D}'_\circ(-w_0\mu)$ at $(\tilde\lambda,\widetilde{-\rho},\tilde\mu)$ with the completion of $\mathfrak{D}$ at $(\tilde\lambda,\tilde\mu)$, that is, with the spectrum of $\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}$. By construction the completion of the sheaf (4.6) at $(\tilde\lambda,\widetilde{-\rho},\tilde\mu)$ is $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$; since this sheaf is locally free this proves that $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ is faithfully projective. In fact, since the ring $\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}$ is local, this module is even free (of rank $p^{2\#\mathfrak{R}^+}$) by [Sta20, Tag 00NZ].

Finally, we consider the natural morphism

$$\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\mu}\to\mathrm{End}_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}(\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}).$$

Here, both sides are finite free as modules over $\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}$. In fact, for the right-hand side this follows from the same property for the module $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$, which we have seen above. For the left-hand side, we observe that $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$ is finite projective over $\mathscr{O}(\mathfrak{C}_{\mathbf{S}})$ by Proposition 4.1; it follows that $\mathcal{U}_{\mathbf{S}\mathfrak{g}}\otimes_{\mathscr{O}(\mathbf{S}^*)}\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}}$ is finite projective over $\mathcal{Z}_{\mathbf{S}}$, and finally that $\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\mu}$ is finite projective, hence finite free (again by [Sta20, Tag 00NZ]), over the local ring $\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}$. Given this property, to prove that our morphism is an isomorphism it suffices to prove that it is invertible after application of the functor $\Bbbk\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}(-)$. Now if we denote by $\chi\in\mathfrak{g}^{*(1)}$ the point corresponding to the image of 0 in $\mathfrak{t}^{*(1)}/W$ under the identification $\mathbf{S}^*\xrightarrow{\sim}\mathfrak{t}^{*(1)}/W$, we have

$$\Bbbk\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\mu}=\mathcal{U}_\chi^\lambda\mathfrak{g}\otimes(\mathcal{U}_\chi^\mu\mathfrak{g})^{\mathrm{op}}.$$

On the other hand, since $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ is a free module we have

$$\Bbbk\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}\mathrm{End}_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}(\mathsf{M}_{\mathbf{S}}^{\lambda,\mu})\cong\mathrm{End}_{\Bbbk}(\Bbbk\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}),$$

and, applying the considerations above with $\xi=\xi'=\overline{-\rho}$, we have

$$\Bbbk\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}\cong\mathsf{Z}_{\chi,B'}(\bar\lambda)\otimes\mathsf{Z}_{\chi,B'}(\bar\mu)^*,\tag{4.8}$$

where $B'\subset G$ is the unique Borel subgroup with unipotent radical U' such that $\chi_{|\mathrm{Lie}(U')^{(1)}}=0$. By (4.2) our morphism is indeed an isomorphism, which finishes the proof. □

4.6 Localization for Harish-Chandra bimodules

The main consequence of Theorem 4.3 that will be used below is the following statement. (See § 3.9 for the definition of $\mathcal{U}_{\mathbf{S}}^{\hat\lambda}$, and § 4.3 for that of $\mathcal{Z}_{\mathbf{S}}^{\hat\lambda}$.)

COROLLARY 4.8. *For any $\lambda,\mu\in\mathbb{X}$ in the lower closure of the fundamental alcove, the functor $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}(-)$ induces an equivalence of abelian categories*

$$\mathscr{L}_{\lambda,\mu}:\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu})\xrightarrow{\sim}\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\mu})$$

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which restricts to an equivalence of abelian subcategories

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}.$$

Moreover, in the case where $\lambda = \mu$, there exists a canonical isomorphism

$$\mathscr{L}_{\lambda,\lambda}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}) \cong \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}}. \tag{4.9}$$

Proof. The properties stated in Theorem 4.3 ensure that the functor

$$\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}} (-)$$

induces an equivalence of abelian categories

$$\mathsf{Mod}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}});$$

see (4.1). Adding the $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-actions in the picture, we obtain the desired equivalence

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}).$$

Let us now identify the subcategory corresponding to completed Harish-Chandra $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$-bimodules under this equivalence. Recall that any object in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ has a canonical action of $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \ltimes G_1$, and that such an object is a Harish-Chandra bimodule if and only if this action factors through the product morphism $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \ltimes G_1 \to \mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$, that is, if and only if the action of the kernel $\mathbb{K}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ of this map is trivial. Now if V is in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, the action of $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \ltimes G_1$ on $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}} V$ is diagonal, induced by the action on $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ and the action on V obtained by pullback under the morphism $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \ltimes G_1 \to \mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ given by projection on the first factor. By construction the module $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ is a completed Harish-Chandra $\mathcal{U}_{\mathbf{S}}\mathfrak{g}$-bimodule, so that the action on this factor does factor though the product morphism $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \ltimes G_1 \to \mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$, and, moreover, $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu}$ is free over $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$. Hence, $\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}} V$ is a completed Harish-Chandra bimodule if and only if the action of $\mathbb{K}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ on V is trivial, or in other words if and only if the action of the subgroup scheme $G_1 \times \mathrm{Spec}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ on V is trivial, or finally if and only if the action of $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ factors through the quotient morphism $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} \to \mathbb{J}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$. This proves that our equivalence restricts to an equivalence $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \xrightarrow{\sim} \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$.

Finally, we consider the special case $\lambda = \mu$, and construct a canonical isomorphism

$$\mathscr{L}_{\lambda,\lambda}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}) \cong \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}}.$$

Adjunction (see Lemma 3.6) provides a canonical morphism

$$\mathsf{P}_{\mathbf{S}}^{\lambda,-\rho} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} \mathsf{P}_{\mathbf{S}}^{-\rho,\lambda} \to \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}},$$

which factors through a morphism

$$\mathscr{L}_{\lambda,\lambda}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}) = \mathsf{M}_{\mathbf{S}}^{\lambda,\lambda} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}} \to \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}}.$$

Here, by the same considerations as in the proof of Theorem 4.3, both sides are finite free modules over the local ring $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$; to prove that this morphism is an isomorphism it therefore suffices to check that the induced morphism

$$\left(\mathsf{M}_{\mathbf{S}}^{\lambda,\lambda} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}\right) \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}} \Bbbk \to \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}} \Bbbk$$

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is invertible. The right-hand side identifies with $\mathcal{U}_\chi^\lambda\mathfrak{g}$, where χ is as at the end of the proof of Theorem 4.3, and by (4.8) the left-hand side identifies with $\mathsf{Z}_{\chi,B'}(\bar{\lambda}) \otimes \mathsf{Z}_{\chi,B'}(\bar{\lambda})^*$, where $B' \subset G$ is the unique Borel subgroup with unipotent radical U' such that $\chi_{|\mathrm{Lie}(U')^{(1)}} = 0$; the desired claim is therefore clear from the isomorphism (4.2). □

Remark 4.9. We will prove later (at least in the special case when μ belongs to the fundamental alcove; see § 5.5) that for any $\lambda, \mu, \nu \in \mathbb{X}$ in the lower closure of the fundamental alcove the equivalences of Corollary 4.8 intertwine the bifunctors

$$\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}} : \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \times \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$$

(see § 3.9) and

$$\hat{\star}_{\mathbf{S}} : \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \times \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}) \to \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$$

(see § 4.3).

5. $\mathcal{U}\mathfrak{g}$ and differential operators on the flag variety

In this section we study the equivalences $\mathscr{L}_{\lambda,\mu}$ of Corollary 4.8 further, using the relation between the algebra $\mathcal{U}\mathfrak{g}$ and differential operators on the flag variety of G.

5.1 Universal twisted differential operators

Set $\mathcal{B} := G/B$, and consider the natural projection morphism

$$\omega : G/U \to \mathcal{B}.$$

Here G/U admits a natural action of T induced by multiplication on the right on G, and ω is a (Zariski locally trivial) T-torsor. The sheaf of universal twisted differential operators on $\mathcal{B}$ is the quasi-coherent sheaf of algebras

$$\tilde{\mathscr{D}} := \omega_*(\mathscr{D}_{G/U})^T,$$

where the exponent means T-invariants. The actions of G and T on G/U induce a canonical algebra morphism

$$\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*) \to \Gamma(\mathcal{B}, \tilde{\mathscr{D}}); \tag{5.1}$$

see [BMR08, Lemma 3.1.5].

Recall the Grothendieck resolution $\tilde{\mathbf{g}}$ for the group $\mathbf{G} = G^{(1)}$ and the morphism $\vartheta : \tilde{\mathbf{g}} \to \mathfrak{t}^{*(1)}$; see §§ 2.3 and 3.8. Consider the Frobenius morphism $\mathrm{Fr}_{\mathcal{B}} : \mathcal{B} \to \mathcal{B}^{(1)}$ and the natural morphism $f : \tilde{\mathbf{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \to \mathcal{B}^{(1)}$. As explained in [BMR08, § 2.3], there exists a canonical algebra morphism

$$f_*\mathscr{O}_{\tilde{\mathbf{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*} \to (\mathrm{Fr}_{\mathcal{B}})_*\tilde{\mathscr{D}}$$

which takes values in the center of $(\mathrm{Fr}_{\mathcal{B}})_*\tilde{\mathscr{D}}$, and which makes $(\mathrm{Fr}_{\mathcal{B}})_*\tilde{\mathscr{D}}$ a locally finitely generated $f_*\mathscr{O}_{\tilde{\mathbf{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$-module. Since all the morphisms involved in this construction are affine, using this morphism one can consider $\tilde{\mathscr{D}}$ as a coherent sheaf of $\mathscr{O}_{\tilde{\mathbf{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$-algebras on $\tilde{\mathbf{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$. (We will not introduce a different notation for this sheaf of algebras.)

Recall also (see § 4.4) that we denote by $\tilde{\mathbf{S}}^*$ the preimage of $\mathbf{S}^*$ under the natural morphism $\pi : \tilde{\mathbf{g}} \to \mathfrak{g}^{*(1)}$, and that the morphism ϑ restricts to an isomorphism $\tilde{\mathbf{S}}^* \xrightarrow{\sim} \mathfrak{t}^{*(1)}$; in particular,

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$\tilde{\mathbf{S}}^*$ is an affine scheme. We set

$$\tilde{\mathscr{D}}_{\mathbf{S}} := \tilde{\mathscr{D}}_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}.$$

We will also set

$$\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}} := \mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*).$$

Lemma 5.1. *The morphism* (5.1) *induces an algebra isomorphism*

$$\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}} \xrightarrow{\sim} \Gamma(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, \tilde{\mathscr{D}}_{\mathbf{S}}).$$

Proof. Consider the natural morphism

$$h : \tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \to \mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet).$$

If we still denote by $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$ the sheaf of $\mathscr{O}_{\mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)}$-algebras associated with the $\mathscr{O}(\mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet))$-algebra $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$, then as in [BMR08, Proposition 5.2.1] the morphism (5.1) induces a canonical isomorphism of sheaves of algebras

$$h^*(\mathcal{U}_{\mathbf{S}\mathfrak{g}}) \xrightarrow{\sim} \tilde{\mathscr{D}}_{\mathbf{S}}.$$

Now h induces an isomorphism

$$\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \to (\mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)) \times_{\mathfrak{t}^*/(W,\bullet)} \mathfrak{t}^*$$

(in fact, both sides identify canonically with $\mathfrak{t}^*$) so that the claim follows by taking global sections. □

Remark 5.2. One can give a different proof of Lemma 5.1 as follows. By [BMR08, Proposition 3.4.1], the morphism (5.1) is an isomorphism; in other words, identifying quasi-coherent sheaves on $\mathfrak{g}^{*(1)}$ and $\mathscr{O}(\mathfrak{g}^{*(1)})$-modules, we have a canonical isomorphism of sheaves of $\mathscr{O}_{\mathfrak{g}^{*(1)}}$-algebras

$$g_* \tilde{\mathscr{D}} \cong \mathcal{U}_{\mathfrak{g}} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*),$$

where $g : \tilde{\mathfrak{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^* \to \mathfrak{g}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ is the morphism induced by π. Restricting this isomorphism first to $\mathfrak{g}^{*(1)}_{\mathrm{reg}} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ and then to $\mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ we deduce the isomorphism of the lemma, since g restricts to an isomorphism on the preimage of $\mathfrak{g}^{*(1)}_{\mathrm{reg}} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$ (see (2.5)).

5.2 Study of some equivariant $\mathcal{U}_{\mathbf{S}\mathfrak{g}}$-bimodules

Given any $\lambda \in \mathbb{X}$, we have a line bundle $\mathscr{O}_{\mathcal{B}}(\lambda)$ on $\mathcal{B}$ attached naturally to λ. (Our normalization is that of [Jan03], so that line bundles attached to dominant weights are ample.) This line bundle identifies with the direct summand of $\omega_* \mathscr{O}_{G/U}$ consisting of sections which have weight λ for the T-action induced by right multiplication on G; it therefore admits a natural action of the sheaf of algebras $\tilde{\mathscr{D}}$. Using this action and the natural action on $\tilde{\mathscr{D}}$, we obtain a left action of $\tilde{\mathscr{D}}$ on the tensor product

$$\mathscr{O}_{\mathcal{B}}(\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}}.$$

As for $\tilde{\mathscr{D}}$ itself, this module can be also considered as a sheaf of modules on $\tilde{\mathfrak{g}} \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$. We can therefore consider the $\Bbbk$-vector space

$$\Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathscr{O}_{\mathcal{B}}(\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big), \tag{5.2}$$

which in view of Lemma 5.1 admits a natural left action of $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}$. The tensor product $\mathscr{O}_{\mathcal{B}}(\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}}$ also admits a natural right action of $\tilde{\mathscr{D}}$, induced by right multiplication on the second

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factor. The action of $\pi_*\mathscr{O}_{\tilde{\mathfrak{g}}}$ on $(\mathrm{Fr}_{\mathcal{B}})_*\mathscr{O}_{\mathcal{B}}(\lambda)$ being trivial, the two actions of this subalgebra on $(\mathrm{Fr}_{\mathcal{B}})_*(\mathscr{O}_{\mathcal{B}}(\lambda)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})$ coincide, and the space (5.2) therefore also admits a right action of $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}$; moreover, these actions combine to provide an action of $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}\otimes_{\mathscr{O}(\mathbf{S}^*)}(\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$. By construction the action of the central subalgebra

$$\mathscr{O}(\mathfrak{t}^*)\otimes_{\mathscr{O}(\mathbf{S}^*)}\mathscr{O}(\mathfrak{t}^*)\cong\mathscr{O}(\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*)$$

factors through an action of the image of the closed embedding $\mathfrak{t}^*\to\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*$ given by

$$\xi\mapsto(\xi+\bar{\lambda},\xi).$$

The object $\mathscr{O}_{\mathcal{B}}(\lambda)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}}$ also admits a natural G-equivariant quasi-coherent sheaf structure, compatible with the actions considered above. The module (5.2) therefore also admits a natural and compatible module structure for the group scheme

$$\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*;$$

see §§ 3.8–3.9.

For $\lambda,\mu\in\mathbb{X}$, we will denote by

$$\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$$

the completion of the $\mathscr{O}(\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*)$-algebra $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}\otimes_{\mathscr{O}(\mathbf{S}^*)}(\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$ at the ideal corresponding to the point $(\bar{\lambda},\bar{\mu})\in\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*$. Copying the constructions in § 3.9 (replacing $\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ by $\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ and $\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet)$ by $\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*$), we define the category $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$. Copying the definition of $\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}$, we obtain, for $\lambda,\mu,\nu\in\mathbb{X}$, a bifunctor

$$(-)\,\widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}}(-):\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})\times\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}})\to\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}}).$$

For any $\lambda,\mu\in\mathbb{X}$ we have a natural 'forgetful' functor

$$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})\to\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}),$$

which we will usually omit from the notation. In the case where λ and μ are regular, this functor is an equivalence by Lemma 3.2. In the case where μ is regular, for $M\in\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ and $N\in\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}})$ we also have a canonical identification

$$M\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}N\xrightarrow{\sim}M\,\widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}}N.$$

For $\lambda,\mu\in\mathbb{X}$, we will denote by $\mathsf{Q}_{\lambda,\mu}$ the completion of the module

$$\Gamma\big(\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*,(\mathscr{O}_{\mathcal{B}}(\lambda-\mu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*}\big)$$

at the ideal of $\mathscr{O}(\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*)$ corresponding to the element $(\bar{\lambda},\bar{\mu})$. In view of the remarks above, this object can equivalently be obtained by completing this module at the ideal of $\mathscr{O}(\mathfrak{t}^*)$ corresponding to $\bar{\lambda}$ for the left action, or at the ideal of $\mathscr{O}(\mathfrak{t}^*)$ corresponding to $\bar{\mu}$ for the right action. This construction provides an object in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, hence a fortiori an object in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$.

LEMMA 5.3. *For $\lambda,\mu,\nu\in\mathbb{X}$, there exists a canonical isomorphism*

$$\mathsf{Q}_{\lambda,\mu}\,\widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}}\mathsf{Q}_{\mu,\nu}\xrightarrow{\sim}\mathsf{Q}_{\lambda,\nu}$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$. *In particular, in the case where μ is regular there exists a canonical isomorphism*

$$\mathsf{Q}_{\lambda,\mu}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\mathsf{Q}_{\mu,\nu}\xrightarrow{\sim}\mathsf{Q}_{\lambda,\nu}$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$.

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Proof. There exist canonical isomorphisms

$$(\mathscr{O}_{\mathcal{B}}(\lambda-\mu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})\otimes_{\tilde{\mathscr{D}}}(\mathscr{O}_{\mathcal{B}}(\mu-\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})$$
$$\xrightarrow{\sim}\mathscr{O}_{\mathcal{B}}(\lambda-\mu)\otimes_{\mathscr{O}_{\mathcal{B}}}\mathscr{O}_{\mathcal{B}}(\mu-\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}}\cong\mathscr{O}_{\mathcal{B}}(\lambda-\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}}.$$

The desired isomorphism follows by restriction to $\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*$ and then completion at $(\bar{\lambda},\bar{\nu})$. □

If $\lambda\in\mathbb{X}$ is regular, Lemmas 3.2 and 5.1 imply that we have

$$\mathcal{U}_{\mathbf{S}}^{\hat{\lambda}}\cong\mathsf{Q}_{\lambda,\lambda},$$

where the left-hand side is as in § 3.9. Hence, the functor of convolution on the left (respectively, right) with $\mathsf{Q}_{\lambda,\lambda}$ is isomorphic to the identity functor of $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ (respectively, $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}})$), for any $\mu\in\mathbb{X}$. Combining this observation with Lemma 5.3, we see that if $\lambda\in\mathbb{X}$ belongs to the fundamental alcove, then for any $w\in W_{\mathrm{ext}}$ the object $\mathsf{Q}_{\lambda,w\bullet\lambda}$ is invertible in the monoidal category $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$, with inverse $\mathsf{Q}_{w\bullet\lambda,\lambda}$.

Recall from (3.16) that we have a canonical morphism of group schemes

$$\mathfrak{t}^{*(1)}\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\to\mathfrak{t}^{*(1)}\times T^{(1)}.$$

Taking the fiber product with the morphism $\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*\to\mathfrak{t}^*\xrightarrow{\mathrm{AS}}\mathfrak{t}^{*(1)}$ (where the first morphism is the first projection), we obtain a morphism of group schemes

$$\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*\to(\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*)\times T^{(1)}.$$

Using this morphism, for any character η of $T^{(1)}$ we obtain a structure of representation of $\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}_{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*$ on $\mathscr{O}(\mathfrak{t}^*\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*)$ defined by this character. Tensoring with this representation we obtain an autoequivalence of $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, which we denote by $M\mapsto M\langle\eta\rangle$.

Recall from § 3.1 that we identify the lattice of characters of $T^{(1)}$ with $p\cdot\mathbb{X}$.

Lemma 5.4. *For any $\lambda,\nu\in\mathbb{X}$, there exists a canonical isomorphism*

$$\mathsf{Q}_{\lambda+p\nu,\lambda}\cong\mathsf{Q}_{\lambda,\lambda}\langle p\nu\rangle$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$.

Proof. By definition, $\mathsf{Q}_{\lambda+p\nu,\lambda}$ is the completion at the ideal corresponding to $(\bar{\lambda},\bar{\lambda})$ of the $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}\otimes_{\mathscr{O}(\mathbf{S}^*)}(\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$-module

$$\Gamma\big(\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*,(\mathscr{O}_{\mathcal{B}}(p\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*}\big).$$

If we denote by U^+ the unipotent radical of the Borel subgroup opposite to B, then $U^+B/B\subset\mathcal{B}$ is an open subvariety isomorphic to U^+, and the projection $\tilde{\mathbf{S}}^*\to\mathcal{B}$ factors through a morphism $\tilde{\mathbf{S}}^*\to U^+B/B$; see [MR18, Lemma 4.8]. As a consequence, the sheaf $(\mathscr{O}_{\mathcal{B}}(p\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*}$ can be obtained as a further restriction of $(\mathscr{O}_{\mathcal{B}}(p\nu)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})_{|U^+B/B}$.

Since $\tilde{\mathscr{D}}$ acts on $\mathscr{O}_{\mathcal{B}}(p\nu)$, we have an action of the algebra $\mathcal{U}\mathfrak{g}\otimes_{Z_{\mathrm{HC}}}\mathscr{O}(\mathfrak{t}^*)$ on the space $\Gamma(U^+B/B,\mathscr{O}_{\mathcal{B}}(p\nu))$, see (5.1). We have

$$\Gamma(U^+B/B,\mathscr{O}_{\mathcal{B}}(p\nu))=\{f:U^+B\to\Bbbk\mid\forall b\in B,\ \forall x\in U^+B,\ f(xb^{-1})=(p\nu)(b)\cdot f(x)\}.$$

In this space we have a canonical vector, namely the function $f:U^+B\to\Bbbk$ defined by $f(u_1tu_2)=(p\nu)^{-1}(t)$ for all $u_1\in U^+$, $t\in T$ and $u_2\in U$. This section does not vanish on U^+B/B, hence

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induces an isomorphism of line bundles $\mathscr{O}_{U^+B/B} \xrightarrow{\sim} \mathscr{O}_{\mathcal{B}}(p\nu)_{|U^+B/B}$. We claim, furthermore, that it is annihilated by the action of $\mathfrak{g} \subset \mathcal{U}\mathfrak{g}$ and $\mathfrak{t} \subset \mathscr{O}(\mathfrak{t}^*)$. In fact, the second case is clear. For the action of $\mathfrak{g}$, in the case where $\nu \in \mathbb{X}^+$ the claim follows from the fact that our vector is the restriction of the unique (up to scalar) vector of weight $p\nu$ in $\Gamma(\mathcal{B}, \mathscr{O}_{\mathcal{B}}(p\nu))$ (see [Jan03, proof of Proposition II.2.6]), and that this vector belongs to the G-submodule $\mathsf{L}(p\nu)$, on which the action of $\mathfrak{g}$ is well known to vanish. From this we deduce the general case by using the Leibniz rule for the action on tensor products of line bundles.

Tensoring this section with the unit in $\tilde{\mathscr{D}}$, we obtain a section of $(\mathscr{O}_{\mathcal{B}}(p\nu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|U^+B/B}$. The right action on this section provides an isomorphism

$$\tilde{\mathscr{D}}_{|U^+B/B} \to (\mathscr{O}_{\mathcal{B}}(p\nu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|U^+B/B},$$

which commutes with the natural left and right actions of $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*)$. Restricting further, we obtain an isomorphism

$$\tilde{\mathscr{D}}_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*} \xrightarrow{\sim} (\mathscr{O}_{\mathcal{B}}(p\nu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^*(1)} \mathfrak{t}^*},$$

and then, taking global sections and completing, an isomorphism of $\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$-modules $\mathsf{Q}_{\lambda,\lambda} \xrightarrow{\sim} \mathsf{Q}_{\lambda+p\nu,\lambda}$. Taking the action of $\mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathbb{I}_{\mathbf{S}}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*$ into account, this provides the desired isomorphism $\mathsf{Q}_{\lambda,\lambda}\langle p\nu \rangle \xrightarrow{\sim} \mathsf{Q}_{\lambda+p\nu,\lambda}$. □

5.3 Relation with translation bimodules

Recall from § 3.5 the notation $\mathfrak{D} = \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$. In our constructions below we will also have to work with variants of this scheme where one of the two copies of $\mathfrak{t}^*/(W, \bullet)$ is replaced by $\mathfrak{t}^*$. We therefore introduce the notation

$$\mathfrak{E} := \mathfrak{t}^* \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet), \quad \mathfrak{E}' := \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*.$$

We now explain the relation between the objects $\mathsf{Q}_{\lambda,\mu}$ and the 'translation bimodules' introduced in § 3.5.

LEMMA 5.5. *Let $\lambda, \mu \in \mathbb{X}$ in the closure of the fundamental alcove, with one of them in the fundamental alcove itself. Then for any $w \in W_{\mathrm{ext}}$ we have*

$$\mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \cong \mathsf{Q}_{w\bullet\mu, w\bullet\lambda}$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}})$.

Proof. To fix notation we assume that λ is in the fundamental alcove; the other case can be obtained similarly. It is clear that we can assume that $w \in W$. Let $\nu \in \mathbb{X}$ be the unique dominant weight which belongs to $W(\mu - \lambda)$. Then by definition, $\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}$ is the completion of the module

$$\mathsf{L}(\nu) \otimes \mathcal{U}_{\mathbf{S}\mathfrak{g}}$$

at the ideal corresponding to the point $(\tilde{\mu}, \tilde{\lambda}) \in \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^*(1)/W} \mathfrak{t}^*/(W, \bullet)$. Now by Lemma 3.2 the quotient morphism $\mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet)$ is étale at $w \bullet \bar{\lambda}$. It follows that $\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}$ can also be obtained as the completion of the $\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathbf{S}^*)} (\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$-module

$$\mathsf{L}(\nu) \otimes \tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}$$

with respect to the ideal of $\mathscr{O}(\mathfrak{E}')$ corresponding to $(\tilde{\mu}, w \bullet \bar{\lambda})$.

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By Lemma 5.1 we have canonical isomorphisms

$$\mathsf{L}(\nu)\otimes\tilde{\mathcal{U}}_{\mathcal{S}\mathfrak{g}}\cong\mathsf{L}(\nu)\otimes\Gamma\big(\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*,\tilde{\mathscr{D}}_{\mathbf{S}}\big)\cong\Gamma(\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*,\mathsf{L}(\nu)\otimes\tilde{\mathscr{D}}_{\mathbf{S}}).$$

It is a classical fact that the coherent sheaf $\mathsf{L}(\nu)\otimes\mathscr{O}_{\mathcal{B}}$ on $\mathcal{B}$ admits a filtration whose subquotients have the form $\mathscr{O}_{\mathcal{B}}(\eta)$, where η runs over the weights of $\mathsf{L}(\nu)$ (counted with multiplicities). We deduce a similar filtration for the sheaf $\mathsf{L}(\nu)\otimes\tilde{\mathscr{D}}$, and then for its restriction to $\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*$. (Here we use the fact that restriction along the closed embedding $\tilde{\mathbf{S}}^*\hookrightarrow\tilde{\mathfrak{g}}$ is exact on the category $\mathsf{QCoh}^G(\tilde{\mathfrak{g}})$, which follows from the same arguments as those used at the beginning of the proof of Proposition 4.1.) In other words, we have obtained a filtration of $\mathsf{L}(\nu)\otimes\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}$ with subquotients

$$\Gamma\big(\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*,(\mathscr{O}_{\mathcal{B}}(\eta)\otimes_{\mathscr{O}_{\mathcal{B}}}\tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^*\times_{\mathfrak{t}^{*(1)}}\mathfrak{t}^*}\big),\tag{5.3}$$

where η runs over the weights of $\mathsf{L}(\nu)$ (counted with multiplicities). This filtration is clearly compatible with the action of $\mathcal{U}_{\mathbf{S}\mathfrak{g}}\otimes_{\mathscr{O}(\mathbf{S}^*)}(\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}})^{\mathrm{op}}$ and the natural module structure over the group scheme $\mathfrak{t}^*/(W,\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathbb{I}^*_{\mathbf{S}}\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*$.

Let us denote by $\varpi:\mathfrak{t}^*\to\mathfrak{t}^*/(W,\bullet)$ the quotient morphism. The irreducible components of $\mathfrak{E}'$ are parametrized by $\mathfrak{t}^*_{\mathbb{F}_p}$, with the component corresponding to $\bar\gamma$ being the image of the closed embedding $\mathfrak{t}^*\to\mathfrak{E}'$ given by $\xi\mapsto(\varpi(\xi+\bar\gamma),\xi)$. The components containing the point $(\tilde\mu,w\bullet\bar\lambda)$ correspond to the elements $\bar\gamma\in\mathfrak{t}^*_{\mathbb{F}_p}$ such that $w\bullet\bar\lambda+\bar\gamma\in W\bullet\bar\mu$, that is, $\bar\lambda+w^{-1}\bar\gamma\in W\bullet\bar\mu$. On the other hand, the module (5.3) is supported on the component corresponding to $\bar\eta$. Hence, after completion at $(\tilde\mu,w\bullet\bar\lambda)$, the only subquotients that survive are those corresponding to the weights η such that $\bar\lambda+w^{-1}\bar\eta\in W\bullet\bar\mu$, namely, $\lambda+w^{-1}\eta\in W_{\mathrm{ext}}\bullet\mu$. Since $w^{-1}\eta$ is a weight of $\mathsf{L}(\nu)$, it belongs to $\mu-\lambda+\mathbb{Z}\mathfrak{R}$, so that $\lambda+w^{-1}\eta\in\mu+\mathbb{Z}\mathfrak{R}$. By Lemma 3.1(i) the condition that $\lambda+w^{-1}\eta\in W_{\mathrm{ext}}\bullet\mu$ is therefore equivalent to $\lambda+w^{-1}\eta\in W_{\mathrm{aff}}\bullet\mu$. Now by [Jan03, Lemma II.7.7] this condition is satisfied only when $\lambda+w^{-1}\eta=\mu$, that is, $\eta=w(\mu-\lambda)$. We deduce the desired isomorphism, since $w\bullet\mu-w\bullet\lambda=w(\mu-\lambda)$. □

Remark 5.6. Let $\lambda,\mu\in\mathbb{X}$ belonging to the closure of the fundamental alcove, and assume that the stabilizer of λ for the dot action of W_{aff} is contained in the stabilizer of μ. Then, if we denote by $\mathscr{O}(\mathfrak{E})^{\hat\lambda,\hat\mu}$ the completion of $\mathscr{O}(\mathfrak{E})$ at the ideal corresponding to $(\bar\lambda,\tilde\mu)$, the same considerations as in the proof of Lemma 5.5 show that there exists an isomorphism

$$\mathsf{Q}_{\lambda,\mu}\cong\mathscr{O}(\mathfrak{E})^{\hat\lambda,\hat\mu}\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat\lambda,\hat\mu}}\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}.$$

Recall that, given a simple reflection $s\in S_{\mathrm{aff}}$, a weight $\lambda\in\mathbb{X}$ belonging to the closure of the fundamental alcove is said to be on the *wall* corresponding to s if $s\bullet\lambda=\lambda$.

Lemma 5.7. *Let $\lambda,\mu\in\mathbb{X}$, with λ belonging to the fundamental alcove and μ on exactly one wall of the fundamental alcove, attached to the simple reflection s. Let also $w\in W$.*

(i) *If $ws\bullet\lambda>w\bullet\lambda$, then there exists an exact sequence*

$$\mathsf{Q}_{w\bullet\lambda,w\bullet\lambda}\hookrightarrow\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\twoheadrightarrow\mathsf{Q}_{ws\bullet\lambda,w\bullet\lambda}$$

in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\lambda})$.

(ii) *If $ws\bullet\lambda<w\bullet\lambda$, then there exists an exact sequence*

$$\mathsf{Q}_{ws\bullet\lambda,w\bullet\lambda}\hookrightarrow\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\twoheadrightarrow\mathsf{Q}_{w\bullet\lambda,w\bullet\lambda}$$

in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat\lambda,\hat\lambda})$.

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Proof. By Lemma 5.5 we have

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \cong \mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\mu,w\bullet\lambda}.$$

Hence, if we denote by ν the unique dominant weight in $W(\lambda-\mu)$, this object can be obtained by completing the bimodule

$$\begin{aligned}&\mathsf{L}(\nu) \otimes \Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathscr{O}_{\mathcal{B}}(w\bullet\mu - w\bullet\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big)\\ &\cong \Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, \mathsf{L}(\nu) \otimes (\mathscr{O}_{\mathcal{B}}(w\bullet\mu - w\bullet\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big)\end{aligned}$$

with respect to the ideal of $\mathscr{O}(\mathfrak{E}')$ corresponding to $(\tilde{\lambda}, w\bullet\bar{\lambda})$. Hence, as in the proof of Lemma 5.5, if we choose an enumeration $\eta_1, \dots, \eta_n$ of the T-weights of $\mathsf{L}(\nu)$ (counted with multiplicities) such that $\eta_i < \eta_j$ implies $i < j$, then this bimodule admits a filtration

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M_n = \Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathscr{O}_{\mathcal{B}}(w\bullet\mu - w\bullet\lambda) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big)$$

such that

$$M_i/M_{i-1} \cong \Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathscr{O}_{\mathcal{B}}(w\bullet\mu - w\bullet\lambda + \eta_i) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big)$$

for any i. The subquotient M_i/M_{i-1} survives after completion at the ideal corresponding to $(\tilde{\lambda}, w\bullet\bar{\lambda})$ if and only if

$$w\bullet\bar{\mu} - w\bullet\bar{\lambda} + \overline{\eta_i} \in W\bullet\bar{\lambda} - w\bullet\bar{\lambda},$$

that is, if and only if

$$\mu + w^{-1}\eta_i \in W_{\mathrm{ext}}\bullet\lambda.$$

Here $w^{-1}\eta_i$ is a weight of $\mathsf{L}(\nu)$, hence $\mu + w^{-1}\eta_i$ belongs to $\lambda + \mathbb{Z}\mathfrak{R}$; in view of Lemma 3.1(i), this condition is therefore equivalent to $\mu + w^{-1}\eta_i \in W_{\mathrm{aff}}\bullet\lambda$. Since the stabilizer of μ for the dot action of W_{aff} is $\{e, s\}$, by [Jan03, Lemma II.7.7] this condition is satisfied for two values of η_i, corresponding to

$$\mu + w^{-1}\eta_i = \lambda \quad \text{and} \quad \mu + w^{-1}\eta_i = s\bullet\lambda,$$

that is,

$$w\bullet\mu + \eta_i = w\bullet\lambda \quad \text{and} \quad w\bullet\mu + \eta_i = ws\bullet\lambda.$$

Hence, $\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}$ admits a two-step filtration, with subquotients isomorphic respectively to $\mathsf{Q}_{w\bullet\lambda,w\bullet\lambda}$ and $\mathsf{Q}_{ws\bullet\lambda,w\bullet\lambda}$. The order in which these subquotients appear depends on whether $ws\bullet\lambda > w\bullet\lambda$ or $ws\bullet\lambda < w\bullet\lambda$, and is as indicated in the statement. □

5.4 Convolution with translation bimodules

Let $\lambda, \mu \in \mathbb{X}$, and assume that λ is regular. Then there exists a canonical algebra morphism

$$\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}} \to \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \tag{5.4}$$

which can be defined as follows. The algebra $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$ is by definition the completion of $\mathscr{O}(\mathfrak{D})$ at the ideal corresponding to $(\tilde{\mu}, \tilde{\lambda})$. Hence, it admits a canonical morphism to the completion $\mathscr{O}(\mathfrak{E})^{\hat{\mu},\hat{\lambda}}$ of $\mathscr{O}(\mathfrak{E})$ at the ideal corresponding to $(\bar{\mu}, \bar{\lambda})$. Now the morphism $\mathfrak{t}^* \to \mathfrak{t}^*$ defined by $\xi \mapsto \xi + \bar{\mu} - \bar{\lambda}$

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induces an automorphism of $\mathfrak{E}$ sending $(\bar{\lambda}, \tilde{\lambda})$ to $(\bar{\mu}, \tilde{\lambda})$, which therefore induces an isomorphism

$$\mathscr{O}(\mathfrak{E})^{\hat{\mu},\hat{\lambda}} \xrightarrow{\sim} \mathscr{O}(\mathfrak{E})^{\hat{\lambda},\hat{\lambda}}, \tag{5.5}$$

where the right-hand side is the completion of $\mathscr{O}(\mathfrak{E})$ at the ideal corresponding to $(\bar{\lambda}, \tilde{\lambda})$. Finally, the natural morphism

$$\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \to \mathscr{O}(\mathfrak{E})^{\hat{\lambda},\hat{\lambda}} \tag{5.6}$$

is an isomorphism by Lemma 3.2, since λ is regular; combining these constructions, we obtain the wished-for morphism (5.4).

Our goal in this subsection is to prove the following claim, which involves the equivalence constructed in Corollary 4.8.

Proposition 5.8. *Let $\lambda, \mu \in \mathbb{X}$, with λ belonging to the fundamental alcove and μ on exactly one wall of the fundamental alcove, attached to a simple reflection s which belongs to W. Then there exists an isomorphism*

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \cong \mathscr{L}_{\lambda,\lambda}\big(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}\big)$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$, where $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$ is regarded as a $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$-module via the morphism (5.4) and $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$ is endowed with the trivial structure as a representation.

Remark 5.9. From the proof below one can check that the isomorphism in Proposition 5.8 is 'canonical' in that it depends only on the choice of an adjunction $(\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}(-), \mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}(-))$, which can be defined by a choice of an isomorphism $\mathsf{L}(\nu)^* \cong \mathsf{L}(-w_0(\nu))$ where ν is the only W-translate of $\mu - \lambda$; see the proof of Lemma 3.6. From this proof it is clear also that the morphism

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \to \mathcal{U}_{\mathbf{S}}^{\hat{\lambda}}$$

defined by this adjunction corresponds under $\mathscr{L}_{\lambda,\lambda}$ to the morphism

$$\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}} \to \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$$

given by the action of $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$ on $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$.

The proof of this proposition will use two preliminary lemmas.

Lemma 5.10. *If s is a simple reflection in W, then $\mathscr{O}(\mathfrak{t}^*)$ is free of rank 2 as a module over $\mathscr{O}(\mathfrak{t}^*/(\{e, s\}, \bullet))$.*

Proof. First, translating by ρ, we can reduce the $\bullet$-action to the standard action; it therefore suffices to prove that $\mathscr{O}(\mathfrak{t}^*)$ is free of rank 2 over the subalgebra $\mathscr{O}(\mathfrak{t}^*)^s$ of s-invariants. Next, recall that we have a W-equivariant isomorphism $\mathfrak{t} \xrightarrow{\sim} \mathfrak{t}^*$, induced by a choice of G-equivariant isomorphism $\mathfrak{g} \xrightarrow{\sim} \mathfrak{g}^*$. We are therefore reduced to proving that $\mathscr{O}(\mathfrak{t})$ is free of rank 2 over $\mathscr{O}(\mathfrak{t})^s$. Now, standard arguments show that $(1, \bar{\rho})$ is a basis of $\mathscr{O}(\mathfrak{t})$ over $\mathscr{O}(\mathfrak{t})^s$; see, for example, [EW16, Claim 3.11]. □

Lemma 5.11. *Let $\lambda, \mu \in \mathbb{X}$, with λ belonging to the fundamental alcove and μ on exactly one wall of the fundamental alcove, attached to a simple reflection s which belongs to W. Then there exist isomorphisms of functors which make the following diagrams commutative, where the upper horizontal arrow on the left-hand side is the restriction-of-scalars functor associated*

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with the morphism (5.4)*:*

$$\begin{array}{ccc} \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}) & \longrightarrow & \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}) \\ \downarrow{\scriptstyle \mathscr{L}_{\lambda,\lambda}} & & \downarrow{\scriptstyle \mathscr{L}_{\mu,\lambda}} \\ \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}) & \xrightarrow{\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}(-)} & \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}) \end{array} \qquad \begin{array}{ccc} \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathcal{S}}^{\hat{\mu},\hat{\lambda}}) & \xrightarrow{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}}(-)} & \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}) \\ \downarrow{\scriptstyle \mathscr{L}_{\mu,\lambda}} & & \downarrow{\scriptstyle \mathscr{L}_{\lambda,\lambda}} \\ \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}) & \xrightarrow{\mathsf{P}_{\mathbf{S}}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}(-)} & \mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}) \end{array}$$

Proof. By definition we have

$$\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathscr{L}_{\lambda,\lambda}(-) \cong (\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{\lambda,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda})\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}}(-).$$

Using Lemmas 5.3 and 5.5, we deduce that

$$\mathsf{P}_{\mathbf{S}}^{\mu,\lambda}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathscr{L}_{\lambda,\lambda}(-) \cong (\mathsf{Q}_{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda})\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}}(-).$$

Hence, to prove the commutativity of the diagram of the left it suffices to construct an isomorphism

$$(\mathsf{P}_{\mathbf{S}}^{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda})\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}}\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \xrightarrow{\sim} \mathsf{Q}_{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda}$$

or, in other words (in view of (5.5) and (5.6)), an isomorphism

$$(\mathsf{P}_{\mathbf{S}}^{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda})\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}}\mathscr{O}(\mathfrak{E})^{\hat{\mu},\hat{\lambda}} \xrightarrow{\sim} \mathsf{Q}_{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda}. \tag{5.7}$$

To construct a morphism as in (5.7) it suffices to construct a morphism

$$\mathsf{P}_{\mathbf{S}}^{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda} \to \mathsf{Q}_{\mu,-\rho}\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\,\mathsf{P}_{\mathbf{S}}^{-\rho,\lambda} \tag{5.8}$$

in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}})$. By Remark 5.6 we have

$$\mathsf{Q}_{\mu,-\rho} \cong \mathscr{O}(\mathfrak{E})^{\hat{\mu},\widehat{-\rho}}\otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\widehat{-\rho}}}\mathsf{P}_{\mathbf{S}}^{\mu,-\rho}; \tag{5.9}$$

in particular, there exists a natural morphism $\mathsf{P}_{\mathbf{S}}^{\mu,-\rho}\to\mathsf{Q}_{\mu,-\rho}$, which allows to define the wished-for morphism (5.8), hence the morphism (5.7).

Now we claim that $\mathscr{O}(\mathfrak{E})^{\hat{\mu},\hat{\lambda}}$ (respectively, $\mathscr{O}(\mathfrak{E})^{\hat{\mu},\widehat{-\rho}}$) is free of rank 2 over the algebra $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$ (respectively, $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\widehat{-\rho}}$), which in view of (5.9) will imply that the morphism (5.7) is an isomorphism. The two cases are similar, so we only consider $\mathscr{O}(\mathfrak{E})^{\hat{\mu},\hat{\lambda}}$. It follows from Lemma 3.2 that $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$ identifies with the completion

$$\mathscr{O}(\mathfrak{t}^*/(\{e,s\},\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))^{\hat{\mu},\hat{\lambda}}$$

of $\mathscr{O}(\mathfrak{t}^*/(\{e,s\},\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))$ with respect to the ideal corresponding to the image of $(\bar{\mu},\tilde{\lambda})$. Now $\mathscr{O}(\mathfrak{E})$ is free of rank 2 over $\mathscr{O}(\mathfrak{t}^*/(\{e,s\},\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))$ by Lemma 5.10, and its completion with respect to the ideal corresponding to $(\bar{\mu},\tilde{\lambda})$ coincides with its completion with respect to the ideal of $\mathscr{O}(\mathfrak{t}^*/(\{e,s\},\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet))$ corresponding to the image of $(\bar{\mu},\tilde{\lambda})$ (because $(\bar{\mu},\tilde{\lambda})$ is the only closed point in the fiber over its image in $\mathfrak{t}^*/(\{e,s\},\bullet)\times_{\mathfrak{t}^{*(1)}/W}\mathfrak{t}^*/(W,\bullet)$). The desired claim follows.

We have finally proved the commutativity of the left diagram of the lemma. The commutativity of the right diagram follows from that of the left one by adjunction, in view of Lemma 3.6. □

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Proof of Proposition 5.8. Lemma 5.11 provides isomorphisms

$$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu,\lambda} \cong \mathscr{L}_{\lambda,\lambda}\big(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathscr{L}_{\mu,\lambda}^{-1}(\mathsf{P}_{\mathbf{S}}^{\mu,\lambda})\big) \cong \mathscr{L}_{\lambda,\lambda}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathscr{L}_{\lambda,\lambda}^{-1}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda}})).$$

The desired claim follows, using the isomorphism (4.9). □

5.5 Monoidality of the functors $\mathscr{L}_{\lambda,\lambda}$

Our goal in this subsection is to prove the following claim, announced in Remark 4.9.

Proposition 5.12. *Let $\lambda, \nu \in \mathbb{X}$ in the lower closure of the fundamental alcove, and let $\mu \in \mathbb{X}$ be in the fundamental alcove. Then for $M \in \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ and $N \in \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}})$ there exists a canonical (in particular, bifunctorial) isomorphism*

$$\mathscr{L}_{\lambda,\nu}(M \,\hat{\star}_{\mathbf{S}}\, N) \cong \mathscr{L}_{\lambda,\mu}(M) \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathscr{L}_{\mu,\nu}(N).$$

In the case where $\lambda = \mu = \nu$, this isomorphism and (4.9) define on $\mathscr{L}_{\lambda,\lambda}$ the structure of a monoidal functor.

Proof. Recall the completion $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$ introduced in § 3.6. By definition and Remark 3.5, if we set $\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge := \mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)} \mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$ we have

$$\mathscr{L}_{\lambda,\mu}(M) = \mathsf{M}_{\mathbf{S}}^{\lambda,\mu} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}} M = (\mathsf{P}_{\mathbf{S}}^{\lambda,-\rho} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{-\rho,\mu}) \otimes_{Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\mu}}} M$$

and

$$\mathscr{L}_{\mu,\nu}(N) = \mathsf{M}_{\mathbf{S}}^{\mu,\nu} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\nu}}} M = (\mathsf{P}_{\mathbf{S}}^{\mu,-\rho} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{-\rho,\nu}) \otimes_{Z_{\mathrm{HC}}^{\hat{\mu}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\nu}}} N.$$

We deduce that

$$\begin{aligned}\mathscr{L}_{\lambda,\mu}(M) \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathscr{L}_{\mu,\nu}(N) \cong (\mathsf{P}_{\mathbf{S}}^{\lambda,-\rho} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{-\rho,\mu} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{\mu,-\rho} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{-\rho,\nu}) \\ \otimes_{Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\mu}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\nu}}} (M \otimes_{Z_{\mathrm{HC}}^{\hat{\mu}}} N).\end{aligned}$$

(Here, $Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\mu}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\nu}}$ identifies with the completion of the algebra $\mathscr{O}(\mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet))$ with respect to the ideal corresponding to $(\tilde{\lambda}, \tilde{\mu}, \tilde{\nu})$.) By Lemmas 5.5 and 5.3 we have

$$\mathsf{P}_{\mathbf{S}}^{-\rho,\mu} \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathsf{P}_{\mathbf{S}}^{\mu,-\rho} \cong \mathsf{Q}_{-\rho,\mu} \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathsf{Q}_{\mu,-\rho} \cong \mathsf{Q}_{-\rho,-\rho}.$$

By Lemma 5.1, $\mathsf{Q}_{-\rho,-\rho}$ identifies with the completion of $\tilde{\mathcal{U}}_{\mathbf{S}\mathfrak{g}}$ with respect to the ideal of $\mathscr{O}(\mathfrak{t}^*)$ corresponding to $\overline{-\rho}$. Via this identification, the action of $Z_{\mathrm{HC}}^{\hat{\mu}}$ is given by the composition

$$Z_{\mathrm{HC}}^{\hat{\mu}} \to \mathscr{O}(\mathfrak{t}^*)^{\hat{\mu}} \xrightarrow{\sim} \mathscr{O}(\mathfrak{t}^*)^{\widehat{-\rho}},$$

where $\mathscr{O}(\mathfrak{t}^*)^{\hat{\mu}}$ and $\mathscr{O}(\mathfrak{t}^*)^{\widehat{-\rho}}$ are the completions of $\mathscr{O}(\mathfrak{t}^*)$ with respect to the ideals corresponding to $\bar{\lambda}$ and $\overline{-\rho}$ respectively, the first map is induced by the embedding $Z_{\mathrm{HC}} \cong \mathscr{O}(\mathfrak{t}^*/(W,\bullet)) \to \mathscr{O}(\mathfrak{t}^*)$, and the second one is defined in terms of translation, in a way similar to that considered at the beginning of § 5.4. Here the first map is an isomorphism by Lemma 3.2, from which we obtain an isomorphism

$$\begin{aligned}&\mathscr{L}_{\lambda,\mu}(M) \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathscr{L}_{\mu,\nu}(N) \\ &\quad\cong (\mathsf{P}_{\mathbf{S}}^{\lambda,-\rho} \otimes_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^\wedge} \mathsf{P}_{\mathbf{S}}^{-\rho,\nu}) \otimes_{Z_{\mathrm{HC}}^{\hat{\lambda}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\widehat{-\rho}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge} Z_{\mathrm{HC}}^{\hat{\nu}}} (M \otimes_{Z_{\mathrm{HC}}^{\hat{\mu}}} N).\end{aligned}$$

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Finally, we use the fact that the morphism $\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge \to Z_{\mathrm{HC}}^{\widehat{-\rho}}$ is an isomorphism (see the proof of Lemma 4.7) to deduce the wished-for isomorphism

$$\mathscr{L}_{\lambda,\mu}(M)\,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}\,\mathscr{L}_{\mu,\nu}(N) \cong \mathscr{L}_{\lambda,\nu}(M\,\hat{\star}_{\mathbf{S}}\,N).$$

In the case where $\lambda=\mu=\nu$, the fact that the relevant isomorphisms define a monoidal structure on $\mathscr{L}_{\lambda,\lambda}$ is clear from constructions. □

Remark 5.13. Proposition 5.12 also holds in the case where μ is singular (in the lower closure of the fundamental alcove). This case can be treated using the methods of § 5.6 below; since it is not needed in this paper, we omit the details.

5.6 Singular analogues

Let $I \subset \mathfrak{R}^{\mathrm{s}}$ be a subset, and let $P_I \subset G$ be the associated standard (i.e. containing B) parabolic subgroup of G. (In practice, only the case $\#I=1$ will be considered below.) Let $U_I \subset P_I$ be the unipotent radical of P_I, and let L_I be the Levi factor containing T, so that $P_I \cong L_I \ltimes U_I$. Let $\mathcal{P}_I := G/P_I$, and consider the natural projection

$$\omega_I : G/U_I \to \mathcal{P}_I.$$

The group L_I acts naturally on G/U_I on the right, via the action induced by multiplication on the right on G; this action makes ω_I a (Zariski locally trivial) L_I-torsor. We set

$$\tilde{\mathscr{D}}_I := (\omega_I)_*(\mathscr{D}_{G/U_I})^{L_I},$$

where the exponent means L_I-invariants. The actions of G and L_I on G/U_I induce a canonical algebra morphism

$$\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*/(W_I,\bullet)) \to \Gamma(\mathcal{P}_I, \tilde{\mathscr{D}}_I); \tag{5.10}$$

see [BMR06, Proposition 1.2.3].

Let also $\mathbf{P}_I := P_I^{(1)}$, a parabolic subgroup of $\mathbf{G} = G^{(1)}$ with unipotent radical $\mathbf{U}_I := U_I^{(1)}$. Let $\tilde{\mathbf{g}}_I$ be the parabolic Grothendieck resolution (for the group $\mathbf{G}$) associated with I, defined as

$$\tilde{\mathbf{g}}_I := \mathbf{G} \times^{\mathbf{P}_I} (\mathbf{g}/\mathrm{Lie}(\mathbf{U}_I))^*.$$

Here $\tilde{\mathbf{g}}_I$ is a vector bundle over $\mathbf{G}/\mathbf{P}_I = \mathcal{P}_I^{(1)}$, and if $\mathbf{L}_I := L_I^{(1)}$ there is a natural morphism

$$\tilde{\mathbf{g}}_I \to \mathrm{Lie}(\mathbf{L}_I)^*/\mathbf{L}_I \cong \mathfrak{t}^{*(1)}/W_I,$$

where $W_I \subset W$ is as in § 3.1 (seen here as the Weyl group of $(\mathbf{L}_I, \mathbf{T})$).

Consider the induced morphism $f_I : \tilde{\mathbf{g}}_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet) \to \mathcal{P}_I^{(1)}$, and the Frobenius morphism $\mathrm{Fr}_{\mathcal{P}_I} : \mathcal{P}_I \to \mathcal{P}_I^{(1)}$. As explained in [BMR06, § 1.2.1], there exists a canonical algebra morphism

$$(f_I)_*\mathscr{O}_{\tilde{\mathbf{g}}_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)} \to (\mathrm{Fr}_{\mathcal{P}_I})_*\tilde{\mathscr{D}}_I,$$

where the morphism $\mathfrak{t}^*/(W_I,\bullet) \to \mathfrak{t}^{*(1)}/W_I$ is induced by the Artin–Schreier map. This morphism takes values in the center of $(\mathrm{Fr}_{\mathcal{P}_I})_*\tilde{\mathscr{D}}_I$, and makes $(\mathrm{Fr}_{\mathcal{P}_I})_*\tilde{\mathscr{D}}_I$ a locally finitely generated $(f_I)_*\mathscr{O}_{\tilde{\mathbf{g}}_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}$-module. Since all the morphisms involved in this construction are affine, using this morphism one can consider $\tilde{\mathscr{D}}_I$ as a coherent sheaf of $\mathscr{O}_{\tilde{\mathbf{g}}_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}$-algebras on $\tilde{\mathbf{g}}_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)$. (We will not introduce a different notation for this sheaf of algebras.)

We also have a canonical morphism $\tilde{\mathbf{g}}_I \to \mathfrak{g}^{*(1)}$, and we denote by $\tilde{\mathbf{S}}_I^*$ the (scheme-theoretic) inverse image of $\mathbf{S}^*$ under this morphism. As in the case $I=\varnothing$, using [Ric17, Remark 3.5.4]

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one can check that the morphism $\tilde{\mathfrak{g}}_I \to \mathfrak{t}^{*(1)}/W_I$ considered above restricts to an isomorphism $\tilde{\mathbf{S}}_I^* \xrightarrow{\sim} \mathfrak{t}^{*(1)}/W_I$; in particular, this scheme is affine. We set

$$\tilde{\mathscr{D}}_{I,\mathbf{S}} := (\tilde{\mathscr{D}}_I)_{|\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}.$$

The following lemma is a parabolic analogue of Lemma 5.1, for which the same proof applies.

LEMMA 5.14. *The morphism* (5.10) *induces an algebra isomorphism*

$$\mathcal{U}_{\mathcal{S}\mathfrak{g}} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*/(W_I,\bullet)) \xrightarrow{\sim} \Gamma(\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet), \tilde{\mathscr{D}}_{I,\mathbf{S}}).$$

Let

$$\mathbb{X}_I := \{\lambda \in \mathbb{X} \mid \forall \alpha \in I,\ \langle \lambda, \alpha^\vee \rangle = 0\},$$

so that $\mathbb{X}_I$ identifies with the character lattice of L_I. Then any $\lambda \in \mathbb{X}_I$ defines a line bundle $\mathscr{O}_{\mathcal{P}_I}(\lambda)$ on $\mathcal{P}_I$, from which one can define the space

$$\Gamma(\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet), (\mathscr{O}_{\mathcal{P}_I}(\lambda) \otimes \tilde{\mathscr{D}}_I)_{|\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}). \tag{5.11}$$

This object admits a natural action of the algebra

$$(\mathcal{U}_{\mathbf{S}\mathfrak{g}} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*/(W_I,\bullet))) \otimes_{\mathscr{O}(\mathbf{S}^*)} (\mathcal{U}_{\mathbf{S}\mathfrak{g}}{}^{\mathrm{op}} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*/(W_I,\bullet)))$$

and of the group scheme

$$\mathfrak{t}^*/(W_I,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathbb{I}_{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W_I,\bullet).$$

Since λ is W_I-invariant, the map $\xi \mapsto \bar{\lambda} + \xi$ factors through an isomorphism

$$\tau_\lambda^I : \mathfrak{t}^*/(W_I,\bullet) \xrightarrow{\sim} \mathfrak{t}^*/(W_I,\bullet),$$

and the action of the subalgebra $\mathscr{O}(\mathfrak{t}^*/(W_I,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W_I,\bullet))$ on (5.11) factors through the morphism induced by the closed embedding

$$\tau_\lambda^I \times \mathrm{id} : \mathfrak{t}^*/(W_I,\bullet) \to \mathfrak{t}^*/(W_I,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W_I,\bullet).$$

Given $\lambda, \mu \in \mathbb{X}$ such that $\lambda - \mu \in \mathbb{X}_I$, one can then define the object

$$\mathsf{Q}_{\lambda,\mu}^I \in \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$$

as the completion of the module

$$\Gamma\big(\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet), (\mathscr{O}_{\mathcal{P}_I}(\lambda-\mu) \otimes \tilde{\mathscr{D}}_I)_{|\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}\big)$$

at the ideal of $\mathscr{O}(\mathfrak{t}^*/(W_I,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W_I,\bullet))$ corresponding to the image of $(\bar{\lambda}, \bar{\mu})$. As for $\mathsf{Q}_{\lambda,\mu}$, this object can be obtained by completing the module at the ideal of $\mathscr{O}(\mathfrak{t}^*/(W_I,\bullet))$ corresponding to the image of $\bar{\lambda}$ with respect to the left action, or at the ideal of $\mathscr{O}(\mathfrak{t}^*/(W_I,\bullet))$ corresponding to the image of $\bar{\mu}$ with respect to the right action.

LEMMA 5.15. *Let* $\lambda, \mu, \nu \in \mathbb{X}$.

(i) *Assume that the stabilizer of* μ *for the dot action of* W_{aff} *is* W_I, *and that* $\nu \in -\rho + \mathbb{X}_I$. *Then there exists a canonical isomorphism*

$$\mathsf{Q}_{\lambda,\mu} \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathsf{Q}_{\mu,\nu}^I \xrightarrow{\sim} \mathsf{Q}_{\lambda,\nu}$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$. *Similarly, if the stabilizer of* μ *for the dot action of* W_{aff} *is* W_I, *and* $\lambda \in -\rho + \mathbb{X}_I$, *then there exists a canonical isomorphism*

$$\mathsf{Q}_{\lambda,\mu}^I \,\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}}\, \mathsf{Q}_{\mu,\nu} \xrightarrow{\sim} \mathsf{Q}_{\lambda,\nu}$$

in $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$.

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(ii) *Assume that the stabilizer of μ for the dot action of W_{aff} is W_I, and that $\lambda, \nu \in -\rho + \mathbb{X}_I$. Then there exists a canonical isomorphism*

$$\mathsf{Q}^I_{\lambda,\mu} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}^I_{\mu,\nu} \xrightarrow{\sim} \mathsf{Q}^I_{\lambda,\nu}$$

in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\nu}})$.

Proof. (i) We only prove the first isomorphism; the proof of the second one is similar. Our assumptions ensure that $\mu - \nu \in \mathbb{X}_I$, so that the object $\mathsf{Q}^I_{\mu,\nu}$ is well defined. Consider the natural morphism $a_I : \mathcal{B} \to \mathcal{P}_I$. By [BMR06, Proposition 1.2.3] there exists a canonical morphism of sheaves of algebras

$$\tilde{\mathscr{D}}_I \to (a_I)_* \tilde{\mathscr{D}}. \tag{5.12}$$

By the projection formula, and since $(a_I)_* \mathscr{O}_{\mathcal{B}} \cong \mathscr{O}_{\mathcal{P}_I}$, we also have

$$(a_I)_* \mathscr{O}_{\mathcal{B}}(\mu - \nu) \cong \mathscr{O}_{\mathcal{P}_I}(\mu - \nu),$$

and via this isomorphism the action of $\tilde{\mathscr{D}}_I$ on $\mathscr{O}_{\mathcal{P}_I}(\mu - \nu)$ is obtained by restriction of scalars along (5.12) from the natural action of $(a_I)_* \tilde{\mathscr{D}}$ on $(a_I)_* \mathscr{O}_{\mathcal{B}}(\mu - \nu)$. We deduce a natural isomorphism

$$(a_I)_*(\mathscr{O}_{\mathcal{B}}(\lambda - \mu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}}) \otimes_{\tilde{\mathscr{D}}_I} (\mathscr{O}_{\mathcal{P}_I}(\mu - \nu) \otimes_{\mathscr{O}_{\mathcal{P}_I}} \tilde{\mathscr{D}}_I) \xrightarrow{\sim} (a_I)_*(\mathscr{O}_{\mathcal{B}}(\lambda - \nu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}}),$$

defined by a formula similar to that considered in the proof of Lemma 5.3. The desired isomorphism follows by restricting to $\tilde{\mathbf{S}}^*_I \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I, \bullet)$ and then completing, using Lemma 5.14 and the fact that the natural morphism $\mathfrak{t}^*/(W_I, \bullet) \to \mathfrak{t}^*/(W, \bullet)$ is étale at the image of $\bar{\mu}$, see Lemma 3.2.

(ii) The proof is similar to that of Lemma 5.3. □

5.7 Conjugation of wall-crossing bimodules

The following proposition will eventually reduce the question of the description of the bimodules realizing wall-crossing functors for G to the case of wall-crossing functors attached to simple reflections which belong to W.

PROPOSITION 5.16. *Consider elements $s \in S_{\mathrm{aff}}$, $s' \in S_{\mathrm{aff}} \cap W$ and $w \in W_{\mathrm{ext}}$ such that $s' = wsw^{-1}$. Let $\lambda, \mu, \mu' \in \mathbb{X}$, with λ belonging to the fundamental alcove, and μ (respectively, μ') belonging to the wall of the fundamental alcove attached to s (respectively, s'), and on no other wall. Then there exists an isomorphism*

$$\mathsf{P}^{\lambda,\mu'}_{\mathbf{S}} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}^{\mu',\lambda}_{\mathbf{S}} \cong \mathsf{Q}_{\lambda,w\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} (\mathsf{P}^{\lambda,\mu}_{\mathbf{S}} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}^{\mu,\lambda}_{\mathbf{S}}) \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\lambda,\lambda}$$

in $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}})$.

Proof. By Lemma 5.5 we have isomorphisms

$$\mathsf{P}^{\lambda,\mu}_{\mathbf{S}} \cong \mathsf{Q}_{w\bullet\lambda,w\bullet\mu}, \quad \mathsf{P}^{\mu,\lambda}_{\mathbf{S}} \cong \mathsf{Q}_{w\bullet\mu,w\bullet\lambda}.$$

Using Lemma 5.3, we deduce isomorphisms

$$\begin{aligned}
&\mathsf{Q}_{\lambda,w\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} (\mathsf{P}^{\lambda,\mu}_{\mathbf{S}} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}^{\mu,\lambda}_{\mathbf{S}}) \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\lambda,\lambda} \\
&\quad \cong \mathsf{Q}_{\lambda,w\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\lambda,w\bullet\mu} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\mu,w\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\lambda,\lambda} \\
&\quad \cong \mathsf{Q}_{\lambda,w\bullet\mu} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{w\bullet\mu,\lambda}.
\end{aligned}$$

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Now the stabilizer of both μ' and $w \bullet \mu$ for the dot action of W_{aff} is $W_{\{\alpha\}}$, where $\alpha \in \mathfrak{R}^{\mathrm{s}}$ is the simple reflection such that $s' = s_\alpha$. By Lemma 5.15(i), it follows that we have isomorphisms

$$\mathsf{Q}_{\lambda, w\bullet\mu} \cong \mathsf{Q}_{\lambda,\mu'} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}^{\{\alpha\}}_{\mu', w\bullet\mu}, \quad \mathsf{Q}_{w\bullet\mu,\lambda} \cong \mathsf{Q}^{\{\alpha\}}_{w\bullet\mu,\mu'} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}_{\mu',\lambda},$$

from which we obtain an isomorphism

$$\mathsf{Q}_{\lambda, w\bullet\mu} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}_{w\bullet\mu,\lambda} \cong \mathsf{Q}_{\lambda,\mu'} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}^{\{\alpha\}}_{\mu', w\bullet\mu} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}^{\{\alpha\}}_{w\bullet\mu,\mu'} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}_{\mu',\lambda}.$$

Then by Lemma 5.15(ii) we have

$$\mathsf{Q}^{\{\alpha\}}_{\mu', w\bullet\mu} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}^{\{\alpha\}}_{w\bullet\mu,\mu'} \cong \mathsf{Q}^{\{\alpha\}}_{\mu',\mu'},$$

which implies (again using Lemma 5.15(i)) that

$$\mathsf{Q}_{\lambda, w\bullet\mu} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}_{w\bullet\mu,\lambda} \cong \mathsf{Q}_{\lambda,\mu'} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{S}}\mathfrak{g}} \mathsf{Q}_{\mu',\lambda}.$$

The desired claim follows, again using Lemma 5.5. □

6. Hecke action on the principal block

In this section we assume that $p > h$, where h is the Coxeter number of G (see Remark 3.3). In particular, this ensures that p is very good for G, so that the results of the previous sections are applicable.

6.1 Categories of G-modules and G-equivariant $\mathcal{U}\mathfrak{g}$-modules

We now take a closer look at the category $\mathsf{Rep}(G)$ of finite-dimensional algebraic G-modules, and review its decomposition into 'blocks'. This will involve the notation introduced in §§ 3.1–3.3.

Recall (see § 3.5) that for any $\lambda \in \mathbb{X}^+$ we have a simple G-module $\mathsf{L}(\lambda)$ of highest weight λ, and that all simple G-modules are of this form. The *linkage principle* (see [Jan03, Corollary II.6.17]) states that for $\lambda, \mu \in \mathbb{X}^+$ we have

$$\mathrm{Ext}^1_{\mathsf{Rep}(G)}(\mathsf{L}(\lambda), \mathsf{L}(\mu)) \neq 0 \ \Rightarrow \ W_{\mathrm{aff}} \bullet \lambda = W_{\mathrm{aff}} \bullet \mu.$$

As a consequence, if for a W_{aff}-orbit $\mathbf{c} \subset \mathbb{X}$ we denote by $\mathsf{Rep}_{\mathbf{c}}(G)$ the Serre subcategory of $\mathsf{Rep}(G)$ generated by the simple objects $\mathsf{L}(\lambda)$ with $\lambda \in \mathbf{c} \cap \mathbb{X}^+$, then we have a direct sum decomposition

$$\mathsf{Rep}(G) = \bigoplus_{\mathbf{c} \in \mathbb{X}/(W_{\mathrm{aff}}, \bullet)} \mathsf{Rep}_{\mathbf{c}}(G). \tag{6.1}$$

For $\lambda \in \mathbb{X}$, we will write $[\lambda]$ for the W_{aff}-orbit of λ. We will also set

$$\mathsf{Rep}_{\langle\lambda\rangle}(G) = \bigoplus_{\substack{\mathbf{c} \in \mathbb{X}/(W_{\mathrm{aff}}, \bullet) \\ \mathbf{c} \subset W_{\mathrm{ext}} \bullet \lambda}} \mathsf{Rep}_{\mathbf{c}}(G).$$

Consider the category $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ of G-equivariant finitely generated $\mathcal{U}\mathfrak{g}$-modules. For $\xi \in \mathfrak{t}^*/(W, \bullet)$, we will denote by

$$\mathsf{Mod}^{G,\xi}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$$

the full subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ whose objects are the modules annihilated by a power of the ideal $\mathfrak{m}^\xi \subset Z_{\mathrm{HC}}$. As for other similar notation, in the case where $\xi = \tilde{\lambda}$ for some $\lambda \in \mathbb{X}$, we will write $\mathsf{Mod}^{G,\lambda}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ for $\mathsf{Mod}^{G,\tilde{\lambda}}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$. If we denote by $\mathsf{Mod}^{G,\wedge}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ the category of G-equivariant finitely generated $\mathcal{U}\mathfrak{g}$-modules annihilated by a power of the ideal $\mathcal{I} \subset Z_{\mathrm{HC}} \cap Z_{\mathrm{Fr}}$ defined in § 3.6,

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then, as for example in (3.13), we have a canonical decomposition

$$\mathsf{Mod}_{\mathrm{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g}) = \bigoplus_{\lambda\in\Lambda} \mathsf{Mod}_{\mathrm{fg}}^{G,\lambda}(\mathcal{U}\mathfrak{g}),$$

where $\Lambda \subset \mathbb{X}$ is as in (3.13).

There is a natural fully faithful functor

$$\mathsf{Rep}(G) \to \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}\mathfrak{g}) \tag{6.2}$$

sending a G-module V to itself, with its G-module structure, and with the $\mathcal{U}\mathfrak{g}$-module structure obtained by differentiating the G-action. The essential image of this functor consists of the finite-dimensional G-equivariant $\mathcal{U}\mathfrak{g}$-modules having the property that their $\mathcal{U}\mathfrak{g}$-module structure is obtained from their G-module structure by differentiation. Since, for any $\lambda \in \mathbb{X}^+$, the action of Z_{HC} on $\mathsf{L}(\lambda)$ factors through the quotient $Z_{\mathrm{HC}}/\mathfrak{m}^\lambda$, the functor (6.2) restricts to a functor

$$\mathsf{Rep}_{[\lambda]}(G) \to \mathsf{Mod}_{\mathrm{fg}}^{G,\lambda}(\mathcal{U}\mathfrak{g})$$

for any $\lambda \in \mathbb{X}$. Since $\mathfrak{m}^\lambda$ only depends on the orbit $W_{\mathrm{ext}} \bullet \lambda$, in this way we also obtain a fully faithful functor

$$\mathsf{Rep}_{\langle\lambda\rangle}(G) \to \mathsf{Mod}_{\mathrm{fg}}^{G,\lambda}(\mathcal{U}\mathfrak{g}). \tag{6.3}$$

6.2 Action of completed bimodules

Recall the category $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^\wedge)$ introduced in § 3.6. There exists a canonical bifunctor

$$(-)\,\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\,(-) : \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^\wedge) \times \mathsf{Mod}_{\mathrm{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g}) \to \mathsf{Mod}_{\mathrm{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g})$$

which can be defined as follows. Consider some M in $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^\wedge)$ and some V in $\mathsf{Mod}_{\mathrm{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g})$. By definition, there exists $m \in \mathbb{Z}_{\geq 1}$ such that $\mathcal{I}^m$ acts trivially on V. Then the tensor product

$$(M/\mathcal{I}^m \cdot M) \otimes_{\mathcal{U}\mathfrak{g}} V$$

is a finitely generated left $\mathcal{U}\mathfrak{g}$-module (where in the tensor product we consider the right $\mathcal{U}\mathfrak{g}$-action on $M/\mathcal{I}^m \cdot M$), which does not depend on the choice of m, and which admits a natural (diagonal) algebraic G-module structure. Moreover, the action of $\mathcal{I}$ on this module is nilpotent. We can therefore take this as the definition of $M\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}V$.

The bifunctor $\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}$ defines on $\mathsf{Mod}_{\mathrm{fg}}^{G,\wedge}(\mathcal{U}\mathfrak{g})$ a module category structure for the monoidal category $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^\wedge)$. It is also easily seen that for $\lambda, \mu \in \mathbb{X}$ this bifunctor restricts to a bifunctor

$$\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}) \times \mathsf{Mod}_{\mathrm{fg}}^{G,\mu}(\mathcal{U}\mathfrak{g}) \to \mathsf{Mod}_{\mathrm{fg}}^{G,\lambda}(\mathcal{U}\mathfrak{g})$$

(where the category $\mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$ is as in § 3.5), which itself restricts to a bifunctor

$$\mathsf{HC}^{\hat{\lambda},\hat{\mu}} \times \mathsf{Rep}_{\langle\mu\rangle}(G) \to \mathsf{Rep}_{\langle\lambda\rangle}(G)$$

under the embeddings $\mathsf{HC}^{\hat{\lambda},\hat{\mu}} \to \mathsf{Mod}_{\mathrm{fg}}^{G}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$ and (6.3).

6.3 Relation with translation functors

Recall the definition of the translation functors for G-modules from [Jan03, Chap. II.7]. Fix $\lambda, \mu \in \mathbb{X}$, and denote by ν the only dominant W-translate of $\lambda - \mu$. Then the translation functor

$$T_\mu^\lambda : \mathsf{Rep}_{[\mu]}(G) \to \mathsf{Rep}_{[\lambda]}(G)$$

is the functor sending an object V to the direct summand of $\mathsf{L}(\nu) \otimes V$ which belongs to $\mathsf{Rep}_{[\lambda]}(G)$ in the decomposition provided by (6.1). We will consider these functors only in the case where

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λ and μ both belong to the closure of the fundamental alcove. In this setting, we have defined in § 3.5 an object $\mathsf{P}^{\lambda,\mu} \in \mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}}$.

Lemma 6.1. *Let $\lambda, \mu \in \mathbb{X}$ belonging to the closure of the fundamental alcove. The composition*

$$\mathsf{Rep}_{[\mu]}(G) \to \mathsf{Rep}_{\langle\mu\rangle}(G) \xrightarrow{\mathsf{P}^{\lambda,\mu}\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}(-)} \mathsf{Rep}_{\langle\lambda\rangle}(G)$$

is canonically isomorphic to the composition

$$\mathsf{Rep}_{[\mu]}(G) \xrightarrow{T^{\lambda}_{\mu}} \mathsf{Rep}_{[\lambda]}(G) \to \mathsf{Rep}_{\langle\lambda\rangle}(G).$$

Proof. By definition, the first functor sends a module V in $\mathsf{Rep}_{[\mu]}(G)$ to the quotient

$$(\mathsf{L}(\nu) \otimes V)/(\mathfrak{m}^{\lambda})^n \cdot (\mathsf{L}(\nu) \otimes V)$$

for $n \gg 0$, that is, to the direct sum of the factors in $\mathsf{L}(\nu) \otimes V$ corresponding to orbits included in $W_{\mathrm{ext}} \bullet \lambda$ in the decomposition provided by (6.1). However, all the T-weights in $\mathsf{L}(\nu) \otimes V$ belong to $\lambda + \mathbb{Z}\mathfrak{R}$. In view of Lemma 3.1(i), this implies that $[\lambda]$ is the only W_{aff}-orbit contained in $W_{\mathrm{ext}} \bullet \lambda$ that can contribute to the direct sum above. □

Remark 6.2. See [Ric10, Lemma 4.3.1] for a different proof of this claim, under more restrictive assumptions which would be sufficient for our present purposes.

6.4 Main result

We now consider the category D_{BS} of § 2.1 associated with the group $\mathbf{G} = G^{(1)}$. We also fix a weight λ in the fundamental alcove. (Such a weight exists since $p \geq h$.) For any $s \in S_{\mathrm{aff}}$, we choose a weight $\mu_s \in \mathbb{X}$ in the closure of the fundamental alcove, which lies on the wall associated with s but on no other wall. (For the existence of such a weight, see [Jan03, §II.6.3].) Once these choices have been made, the *wall-crossing functor* associated with $s \in S_{\mathrm{aff}}$ is the composition

$$\Theta_s := T^{\lambda}_{\mu_s} \circ T^{\mu_s}_{\lambda} : \mathsf{Rep}_{[\lambda]}(G) \to \mathsf{Rep}_{[\lambda]}(G).$$

The main result of the present section (and of this paper) is the following theorem.

Theorem 6.3. *There exists a monoidal functor*

$$\Psi^{\lambda} : \mathsf{D}_{\mathrm{BS}} \to \mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$$

such that

$$\Psi^{\lambda}(B_s) \cong \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda}$$

for any $s \in S_{\mathrm{aff}}$.

We explain the proof of this theorem in § 6.5. Before we do so, we show that (as explained in the introduction) this theorem implies the main conjecture of [RW18].

Corollary 6.4. *There exists a $\Bbbk$-linear right action of the monoidal category D_{BS} on $\mathsf{Rep}_{[\lambda]}(G)$ such that for any $s \in S_{\mathrm{aff}}$ the action of the object B_s is isomorphic to Θ_s.*

Proof. As explained in § 6.2, there exists a canonical (left) action of the category $\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$ on the category $\mathsf{Rep}_{\langle\lambda\rangle}(G)$. The category D_{BS} admits a canonical autoequivalence $\imath$ which satisfies $\imath(X \cdot Y) = \imath(Y) \cdot \imath(X)$ for any $X, Y \in \mathsf{D}_{\mathrm{BS}}$; see, for example, [RW18, § 4.2]. Using this autoequivalence, the functor of Theorem 6.3 therefore provides a right action of D_{BS} on $\mathsf{Rep}_{\langle\lambda\rangle}(G)$ such that B_s acts via the bimodule $\mathsf{P}^{\lambda,\mu_s}\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\mathsf{P}^{\mu_s,\lambda}$, for any $s \in S_{\mathrm{aff}}$. By Lemma 6.1, the action of this bimodule stabilizes the subcategory $\mathsf{Rep}_{[\lambda]}(G)$, and its action on this summand is isomorphic to Θ_s. We have therefore constructed the desired action. □

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Remark 6.5. It is clear from the proof of Corollary 6.4 that the existence of a *right* action of D_{BS} with the required action of each B_s is equivalent to the existence of a *left* action with the same property. The reason why Conjecture 1.1 mentions a right action is that it makes the comparison with the combinatorics of the category $\mathsf{Rep}_{[\lambda]}(G)$ easier.

See § 6.6 for a discussion of what can be said about the images under Ψ^λ of the generating morphisms of D_{BS}.

6.5 Proof of Theorem 6.3

Recall that Theorem 2.10 provides a monoidal functor

$$\mathsf{D}_{\mathrm{BS}} \to \mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{J}_{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}). \tag{6.4}$$

Since λ belongs to the fundamental alcove, its stabilizer for the dot action on $\mathbb{X}$ is trivial, so that the quotient morphism

$$\mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet)$$

is étale at $\bar{\lambda}$; see Lemma 3.2. Similarly, the Artin–Schreier map

$$\mathfrak{t}^* \to \mathfrak{t}^{*(1)}$$

is étale (everywhere, hence in particular at $\bar{\lambda}$), and sends $\bar{\lambda}$ to 0. Using these maps, we obtain morphisms

$$\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} \leftarrow \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$$

étale at $(\bar{\lambda}, \bar{\lambda})$, which identify the algebra $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$ from § 3.9 with the completion $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}}$ of $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ with respect to the maximal ideal corresponding to $(0, 0)$. Using this identification, the pullback functor associated with the natural morphism

$$\mathrm{Spec}(\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}}) \to \mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}$$

induces a monoidal functor

$$\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{J}_{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)}) \to \mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}), \tag{6.5}$$

where the category on the right-hand side is as in § 4.3. Precomposing this functor with (6.4), and then composing with the equivalence of Corollary 4.8, we obtain a monoidal functor

$$\Psi_{\mathbf{S}}^{\lambda} : \mathsf{D}_{\mathrm{BS}} \to \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}. \tag{6.6}$$

PROPOSITION 6.6. *For any $s \in S_{\mathrm{aff}} \cap W$, there exists an isomorphism*

$$\Psi_{\mathbf{S}}^{\lambda}(B_s) \cong \mathsf{P}_{\mathbf{S}}^{\lambda,\mu_s} \mathbin{\widehat{\otimes}}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_s,\lambda}.$$

Proof. In the course of the proof of Lemma 2.9 we have seen that the image of B_s in the category $\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathbb{J}_{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ is $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e,s\}} \mathfrak{t}^{*(1)})$, endowed with the trivial structure as a representation. On the other hand, by Proposition 5.8 the wall-crossing bimodule $\mathsf{P}_{\mathbf{S}}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_s,\lambda}$ corresponds to the object $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$ (again endowed with the trivial structure as a representation) under the equivalence $\mathscr{L}_{\lambda,\lambda}$. Recall that $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$ identifies with the completion of $\mathscr{O}(\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$ at the ideal corresponding to $(\bar{\lambda}, \bar{\lambda})$. The considerations in the proof of Lemma 5.11, together with the fact that the quotient morphism $\mathfrak{t}^* \to \mathfrak{t}^*/(W, \bullet)$ is étale at $\bar{\lambda}$, imply that the algebra $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}$ identifies with the completion of $\mathscr{O}(\mathfrak{t}^*/(\{e, s\}, \bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*)$

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at the ideal corresponding to the image of $(\bar{\mu}, \bar{\lambda})$, and that via this identification the morphism $\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}} \to \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}}$ is induced by the natural morphism

$$\mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* \to \mathfrak{t}^*/(\{e,s\},\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$$

sending $(\bar{\lambda}, \bar{\lambda})$ to the image of $(\bar{\mu}, \bar{\lambda})$. This morphism fits in a natural commutative diagram

$$\begin{array}{ccc} \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} \\ \downarrow & & \downarrow \\ \mathfrak{t}^*/(\{e,s\},\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^* & \longrightarrow & \mathfrak{t}^{*(1)}/\{e,s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)} \end{array}$$

where the right vertical arrow is induced by the natural quotient morphism $\mathfrak{t}^{*(1)} \to \mathfrak{t}^{*(1)}/\{e,s\}$ and the horizontal arrows are induced by the Artin–Schreier map. Here the morphism on the upper row is étale at $(\bar{\lambda}, \bar{\lambda})$, and that on the lower row is étale at the image of $(\bar{\mu}, \bar{\lambda})$ by the same arguments as for Lemma 4.7. This observation shows that $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\lambda}} \otimes_{\mathcal{Z}_{\mathbf{S}}^{\hat{\mu},\hat{\lambda}}} \mathcal{Z}_{\mathbf{S}}^{\hat{\lambda}}$ identifies with the $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}}$-module

$$\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}} \otimes_{\mathscr{O}(\mathfrak{t}^{*(1)}/\{e,s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}}} \mathscr{O}(\mathfrak{t}^{*(1)})^{\hat{0}},$$

where $\mathscr{O}(\mathfrak{t}^{*(1)}/\{e,s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})^{\hat{0},\hat{0}}$ is the completion of $\mathscr{O}(\mathfrak{t}^{*(1)}/\{e,s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$ at the ideal corresponding to the image of $(0,0)$, and $\mathscr{O}(\mathfrak{t}^{*(1)})^{\hat{0}}$ is the completion of $\mathscr{O}(\mathfrak{t}^{*(1)})$ (seen as an $\mathscr{O}(\mathfrak{t}^{*(1)}/\{e,s\} \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^{*(1)})$-module in the natural way) at the ideal corresponding to 0. Using the same considerations as in the proof of Lemma 5.11, it is easily seen that this module identifies with the completion of $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e,s\}} \mathfrak{t}^{*(1)})$, which finishes the proof of our claim. □

Remark 6.7. The isomorphism constructed in the proof of Proposition 6.6 is essentially canonical, in the sense that it only depends on the isomorphism of Proposition 5.8 (for the pair (λ, μ_s)), which itself only depends on a choice of isomorphism $\mathsf{L}(\nu_s)^* \cong \mathsf{L}(-w_0(\nu_s))$ where ν_s is the only dominant W-translate of $\mu_s - \lambda$; see Remark 5.9.

Recall the objects $(\Delta_w^{\mathbb{J}} : w \in W_{\mathrm{ext}})$ introduced at the end of § 2.4, and the objects $(\mathsf{Q}_{\nu,\eta} : \nu, \eta \in \mathbb{X})$ introduced in § 5.2.

Lemma 6.8. *For any $w \in W_{\mathrm{ext}}$, the object $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{\lambda,w\bullet\lambda})$ is isomorphic to the image of $\Delta_w^{\mathbb{J}}$ under the functor* (6.5).

Proof. Write $w = t_\nu x$ with $\nu \in p\mathbb{X}$ and $x \in W$. Then by Lemma 5.3 we have

$$\mathsf{Q}_{\lambda,w\bullet\lambda} = \mathsf{Q}_{\lambda,x\bullet\lambda+p\nu} \cong \mathsf{Q}_{\lambda,x\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}\mathfrak{s}\mathfrak{g}} \mathsf{Q}_{x\bullet\lambda,x\bullet\lambda+p\nu}.$$

It follows from Lemma 5.4 and the proof of Lemma 2.8 that $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{x\bullet\lambda,x\bullet\lambda+p\nu})$ is the image of $\Delta_{t_\nu}^{\mathbb{J}}$. In view of (2.3) and the monoidality of $\mathscr{L}_{\lambda,\lambda}^{-1}$ (see Proposition 5.12), to conclude it therefore suffices to prove that $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{\lambda,x\bullet\lambda})$ is the image of $\Delta_x^{\mathbb{J}}$. In turn, if $x = s_1 \cdots s_r$ is a reduced expression (with each s_i in $W \cap S_{\mathrm{aff}}$) then again by Lemma 5.3 we have

$$\mathsf{Q}_{\lambda,x\bullet\lambda} \cong \mathsf{Q}_{\lambda,s_1\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}\mathfrak{s}\mathfrak{g}} \mathsf{Q}_{s_1\bullet\lambda,s_1s_2\bullet\lambda} \mathbin{\widehat{\otimes}}_{\mathcal{U}\mathfrak{s}\mathfrak{g}} \cdots \mathbin{\widehat{\otimes}}_{\mathcal{U}\mathfrak{s}\mathfrak{g}} \mathsf{Q}_{(s_1\cdots s_{r-1})\bullet\lambda,x\bullet\lambda}.$$

If we write $y_j = s_1 \cdots s_j$ for $j \in \{0,\dots,r\}$ then, by monoidality of $\mathscr{L}_{\lambda,\lambda}$, to conclude it suffices to prove that $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{y_{i-1}\bullet\lambda,y_i\bullet\lambda})$ is the image of $\Delta_{s_i}^{\mathbb{J}}$ for any $i \in \{1,\dots,r\}$.

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Fix $i \in \{1, \dots, r\}$. We have $y_i \bullet \lambda < y_{i-1} \bullet \lambda$. By Lemma 5.7 we therefore have an exact sequence

$$\mathsf{Q}_{y_i \bullet \lambda, y_{i-1} \bullet \lambda} \hookrightarrow \mathsf{P}_{\mathbf{S}}^{\lambda, \mu_{s_i}} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_{s_i}, \lambda} \twoheadrightarrow \mathsf{Q}_{y_{i-1} \bullet \lambda, y_{i-1} \bullet \lambda}.$$

As seen in the course of the proof of Proposition 6.6, $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{P}_{\mathbf{S}}^{\lambda, \mu_{s_i}} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_{s_i}, \lambda})$ corresponds to the completion of $\mathscr{O}(\mathfrak{t}^{*(1)} \times_{\mathfrak{t}^{*(1)}/\{e, s_i\}} \mathfrak{t}^{*(1)})$ (with the trivial structure as a representation of the appropriate group scheme), and it is clear that $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{y_{i-1} \bullet \lambda, y_{i-1} \bullet \lambda})$ is the completion of $\mathscr{O}(\mathfrak{t}^{*(1)})$. The object $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{y_i \bullet \lambda, y_{i-1} \bullet \lambda})$ is therefore the kernel of a surjection from the former completion to the latter completion. However, up to an automorphism of the completion of $\mathscr{O}(\mathfrak{t}^{*(1)})$ there exists only one such surjection, and its kernel corresponds to the completion of $\Delta_{s_i}^{\mathbb{J}}$ by Lemma 2.5. □

Once Lemma 6.8 is proved, one also obtains that $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{w \bullet \lambda, \lambda})$, which is the inverse of $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{\lambda, w \bullet \lambda})$ (see § 5.2), is isomorphic to the image of $\Delta_{w^{-1}}^{\mathbb{J}}$ under (6.5). (In fact, one can then check that, for any $\mu \in W_{\mathrm{ext}} \bullet \lambda$, $\mathscr{L}_{\lambda,\lambda}^{-1}(\mathsf{Q}_{\mu, w \bullet \mu})$ is isomorphic to the image of $\Delta_w^{\mathbb{J}}$.)

We can finally complete the proof of Theorem 6.3.

Proof of Theorem 6.3. Consider the functor $\Psi_{\mathbf{S}}^{\lambda}$ from (6.6). We claim that for any $s \in S_{\mathrm{aff}}$ we have

$$\Psi_{\mathbf{S}}^{\lambda}(B_s) \cong \mathsf{P}_{\mathbf{S}}^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_s, \lambda}. \tag{6.7}$$

In fact, if $s \in W$ this is the content of Proposition 6.6. Otherwise, as already seen in the course of the proof of Lemma 2.9, there exist $x \in W_{\mathrm{ext}}$ and $t \in S_{\mathrm{aff}} \cap W$ such that $s = xtx^{-1}$. Then, by Lemma 2.4, the image of B_s in $\mathsf{C}_{\mathrm{ext}}$ is

$$B_s^{\mathrm{Bim}} \cong \Delta_x \otimes_R B_t^{\mathrm{Bim}} \otimes_R \Delta_{x^{-1}};$$

using Lemma 6.8 (together with the remark following it) and the known description of $\Psi_{\mathbf{S}}^{\lambda}(B_t)$, we deduce that

$$\Psi_{\mathbf{S}}^{\lambda}(B_s) \cong \mathsf{Q}_{x^{-1} \bullet \lambda, \lambda} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\lambda, \mu_t} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{P}_{\mathbf{S}}^{\mu_t, \lambda} \widehat{\otimes}_{\mathcal{U}_{\mathbf{S}\mathfrak{g}}} \mathsf{Q}_{\lambda, x^{-1} \bullet \lambda}.$$

In view of Proposition 5.16, this implies (6.7).

Since each object of D_{BS} is isomorphic to a shift of a product of objects B_s, and since both of the functors involved are monoidal, our claim implies that $\Psi_{\mathbf{S}}^{\lambda}$ takes values in the essential image of the fully faithful functor

$$\mathsf{HC}_{\mathrm{diag}}^{\hat{\lambda}, \hat{\lambda}} \to \mathsf{HC}_{\mathbf{S}}^{\hat{\lambda}, \hat{\lambda}}$$

(see Proposition 3.7). It follows that $\Psi_{\mathbf{S}}^{\lambda}$ factors in a canonical way through a monoidal functor

$$\Psi^{\lambda} : \mathsf{D}_{\mathrm{BS}} \to \mathsf{HC}^{\hat{\lambda}, \hat{\lambda}}$$

sending B_s to $\mathsf{P}^{\lambda, \mu_s} \widehat{\otimes}_{\mathcal{U}_{\mathfrak{g}}} \mathsf{P}^{\mu_s, \lambda}$ for any $s \in S_{\mathrm{aff}}$, which finishes the proof. □

6.6 Images of generating morphisms

The category D_{BS} is defined in [EW16] in terms of generators and relations. In Theorem 6.3 we have explained what are the images of the generating objects under Ψ^{λ} (at least, up to isomorphism); by monoidality this determines the image of any object in D_{BS} (again, up to isomorphism). We finish the paper with a discussion of what can be said about the image under Ψ^{λ} of the generating morphisms of D_{BS}.

Remark 6.9. As explained in [RW18, Remark 5.1.2(3)], although the original version of Conjecture 1.1 contained information about these images, this information is not needed for

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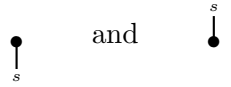
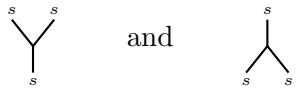
the applications considered in [RW18, Part I], and in particular the character formula for tilting modules in $\mathsf{Rep}_{[\lambda]}(G)$.

Recall that these generating morphisms fall into four families:

– the polynomials (morphisms from $B_\varnothing$ to a shift of $B_\varnothing$, determined by homogeneous elements in $R = \mathscr{O}(\mathfrak{t}^{*(1)})$);
– the 'dot' morphisms for $s \in S_{\mathrm{aff}}$



(morphisms from B_s to a shift of $B_\varnothing$ and from $B_\varnothing$ to a shift of B_s);
– the 'trivalent' morphisms for $s \in S_{\mathrm{aff}}$



(morphisms from B_s to a shift of B_{ss} and from B_{ss} to a shift of B_s);
– the '$2m_{s,t}$-valent' morphisms, for pairs (s,t) of distinct elements of S_{aff} generating a finite subgroup of W_{aff}.

The information we can give concerns only the first two families of morphisms.

The image of polynomials is easy to describe: we have $\Psi^\lambda(B_\varnothing) = \mathcal{U}^{\hat{\lambda}}$. If $Z^{\hat{\lambda}}_{\mathrm{HC}}$ is as in Remark 3.5, any element in $Z^{\hat{\lambda}}_{\mathrm{HC}}$ determines an endomorphism of $\mathcal{U}^{\hat{\lambda}}$. Now the natural morphisms

$$\mathfrak{t}^*/(W,\bullet) \leftarrow \mathfrak{t}^* \rightarrow \mathfrak{t}^{*(1)}$$

are étale at $\bar{\lambda}$; they therefore determine an isomorphism between $Z^{\hat{\lambda}}_{\mathrm{HC}}$ and the completion $\mathscr{O}(\mathfrak{t}^{*(1)})^{\hat{0}}$ of $\mathscr{O}(\mathfrak{t}^{*(1)})$ with respect to the ideal of 0. The image under Ψ^λ of a homogenous polynomial in R is the endomorphism of $\mathcal{U}^{\hat{\lambda}}$ determined by the corresponding element in $Z^{\hat{\lambda}}_{\mathrm{HC}}$.

To describe the image of the other morphisms, we must be more specific about the isomorphism $\Psi^\lambda(B_s) \cong \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda}$. First, assume that $s \in W$. In this case, after choosing an isomorphism $\mathsf{L}(\nu_s)^* \cong \mathsf{L}(-w_0(\nu_s))$ we obtain a canonical such isomorphism; see Remark 6.7. Using this isomorphism, Remark 5.9 shows that the image of the upper dot morphism

is the morphism $\varphi_s : \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda} \to \mathcal{U}^{\hat{\lambda}}$ determined by the adjunction

$$(\mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} (-),\ \mathsf{P}^{\mu_s,\lambda} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} (-))$$

defined by our choice of isomorphism $\mathsf{L}(\nu_s)^* \cong \mathsf{L}(-w_0(\nu_s))$. (Here we use an obvious variant of Lemma 3.6 for $\mathcal{U}\mathfrak{g}$ in place of $\mathcal{U}_{\mathfrak{S}}\mathfrak{g}$.)

Proposition 3.7, Corollary 4.8 and Proposition 5.8 (together with the various étale maps considered above) show that the $\mathscr{O}(\mathfrak{t}^{*(1)})^{\hat{0}}$-modules

$$\mathrm{Hom}_{\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}}(\mathcal{U}^{\hat{\lambda}},\mathcal{U}^{\hat{\lambda}}) \quad \text{and} \quad \mathrm{Hom}_{\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}}(\mathcal{U}^{\hat{\lambda}}, \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda})$$

are both free of rank 1; from this one can check that there exists a unique morphism

$$\psi_s : \mathcal{U}^{\hat{\lambda}} \to \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda}$$

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whose composition with φ_s is the differential of the coroot of $(\mathbf{G}, \mathbf{T})$ associated with s (seen as an endomorphism of $\mathcal{U}^{\hat{\lambda}}$), and that this morphism is a generator of $\mathrm{Hom}_{\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}}(\mathcal{U}^{\hat{\lambda}}, \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda})$. In view of the 'barbell relation' in D_{BS}, the image of the lower dot morphism

is ψ_s.

In the case where $s \notin W_{\mathrm{aff}}$, we do not have a canonical choice of isomorphism $\Psi^{\lambda}(B_s) \cong \mathsf{P}^{\lambda,\mu_s} \widehat{\otimes}_{\mathcal{U}\mathfrak{g}} \mathsf{P}^{\mu_s,\lambda}$. What we can say is that there exists a choice of such an isomorphism (not unique) such that the images of the dot morphisms are described by the same rules as above.

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While we were in the final stages of our work, we were informed that J. Ciappara had obtained a radically different proof of Conjecture 1.1. His construction of the action relies on the geometric Satake equivalence and the Smith–Treumann theory of [RW20]; these results have now appeared in [Cia21]. We thank G. Williamson for keeping us informed of this work, and for useful comments.

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Appendix A. Index of notation

Below is a list of the main notation used in the paper, listed by section of appearance. (We sometimes omit notation used only in one specific subsection.)

A.1 Section 2

$\Bbbk$: algebraically closed field of characteristic p, § 2.1.
$\mathbf{G}$: connected reductive algebraic group over $\Bbbk$, § 2.1.
$\mathbf{B}$, $\mathbf{T}$: Borel subgroup and maximal torus in $\mathbf{G}$, § 2.1.
$\mathbf{g}$, $\mathbf{b}$, $\mathbf{t}$: Lie algebras of $\mathbf{G}$, $\mathbf{B}$, $\mathbf{T}$, § 2.1.
$\mathbf{X}$, $\mathbf{X}^\vee$: weight and coweight lattices of $\mathbf{T}$, § 2.1.
Φ, $\Phi^\vee$, Φ^+, Φ^{s}: roots, coroots, positive roots, simple roots of $(\mathbf{G}, \mathbf{B}, \mathbf{T})$, § 2.1.
κ: choice of isomorphism $\mathbf{g} \xrightarrow{\sim} \mathbf{g}^*$, § 2.1.
$\mathbf{W}$: Weyl group of $(\mathbf{G}, \mathbf{T})$, § 2.1.
W_{aff}, W_{ext}: affine and extended affine Weyl groups of $(\mathbf{G}, \mathbf{T})$, § 2.1.
S_{aff}, W_{ext}: simple reflections in W_{aff}, § 2.1.
D_{BS}: diagrammatic Hecke category attached to $\mathbf{G}$, § 2.1.

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$B_{\underline{w}}$: object of D_{BS} attached to $\underline{w}$, § 2.1.
$R = \mathscr{O}(\mathbf{t}^*)$, § 2.1.
Q: fraction field of R, § 2.2.
C, C', $\mathsf{C}_{\mathrm{ext}}$, $\mathsf{C}'_{\mathrm{ext}}$: Abe's categories, § 2.2.
B_s^{Bim}: bimodule in C attached to s, § 2.2.
Δ_x: 'standard' object in $\mathsf{C}_{\mathrm{ext}}$ attached to x, § 2.2.
$\mathbf{U}$: unipotent radical in $\mathbf{B}$, § 2.3.
$\mathbf{n}$: Lie algebra of $\mathbf{U}$, § 2.3.
$\mathbf{g}_{\mathrm{reg}}$, $\mathbf{g}^*_{\mathrm{reg}}$: regular parts in $\mathbf{g}$ and $\mathbf{g}^*$, § 2.3.
$\mathbb{J}_{\mathrm{reg}}$, $\mathbb{J}^*_{\mathrm{reg}}$: universal centralizers over $\mathbf{g}_{\mathrm{reg}}$ and $\mathbf{g}^*_{\mathrm{reg}}$, § 2.3.
$\tilde{\mathbf{g}}$: Grothendieck resolution attached to $\mathbf{G}$, § 2.3.
$\tilde{\mathbf{g}}_{\mathrm{reg}}$: regular part in $\tilde{\mathbf{g}}$, § 2.3.
π: natural morphism $\tilde{\mathbf{g}} \to \mathbf{g}^*$, § 2.3.
ϑ: natural morphism $\tilde{\mathbf{g}} \to \mathbf{t}^*$, § 2.3.
$\mathbf{g}_{\mathrm{rs}}$, $\mathbf{g}^*_{\mathrm{rs}}$: regular semisimple parts in $\mathbf{g}$ and $\mathbf{g}^*$, § 2.3.
$\mathbb{J}_{\mathrm{rs}}$, $\mathbb{J}^*_{\mathrm{rs}}$: restrictions of $\mathbb{J}_{\mathrm{reg}}$, $\mathbb{J}^*_{\mathrm{reg}}$ to $\mathbf{g}_{\mathrm{rs}}$, $\mathbf{g}^*_{\mathrm{rs}}$, § 2.3.
$\mathbf{S}$, $\mathbf{S}^*$: Kostant sections in $\mathbf{g}$ and $\mathbf{g}^*$, § 2.3.
$\mathbb{J}^*_{\mathbf{S}}$: restriction of $\mathbb{J}^*_{\mathrm{reg}}$ to $\mathbf{S}^*$, § 2.3.
$\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$: category of $\mathbb{G}_{\mathrm{m}}$-equivariant representations of $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$ on coherent sheaves on $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*$, § 2.4.
$\star$: monoidal product on $\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$, § 2.4.
$\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}_{\mathrm{fl}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$: subcategory of $\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ consisting of modules flat with respect to the second projection $\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^* \to \mathbf{t}^*$, § 2.4.
$\Delta^{\mathbb{J}}_w$: object in $\mathsf{Rep}^{\mathbb{G}_{\mathrm{m}}}(\mathbf{t}^* \times_{\mathbf{t}^*/\mathbf{W}} \mathbb{J}^*_{\mathbf{S}} \times_{\mathbf{t}^*/\mathbf{W}} \mathbf{t}^*)$ corresponding to Δ_w, § 2.4.

A.2 Section 3

G: connected reductive algebraic group such that $\mathbf{G} = G^{(1)}$, § 3.1.
Fr: Frobenius morphism of G, § 3.1.
B, T, U: subgroups of G corresponding to $\mathbf{B}$, $\mathbf{T}$, $\mathbf{U}$, § 3.1.
$\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{n}$: Lie algebras of G, B, T, U, § 3.1.
W: Weyl group of (G, T), § 3.1.
$\mathbf{X}$, $\mathbf{X}^\vee$: weight and coweight lattices of T, § 3.1.
$\mathfrak{R}$, $\mathfrak{R}^\vee$, $\mathfrak{R}^+$, $\mathfrak{R}^{\mathrm{s}}$: roots, coroots, positive roots, simple roots of (G, B, T), § 3.1.
w_0: longest element in W, § 3.1.
ρ: half-sum of the positive roots, § 3.1.
$\bullet$: dot action of W_{ext} on $\mathbb{X}$ and W on $\mathfrak{t}^*$, § 3.1.
$\bar{\lambda}$: element of $\mathfrak{t}^*$ associated with $\lambda \in \mathbb{X}$, § 3.1.
$\tilde{\lambda}$: image of $\bar{\lambda}$ in $\mathfrak{t}^*/(W, \bullet)$, § 3.1.
$\mathfrak{t}^*_{\mathbb{F}_p}$: 'integral' part of $\mathfrak{t}^*$, § 3.1.
W_I: subgroup of W associated with $I \subset \mathfrak{R}^{\mathrm{s}}$, § 3.1.
$\mathcal{U}\mathfrak{g}$: universal enveloping algebra of $\mathfrak{g}$, § 3.2.
Z_{HC}, Z_{Fr}: Harish-Chandra and Frobenius centers of $\mathcal{U}\mathfrak{g}$, § 3.2.
AS: Artin–Schreier morphism, § 3.2.
$\mathfrak{C}$: spectrum of $Z(\mathcal{U}\mathfrak{g})$, § 3.2.
$\mathfrak{m}_\eta$: ideal in Z_{Fr} associated with $\eta \in \mathfrak{g}^{*(1)}$, § 3.3.
$\mathfrak{m}^\xi$: ideal in Z_{HC} associated with $\xi \in \mathfrak{t}^*/(W, \bullet)$, § 3.3.

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$\mathcal{U}_\eta\mathfrak{g}$, $\mathcal{U}^\xi\mathfrak{g}$, $\mathcal{U}^\xi_\eta\mathfrak{g}$: central reductions of $\mathcal{U}\mathfrak{g}$, § 3.3.
$\mathcal{N}^*$: nilpotent cone in $\mathfrak{g}^*$, § 3.3.
HC: category of Harish-Chandra bimodules for G, § 3.4.
$\mathcal{Z} = Z(\mathcal{U}\mathfrak{g}) \otimes_{Z_{\mathrm{Fr}}} Z(\mathcal{U}\mathfrak{g})$, § 3.4.
$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}})$: category of G-equivariant finitely generated $\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}$-modules, § 3.4.
$\mathfrak{D} = \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)$, § 3.5.
$\mathcal{I}^{\lambda,\mu}$: ideal in $\mathscr{O}(\mathfrak{D})$ associated with $\lambda,\mu \in \mathbb{X}$, § 3.5.
$\mathfrak{D}^{\hat{\lambda},\hat{\mu}}$: spectrum of the completion of $\mathscr{O}(\mathfrak{D})$ with respect to $\mathcal{I}^{\lambda,\mu}$, § 3.5.
$\mathcal{U}^{\hat{\lambda},\hat{\mu}} = (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}})$, § 3.5.
$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$: category of G-equivariant finitely generated $\mathcal{U}^{\hat{\lambda},\hat{\mu}}$-modules, § 3.5.
$\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$: subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$ of Harish-Chandra bimodules, § 3.5.
$\mathsf{HC}^{\hat{\lambda},\hat{\mu}}_{\mathrm{diag}}$: subcategory of $\mathsf{HC}^{\hat{\lambda},\hat{\mu}}$ of diagonally induced bimodules, § 3.5.
$\mathsf{C}^{\lambda,\mu}(-) = \mathscr{O}(\mathfrak{D}^{\hat{\lambda},\hat{\mu}}) \otimes_{\mathscr{O}(\mathfrak{D})} (-)$, § 3.5.
$\mathcal{U}^{\hat{\lambda}} = \mathsf{C}^{\lambda,\lambda}(\Bbbk \otimes \mathcal{U}\mathfrak{g})$, § 3.5.
h: Coxeter number of G, § 3.5.
$\mathsf{P}^{\lambda,\mu}$: translation bimodule attached to $\lambda,\mu \in \mathbb{X}$, § 3.5.
Λ: set of representatives for the $(W_{\mathrm{ext}},\bullet)$-orbits on $\mathbb{X}$, § 3.6.
$\mathcal{I}$: ideal of $\mathscr{O}(\mathfrak{t}^{*(1)}/W)$ corresponding to the image of 0, § 3.6.
$\mathfrak{D}^\wedge$: spectrum of the completion of $\mathscr{O}(\mathfrak{D})$ with respect to $\mathcal{I}\cdot\mathscr{O}(\mathfrak{D})$, § 3.6.
$\mathscr{O}(\mathfrak{t}^{*(1)}/W)^\wedge$: completion of $\mathscr{O}(\mathfrak{t}^{*(1)}/W)$ with respect to $\mathcal{I}$, § 3.6.
$Z^\wedge_{\mathrm{HC}}$: completion of Z_{HC} with respect to $\mathcal{I}\cdot Z_{\mathrm{HC}}$, § 3.6.
$Z^{\hat{\lambda}}_{\mathrm{HC}}$: completion of Z_{HC} with respect to $\mathfrak{m}^\lambda$, § 3.6.
$\mathcal{U}^\wedge = (\mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathcal{U}\mathfrak{g}^{\mathrm{op}}) \otimes_{\mathscr{O}(\mathfrak{D})} \mathscr{O}(\mathfrak{D}^\wedge)$, § 3.6.
$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$: category of G-equivariant finitely generated $\mathcal{U}^\wedge$-modules, § 3.6.
$\mathsf{HC}^\wedge$: subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$ of Harish-Chandra bimodules, § 3.6.
$\mathsf{HC}^\wedge_{\mathrm{diag}}$: subcategory of $\mathsf{HC}^\wedge$ of diagonally induced bimodules, § 3.6.
$\mathsf{C}^\wedge(-) = \mathscr{O}(\mathfrak{D}^\wedge) \otimes_{\mathscr{O}(\mathfrak{D})} (-)$, § 3.6.
$(-)\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}(-)$: monoidal product for the categories $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}})$, § 3.7.
$\mathbb{I}^*_{\mathbf{S}}$: group scheme over $\mathbf{S}^*$, § 3.8.
$\mathcal{U}_{\mathbf{S}}\mathfrak{g} := \mathcal{U}\mathfrak{g} \otimes_{Z_{\mathrm{Fr}}} \mathscr{O}(\mathbf{S}^*)$, § 3.8.
$\mathfrak{C}_{\mathbf{S}} := \mathbf{S}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W,\bullet)$, § 3.8.
$\mathcal{Z}_{\mathbf{S}} = \mathcal{Z} \otimes_{Z_{\mathrm{Fr}}} \mathscr{O}(\mathbf{S}^*)$, § 3.9.
$\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})$: category of $\mathbb{I}^*_{\mathbf{S}}$-equivariant finitely generated $\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}}$-modules, § 3.9.
$\mathsf{HC}_{\mathbf{S}}$: subcategory of $\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})$ of Harish-Chandra bimodules, § 3.9.
$\mathcal{I}^{\lambda,\mu}_{\mathbf{S}} = \mathcal{I}^{\lambda,\mu} \cdot \mathcal{Z}_{\mathbf{S}}$, § 3.9.
$\mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$: completion of $\mathcal{Z}_{\mathbf{S}}$ with respect to $\mathcal{I}^{\lambda,\mu}_{\mathbf{S}}$, § 3.9.
$\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} = \mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} \otimes_{\mathcal{Z}_{\mathbf{S}}} (\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})$, § 3.9.
$\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}} = \mathrm{Spec}(\mathcal{Z}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}) \times_{\mathbf{S}^*} \mathbb{I}^*_{\mathbf{S}}$, § 3.9.
$\mathsf{Mod}^{\mathbb{I}}_{\mathrm{fg}}(\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}})$: category of $\mathbb{I}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$-equivariant finitely generated $\mathcal{U}^{\hat{\lambda},\hat{\mu}}_{\mathbf{S}}$-modules, § 3.9.

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$\mathsf{HC}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$: subcategory of $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$ of Harish-Chandra bimodules, § 3.9.
$\mathsf{P}_{\mathbf{S}}^{\lambda,\mu} = \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathsf{P}^{\lambda,\mu}$, § 3.9.
$(-)\widehat{\otimes}_{\mathcal{U}_{\mathbf{S}}\mathfrak{g}}(-)$: monoidal product for the categories $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, § 3.9.
$\mathcal{Z}_{\mathbf{S}}^{\wedge}$: completion of $\mathcal{Z}_{\mathbf{S}}$ with respect to $\mathcal{I} \cdot \mathcal{Z}_{\mathbf{S}}$, § 3.9.
$\mathcal{U}_{\mathbf{S}}^{\wedge} = \mathcal{Z}_{\mathbf{S}}^{\wedge} \otimes_{\mathcal{Z}_{\mathbf{S}}} (\mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} \mathcal{U}_{\mathbf{S}}\mathfrak{g}^{\mathrm{op}})$, § 3.9.
$\mathbb{I}_{\mathbf{S}}^{\wedge} = \mathrm{Spec}(\mathcal{Z}_{\mathbf{S}}^{\wedge}) \times_{\mathbf{S}^*} \mathbb{I}_{\mathbf{S}}^*$, § 3.9.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{U}_{\mathbf{S}}^{\wedge})$: category of $\mathbb{I}_{\mathbf{S}}^{\wedge}$-equivariant finitely generated $\mathcal{U}_{\mathbf{S}}^{\wedge}$-modules, § 3.9.
$\mathsf{C}_{\mathbf{S}}^{\wedge}(-) = \mathscr{O}(\mathbf{S}^*) \otimes_{Z_{\mathrm{Fr}}} \mathsf{C}^{\wedge}(-)$, § 3.9.
$\mathcal{U}_{\mathbf{S}}^{\hat{\lambda}} = \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathcal{U}^{\hat{\lambda}}$, § 3.9.

A.3 Section 4

$\mathsf{Z}_{\eta,B'}(\xi)$: baby Verma module, § 4.2.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\wedge})$: category of representations of $\mathbb{I}_{\mathbf{S}}^{\wedge}$ on finitely generated $\mathcal{Z}_{\mathbf{S}}^{\wedge}$-modules, § 4.3.
$(-)\,\hat{\star}_{\mathbf{S}}\,(-)$: monoidal product on $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\wedge})$, § 4.3.
$\mathbb{J}_{\mathbf{S}}^{\wedge} = \mathrm{Spec}(\mathcal{Z}_{\mathbf{S}}^{\wedge}) \times_{\mathbf{S}^*} \mathbb{J}_{\mathbf{S}}^*$, § 4.3.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\wedge})$: category of representations of $\mathbb{J}_{\mathbf{S}}^{\wedge}$ on finitely generated $\mathcal{Z}_{\mathbf{S}}^{\wedge}$-modules, § 4.3.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$: category of representations of $\mathbb{I}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ on finitely generated $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-modules, § 4.3.
$\mathbb{J}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}} = \mathrm{Spec}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}) \times_{\mathbf{S}^*} \mathbb{J}_{\mathbf{S}}^*$, § 4.3.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{J}}(\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$: category of representations of $\mathbb{J}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$ on finitely generated $\mathcal{Z}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-modules, § 4.3.
$\mathsf{M}^{\lambda,\mu} = \mathsf{P}^{\lambda,-\rho}\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}\mathsf{P}^{-\rho,\mu}$, § 4.3.
$\mathsf{M}_{\mathbf{S}}^{\lambda,\mu} = \mathscr{O}(\mathbf{S}^*) \otimes_{\mathscr{O}(\mathfrak{g}^{*(1)})} \mathsf{M}^{\lambda,\mu}$, § 4.3.
$\tilde{\mathfrak{D}} = \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*$, § 4.4.
$\tilde{\mathfrak{D}}(\lambda)$, $\mathfrak{D}(\lambda)$: irreducible components associated with $\lambda \in \mathbb{X}$, § 4.4.
$\tilde{\Lambda}$: set of representatives for $\mathbb{X}/p\mathbb{X}$, § 4.4.
$\mathfrak{t}^*_{\circ}$, open subset in $\mathfrak{t}^*$, § 4.4.
$\tilde{\mathbf{S}}^*$: preimage of $\mathbf{S}^*$ in $\tilde{\mathfrak{g}}$, § 4.4.
$\mathscr{L}_{\lambda,\mu}$: localization equivalence, § 4.6.

A.4 Section 5

$\mathcal{B} = G/B$, § 5.1.
ω: natural morphism $G/U \to \mathcal{B}$, § 5.1.
$\tilde{\mathscr{D}}$: universal twisted differential operators on $\mathcal{B}$, § 5.1.
$\tilde{\mathscr{D}}_{\mathbf{S}} := \tilde{\mathscr{D}}_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}$, § 5.1.
$\tilde{\mathcal{U}}_{\mathbf{S}}\mathfrak{g} := \mathcal{U}_{\mathbf{S}}\mathfrak{g} \otimes_{Z_{\mathrm{HC}}} \mathscr{O}(\mathfrak{t}^*)$, § 5.1.
$\mathscr{O}_{\mathcal{B}}(\lambda)$: line bundle on $\mathcal{B}$ attached to $\lambda \in \mathbb{X}$, § 5.2.
$\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$: completion of $\tilde{\mathcal{U}}_{\mathbf{S}}\mathfrak{g} \otimes_{\mathscr{O}(\mathbf{S}^*)} (\tilde{\mathcal{U}}_{\mathbf{S}}\mathfrak{g})^{\mathrm{op}}$ at the ideal corresponding to $(\bar{\lambda}, \bar{\mu}) \in \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$, § 5.2.
$\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$: category of equivariant finitely generated $\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}}$-modules, § 5.2.
$(-)\widehat{\otimes}_{\tilde{\mathcal{U}}_{\mathbf{S}}\mathfrak{g}}(-)$: monoidal product for the categories $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, § 5.2.
$\mathsf{Q}_{\lambda,\mu}$: completion of $\Gamma\big(\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*, (\mathscr{O}_{\mathcal{B}}(\lambda-\mu) \otimes_{\mathscr{O}_{\mathcal{B}}} \tilde{\mathscr{D}})_{|\tilde{\mathbf{S}}^* \times_{\mathfrak{t}^{*(1)}} \mathfrak{t}^*}\big)$, § 5.2.
$\langle\eta\rangle$: twist functor on $\mathsf{Mod}_{\mathrm{fg}}^{\mathbb{I}}(\tilde{\mathcal{U}}_{\mathbf{S}}^{\hat{\lambda},\hat{\mu}})$, § 5.2.
$\mathfrak{E} = \mathfrak{t}^* \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*/(W, \bullet)$, § 5.3.

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$\mathfrak{E}' := \mathfrak{t}^*/(W,\bullet) \times_{\mathfrak{t}^{*(1)}/W} \mathfrak{t}^*$, § 5.3.
$\mathscr{O}(\mathfrak{E})^{\hat{\lambda},\hat{\mu}}$: completion of $\mathscr{O}(\mathfrak{E})$ at the ideal corresponding to $(\bar{\lambda},\tilde{\mu})$, § 5.3.
P_I, $\mathbf{P}_I$: standard parabolic subgroups in G and $\mathbf{G}$ associated with $I \subset \mathfrak{R}^{\mathrm{s}}$, § 5.6.
L_I, $\mathbf{L}_I$, U_I, $\mathbf{U}_I$: Levi factor and unipotent radical of P_I and $\mathbf{P}_I$, § 5.6.
$\mathcal{P}_I = G/P_I$, § 5.6.
ω_I: natural morphism $G/U_I \to \mathcal{P}_I$, § 5.6.
$\tilde{\mathscr{D}}_I$: twisted differential operators on $\mathcal{P}_I$, § 5.6.
$\tilde{\mathbf{g}}_I$: parabolic Grothendieck resolution for $\mathbf{G}$, § 5.6.
$\tilde{\mathbf{S}}_I^*$: preimage of $\mathbf{S}^*$ in $\tilde{\mathbf{g}}_I$, § 5.6.
$\tilde{\mathscr{D}}_{I,\mathbf{S}} = (\tilde{\mathscr{D}}_I)_{|\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}$, § 5.6.
$\mathsf{Q}^I_{\lambda,\mu}$: completion of $\Gamma\big(\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet), (\mathscr{O}_{\mathcal{P}_I}(\lambda-\mu) \otimes \tilde{\mathscr{D}}_I)_{|\tilde{\mathbf{S}}_I^* \times_{\mathfrak{t}^{*(1)}/W_I} \mathfrak{t}^*/(W_I,\bullet)}\big)$, § 5.6.

A.5 Section 6

$\mathsf{Rep}_{\mathbf{c}}(G)$: subcategory of $\mathsf{Rep}(G)$ associated with a W_{aff}-orbit $\mathbf{c} \subset \mathbb{X}$, § 6.1.
$[\lambda]$: W_{aff}-orbit of $\lambda \in \mathbb{X}$, § 6.1.
$\mathsf{Rep}_{\langle\lambda\rangle}(G)$: sum of the categories $\mathsf{Rep}_{\mathbf{c}}(G)$ with $\mathbf{c} \subset W_{\mathrm{ext}} \bullet \lambda$, § 6.1.
$\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$: category of G-equivariant finitely generated $\mathcal{U}\mathfrak{g}$-modules, § 6.1.
$\mathsf{Mod}^{G,\xi}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$: full subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ of modules annihilated by a power of $\mathfrak{m}^\xi$, § 6.1.
$\mathsf{Mod}^{G,\wedge}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$: full subcategory of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$ of modules annihilated by a power of $\mathcal{I}$, § 6.1.
$(-)\widehat{\otimes}_{\mathcal{U}\mathfrak{g}}(-)$: bifunctor defining the action of $\mathsf{Mod}^G_{\mathrm{fg}}(\mathcal{U}^\wedge)$ on $\mathsf{Mod}^{G,\wedge}_{\mathrm{fg}}(\mathcal{U}\mathfrak{g})$, § 6.2.
T^μ_λ: translation functor (for G-modules) associated with $\lambda,\mu \in \mathbb{X}$, § 6.3.
Θ_s: wall-crossing functor associated with $s \in S_{\mathrm{aff}}$, § 6.4.
Ψ^λ: functor from D_{BS} to $\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}$, § 6.4.
$\Psi^\lambda_{\mathbf{S}}$: functor from D_{BS} to $\mathsf{HC}^{\hat{\lambda},\hat{\lambda}}_{\mathbf{S}}$, § 6.5.

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