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A STABILITY PROPERTY OF A CLASS OF BANACH SPACES NOT CONTAINING c_0

PATRICK N. DOWLING

ABSTRACT. Let E be a Banach ideal space and X be a Banach space. The Banach function space $E(X)$ does not contain a copy of c_0 if and only if neither E nor X contains a copy of c_0. Some extensions of this result are also noted.

1. **Introduction.** A Banach ideal space on a σ-finite measure space (Ω, Σ, μ) is a Banach space E of complex measurable functions on (Ω, Σ, μ) satisfying the following: If $f \in E$ and $g: \Omega \longrightarrow \mathbb{C}$ is a μ-measurable function with $|g| \leq |f|$, then $g \in E$ and $\|g\|_E \leq \|f\|_E$.

For a Banach space X, we denote by $E(X)$ the Banach space of all measurable functions $F: \Omega \longrightarrow X$ such that $\|F(\cdot)\|_X \in E$ and with the norm

$$\|F\|_{E(X)} = \Big\| \|F(\cdot)\|_X \Big\|_E.$$

The purpose of this note is to show that $E(X)$ does not contain a subspace isomorphic to c_0 if and only if neither E nor X contain a subspace isomorphic to c_0. This result generalizes results of Kwapien [8] and Bukhvalov [2]. The method of proof is quite different from the usual proofs concerning the noncontainment of c_0 in a Banach space. We will use a new characterization of Banach space not containing a subspace isomorphic to c_0 in terms of Radon-Nikodym-type properties [5].

2. **Preliminaries and results.** Let G denote a compact metrizable abelian group, $\mathcal{B}(G)$ the σ-algebra of Borel subsets of G and λ the normalized Haar measure on G. We let Γ denote the dual group of G and let Λ be a subset of G. For a complex Banach space X, we say that a measure $\mu: \mathcal{B}(G) \longrightarrow X$ is a Λ-measure if

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma(g)}\, d\mu(g) = 0 \text{ for all } \gamma \notin \Lambda.$$

DEFINITION 1. A Banach space X is said to have type I-Λ-Radon-Nikodym property (type I-Λ-RNP) if every X-valued Λ-measure of bounded average range has a Radon-Nikodym derivative with respect to λ.

DEFINITION 2. A Banach space X is said to have type II-Λ-Radon-Nikodym property (type II-Λ-RNP) if every X-valued Λ-measure of bounded variation, which is absolutely continuous with respect to λ, has a Radon-Nikodym derivative with respect to λ.

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DEFINITION 3. A sequence $\{i_n\}_{n=1}^{\infty}$ of measurable functions $i_n: G \longrightarrow \mathbb{R}$ is called a *good approximate identity on G* if

(a) $i_n \geq 0$ for all $n \in \mathbb{N}$,
(b) $\int_G i_n(g)\, d\lambda(g) = 1$ for all $n \in \mathbb{N}$,
(c) $\sum_{\gamma \in \Gamma} i_n(\gamma) < \infty$ for all $n \in \mathbb{N}$, and
(d) $\lim_{n\to\infty} \int_U i_n(g)\, d\lambda(g) = 1$ for all neighborhoods U of 1 in G.

PROPOSITION 1 ([7]). *Let G be a compact metrizable abelian group, let Λ be a subset of Γ and let $\{i_n\}_{n=1}^{\infty}$ be a good approximate identity on G. For a complex Banach space X the following conditions are equivalent;*

(i) X has type I-Λ-RNP,
(ii) If $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and $f_n = \sum_{\gamma\in\Lambda} \hat{\imath}_n(\gamma)a_\gamma\, \gamma$ is bounded in $L^\infty_\Lambda(G,X)$, then there exists a function $f \in L^\infty_\Lambda(G,X)$ with $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$,
(iii) If $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and the sequence $\{f_n\}_{n=1}^{\infty}$, as in (ii), is bounded in $L^\infty_\Lambda(G,X)$, then $\{f_n\}_{n=1}^{\infty}$ converges in $L^1(G,X)$-norm.

PROPOSITION 2 ([6]). *Let G be a compact metrizable abelian group, let Λ be a Riesz subset of Γ and let $\{i_n\}_{n=1}^{\infty}$ be a good approximate identity on G. For a complex Banach space X the following conditions are equivalent;*

(i) X has type II-Λ-RNP,
(ii) If $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and $f_n = \sum_{\gamma\in\Lambda} \hat{\imath}_n(\gamma)a_\gamma\, \gamma$ is bounded in $L^1_\Lambda(G,X)$, then there exists a function $f \in L^1_\Lambda(G,X)$ with $\hat{f}(\gamma) = a_\gamma$ for all $\gamma \in \Lambda$,
(iii) If $\{a_\gamma\}_{\gamma\in\Lambda} \subset X$ and the sequence $\{f_n\}_{n=1}^{\infty}$, as in (ii), is bounded in $L^1_\Lambda(G,X)$, then $\{f_n\}_{n=1}^{\infty}$ converges in $L^1(G,X)$-norm.

(A subset Λ of Γ is a Riesz set if every Radon measure μ, on G with $\hat{\mu}(\gamma) = 0$ for all $\gamma \notin \Lambda$, is absolutely continuous with respect to λ.)

PROPOSITION 3 ([5]). *Let G be a compact metrizable abelian group and let Λ be an infinite Sidon subset of Γ. For a complex Banach space X the following conditions are equivalent:*

(i) X has type I-Λ-RNP,
(ii) X has type II-Λ-RNP,
(iii) X does not contain a subspace isomorphic to c_0.

(A subset Λ of Γ is a Sidon set if there is a constant C such that for all $f \in C_\Lambda(G)$, $\sum_{\gamma\in\Lambda} |\hat{f}(\gamma)| \leq C\|f\|_\infty$, where $\|f\|_\infty = \sup\{|f(g)| : g \in G\}$.)

THEOREM 1. *Let E be a Banach ideal space and X a complex Banach space with $E \neq \{0\}$ and $X \neq \{0\}$. Then E(X) does not contain a subspace isomorphic to c_0 if and only if neither E nor X contains a subspace isomorphic to c_0.*

PROOF. If $E(X)$ does not contains a subspace isomorphic to c_0 then neither does E nor X since $E(X)$ contains isometric copies of both E and X.

Conversely, suppose neither E nor X contain a subspace isomorphic to c_0. To show that $E(X)$ does not contain a subspace isomorphic to c_0 we may assume, without loss of generality, that E and X are both separable [2]. Let $G = \mathbb{T}$, the circle group. Then $\Gamma = \mathbb{Z}$. The set $\Lambda = \{2^j\}_{j=1}^\infty \subset \mathbb{Z}$ is an infinite Sidon subset of $\mathbb{Z}$ (see Rudin [10]). To show that $E(X)$ does not contain a subspace isomorphic to c_0 it suffices, by Proposition 3, to show that $E(X)$ has type I-Λ-RNP. For each $n \in \mathbb{N}$, let $r_n = 1 - \frac{1}{n}$ and $i_n = P_{r_n}$ where

$$P_{r_n}(t) = \frac{1 - r_n^2}{1 - 2r_n \cos t + r_n^2} \quad \text{for } 0 \leq t \leq 2\pi.$$

Then $\{i_n\}_{n=1}^\infty$ is a good approximate identity on $\mathbb{T}$. Suppose that $\{a_m\}_{m\in\Lambda} \subset E(X)$ and define

$$f_n(t) = \sum_{m\in\Lambda} \hat{\imath}_n(m) a_m e^{imt}.$$

Now suppose that $\{f_n\}_{n=1}^\infty$ is bounded in $L_\Lambda^\infty\big(\mathbb{T}, E(X)\big)$; that is, $\sup_n \|f_n\|_{L_\Lambda^\infty(\mathbb{T},E(X))} < \infty$. By Proposition 1, to show that $E(X)$ has type I-Λ-RNP it suffices to show that $\{f_n\}_{n=1}^\infty$ converges in $L^1(\mathbb{T}, E(X))$-norm. For $\omega \in \Omega$ we define $F_n(\omega, t) = \big(f_n(t)\big)(\omega)$. We note that since $P_{r_n/r_{n+1}} * f_{n+1} = f_n$ and $\|P_{r_n/r_{n+1}}\|_1 = 1$ we have $\|f_n\|_{L_\Lambda^\infty(\mathbb{T},E(X))} \leq \|f_{n+1}\|_{L_\Lambda^\infty(\mathbb{T},E(X))}$ and so we can apply the same method of proof as Theorem 1 of [4] to obtain that for almost all $\omega \in \Omega$ and for all $n \in \mathbb{N}$, $F_n(\omega, \cdot)\colon \mathbb{T} \longrightarrow X$, defined by $\big(F_n(\omega, \cdot)\big)(t) = F_n(\omega, t)$, has its Fourier transform supported on Λ. Also, it can be shown, again using Theorem 1 of [4] that $e_0 \in E$ where $e_0(\omega) = \sup_n \int_{\mathbb{T}} \|F_n(\omega, t)\|_X \frac{dt}{2\pi}$. In particular, for almost all $\omega \in \Omega$, $e_0(\omega) < \infty$ and so for almost all $\omega \in \Omega$, $\sup_n \|F_n(\omega, \cdot)\|_{L_\Lambda^1(\mathbb{T},X)} < \infty$. Notice also that $F_n(\omega, t) = \sum_{m\in\Lambda} \hat{\imath}_n(m) a_m(\omega) e^{imt}$. Since X does not contain a subspace isomorphic to c_0, X has type II-Λ-RNP, by Proposition 3. Hence, by Proposition 2, we have that for almost all $\omega \in \Omega$, $\{F_n(\omega, \cdot)\}_{n=1}^\infty$ converges in $L_\Lambda^1(\mathbb{T}, X)$-norm. Thus, for almost all $\omega \in \Omega$, there exists $g_\omega \in L_\Lambda^1(\mathbb{T}, X)$ such that

$$\lim_{n\to\infty} \int_{\mathbb{T}} \|F_n(\omega, t) - g_\omega(t)\|_X \frac{dt}{2\pi} = 0.$$

It is easily seen that for almost all $\omega \in \Omega$,

$$e_0(\omega) = \int_{\mathbb{T}} \|g_\omega(t)\|_X \frac{dt}{2\pi}.$$

From the above results we see that for almost all $\omega \in \Omega$, $\|F_n(\omega, t) - g_\omega(t)\|_X \longrightarrow 0$ as $n \longrightarrow \infty$ for almost all $t \in \mathbb{T}$.

Now, for almost all $\omega \in \Omega$ and for all $n \in \mathbb{N}$ the X-valued function, $F_n(\omega, t)$ is continuous in the t variable and so is measurable in the t variable. As in the proof of Theorem 7 of [3], by passing to a weighted L^1-space we may assume that $E \subset L^1$. Hence by the Dominated Convergence Theorem

$$\lim_{n,m\to\infty} \int_\Omega \int_{\mathbb{T}} \|F_n(\omega, t) - F_m(\omega, t)\|_X \frac{dt}{2\pi} d\mu(\omega) = 0.$$

That is, $\{F_n\}_{n=1}^\infty$ is a Cauchy sequence in $L^1(\Omega \times \mathbb{T}, X)$ and so there exists a function $g \in L^1(\Omega \times \mathbb{T}, X)$ such that

$$\lim_{n\to\infty} \int_\Omega \int_{\mathbb{T}} \|F_n(\omega, t) - g(\omega, t)\|_X \frac{dt}{2\pi} d\mu(\omega) = 0.$$

Therefore, (by passing to a subsequence if necessary) we have that for almost all $\omega \in \Omega$,

$$\lim_{n\to\infty} \|F_n(\omega, t) - g(\omega, t)\|_X = 0 \text{ for almost all } t \in \mathbb{T}.$$

Hence, for almost all $\omega \in \Omega$, $g(\omega, t) = g_\omega(t)$ for almost all $t \in \mathbb{T}$. We note also that (by passing to a subsequence if necessary) we have for almost all $t \in \mathbb{T}$,

$$\lim_{n\to\infty} \|F_n(\omega, t) - g(\omega, t)\|_X = 0 \text{ for almost all } \omega \in \Omega.$$

Since $\sup_n \|f_n\|_{L^\infty_\Lambda(\mathbb{T}, E(X))} < \infty$ we have for λ-almost all $t \in \mathbb{T}$ that $\sup_n \big\| \|F_n(\cdot, t)\|_X \big\|_E < \infty$. Hence the mapping $g(\cdot, t): \Omega \longrightarrow X$ given by $\big(g(\cdot, t)\big)(\omega) = g(\omega, t)$ is an element of $E(X)$, because E does not contain a subspace isomorphic to c_0 (see [9, p. 34]), for almost all $t \in \mathbb{T}$. Now we need to show that the function $g: \mathbb{T} \longrightarrow E(X)$ defined by $\big(g(t)\big)(\omega) = g(\omega, t)$ is measurable. Since E and X are separable it suffices to check that g is scalarly measurable on a total set in $\big(E(X)\big)^*$. By [1], the set $\{e^* \otimes x^* : e^* \in E^*$ and $x^* \in X^*\}$ is total in $\big(E(X)\big)^*$. For $e^* \in E^*$, $x^* \in X^*$ and $t \in \mathbb{T}$ we have

$$(e^* \otimes x^*)\big(e(t)\big) = \int_\Omega x^*\big(g(\omega, t)\big) e^*(\omega)\, d\mu(\omega).$$

This integral is a measurable function of t since $g(\omega, t)$ is measurable in both variables. Therefore $t \longrightarrow (e^* \otimes x^*)\big(g(t)\big)$ is measurable and consequently so is g. Finally, we need to show that $\{f_n\}_{n=1}^\infty$ converges to g in $L^1(\mathbb{T}, E(X))$-norm. We note from Proposition 1, that this is equivalent to showing that $f_n = i_n * g$ for all $n \in \mathbb{N}$. Since for almost all $\omega \in \Omega$, $\{F_n(\omega, \cdot)\}_{n=1}^\infty$ converges in $L^1(\mathbb{T}, X)$-norm to g_ω we have $F_n(\omega, \cdot) = i_n * g_\omega$. But for almost all $\omega \in \Omega$, $g_\omega(t) = g(\omega, t)$ for almost all $t \in \mathbb{T}$ we have $F_n(\omega, \cdot) = i_n * g(\omega, \cdot)$; that is, $\big(f_n(\cdot)\big)(\omega) = i_n * \big(g(\cdot)\big)(\omega)$. Hence $f_n = i_n * g$ and so $E(X)$ has type I-Λ-RNP which completes the proof.

REMARK 1. A special case of the above result is that $L^1(\mathbb{T}, X)$ does not contain a subspace isomorphic to c_0 if and only if X does not contain a subspace isomorphic to c_0. This special case was proved by Kwapien [8]. We have indirectly used this result in proving our Theorem because the equivalence of conditions (ii) and (iii) of Proposition 3 uses Kwapien's result.

REMARK 2. In [6], it is shown that if Λ is a Riesz subset of Γ, then Banach lattices not containing subspaces isomorphic to c_0 have type I-Λ-RNP. A close analysis of Theorem 1 combined with this Remark yields the following generalization of Theorem 1;

THEOREM 2. *Let G be a compact metrizable abelian group and let Λ be a Riesz subset of Γ. If type* I-Λ-RNP *and type* II-Λ-RNP *are equivalent properties then $E(X)$ has type* I-Λ-RNP *if and only if X has type* I-Λ-RNP *and E does not contain a subspace isomorphic to c_0.*

Applying Theorem 2 and Proposition 2 of [6] we also get

COROLLARY. *Let G be a compact abelian group and let Λ be a Riesz subset of Γ. If $L^1(G, X)$ has type* I-Λ-RNP *whenever X has type* I-Λ-RNP, *then $E(X)$ has type* I-Λ-RNP *whenever X has type* I-Λ-RNP *and E does not contain a subspace isomorphic to c_0.*

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Department of Mathematics and Statistics
Miami University
Oxford, Ohio 45056
U.S.A.