Proceedings of the Royal Society of Edinburgh, **150**, 1127–1138, *2020*
DOI:10.1017/prm.2018.96

Some congruences involving fourth powers of central q-binomial coefficients

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(MS received 18 June 2017; accepted 16 November 2017)

We prove some congruences on sums involving fourth powers of central q-binomial coefficients. As a conclusion, we confirm the following supercongruence observed by Long [Pacific J. Math. 249 (2011), 405–418]:

$$\sum_{k=0}^{((p^r-1)/(2))} \frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p^r \pmod{p^{r+3}},$$

where $p \geqslant 5$ is a prime and r is a positive integer. Our method is similar to but a little different from the WZ method used by Zudilin to prove Ramanujan-type supercongruences.

Keywords: q-binomial coefficients; q-WZ method; cyclotomic polynomials; q-analogue of Wolstenholme's binomial congruence; q-analogue of Morley's congruence.

2010 *Mathematics subject classification:* Primary: 11B65; Secondary: 05A10; 05A30

1. Introduction

Congruences on binomial coefficients modulo prime powers have been widely studied. In 1819, Charles Babbage [**2**] proved that, for any prime $p \geqslant 3$,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.$$

In 1862, J. Wolstenholme [**25**] further showed that the above congruence holds modulo p^3 for all primes $p \geqslant 5$, or equivalently,

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2} \quad \text{for} \quad p \geqslant 5.$$

There are many variations and generalizations of Wolstenholme's congruences in the literature. See [12] for a historical survey on these congruences. In 1895, another interesting binomial congruence was given by Morley [13]:

$$\binom{p-1}{((p-1)/(2))} \equiv (-1)^{((p-1)/(2))} 4^{p-1} \pmod{p^3}$$

for any prime $p \geqslant 5$. Moreover, in 1986, Chowla, Dwork, and Evans [3] proved the congruence:

$$\binom{((p-1)/(2))}{((p-1)/(4))} \equiv \frac{2^{p-1}+1}{2}(2x - ((p)/(2x))) \pmod{p^2},$$

where p is a prime such that $p \equiv 1 \pmod 4$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$, proposed by F. Beukers which refines the well-known congruence $\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p$. See [16] for an elementary proof of this congruence.

In 1997, van Hamme [23] conjectured that, for any prime $p \geqslant 3$,

$$\sum_{k=0}^{((p-1)/(2))} \frac{4k+1}{(-64)^k}\binom{2k}{k}^3 \equiv p(-1)^{((p-1)/(2))} \pmod{p^3}, \tag{1.1}$$

which was later proved by Mortenson [14] and Zudilin [26]. Recall that the Pochhammer symbol $(a)_k$ is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1)$ for $k = 1, 2, \ldots$. Then we have $((1)/(4^k))\binom{2k}{k} = (((1/2)_k/((1)_k))$. In a previous paper, motivated by Zudilin's work [26], the first author [8] used the q-WZ method to prove that, for any odd prime p,

$$\sum_{k=0}^{((p-1)/(2))} (-1)^k q^{k^2}[4k+1]\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} \equiv [p]q^{(((p-1)^2)/(4))}(-1)^{((p-1)/(2))} \pmod{[p]^3}, \tag{1.2}$$

where the *q-shifted factorial* is defined by $(a;q)_0 = 1$ and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ for $n \geqslant 1$, while the *q-integer* is defined as $[n] = [n]_q = 1 + q + \cdots + q^{n-1}$ (see [4]). It is clear that (1.2) is a q-analogue of (1.1).

On the other hand, applying hypergeometric identities, Long [11, theorem 1.1] proved that, for any prime $p \geqslant 5$,

$$\sum_{k=0}^{((p-1)/(2))} \frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p \pmod{p^4}. \tag{1.3}$$

Recently, by using the Zeilberger algorithm, the second author [24] has given some generalizations of (1.3), such as

$$\sum_{k=0}^{((p-1)/(2))} \frac{(4k+1)^3}{256^k}\binom{2k}{k}^4 \equiv -p \pmod{p^4}$$

for any odd prime p with $p \equiv 2 \pmod 3$.

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The first purpose of this paper is to give a q-analogue of (1.3) as follows.

THEOREM 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{((p-1)/(2))}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv [p]q^{((1-p)/(2))}+\frac{(p^2-1)(1-q)^2}{24}[p]^3q^{((1-p)/(2))} \pmod{[p]^4}. \tag{1.4}$$

Note that the congruence (1.4) modulo $[p]^3$ confirms a congruence in [8, conjecture 5.2]. Recall that the *q-binomial coefficients* $\begin{bmatrix}x\\k\end{bmatrix}_{q^n}$ are defined by

$$\begin{bmatrix}x\\k\end{bmatrix}_{q^n}=\begin{cases}\dfrac{(q^{(x-k+1)n};q^n)_k}{(q^n;q^n)_k} & \text{if } k\geqslant 0,\\ 0 & \text{otherwise.}\end{cases}$$

For convenience, when $n=1$, the subscript of the q-binomial coefficients will be omitted in the later discussion. It is easy to see that

$$\frac{(q;q^2)_k}{(q^2;q^2)_k}=\frac{1}{(-q;q)_k^2}\begin{bmatrix}2k\\k\end{bmatrix},$$

and so the congruence (1.4) may be restated as

$$\sum_{k=0}^{((p-1)/(2))}\frac{[4k+1]}{(-q;q)_k^8}\begin{bmatrix}2k\\k\end{bmatrix}^4\equiv [p]q^{((1-p)/(2))}+\frac{(p^2-1)(1-q)^2}{24}[p]^3q^{((1-p)/(2))} \pmod{[p]^4}.$$

Let $\Phi_n(q)$ be the n-th *cyclotomic polynomial* in q, which may be defined as

$$\Phi_n(q):=\prod_{\substack{1\leqslant k\leqslant n\\ \gcd(k,n)=1}}(q-e^{2\pi i((k)/(n))}),$$

where i is the imaginary unit. It is well known that $\Phi_n(q)$ is the minimal polynomial over the field of rational numbers of any primitive n-th root of unity. Our second purpose is to give a further generalization of (1.4).

THEOREM 1.2. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{((n-1)/(2))}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4}\equiv [n]q^{((1-n)/(2))}+\frac{(n^2-1)(1-q)^2}{24}[n]^3q^{((1-n)/(2))} \pmod{[n]\Phi_n(q)^3}. \tag{1.5}$$

Letting $n=p^r$ be an odd prime power, and noticing that $\Phi_{p^r}(q)=[p]_{q^{p^{r-1}}}$, we immediately get the following conclusion from theorem 1.2.

COROLLARY 1.3. *Let p be an odd prime and r a positive integer. Then*

$$\sum_{k=0}^{((p^r-1)/(2))}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv [p^r]q^{((1-p^r)/(2))} + \frac{(p^{2r}-1)(1-q)^2}{24}[p^r]^3 q^{((1-p^r)/(2))} \pmod{[p^r][p]^3_{q^{p^{r-1}}}}. \tag{1.6}$$

In particular, if $p \geqslant 5$, then the $q=1$ case gives

$$\sum_{k=0}^{((p^r-1)/(2))}\frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p^r \pmod{p^{r+3}}. \tag{1.7}$$

Note that the congruence (1.7) has already appeared in [**11**, theorem 1.1]. However, the proof given there is not correct for $r>1$ (see [**6**, p. 897]). Here we indeed confirm [**11**, theorem 1.1].

By using the WZ method, Z.-W. Sun [**20**] proves the following generalization of (1.1):

$$\sum_{k=0}^{n}(4k+1)\binom{2k}{k}^3(-64)^{n-k} \equiv 0 \left(\bmod\ 4(2n+1)\binom{2n}{n}\right). \tag{1.8}$$

Recently, a q-analogue of (1.8) has been given by the first author [**8**]:

$$\sum_{k=0}^{n}(-1)^k q^{k^2}[4k+1]{\begin{bmatrix}2k\\k\end{bmatrix}}^3(-q^{k+1};q)_{n-k}^6 \equiv 0 \left(\bmod\ (1+q^n)^2[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}\right).$$

The last purpose of this paper is to give the following similar congruence, which confirms the $r=2$ case of the second congruence in [**8**, conjecture 5.4].

THEOREM 1.4. *Let n be a positive integer. Then*

$$\sum_{k=0}^{n}[4k+1]{\begin{bmatrix}2k\\k\end{bmatrix}}^4(-q^{k+1};q)_{n-k}^8 \equiv 0 \left(\bmod\ (1+q^n)^3[2n+1]\begin{bmatrix}2n\\n\end{bmatrix}\right). \tag{1.9}$$

It should be mentioned that the following $q=1$ case of (1.9) is also new:

$$\sum_{k=0}^{n}(4k+1)\binom{2k}{k}^4 256^{n-k} \equiv 0 \left(\bmod\ 8(2n+1)\binom{2n}{n}\right).$$

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2. Proof of Theorem 1.1

We define two rational functions in q^n and q^k:

$$F(n,k) = (-1)^k q^{k(k-2n-1)} \frac{[4n+1](q;q^2)_n^3 (q;q^2)_{n+k}}{(q^2;q^2)_n^3 (q^2;q^2)_{n-k} (q;q^2)_k^2},$$

$$G(n,k) = \frac{(-1)^{k-1} q^{k(k-2n+1)} (q;q^2)_n^3 (q;q^2)_{n+k-1}}{(1-q)^2 (q^2;q^2)_{n-1}^3 (q^2;q^2)_{n-k} (q;q^2)_k^2},$$

where we use the convention that $1/(q^2;q^2)_m = 0$ for $m = -1, -2, \ldots$. The functions $F(n,k)$ and $G(n,k)$ satisfy the relation

$$[2k-1]F(n,k-1) - [2k]F(n,k) = G(n+1,k) - G(n,k). \tag{2.1}$$

Indeed, we have the following expressions:

$$\frac{F(n,k-1)}{G(n,k)} = \frac{q^{2n-4k+2}(1-q)(1-q^{4n+1})(1-q^{2k-1})^2}{(1-q^{2n-2k+2})(1-q^{2n})^3},$$

$$\frac{F(n,k)}{G(n,k)} = -\frac{q^{-2k}(1-q)(1-q^{4n+1})(1-q^{2n+2k-1})}{(1-q^{2n})^3},$$

$$\frac{G(n+1,k)}{G(n,k)} = \frac{q^{-2k}(1-q^{2n+1})^3(1-q^{2n+2k-1})}{(1-q^{2n})^3(1-q^{2n-2k+2})}.$$

Then it is routine to verify the identity

$$\frac{q^{2n-4k+2}(1-q^{4n+1})(1-q^{2k-1})^3}{(1-q^{2n-2k+2})(1-q^{2n})^3} + \frac{q^{-2k}(1-q^{2k})(1-q^{4n+1})(1-q^{2n+2k-1})}{(1-q^{2n})^3}$$
$$= \frac{q^{-2k}(1-q^{2n+1})^3(1-q^{2n+2k-1})}{(1-q^{2n})^3(1-q^{2n-2k+2})} - 1,$$

which is equivalent to (2.1) (dividing both sides by $G(n,k)$).

Summing (2.1) over n from 0 to $((p-1)/(2))$, we obtain

$$[2k-1]\sum_{n=0}^{((p-1)/(2))} F(n,k-1) - [2k]\sum_{n=0}^{((p-1)/(2))} F(n,k) = G\left(\frac{p+1}{2},k\right) - G(0,k)$$
$$= G\left(\frac{p+1}{2},k\right). \tag{2.2}$$

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It is easy to see that, for $k = 1, 2, \ldots, ((p-1)/(2))$, we have

$$\begin{aligned} G\left(\frac{p+1}{2}, k\right) &= (-1)^{k-1} q^{k(k-p)} \frac{(q;q^2)^3_{(p+1)/2}(q;q^2)_{(p+1)/2+k-1}}{(1-q)^2(q^2;q^2)^3_{(p-1)/2}(q^2;q^2)_{(p+1)/2-k}(q;q^2)^2_k} \\ &= (-1)^{k-1} q^{k(k-p)} \frac{(1-q)[p]^3(q;q^2)^3_{(p-1)/2}(q;q^2)_{(p+1)/2+k-1}}{(q^2;q^2)^3_{(p-1)/2}(q^2;q^2)_{(p+1)/2-k}(q;q^2)^2_k} \\ &\equiv 0 \pmod{[p]^4}, \end{aligned} \tag{2.3}$$

since $(q;q^2)_{(p+1)/2+k-1}$ is divisible by $[p]$, while the denominator is relatively prime to $[p]$. In view of (2.2) and (2.3), we have

$$\begin{aligned} \sum_{n=0}^{((p-1)/(2))} F(n,0) &\equiv \frac{[2]}{[1]} \sum_{n=0}^{((p-1)/(2))} F(n,1) \equiv \frac{[2][4]}{[1][3]} \sum_{n=0}^{((p-1)/(2))} F(n,2) \\ &\equiv \cdots \equiv \frac{[2][4]\cdots[p-1]}{[1][3]\cdots[p-2]} \sum_{n=0}^{((p-1)/(2))} F\left(n, \frac{p-1}{2}\right) \pmod{[p]^4}. \end{aligned} \tag{2.4}$$

Furthermore,

$$\begin{aligned} &\sum_{n=0}^{((p-1)/(2))} F\left(n, \frac{p-1}{2}\right) = F\left(\frac{p-1}{2}, \frac{p-1}{2}\right) \\ &= (-1)^{((p-1)/(2))} q^{((1-p^2)/(4))} [2p-1] \frac{(q;q^2)_{(p-1)/2}(q;q^2)_{p-1}}{(q^2;q^2)^3_{(p-1)/2}} \\ &= \frac{(-1)^{((p-1)/(2))} q^{((1-p^2)/(4))} [p] (q;q^2)_{(p-1)/2}}{(-q;q)^2_{p-1}(q^2;q^2)_{(p-1)/2}} {p-1 \brack ((p-1)/(2))}_{q^2} {2p-1 \brack p-1}. \end{aligned} \tag{2.5}$$

By the proof of [19, lemma 5], we have

$$\begin{aligned} {2p-1 \brack p-1} &\equiv 1 - \frac{(p-1)(1-q)}{2}[p] + \frac{(p-1)(5p-7)(1-q)^2}{24}[p]^2 \\ &\equiv q^{((p(p-1))/(2))} + \frac{(p^2-1)(1-q)^2}{12}[p]^2 \pmod{[p]^3}, \end{aligned} \tag{2.6}$$

which may be deemed a q-analogue of Wolstenholme's binomial congruence, and a q-analogue of Morley's congruence due to Pan [15, theorem 1.2] gives

$$\begin{aligned} {p-1 \brack \frac{p-1}{2}}_{q^2} &\equiv (-1)^{((p-1)/(2))} q^{((1-p^2)/(4))} (-q;q)^2_{p-1} \\ &\quad - (-1)^{((p-1)/(2))} q^{((1-p^2)/(4))} \frac{(p^2-1)(1-q)^2}{24}[p]^2 \pmod{[p]^3}. \end{aligned} \tag{2.7}$$

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Substituting (2.6) and (2.7) into (2.5) and noticing (2.4), we get

$$\sum_{k=0}^{((p-1)/(2))}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv [p]q^{((1-p)/(2))}$$
$$+ (p^2-1)(1-q)^2[p]^3\left(\frac{q^{((1-p^2)/(2))}}{12} - \frac{q^{((1-p)/(2))}}{24(-q;q)_{p-1}^2}\right) \pmod{[p]^4}.$$

The proof then follows from the fact that $(-q;q)_{p-1} \equiv 1 \pmod{[p]}$ (see, e.g., [**5**, (1.6)]) and $q^{((1-p^2)/(2))} \equiv q^{((1-p)/(2))} \pmod{[p]}$.

Remark. The method we use to prove theorem 1.1 is similar to the WZ method used by Zudilin [**26**], or the q-WZ method used by Tauraso [**22**] and the first author [**8**]. In general, the functions $F(n,k)$ and $G(n,k)$ are difficult to find. Here these functions in the $q=1$ case have already been given by the second author [**24**] using the Zeilberger algorithm (see [**9**, **17**]). This enables us to guess them out, fortunately.

3. Proof of Theorem 1.2

We first give a generalization of (2.6) indicated by the proof of [**19**, lemma 5]. The proof of this generalization is similar to that of [**19**, lemma 5]. The difference is we need to use a general form of Shi and Pan's q-analogue of Wolstenholme's harmonic series congruence [**18**]. However, the proof from [**18**] also works for this general form. We provide a short proof here for the sake of completeness.

Lemma 3.1. *Let n be a positive integer. Then*

$$\begin{bmatrix}2n-1\\n-1\end{bmatrix} \equiv (-1)^{n-1}q^{\binom{n}{2}} + \frac{(n^2-1)(1-q)^2}{12}[n]^2 \pmod{\Phi_n(q)^3}. \tag{3.1}$$

Proof. Noticing that $[a+b]=[a]+q^a[b]$, we compute

$$\begin{bmatrix}2n-1\\n-1\end{bmatrix} = \frac{[2n-1][2n-2]\cdots[n+1]}{[n-1][n-2]\cdots[1]} = \frac{1}{\prod_{1\leqslant j\leqslant n-1}[j]}\prod_{k=1}^{n-1}([n]+q^n[n-k])$$

which modulo $\Phi_n(q)^3$ reduces to

$$q^{n(n-1)} + q^{n(n-2)}\sum_{j=1}^{n-1}\frac{[n]}{[j]} + q^{n(n-3)}\sum_{1\leqslant j<k\leqslant n-1}\frac{[n]^2}{[j][k]}, \tag{3.2}$$

since $[n]$ is divisible by $\Phi_n(q)$ and $[n-1]!$ is relatively prime to $\Phi_n(q)^3$. On the other hand, the proofs given by Shi and Pan [**18**] imply that (although their theorems

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are only concerning the case where $n \geqslant 5$ is a prime)

$$\sum_{j=1}^{n-1} \frac{1}{[j]} \equiv \frac{(n-1)(1-q)}{2} + \frac{(n^2-1)(1-q)^2}{24}[n] \pmod{\Phi_n(q)^2}, \tag{3.3}$$

$$\sum_{j=1}^{n-1} \frac{1}{[j]^2} \equiv -\frac{(n-1)(n-5)(1-q)^2}{12} \pmod{\Phi_n(q)}. \tag{3.4}$$

Combining (3.3) and (3.4), we deduce that

$$\sum_{1 \leqslant j<k \leqslant n-1} \frac{1}{[j][k]} \equiv \frac{(n-1)(n-2)(1-q)^2}{6} \pmod{\Phi_n(q)}. \tag{3.5}$$

In view of (3.3) and (3.5), we may rewrite (3.2) modulo $\Phi_n(q)^3$ as

$$\begin{aligned} &q^{n(n-1)} + q^{n(n-2)}\left(\frac{(n-1)(1-q^n)}{2} + \frac{(n^2-1)(1-q^n)^2}{24}\right) \\ &\quad + q^{n(n-3)}\frac{(n-1)(n-2)(1-q^n)^2}{6}. \end{aligned}$$

Using the binomial theorem

$$q^{mn} = ((q^n-1)+1)^m = \sum_{k=0}^{m} \binom{m}{k}(q^n-1)^k$$

to reduce the term q^{mn} modulo the approximate power of $\Phi_n(q)$, we obtain

$$\begin{bmatrix} 2n-1 \\ n-1 \end{bmatrix} \equiv 1 - \frac{(n-1)(1-q^n)}{2} + \frac{(n-1)(5n-7)(1-q^n)^2}{24} \pmod{\Phi_n(q)^3},$$

which is equivalent to (3.1) by noticing that $q^{((n)/(2))} \equiv -1 \pmod{\Phi_n(q)}$ if n is even. □

We also need another auxiliary result.

Lemma 3.2. *Let n be a positive odd integer. Then*

$$(-q;q)_{n-1} \equiv 1 \pmod{\Phi_n(q)}. \tag{3.6}$$

Proof. By the q-binomial theorem (see, e.g., [1, p. 36, (3.3.6)]), we have

$$(-q;q)_{n-1} = \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{\binom{k+1}{2}} \equiv \sum_{k=0}^{n-1} (-1)^k = 1 \pmod{\Phi_n(q)},$$

since

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} = \prod_{j=1}^{k} \frac{1-q^{n-j}}{1-q^j} \equiv \prod_{j=1}^{k} \frac{1-q^{-j}}{1-q^j} = (-1)^k q^{-\binom{k+1}{2}} \pmod{\Phi_n(q)}. \qquad \square$$

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Proof of Theorem 1.2.. The proof is similar to that of theorem 1.1. Let $m > 1$ be an odd integer. Summing (2.1) over n from 0 to $((m-1)/(2))$, and noticing that

$$\frac{(1-q)[m]^3(q;q^2)^3_{(m-1)/2}(q;q^2)_{(m+1)/2+k-1}}{(q^2;q^2)^3_{(m-1)/2}(q^2;q^2)_{(m+1)/2-k}(q;q^2)^2_k} \equiv 0 \pmod{[m]^2\Phi_m(q)^2}$$

for $k = 1, 2, \dots, ((m-1)/(2))$ (see [**7**, lemmas 2.1 and 2.2 with $(r,d) = (1,2)$]), we obtain

$$\begin{aligned}\sum_{n=0}^{((m-1)/(2))} F(n,0) &\equiv \frac{[2][4]\cdots[m-1]}{[1][3]\cdots[m-2]} \sum_{n=0}^{((m-1)/(2))} F\left(n, \frac{m-1}{2}\right) \\ &= \frac{(-1)^{((m-1)/(2))}q^{((1-m^2)/(4))}[m]}{(-q;q)^2_{m-1}} \begin{bmatrix} m-1 \\ ((m-1)/(2)) \end{bmatrix}_{q^2} \begin{bmatrix} 2m-1 \\ m-1 \end{bmatrix} \\ &\quad \pmod{[m]^2\Phi_m(q)^2}. \end{aligned} \tag{3.7}$$

Substituting (3.1) and the following congruence [**10**, (1.5)]

$$\begin{aligned}\begin{bmatrix} m-1 \\ \frac{m-1}{2} \end{bmatrix}_{q^2} &\equiv (-1)^{((m-1)/(2))}q^{((1-m^2)/(4))}(-q;q)^2_{m-1} \\ &\quad - (-1)^{((m-1)/(2))}q^{((1-m^2)/(4))}\frac{(m^2-1)(1-q)^2}{24}[m]^2 \pmod{\Phi_m(q)^3}\end{aligned}$$

into (3.7), then using (3.6) and $q^{((1-m^2)/(2))} \equiv q^{((1-m)/(2))} \pmod{\Phi_m(q)}$, we immediately obtain (1.5) for $n = m$. □

4. Proof of Theorem 1.4

We need two divisibility results on q-binomial coefficients. The first one is just [**7**, lemma 4.1], and the proof of the second one is similar to that of [**7**, lemma 4.2].

Lemma 4.1. *Let n be a positive integer. Then*

$$(-q;q)^3_n \begin{bmatrix} 4n+1 \\ 2n \end{bmatrix} \equiv 0 \pmod{(1+q^n)^2(-q;q)_{2n}}.$$

Lemma 4.2. *Let n and k be positive integers with $k \leqslant n+1$. Then*

$$\frac{(q;q^2)^3_{n+1}(q;q^2)_{n+k}(-q;q)^8_n}{(1-q^{2k-1})(q^2;q^2)^3_n(q^2;q^2)_{n-k+1}(q;q^2)^2_k} \equiv 0 \left(\text{mod } (1+q^n)^3[2n+1]\begin{bmatrix} 2n \\ n \end{bmatrix}\right).$$

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Similarly as before, summing (2.1) over n from 0 to N, we obtain

$$[2k-1]\sum_{n=0}^{N}F(n,k-1)-[2k]\sum_{n=0}^{N}F(n,k)=G(N+1,k). \tag{4.1}$$

By lemma 4.2, for $1\leqslant k\leqslant N$, we have

$$\frac{G(N+1,k)(-q;q)_N^8}{[2k-1]}\equiv 0\left(\bmod\ (1+q^N)^3[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}\right). \tag{4.2}$$

Multiplying both sides of (4.1) by $(-q;q)_N^8/[2k-1]$ and applying (4.2), we get

$$\begin{aligned}&\sum_{n=0}^{N}F(n,k-1)(-q;q)_N^8\\&\quad\equiv\frac{[2k]}{[2k-1]}\sum_{n=0}^{N}F(n,k)(-q;q)_N^8\quad\left(\bmod\ (1+q^N)^3[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}\right),\end{aligned}$$

and, therefore,

$$\begin{aligned}&\sum_{n=0}^{N}F(n,0)(-q;q)_N^8\\&\equiv\frac{[2]}{[1]}\sum_{n=0}^{N}F(n,1)(-q;q)_N^8\equiv\frac{[2][4]}{[1][3]}\sum_{n=0}^{N}F(n,2)(-q;q)_N^8\\&\equiv\cdots\equiv\frac{[2][4]\cdots[2N]}{[1][3]\cdots[2N-1]}\sum_{n=0}^{N}F(n,N)(-q;q)_N^8\left(\bmod\ (1+q^N)^3[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}\right).\end{aligned} \tag{4.3}$$

It is easy to see that

$$\begin{aligned}\sum_{n=0}^{N}F(n,N)=F(N,N)&=(-1)^Nq^{-N(N+1)}[4N+1]\frac{(q;q^2)_N(q;q^2)_{2N}}{(q^2;q^2)_N^3}\\&=(-1)^Nq^{-N(N+1)}\frac{(q;q^2)_N[4N+1]}{(q^2;q^2)_N(-q;q)_{2N}(-q;q)_N^2}\begin{bmatrix}2N\\N\end{bmatrix}\begin{bmatrix}4N\\2N\end{bmatrix}.\end{aligned}$$

By lemma 4.1, we have

$$\begin{aligned}&\frac{[2][4]\cdots[2N]}{[1][3]\cdots[2N-1]}\sum_{n=0}^{N}F(n,N)(-q;q)_N^8\\&=(-1)^Nq^{-N(N+1)}(-q;q)_N^6\frac{[2N+1]}{(-q;q)_{2N}}\begin{bmatrix}2N\\N\end{bmatrix}\begin{bmatrix}4N+1\\2N\end{bmatrix}\\&\equiv 0\left(\bmod\ (1+q^N)^3[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}\right).\end{aligned} \tag{4.4}$$

https://doi.org/10.1017/prm.2018.96 Published online by Cambridge University Press

Combining (4.3) and (4.4), we obtain

$$\sum_{n=0}^{N} F(n,0)(-q;q)_N^8 \equiv 0 \left(\text{mod } (1+q^N)^3[2N+1]\begin{bmatrix}2N\\N\end{bmatrix}\right).$$

That is, the congruence (1.9) holds for $n=N$. □

5. Two open problems

Numerical calculation suggests that the range of summation in (1.5) can be modified but the result modulo $[n]\Phi_n(q)^3$ remains unchanged. Specifically, we have the following conjecture.

CONJECTURE 5.1. *Let n be a positive odd integer. Then*

$$\sum_{k=0}^{n-1}[4k+1]\frac{(q;q^2)_k^4}{(q^2;q^2)_k^4} \equiv [n]q^{((1-n)/(2))} + \frac{(n^2-1)(1-q)^2}{24}[n]^3 q^{((1-n)/(2))} \quad (\text{mod } [n]\Phi_n(q)^3). \tag{5.1}$$

In particular, if $p \geqslant 5$ is a prime and r is a positive integer, then

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p^r \pmod{p^{r+3}}. \tag{5.2}$$

Note that, the congruence (5.1) is true for $n=p$ by theorem 1.1, since $(((q;q^2)_k)/((q^2;q^2)_k)) \equiv 0 \pmod{[p]}$ for $((p+1)/(2)) \leqslant k \leqslant p-1$. Swisher [**21**] has made many conjectural supercongruences that generalize van Hamme's 13 Ramanujan type supercongruences. In particular, her conjectural supercongruence (C.3) is as follows: for any prime $p \geqslant 5$,

$$\sum_{k=0}^{((p^r-1)/(2))}\frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p\sum_{k=0}^{((p^{r-1}-1)/(2))}\frac{4k+1}{256^k}\binom{2k}{k}^4 \pmod{p^{4r}}.$$

Inspired by the above conjecture of Swisher, the first author (see [**6**, conjecture 4.6]) has proposed the following conjecture, which is clearly a refinement of (5.2).

CONJECTURE 5.2. *Let $p \geqslant 5$ be a prime and r a positive integer. Then*

$$\sum_{k=0}^{p^r-1}\frac{4k+1}{256^k}\binom{2k}{k}^4 \equiv p\sum_{k=0}^{p^{r-1}-1}\frac{4k+1}{256^k}\binom{2k}{k}^4 \pmod{p^{4r}}.$$

Acknowledgements

The authors would like to thank the referee for a careful reading of a previous version of this paper. The first author was partially supported by the National Natural Science Foundation of China (grant 11771175), the Natural Science Foundation of Jiangsu Province (grant BK20161304), and the Qing Lan Project of Education Committee of Jiangsu Province.

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