A GENERALIZATION OF SUSPENSION THEOREMS

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Our purpose in this note is to establish a classification theorem for fiberings with a loop-space as fibre. This is deduced by applying a generalized suspension theorem which will be proved in §1. As a by-product we obtain a proposition concerning fiberings with a loop-space as the total. Throughout this note we shall denote by $\mathfrak{W}$ the category of spaces having the based homotopy type of a CW-complex.

§1. Generalized suspension theorems

For a given map $f: X \to Y$, let E_f denote the subspace of $X \times EY$ consisting of the pairs (x, β) such that $f(x) = \beta(1)$ where EY is the space of paths in Y emanating from the base-point y_0, and let C_f denote the space obtained by attaching the reduced cone over X to Y by means of f. Denoting the loop and suspension functors respectively by Ω and S, we have defined in [6, p. 136] $\eta': SE_f \to C_f$ and $\eta: E_f \to \Omega C_f$ by setting, for $(x, \beta) \in E_f$,

$$
\eta'((x, \beta), s) = \begin{cases} \beta(2s) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ (x, 2-2s) & \text{if } \frac{1}{2} \leq s \leq 1, \end{cases}
$$

$$
\{\eta(x, \beta)\}(s) = \begin{cases} (x, 2s) & \text{if } 0 \leq s \leq \frac{1}{2}, \\ \beta(2-2s) & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}
$$

These induce *suspensions* $\sigma^* = (\eta')^*: \pi(C_f, Z) \to \pi(SE_f, Z)$ and $\sigma_* = \eta_*: \pi(Z, E_f) \to \pi(Z, \Omega C_f)$ for any space Z, where $\pi(A, B)$ denotes the set of homotopy classes of maps $A \to B$. The following has been established there:

THEOREM 1. *If Y is r-connected and E_f s-connected, then $\sigma^*: H^q(C_f) \to H^q(SE_f)$ is an isomorphism for $q \leq r+s+1$ and a monomorphism for $q = r+s+2$.*

THEOREM 1'. *Let X and C_f be r- and s-connected respectively. Then σ_*: $\pi_q(E_f) \to \pi_q(\Omega C_f)$ is an isomorphism for $1 \leq q \leq r+s-1$ and an epimorphism for $q = r+s$.*

These theorems may be generalized as follows:

THEOREM 2. *Let X and Y both belong to $\mathfrak{W}$, and let Y and E_f be r- and s-connected respectively ($r \geq 1$, $s \geq 0$). Suppose that Z is n-simple for all $n > 0$. Then*

(a) σ^*: $\pi(C_f, Z) \to \pi(SE_f, Z)$ *is onto if $\pi_q(Z) = 0$ for $q \geq r+s+2$.*

(b) σ^*: $\pi(C_f, Z) \to \pi(SE_f, Z)$ *is $1-1$ if $\pi_q(Z) = 0$ for $q \geq r+s+3$.*

THEOREM 2'. *Let Z be in $\mathfrak{W}$ and let X aud C_f be r- and s-connected respectively ($r \geq 1$, $s \geq 2$). Then*

(a) σ_*; $\pi(Z, E_f) \to \pi(Z, \Omega C_f)$ *is $1-1$ if the integral cohomology groups $H^q(Z)$ are trivial for $q \geq r+s$.*

(b) σ_*: $\pi(Z, E_f) \to \pi(Z, \Omega C_f)$ *is onto if $H^q(Z) = 0$ for $q \geq r+s+1$.*

The proofs will be based on the following simple:

LEMMA. *Let X and Y be in $\mathfrak{W}$ and let f: $X \to Y$ be a given map. Then*
(i) *f is equivalent to an inclusion $X' \subset Y'$ where X' is a subcomplex of a CW-complex Y'.*
(ii) *E_f and C_f also belong to $\mathfrak{W}$.*

Proof of Lemma. By using the mapping cylinder argument we may assume that f is an inclusion. Then we have a commutative diagram

$$
\begin{array}{ccc}
|X| & \xrightarrow{\;|f|\;} & |Y| \\
{\scriptstyle j_1}\downarrow & & \downarrow{\scriptstyle j_2} \\
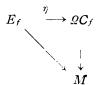X & \xrightarrow{\;f\;} & Y
\end{array}
$$

where $|X|$ and $|Y|$ are respectively the geometric realizations (see [4]) of the singular complexes of X, Y and $|f|$ is induced by f. j_1 and j_2 are canonical maps which induce isomorphisms of homotopy groups [4, Theorem 4], and hence homotopy equivalences by [8, Theorem 1]. Taking X', Y' to be $|X|$, $|Y|$, we obtain the first assertion. We shall now prove the second half. In view of [6, Lemmas 6, 9] and (i), we may assume that X is a subcomplex of a CW-complex Y and that f is an inclusion. Obviously, $C_f = Y \cup CX$ is then

a CW-complex. On the other hand, E_f is the space of paths in Y emanating from the base-point and ending in X. Milnor's result [5, Theorem 3] implies that E_f belongs to $\mathfrak{W}$. This completes the proof of the lemma.

Proof of Theorem 2. By the preceding lemma, η' is equivalent to an inclusion $i: A \subset B$ where A is a subcomplex of a CW-complex B. Hence, to prove our theorem, it suffices to show that i induces $i^*: \pi(B, Z) \to \pi(A, Z)$ having the property stated in Theorem 2. Theorem 1 and the exactness of the cohomology sequence of the pair (B, A) now imply that $H^q(B, A; G) = 0$ for $q \leq r + s + 2$ and for all coefficient groups G.

Firstly, let $\pi_q(Z) = 0$ for $q \geq r + s + 2$; then $H^{q+1}(B, A; \pi_q(Z)) = 0$ for all $q \geq 1$. Thus a standard obstruction argument shows that i^* is onto. Secondly, suppose $\pi_q(Z) = 0$ for $q \geq r + s + 3$; in this case $H^q(B, A; \pi_q(Z))$ vanishes for every $q \geq 1$. Therefore, by the same reasoning, we see that i^* is $1 - 1$.

Proof of Theorem 2'. Consider the mapping cylinder M of $\eta: E_f \to \varOmega C_f$. We have a homotopy-commutative diagram:

$$
\begin{array}{ccc}
E_f & \overset{\eta}{\longrightarrow} & \varOmega C_f \\
& \searrow & \downarrow \\
& & M
\end{array}
$$

where the unlabelled arrows are inclusions and the vertical one is a homotopy equivalence. According to Theorem 1', we have $\pi_q(M, E_f) = 0$ for $1 \leq q \leq r + s$. Let $|Z|$ be the geometric realization of the singular complex of Z and $j: |Z| \to Z$ be the canonical map. In case (a), $H^{q-1}(|Z|; \pi_q(M, E_f)) = 0$ for all $q \geq 2$, and, in case (b), $H^q(|Z|; \pi_q(M, E_f)) = 0$ for all $q \geq 2$. Observe also that our assumption implies $\pi_1(E_f) = 0$ and hence the pair (M, E_f) is n-simple for every $n \geq 2$. Noting that j is an equivalence and using the theory of obstructions to compressions as outlined in [2, Theorem 4.4.2], we obtain the desired conclusions.

COROLLARY 3. (J. Stasheff [7, Theorem 2]) *Suppose Y is a r-connected space belonging to $\mathfrak{W}$, and that Z satisfies the condition $\pi_q(Z) = 0$ for $q \geq 2r + 1$ ($r \geq 1$). Then $\pi(Y, Z)$ is in $1 - 1$ correspondence with $\pi(\varOmega Y, \varOmega Z)$.*

Proof. Taking X to be a point in Theorem 2, we have $E_f = \varOmega Y$, $C_f = Y$,

and $s = r - 1$. The assertion then follows at once. Note that $\sigma^* : \pi(Y, Z)$ $\to \pi(S\Omega Y, Z) \approx \pi(\Omega Y, \Omega Z)$ is seen to be the mapping induced by assigning to $v : Y \to Z$ $\Omega v : \Omega Y \to \Omega Z$.

Similarly, by taking Y to be a point in Theorem 2′, we have

COROLLARY 3′. *Let X be r-connected and let Z be a CW-complex with* $\dim Z \leqq 2r$. *Then $\pi(Z, X)$ is in $1 - 1$ correspondence with $\pi(SZ, SX)$.*

§ 2. Applications

First, the notations introduced in [6] will be used throughout; in particular, we consider $\bar{\sigma}^* : \pi(C_f, Z) \to \pi(E_f, \Omega Z)$ given by $\bar{\sigma}^*\{v\} = \{\eta_{f,u}\}$, where $v : C_f \to Z$, $u = v \circ P' f : Y \to C_f \to Z$, and $\eta_{f,u} : E_f \to \Omega Z$ is defined by

$$\{\eta_{f,u}(x, \beta)\}(s) = \begin{cases} v(x, 2s) & \text{if } 0 \leqq s \leqq \frac{1}{2}, \\[2mm] u\beta(2 - 2s) & \text{if } \frac{1}{2} \leqq s \leqq 1. \end{cases}$$

Note that $\bar{\sigma}^*$ corresponds to $-\sigma^*$ (see [6, Lemma 14]).

Next, let $F \xrightarrow{i} X \xrightarrow{f} Y$ and $F' \xrightarrow{i'} X' \xrightarrow{f'} Y'$ be two fiberings with fibres F, F' respectively, and let the following diagram be commutative:

$$\begin{array}{ccccc} F & \xrightarrow{i} & X & \xrightarrow{f} & Y \\ \downarrow{h} & & \downarrow{\bar{h}} & & \downarrow{g} \\ F' & \xrightarrow{i'} & X' & \xrightarrow{f'} & Y' \end{array}$$

If g, $\bar{h}$, h are all (weak) homotopy equivalences, we say that $(h, \bar{h}, g) : f \to f'$ is a (weak) *equivalence* between two fiberings.

Finally, we define $\langle Y, Z \rangle$ to be the set of equivalence classes of elements in $\pi(Y, Z)$, in which we call u_1, $u_2 : Y \to Z$ *equivalent* if and only if there exist homotopy equivalences $h : Z \to Z$, $k : Y \to Y$ satisfying $u_2 \simeq h u_1 k$.

Under these definitions, the main result obtained by applying Theorem 2 is stated as follows:

THEOREM 4. *Let Y and Z belong to $\mathfrak{W}$, and let Y be $(r-1)$-connected* $(r \geqq 2)$. *Suppose further that $\pi_q(Z) \neq 0$ only for $s + 1 \leqq q \leqq r + s - 1$ $(s \geqq 1)$ and that $s + 1 \geqq r$. Then the equivalence classes of fiberings in $\mathfrak{W}$, with Y as base and with fibre ΩZ, are in $1 - 1$ correspondence with $\langle Y, Z \rangle$.*

We list an immediate consequence of this theorem which seems to be well

known:

COROLLARY. *Let Y be a 1-connected space in $\mathfrak{W}$, and let Z be a space of type $(\pi, n+1)$ which also belongs to $\mathfrak{W}$. Then the classes of fiberings in $\mathfrak{W}$ having Y as base and ΩZ as fibre are in $1-1$ correspondence with the equivalence classes in $H^{n+1}(Y, \pi)$ under Θ, where Θ is the group of automorphisms of $H^{n+1}(Y, \pi)$ determined by homotopy equivalences of Y and automorphisms of π.*

The $1-1$ correspondence in Theorem 4 is established by assigning to $u: Y \to Z$ the class of the fibering $Pu: E_u \to Y$ induced by u from $EZ \to Z$. The fact that this is really $1-1$ will be readily seen by combining several lemmas below.

The following result is due to T. Ganea [3, Lemma 2.1]:

LEMMA 1. *Let $F \xrightarrow{i} X \xrightarrow{f} Y$ be a fibering with X and Y both in $\mathfrak{W}$. Suppose Y is $(r-1)$-connected and that $\pi_q(F) \neq 0$ only for $s \leq q \leq r+s-2$ ($r \geq 2, s \geq 1$). Suppose further that there exists a weak equivalence $\theta: F \to \Omega Z$, where Z is a 1-connected space. Then we can find a map $u: Y \to Z$ such that there exists a weak equivalence $(\xi, \bar{\xi}, 1): f \to Pu$ with $\hat{\xi} \simeq \theta$.*

Proof. By assumption, $\pi_q(Z) \approx \pi_{q-1}(\Omega Z) \approx \pi_{q-1}(F) = 0$ for $q \geq r+s$. Further, since F and E_f are equivalent to each other (see [6, Theorem 1]), E_f is $(s-1)$-connected. Thus Theorem 2, (a) implies that $\bar{\sigma}^*: \pi(C_f, Z) \to \pi(E_f, \Omega Z)$ is onto, i.e. there exists a map $v: C_f \to Z$ such that $\eta_{f,u} \simeq \theta \Psi$ where u is the composition $Y \xrightarrow{P'f} C_f \xrightarrow{v} Z$ and $\Psi: E_f \to F$ the canonical equivalence. We have a homotopy-commutative diagram:

$$
\begin{array}{ccccccc}
E_f & \xrightarrow{Pf} & X & \xrightarrow{f} & Y & \xrightarrow{P'f} & C_f \\
{\scriptstyle \theta\Psi}\downarrow\downarrow{\scriptstyle \eta_{f,u}} & & \downarrow{\scriptstyle \bar{\xi}} & & \vdots & & \downarrow{\scriptstyle v} \\
\Omega Z & \xrightarrow{Iu} & E_u & \xrightarrow{Pu} & Y & \xrightarrow{u} & Z
\end{array}
$$

in which the middle square is commutative, and so $\bar{\xi}$ induces $\hat{\xi}: F \to \Omega Z$. Applying the five lemma to the diagram of homotopy groups derived from the above one, we conclude that $\bar{\xi}$ is a weak equivalence. Since a simple computation shows that

$$\xi(x)(s) = v(x, s) \qquad\qquad \text{for } x \in F,\ 0 \leqq s \leqq 1,$$

$$\{\eta_{f,u} \circ \varPhi(x)\}(s) = \begin{cases} v(x, 2s) & \text{for } 0 \leqq s \leqq \dfrac{1}{2}, \\[2mm] z_0 = \text{base-point of } Z & \text{for } \dfrac{1}{2} \leqq s \leqq 1, \end{cases}$$

where $\varPhi\colon F \to E_f$ is a canonical inverse of $\varPsi$ given by $\varPhi(x) = (x, e)$, e being the constant path at y_0, we have $\theta \simeq \xi$, as asserted.

Next, in order to examine the extent to which u is determined, we need

LEMMA 2. *Let $h\colon Z \to Z'$ be a map; then* $\sigma^* h_* = h_* \sigma^*,\ \bar{\sigma}^* h_* = (\varOmega h)_* \bar{\sigma}^*.$

LEMMA 3. *In Lemma 1 we assume moreover that Z belongs to $\mathfrak{W}$ and that $s + 1 \geqq r$ (this is always the case when $r = 2$). Suppose there is given the commutative diagram*

$$\begin{array}{ccccc} F & \xrightarrow{\ i\ } & X & \xrightarrow{\ f\ } & Y \\ & & \big\downarrow{\xi_j} & \big\downarrow{\bar{\xi}_j} & \qquad j = 1,\, 2, \\ \varOmega Z & \xrightarrow[Iu_j]{} & E_{u_j} & \xrightarrow[Pu_j]{} Y \xrightarrow[u_j]{} Z \end{array}$$

in which vertical maps are (weak) equivalences. Then there exists an equivalence $h\colon Z \to Z$ such that $u_2 \simeq h u_1$ and $\xi_2 \simeq \varOmega h \circ \xi_1$.

Proof. Let $\tilde{\xi}_1$ be a homotopy inverse of ξ_1; since Z is s-connected and $r + s \leqq 2s + 1$, we can apply Corollary 3 to obtain a map $h\colon Z \to Z$ such that $\varOmega h \simeq \xi_2 \circ \tilde{\xi}_1\colon \varOmega Z \to \varOmega Z$. Consider $v_j\colon C_f \to Z$ given by $(j = 1,\, 2)$

$$v_j(y) = u_j(y) \qquad\quad \text{for } y \in Y,$$

$$v_j(x, s) = \gamma_j(x)(s) \qquad \text{for } x \in X,\ 0 \leqq s \leqq 1,$$

where $\bar{\xi}_j(x) = (f(x), \gamma_j(x))$. Then it follows from the proof of Lemma 1 that $\varPhi^* \bar{\sigma}^* \{v_j\} = \{\xi_j\}$. This leads to the following:

$$\begin{aligned} \varPhi^* \bar{\sigma}^* h_* \{v_1\} &= \varPhi^* (\varOmega h)_* \bar{\sigma}^* \{v_1\} \qquad\qquad \text{by Lemma 2,} \\ &= (\varOmega h)_* \varPhi^* \bar{\sigma}^* \{v_1\} = (\varOmega h)_* \{\xi_1\} \\ &= \{\xi_2\} = \varPhi^* \bar{\sigma}^* \{v_2\}. \end{aligned}$$

Because $\bar{\sigma}^*$ is $1-1$ by Theorem 2, (b) and $\varPhi^*$ is an isomorphism, we have $\{hv_1\} = \{v_2\}$, whence, composing $P'f$ to the right, $\{hu_1\} = \{u_2\}$.

LEMMA 4. *Let $u\colon Y \to Z$ be a map and let $h\colon Z \to Z',\ k\colon Y' \to Y$ be homotopy equivalences. Then $Pu\colon E_u \to Y$ and $P(huk)\colon E_{huk} \to Y'$ are equivalent.*

Proof. Let $\tilde{k}$ be a homotopy inverse of k. Consider then the following homotopy-commutative diagram

$$
\begin{array}{ccc}
Y & \xrightarrow{\;u\;} & Z \\
{\scriptstyle \tilde{k}}\downarrow & & \downarrow{\scriptstyle h} \\
Y' & \dashrightarrow{\;huk\;} & Z'
\end{array}
$$

This transformation gives rise to the map $\chi : E_u \to E_{huk}$ which is defined by

$$
\chi(y,\,\gamma) = (\tilde{k}(y),\,\gamma'(y)) \qquad \text{for } y \in Y,\; \gamma \in EZ,\; u(y) = \gamma(1),
$$

where $\gamma'(y)$ is the path in Z' given by, using a homotopy $H_t : Y \to Z'$ with $H_0 = hu$, $H_1 = huk\tilde{k}$, $\gamma'(y)(s) = h\gamma(2\,s)$ for $0 \leqq s \leqq \frac{1}{2}$, $= H_{2s-1}(y)$ for $\frac{1}{2} \leqq s \leqq 1$. It follows from Lemma 6 in [6] that χ is a homotopy equivalence. Thus $(\Omega h,\, \chi,\, \tilde{k}) : Pu \to P(huk)$ is a desired equivalence.

LEMMA 5. *Suppose there is given the commutative diagram*

$$
\begin{array}{ccccc}
F & \xrightarrow{\;i\;} & X & \xrightarrow{\;f\;} & Y \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \bar{\xi}} & & \downarrow{\scriptstyle \varsigma} \\
\Omega Z & \xrightarrow[Iu]{} & E_u & \xrightarrow[Pu]{} & Y' \xrightarrow[u]{} Z
\end{array}
$$

where the top line is a fibering. Then the above diagram admits a factorization:

$$
\begin{array}{ccccc}
F & \xrightarrow{\;i\;} & X & \dashrightarrow{\;f\;} & Y \\
\downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \bar{\varphi}} & & \\
\Omega Z & \xrightarrow[I(u\varphi)]{} & E_{u\varphi} & \xrightarrow[P(u\varphi)]{} & Y \xrightarrow{u\varphi} Z \\
\vdots & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \varphi} \\
\Omega Z & \xrightarrow[Iu]{} & E_u & \xrightarrow[Pu]{} & Y' \xrightarrow[u]{} Z
\end{array}
$$

Proof. If we write down $\bar{\xi}$ as $\bar{\xi}(x) = (\varphi f(x),\, \gamma(x))$ for $x \in X$, $\gamma(x) \in EZ$ with $u\varphi f(x) = \gamma(x)(1)$, then it suffices to consider $\bar{\psi}$, h given by $\bar{\psi}(x) = (f(x), \gamma(x))$, and $h(y, \gamma) = (\varphi(y), \gamma)$, $y \in Y$, $\gamma \in EZ$.

Finally, we shall prove

THEOREM 5. *Let $f : \Omega W \to \Omega V$ be a map with W, V in $\mathfrak{W}$. Suppose further that $\pi_i(W) \neq 0$ only for $n + 1 \leqq i \leqq 2\,n$ and $\pi_j(V) \neq 0$ only for $q + 2 \leqq j \leqq n + q + 1$ $(2 \leqq n \leqq q + 1)$. If E_f is of the same homotopy type as a loop-space of a space in $\mathfrak{W}$, then f is homotopic to Ωg for some $g : W \to V$.*

Proof. Let $\theta\colon E_f \to \Omega Z$ be an equivalence. For $q = n-1$, the assertion is trivial by Corollary 3. Hence we may assume that $q \geqq n$. Since E_f is $(n-1)$-connected and $\pi_i(Z) = 0$ for $i \geqq n+q+1$, we can apply Theorem 2, (a) to find a map $v\colon C_f \to Z$ such that there exists a homotopy-commutative diagram

$$
\begin{array}{ccccccc}
E_f & \xrightarrow{Pf} & \Omega W & \xrightarrow{f} & \Omega V & \xrightarrow{P'f} & C_f \\
\theta\downarrow\;\;\downarrow\eta_{f,u} & & \downarrow\xi_{f,u} & & & & \downarrow v \\
\Omega Z & \xrightarrow[Iu]{} & E_u & \xrightarrow[Pu]{} & \Omega V & \xrightarrow[u]{} & Z
\end{array}
$$

in which $\theta \simeq \eta_{f,u}$, $u = v \circ P'f$. By the five lemma we see that $\xi_{f,u}$ is a homotopy equivalence. Choose a homotopy inverse $\rho\colon E_u \to \Omega W$ of $\xi_{f,u}$. Observe that Z and E_u are n- and $(n-1)$-connected respectively. Thus Theorem 2 yields again a map $v'\colon C_u \to W$ such that $\eta_{u,w} \simeq \rho$ holds in the homotopy-commutative diagram

$$
\begin{array}{ccccccc}
E_u & \xrightarrow{Pu} & \Omega V & \xrightarrow{u} & Z & \xrightarrow{P'u} & C_u \\
\rho\downarrow\;\;\downarrow\eta_{u,w} & & \downarrow\xi_{u,w} & & & & \downarrow v' \\
\Omega W & \xrightarrow[Iw]{} & E_w & \xrightarrow[Pw]{} & Z & \xrightarrow[w]{} & W
\end{array}
$$

where $w = v' \circ P'u$. As above, $\xi_{u,w}$ is seen to be an equivalence. Let $\tau\colon E_w \to \Omega V$ be a homotopy inverse of $\xi_{u,w}$. Since E_w is q-connected, Theorem 2 shows that there exists a homotopy-commutative diagram

$$
\begin{array}{ccccccc}
\Omega W & \xrightarrow{Iw} & E_w & \xrightarrow{Pw} & Z & \xrightarrow{w} & W & \xrightarrow{P'w} & C_w \\
& & \tau\downarrow\;\;\downarrow\eta_{w,t} & & \downarrow\xi_{w,t} & & & & \downarrow \\
\Omega W & \xrightarrow[\omega\circ\Omega t]{} & \Omega V & \xrightarrow[It]{} & E_t & \xrightarrow[Pt]{} & W & \xrightarrow[t]{} & V
\end{array}
$$

where $\tau \simeq \eta_{w,t}$ and ω is the involution induced by reversing loop-parameter. Combining the above three diagrams with one another we see that $f \simeq \omega \circ \Omega t$. Corollary 3 implies that $\omega \simeq \Omega s$ for some $s\colon V \to V$, whence $f \simeq \Omega g$ for $g = s \circ t$.

Remark. By a similar argument as in Theorem 5 we may obtain a result due to Berstein and Hilton ([1, Lemma 3.6]): *Let $f\colon S^{m-1} \to S^q$, $m \geqq q+1 \geqq 3$. If $C_f = S^q \bigcup_f e^m$ is of the same homotopy type as a suspension, then f is homotopic to Sg for some $g\colon S^{m-2} \to S^{q-1}$.*

Added in proof. After the submission of this note, Hilton's paper "On excision and principal fibrations" (Comment. Math. Helv. **35**, 1961, Fasc. 2)

appeared, where he has obtained essentially the same results as our Theorems 2 and 2′.

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