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Generating the mapping class group by torsion elements of small order

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Abstract

We show that the mapping class group of a closed, connected, oriented surface of genus at least three is generated by 3 elements of order 3. Moreover, we show that the mapping class group of a closed, connected, oriented surface of genus at least three is generated and by 4 elements of order 4.

1. *Introduction*

Let Σ_g denote a closed, connected, oriented surface of genus g, and let $\mathcal{M}_g$ denote its mapping class group, which is the group of homotopy classes of orientation-preserving homeomorphisms of Σ_g.

The study of generators of $\mathcal{M}_g$ was pioneered by Dehn. In [**2**] Dehn proved that $\mathcal{M}_g$ is generated by a finite set of Dehn twists. Lickorish [**10**] showed that $3g - 1$ Dehn twists generate $\mathcal{M}_g$. For $g \geqslant 2$ this number was improved to $2g+1$ by Humphries [**6**]. Humphries proved, moreover, that in fact the number $2g + 1$ is minimal; i.e. $\mathcal{M}_g$ cannot be generated by $2g$ (or less) Dehn twists.

It is classical problem to find small generating sets and torsion generating sets for $\mathcal{M}_g$. Maclachlan [**12**] proved that the moduli space is simply connected as a topological space by showing that $\mathcal{M}_g$ is generated by torsion elements. McCarthy and Papadopoulos [**13**] proved that $\mathcal{M}_g$ is generated by infinitely many conjugates of a single involution for $g \geqslant 3$. Luo [**11**] discovered a first finite set of involutions which generate $\mathcal{M}_g$ for $g \geqslant 3$. He posed the question of whether there is a universal upper bound, independent of g, for the number of torsion elements needs to generate $\mathcal{M}_g$. Brendle and Farb answered Luo's question. They proved that $\mathcal{M}_g$ is generated by 3 elements of order $2g + 2$, $4g + 2$ and 2 (or g) in [**1**]. Korkmaz [**9**] showed that $\mathcal{M}_g$ is generated by 2 torsion elements, each of order $4g + 2$. Brendle and Farb [**1**] also constructed a generating set of $\mathcal{M}_g$ for $g \geqslant 3$ consisting of 6 involutions. Kassabov [**8**] improved their method to show that $\mathcal{M}_g$ is generated by 4 involutions if $g \geqslant 7$, 5 involutions if $g \geqslant 5$ and 6 involutions if $g \geqslant 3$.

It is known that for all $g \geqslant 1$ there exist elements of order 2, 3 and 4 are in $\mathcal{M}_g$. In this paper, we show the following results.

THEOREM 1. *For* $g \geqslant 3$,
(i) $\mathcal{M}_g$ *is generated by* 3 *elements of order* 3,
(ii) $\mathcal{M}_g$ *is generated by* 4 *elements of order* 4.

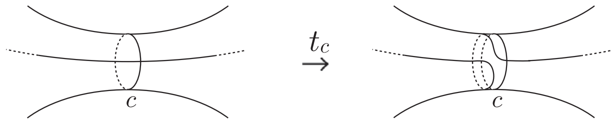

Fig. 1. The Dehn twist.

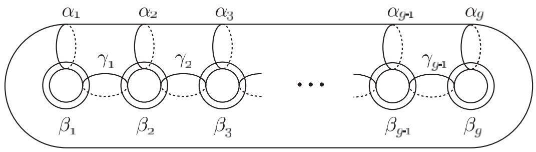

Fig. 2. The curves $\alpha_i, \beta_i, \gamma_i$.

2. *Preliminaries*

Let c be the isotopy class of a simple closed curve on Σ_g. Then the (right-hand) Dehn twist T_c about c is the homotopy class of the homeomorphism obtained by cutting Σ_g along c, twisting one of the sides by 360° to the right and gluing two sides of c back to each other. Figure 1 shows the Dehn twist about the curve c. We will denote by T_c the Dehn twist about the curve c. If f and h are two elements in $\mathcal{M}_g$, then the composition fh means that h is applied first.

We recall the following lemmas (see, for instance, [**4**]) and theorems.

LEMMA 2. *For all $h \in \mathcal{M}_g$, the Dehn twists about c and $h(c)$ are conjugate in $\mathcal{M}_g$,*

$$T_{h(c)} = hT_ch^{-1}.$$

LEMMA 3. *Let c and d be two simple closed curves on Σ_g. If c is disjoint from d, then*

$$T_cT_d = T_dT_c.$$

LEMMA 4. *If the geometric intersection number of c and d is one, then*

$$T_cT_dT_c = T_dT_cT_d.$$

THEOREM 5 ([**6**]). *We denote the curves α_i, β_i, γ_i as shown in Figure 2. $\mathcal{M}_g$ is generated by T_{α_1}, T_{α_2}, $T_{\beta_1}, \ldots, T_{\beta_g}$, $T_{\gamma_1}, \ldots, T_{\gamma_{g-1}}$.*

We call $T_{\alpha_1}, T_{\alpha_2}, T_{\beta_1}, \ldots, T_{\beta_g}, T_{\gamma_1}, \ldots, T_{\gamma_{g-1}}$ Humphries's generators.

We recall the *chain relation*. We say that an ordered set of $c_1, \ldots, c_n$ of simple closed curves on Σ_g forms an *n-chain* if the geometric intersection $(c_k, c_{k+1}) = 1$ for $k = 1, \ldots, n-1$ and $(c_k, c_l) = 0$ if $|k-l| \geqslant 2$. If n is odd, the boundary of a regular neighbourhood of any n-chain has two components d_1 and d_2. The *chain relation* is read as follows : For a given n-chain $c_1, \ldots, c_n$, if n is odd we have

$$(T_{c_n}T_{c_{n-1}} \cdots T_{c_2}T_{c_1})^{n+1} = T_{d_1}T_{d_2}$$

We note that $T_{c_n}T_{c_{n-1}} \cdots T_{c_1}(c_i) = c_{i-1}$ for $i = 2, \ldots, n$. Given an n-chain $c_1, \ldots, c_n$ and an m-chain $c'_1, \ldots, c'_m$, if c_i is disjoint from c'_j for all i, j, then, by Lemma 3, $T_{c_n} \cdots T_{c_1}$ commutes with $T_{c'_m} \cdots T_{c'_1}$.

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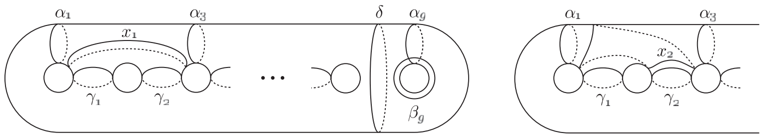

Fig. 3. The curves x_1, x_2, δ.

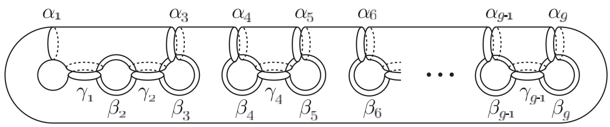

Fig. 4. Cutting the surface of odd genus, I.

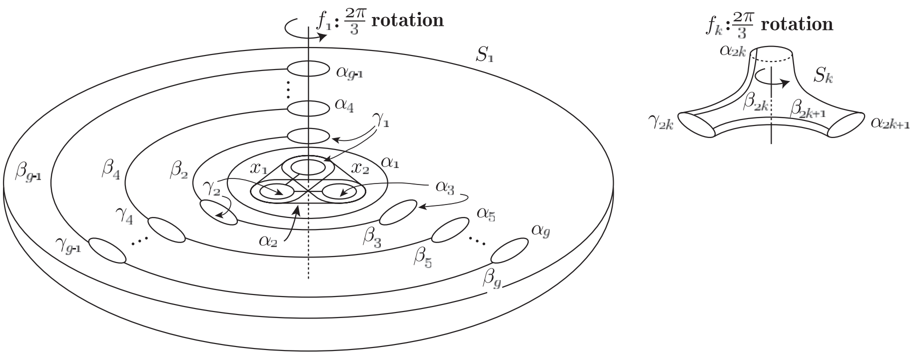

Fig. 5. $\mathbb{Z}_3$-symmetry of Σ_g for odd g, 1.

3. *Generating the mapping class group by* 3 *elements of order* 3

In this section we prove that the mapping class group $\mathcal{M}_g$ is generated by 3 elements of order 3. We assume that $g \geqslant 3$.

3·1. *Construction of elements of order* 3

We construct two elements of order 3 by cutting and gluing surfaces for each $g \geqslant 3$. We take the curves x_1, x_2 and the separating curve δ as in Figure 3.

3·1·1. *Odd genus*

We assume that g is odd.

We construct an element $f \in \mathcal{M}_g$ of order 3 for each $g \geqslant 3$. We cut Σ_g along the curves $\gamma_1, \gamma_2, \alpha_3$ and $\alpha_{2k}, \gamma_{2k}, \alpha_{2k+1}$ $(k = 2, \ldots, (g-1)/2)$ to obtain $(g-1)/2$ surfaces S_1, $S_2, \ldots$, $S_{(g-1)/2}$ as shown in Figure 4. S_1 is a sphere with $3(g+1)/2$ boundary components and S_k $(k = 2, \ldots, (g-1)/2)$ is a pair of pants bounded by $\alpha_{2k}, \gamma_{2k}, \alpha_{2k+1}$.

Let f_1 and f_k $(k = 2, \ldots, (g-1)/2)$ denote the homeomorphisms of S_1 and S_k which are rotation by $2\pi/3$ about the axis indicated in Figure 5. When we embed S_1, $S_2, \ldots$, $S_{(g-1)/2}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $f_1, \ldots, f_{(g-1)/2}$. When f denotes the homotopy class of the homeomorphism of Σ_g, $f \in \mathcal{M}_g$ has order 3.

We construct a second element $h \in \mathcal{M}_g$ of order 3 for each $g \geqslant 3$. We cut Σ_g along the curves $\alpha_{2j-1}, \gamma_{2j-1}$ and α_{2j} $(j = 1, \ldots, (g-1)/2)$ to obtain $(g+1)/2$ surfaces S'_1, $S'_2, \ldots$, $S'_{(g+1)/2}$ as shown in Figure 6. S'_1 is a torus with $3(g-1)/2$ boundary components, and S'_j $(j = 1, \ldots, (g-1)/2)$ is a pair of pants bounded by $\alpha_{2j-1}, \gamma_{2j-1}, \alpha_{2j}$.

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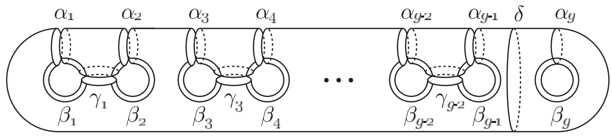

Fig. 6. Cutting the surface of odd genus, II.

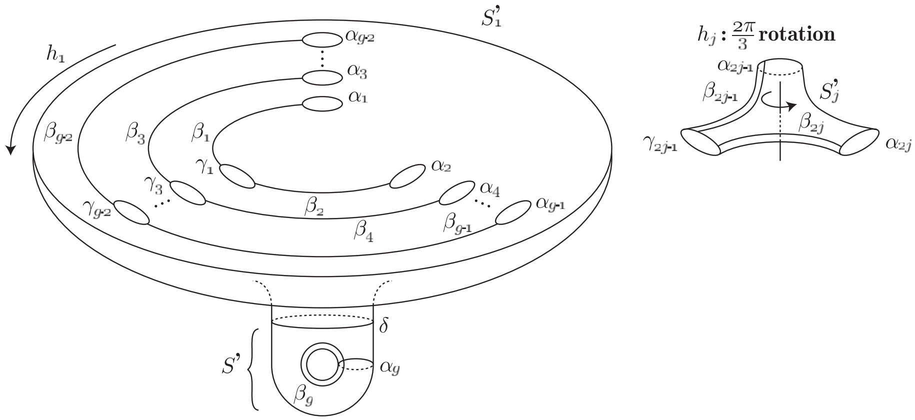

Fig. 7. $\mathbb{Z}_3$-symmetry of Σ_g for odd g, 2.

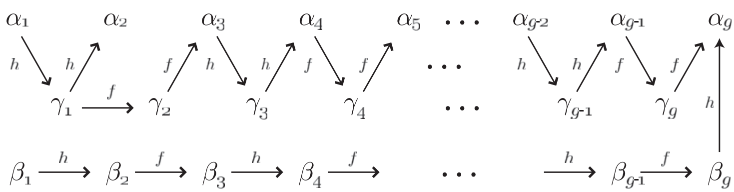

Fig. 8.

We define that the homeomorphism h_j of S'_j $(j = 2, \ldots, (g-1)/2)$ is rotation by $2\pi/3$ about the axis indicated in Figure 7. Let S' be a torus bounded by δ. Then, there exists a cube root $h_1 : S'_1 \to S'_1$ of a twist in a neighbourhood of δ in S' as shown in Figure 7. We define $\rho = (T_{\beta_g} T_{\alpha_g})^{-2}$. Note that $(h_1|_\delta)^3$ is homotopic to T_δ, and that by chain relation $\rho^3 = T_\delta^{-1}$ and $\rho(\beta_g) = \alpha_g$. When we embed $S'_1, S'_2, \ldots, S'_{(g-1)/2}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $h_1, \ldots, h_{(g-1)/2}$ and a representative of ρ. When h denotes the homotopy class of the homeomorphism of Σ_g, $h \in \mathcal{M}_g$ has order 3. Since $h|_{S'} = \rho$, we see $h(\beta_g) = \alpha_g$.

Let α and β be simple closed curves on Σ_g. The symbol

$$\alpha \xrightarrow{f} \beta \quad (\text{resp. } \alpha \xrightarrow{h} \beta)$$

means that $f(\alpha) = \beta$ (resp. $h(\alpha) = \beta$). By the constructions of f and h we can send α_1 to all α_i and γ_i by f and h as shown in Figure 8. Moreover, we find that α_i can be send to all β_i by f and h.

3·1·2. *Even genus*

We assume that g is even. By the similar arguments of the case of odd genus we construct f and h $(\in \mathcal{M}_g)$ for each $g \geqslant 4$ which are order 3.

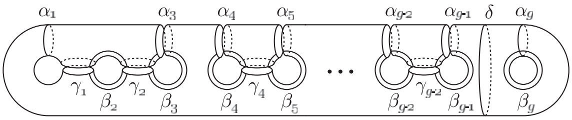

Fig. 9. Cutting the surface of even genus, I.

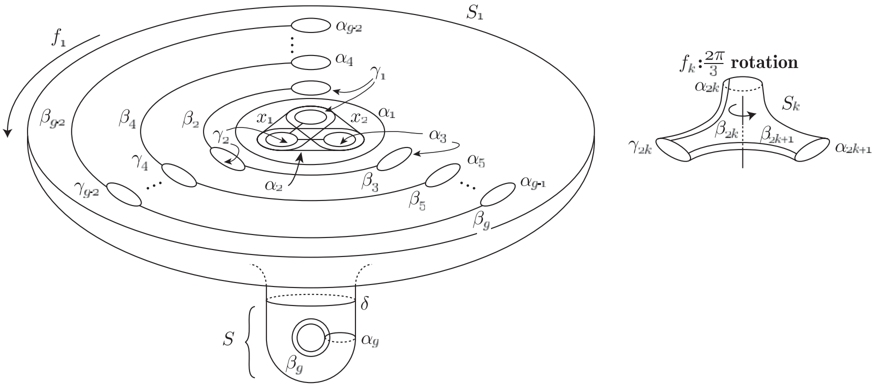

Fig. 10. $\mathbb{Z}_3$-symmetry of Σ_g for even g, 1.

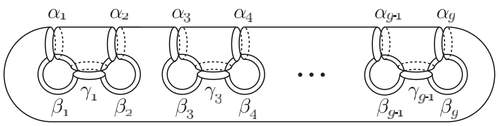

Fig. 11. Cutting the surface of even genus, II.

We construct $f \in \mathcal{M}_g$ of order 3 for each $g \geqslant 4$. We cut Σ_g along the curves $\gamma_1, \gamma_2, \alpha_3$ α_{2k}, γ_{2k} and α_{2k+1} $(k = 2, \cdots, (g-2)/2)$ to obtain $(g-2)/2$ surfaces $S_1, S_2, \ldots, S_{(g-2)/2}$ as shown in Figure 9. S_1 is a torus with $3g/2$ boundary components, S_k $(k = 2, \cdots, (g-2)/2)$ is a pair of pants bounded by $\alpha_{2k}, \gamma_{2k}, \alpha_{2k+1}$.

We define that the homeomorphism of S_k $(j = 2, \ldots, (g-2)/2)$ is rotation by $2\pi/3$ about the axis indicated in Figure 10. Let S be a torus bounded by δ. Then, there exists a cube root $h_1 : S_1 \to S_1$ of a twist in a neighbourhood of δ in S as shown in Figure 10. We define $\rho = (T_{\beta_g} T_{\alpha_g})^{-2}$. Note that $(f_1|_\delta)^3$ is homotopic to T_δ, and that by chain relation $\rho^3 = T_\delta^{-1}$ and $\rho(\beta_g) = \alpha_g$. When we embed $S_1, S_2, \ldots, S_{(g-2)/2}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $f_1, \ldots, f_{(g-2)/2}$ and a representative of ρ. When f denotes the homotopy class of the homeomorphism of Σ_g, $f \in \mathcal{M}_g$ has order 3. Since $f|_{S'} = \rho$, we see $f(\beta_g) = \alpha_g$.

We construct a second element $h \in \mathcal{M}_g$ of order 3 for each $g \geqslant 4$. We cut Σ_g along the curves $\alpha_{2j-1}, \gamma_{2j-1}, \alpha_{2j}$ $(j = 1, \ldots, \frac{g}{2})$ to obtain $\frac{g}{2}$ surfaces $S_1', S_2', \ldots, S_{\frac{g}{2}}'$ as shown in Figure 11. S_1' is a sphere with $3g/2$ boundary components and S_{j+1}' $(j = 1, \ldots, g/2)$ is a pair of pants bounded by $\alpha_{2j-1}, \gamma_{2j-1}, \alpha_{2j}$.

Let h_k $(j = 1, \ldots, g/2)$ denote the homeomorphisms of S_k' which are rotation by $2\pi/3$ about the axis indicated in Figure 12. When we embed $S_1', S_2', \ldots, S_{g/2}'$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $h_1, \ldots, h_{g/2}$. When h denotes the homotopy class of the homeomorphism of Σ_g, $h \in \mathcal{M}_g$ has order 3.

By the constructions of f and h we can send α_1 to all α_i and γ_i by f and h as shown in Figure 13. Moreover, we find that α_1 can be send to all β_i by f and h.

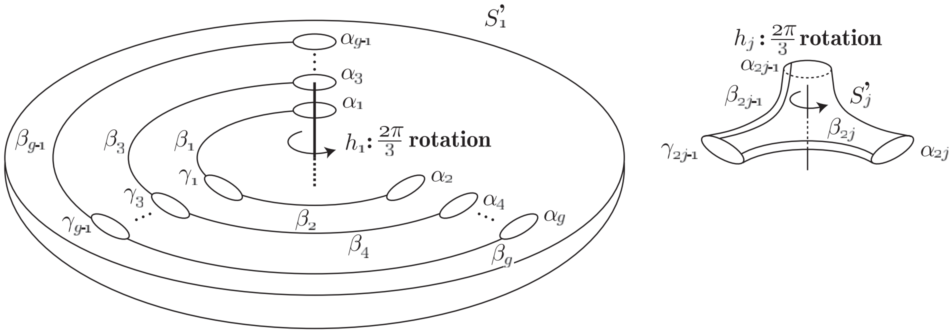

Fig. 12. $\mathbb{Z}_3$-symmetry of Σ_g for even g, 2.

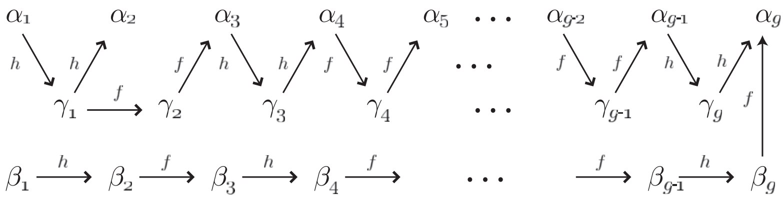

Fig. 13.

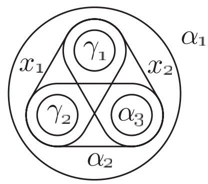

Fig. 14. The Lantern Relation.

3·2. *Generating the Dehn twist by* 3 *elements of order* 3

We generate the Dehn twist by 3 elements of order 3. The basic idea is to use the *lantern relation* which was discovered by Dehn and rediscovered by Johnson (see [**7**]).

The *lantern relation* is read as follows:

$$T_{\gamma_2}T_{\gamma_1}T_{\alpha_3}T_{\alpha_1} = T_{x_1}T_{x_2}T_{\alpha_2}.$$

where the curves $\alpha_1, \alpha_2, \alpha_3, \gamma_1, \gamma_2, x_1$ and x_2 are shown in Figure 3 and Figure 14.

Since $\alpha_1, \gamma_1, \gamma_2$ and α_3 are disjoint each other and α_2, x_1 and x_2, by Lemma 3 we can rewrite the relation as

$$T_{\alpha_1} = (T_{x_1}T_{\alpha_3}^{-1})(T_{x_2}T_{\gamma_2}^{-1})(T_{\alpha_2}T_{\gamma_1}^{-1}). \tag{3·1}$$

From the argument of Section 3·1 we have $f^2(\alpha_2) = x_1$, $f^2(\gamma_1) = \alpha_3$, $f(\alpha_2) = x_2$ and $f(\gamma_1) = \gamma_2$. By using Lemma 2 we see that

$$(T_{x_1}T_{\alpha_3}^{-1}) = f^2(T_{\alpha_2}T_{\gamma_1}^{-1})f^{-2}$$
$$(T_{x_2}T_{\gamma_2}^{-1}) = f(T_{\alpha_2}T_{\gamma_1}^{-1})f^{-1}.$$

Since h maps γ_1 to α_2, we see that $T_{\alpha_2} = hT_{\gamma_1}h^{-1}$ and

$$T_{\alpha_2}T_{\gamma_1}^{-1} = hT_{\gamma_1}h^{-1}T_{\gamma_1}^{-1} = h(T_{\gamma_1}h^{-1}T_{\gamma_1}^{-1}).$$

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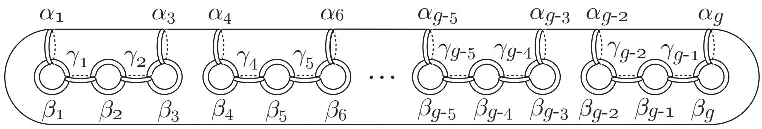

Fig. 15. Cutting the surface of genus $3m$, I.

Let $\bar{h}$ denote $T_{\gamma_1} h^{-1} T_{\gamma_1}^{-1}$. We can now rewrite (3·1) as

$$T_{\alpha_1} = (f^2 h\bar{h} f^{-2})(f h\bar{h} f^{-1})(h\bar{h}). \tag{3·2}$$

and hence T_{α_1} is generated by f, h and $\bar{h}$, each of which has order 3.

Proof of Main Theorem (i). We prove that $\mathcal{M}_g$ is generated by f, h and $\bar{h}$ in the case of odd genus.

Let G denote the group generated by f, h and $\bar{h}$. By the relation (3·2) T_{α_1} is in G. Since we can send α_1 to all α_i, γ_i and β_i by f and h (see Figure 8), by Lemma 2 we have T_{α_i}, T_{γ_i} and $T_{\beta_i} \in G$ for all i.

By a similar arguments we can prove the result in the case of even genus.

4. *Generating the mapping class group by* 4 *elements by order* 4

In this section we prove that $\mathcal{M}_g$ can be generated by 4 elements of order 4. The key point is to use the *chain relation*.

4·1. *Construction of elements of order* 4

4·1·1. *The genus is* $3m$

We assume that $g = 3m$ $(m \geqslant 2)$.

We construct an element ϕ of order 4 for each $g = 3m$. We cut Σ_g along the curves α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k} $(k = 1, \ldots, g/3)$ to obtain $(g+3)/3$ surfaces L_1, $L_2, \ldots$, $L_{(g+3)/3}$ as shown in Figure 15. $L_{(g+3)/3}$ is a sphere with $4g/3$ boundary components, and L_k $(k = 1, \cdots, g/3)$ is a pair of spheres bounded by α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k}.

We define that the homeomorphism ϕ_k of L_k $(k = 1, \ldots, (g+3)/3)$ is rotation by $\pi/2$ about the axis indicated in Figure 16. When we embed L_1, $L_2, \ldots$, $L_{(g+3)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\phi_1, \ldots, \phi_{(g+3)/3}$. When ϕ denote the homotopy class of the homeomorphism of Σ_g, $\phi \in \mathcal{M}_g$ has order 4. From the construction of ϕ we find that $\phi(\alpha_2) = x_1$ and $\phi(\gamma_2) = \alpha_3$.

We construct a second element ψ of order 4 for each $g = 3m$. We take the curves α'_{g-2}, α'_{g-1} and the separating curves δ_{g-3}, δ_{g-2}, δ_{g-1} like Figure 17. We cut Σ_g along the curves α_{3i-2}, γ_{3i-2}, γ_{3i-1} and α_{3i} $(i = 1, \ldots, (g-3)/3)$ to obtain $g/3$ surfaces L_1, $L_2, \ldots$, $L_{(g-3)/3}$, $L''_{g/3}$ as shown in Figure 17. $L''_{g/3}$ is a surface of genus 3 with $4(g-3)/3$ boundary components, and L_i $(i = 1, \cdots, (g-3)/3)$ is a sphere bounded by α_{3i-2}, γ_{3i-2}, γ_{3i-1} and α_{3i}.

We define that the homeomorphism ψ_1 of L_1 is rotation by $\pi/2$ about the axis indicated in Figure 18 and that the homeomorphism ψ_i of L_i $i = 2, \ldots, (g-3)/3$ is ϕ_i. Let S_1 be a surface of genus 3 bounded by δ_{g-2}. Then, there exists a 4th root $\psi_{\frac{g}{3}} : L''_{\frac{g}{3}} \to L''_{\frac{g}{3}}$ of a twist in a neighbourhood of δ_{g-3} in S_1 as shown in Figure 18. Note that $(\psi_{\frac{g}{3}}|_{\delta_{g-3}})^4$ is homotopic to $T_{\delta_{g-3}}$. We define $\rho_1 = (T_{\alpha_{g-2}} T_{\beta_{g-2}} T_{\alpha'_{g-2}})^{-1} (T_{\alpha_{g-1}} T_{\beta_{g-1}} T_{\alpha'_{g-1}}) (T_{\alpha_g} T_{\beta_g} T_{\alpha_g})^{-1}$. By the chain relation

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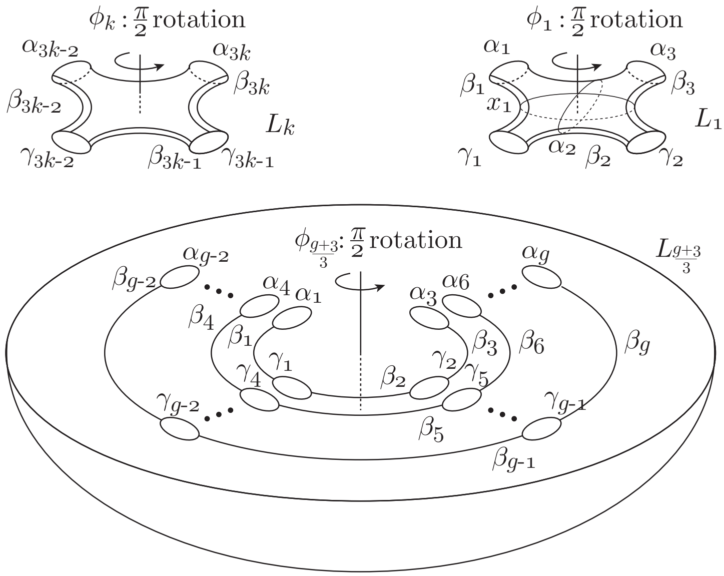

Fig. 16. $\mathbb{Z}_4$-symmetry of Σ_{3m}, I.

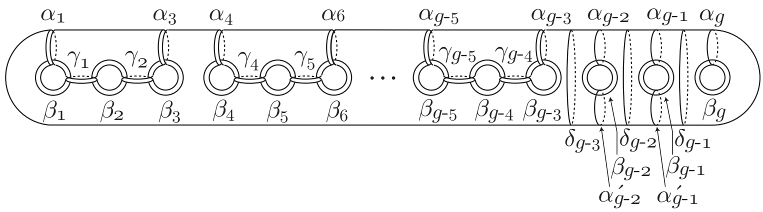

Fig. 17. Cutting the surface of genus $3m$, II.

and Lemma 3 we have

$$\begin{aligned}\rho_1^4 &= (T_{\alpha_{g-2}} T_{\beta_{g-2}} T_{\alpha'_{g-2}})^{-4} (T_{\alpha_{g-1}} T_{\beta_{g-1}} T_{\alpha'_{g-1}})^{4} (T_{\alpha_g} T_{\beta_g} T_{\alpha_g})^{-4} \\ &= (T_{\delta_{g-3}}^{-1} T_{\delta_{g-2}}^{-1})(T_{\delta_{g-2}} T_{\delta_{g-1}})(T_{\delta_{g-1}}^{-1}) = T_{\delta_{g-3}}^{-1}.\end{aligned}$$

Moreover, we find $\rho_1(\alpha_g) = \beta_g$. When we embed $L_1, L_2, \ldots, L_{(g-3)/3}, L''_{g/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\psi_1, \ldots, \psi_{g/3}$ and a representative of ρ_1. When ψ denote the homotopy class of the homeomorphism of Σ_g, $\psi \in \mathcal{M}_g$ has order 4. From the construction of ψ we find that $\psi(x_1) = x_2$, $\psi(\alpha_3) = \gamma_1$ and $\psi(\alpha_g) = \beta_g$.

We construct a third element ω of order 4 for each $g = 3m$. We take the curves ϵ like Figure 19. We cut Σ_g along the curves $\alpha_{3j-1}, \gamma_{3j-1}, \gamma_{3j}, \alpha_{3j+1}$ ($j = 1, \ldots, (g-3)/3$), α_1, ϵ, γ_{g-1} and α_{g-1} to obtain $(g+3)/3$ surfaces $L'_1, L'_2, \ldots, L'_{(g-3)/3}, L'_{g/3}$ as shown in Figure 19. $L'_{g/3}$ is a sphere with $4g/3$ boundary components, $L_{g/3}$ is a sphere bounded by $\alpha_1, \epsilon, \gamma_{g-1}$ and α_{g-1}, and L_j ($j = 1, \cdots, (g-3)/3$) is a sphere bounded by $\alpha_{3j-1}, \gamma_{3j-1}, \gamma_{3j}$ and α_{3j+1}. We define that the homeomorphism ω_j of L_j ($j = 1, \ldots, (g+3)/3$) is rotation by $\pi/2$ about the axis indicated in Figure 20.

When we embed $L'_1, L'_2, \ldots, L'_{(g+3)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\omega_1, \ldots, \omega_{(g+3)/3}$. When ω denotes the homotopy class

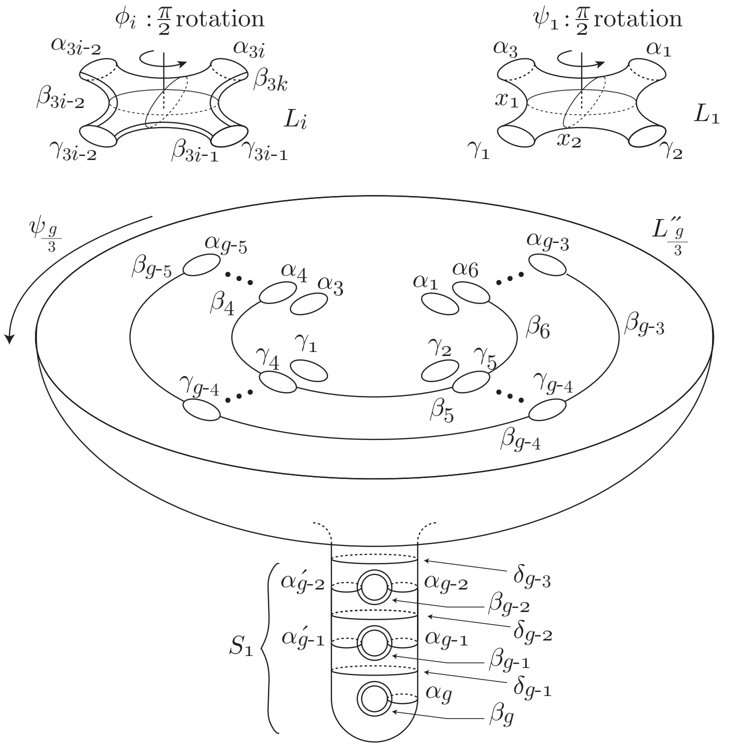

Fig. 18. $\mathbb{Z}_4$-symmetry of Σ_{3m}, 2.

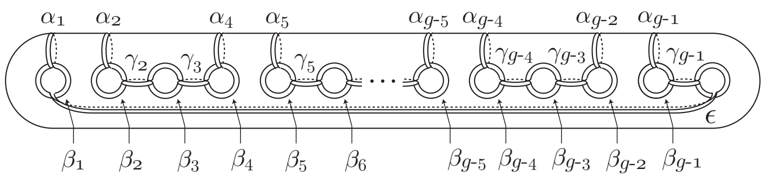

Fig. 19. Cutting the surface of genus $3m$, Ⅲ.

of the homeomorphism of Σ_g, $\omega \in \mathcal{M}_g$ has order 4. From the construction of ω we find that $\omega(\alpha_2) = \gamma_2$.

Let α and β be simple closed curves on Σ_g. The symbol

$$\alpha \xrightarrow{\phi} \beta \quad (\text{resp. } \alpha \xrightarrow{\psi} \beta,\ \alpha \xrightarrow{\omega} \beta)$$

means that $\phi(\alpha) = \beta$ (resp. $\psi(\alpha) = \beta$, $\omega(\alpha) = \beta$). By the constructions of ϕ, ψ and ω we can send α_1 to all γ_i and β_i by ϕ, ψ and ω as shown in Figure 21. Moreover, we can send α_1 to α_2 by ϕ, ψ and ω.

4·1·2. *The genus is* 3

We assume $g = 3$.

The consturctions of ψ and ω in Section 4·1·2 are not applicable in this section. In the case of $g = 3$, the construction of ϕ is the same as the previous argument. Therefore, ϕ satisfies that $\phi(\alpha_2) = x_1$ and $\phi(\gamma_2) = \alpha_3$.

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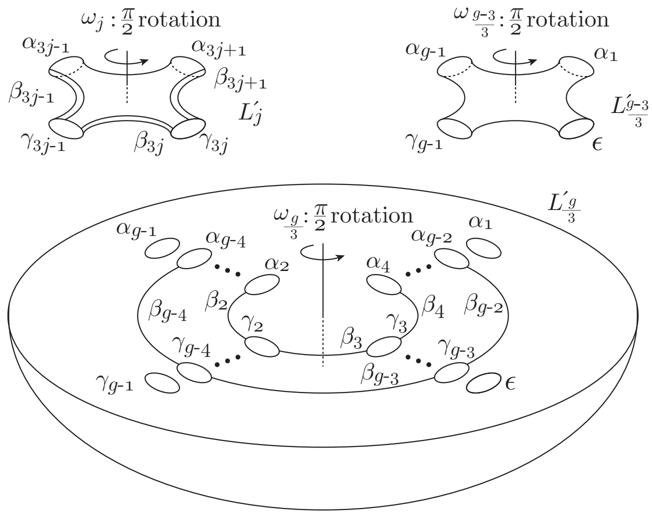

Fig. 20. $\mathbb{Z}_4$-symmetry of Σ_{3m}, 3.

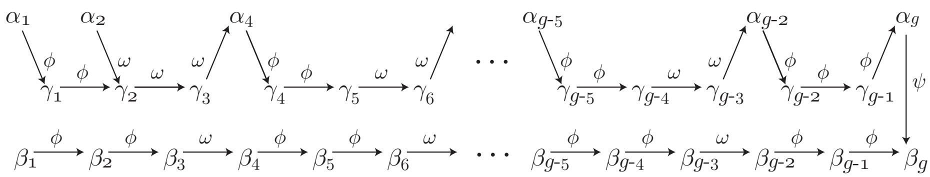

Fig. 21.

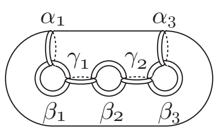

Fig. 22. Cutting the surface of genus 3.

We construct a second element ψ of order 4. We cut Σ_3 along the curves α_1, γ_1, γ_2 and α_3 to obtain two surfaces L_1, L_2 as shown in Figure 22. L_1 and L_2 are spheres bounded by $\alpha_1, \gamma_1, \gamma_2$ and α_3.

We define that the homeomorphism ψ_i of L_i $(i = 1, 2)$ is rotation by $\pi/2$ about the axis indicated in Figure 23. When we embed L_1, L_2 in Σ_3, we can define a homeomorphism of Σ_3 by gluing together the homeomorphisms ψ_1, ψ_2. When ψ denotes the homotopy class of the homeomorphism of Σ_3, $\psi \in \mathcal{M}_3$ has order 4. From the construction of ψ we find that $\psi(x_1) = x_2$, $\psi(\alpha_3) = \gamma_1$.

We construct a third element ω of order 4. Let α'_3 and ϵ be the curves as shown in Figure 24. We define

$$\omega = (T_{\gamma_2} T_{\beta_2} T_{\alpha_2})(T_\epsilon T_{\beta_1} T_{\alpha_1})^{-1}.$$

By the chain relation and Lemma 3 we have

$$\begin{aligned}\omega^4 &= (T_{\gamma_2} T_{\beta_2} T_{\alpha_2})^4 (T_\epsilon T_{\beta_1} T_{\alpha_1})^{-4} \\ &= (T_{\alpha_3} T_{\alpha'_3})(T_{\alpha_3}^{-1} T_{\alpha'_3}^{-1}) = 1.\end{aligned}$$

Hence, ω has order 4. By the construction of ω we find that $\omega(\alpha_2) = \beta_2$ and $\omega^2(\alpha_2) = \gamma_2$.

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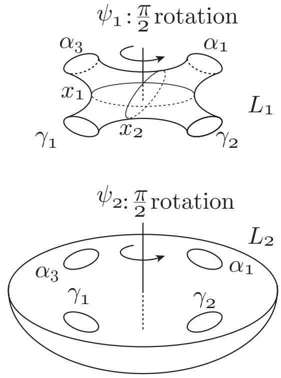

Fig. 23. $\mathbb{Z}_4$-symmetry of Σ_3.

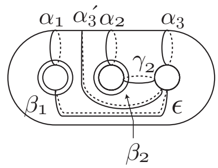

Fig. 24. The curves α_3' and ϵ.

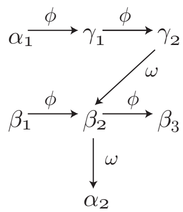

Fig. 25.

By the constructions of ϕ, ψ and ω we can send α_1 to all γ_i and β_i by ϕ, ψ and ω as shown in Figure 25. Moreover, we find that α_1 can be send to α_2 by ϕ, ψ and ω.

4·1·3. *The genus is* $3m + 1$

We assume that $g = 3m + 1$ ($m \geqslant 1$). The construction of ψ is different from the construction of that in Section 4·1·1.

We construct an element ϕ of order 4 for each $g = 3m + 1$. We cut Σ_g along the curves α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k} ($k = 1, \ldots, (g-1)/3$) to obtain $(g+2)/3$ surfaces $L_1, L_2, \ldots, L_{(g+2)/3}$ as shown in Figure 26. $L_{(g+2)/3}$ is a torus with $4(g-1)/3$ boundary components, and L_k ($i = 1, \cdots, (g-1)/3$) is a sphere bounded by α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k}.

We define that the homeomorphism ϕ_k of L_k ($k = 1, \ldots, (g-1)/3$) is rotation by $\pi/2$ about the axis indicated in Figure 27. Let S_2 be a torus bounded by δ_{g-1}. Then, there exists a 4th root $\psi_{(g+2)/3} : L_{(g+2)/3} \to L_{(g+2)/3}$ of a twist in a neighbourhood of δ_{g-1} in S_2 as shown in Figure 27. We define $\rho_2 = (T_{\alpha_g} T_{\beta_g} T_{\alpha_g})^{-1}$. Note that $(\phi_{(g+2)/3}|_{\delta_{g-1}})^4$ is homotopic to $T_{\delta_{g-1}}$, and that $\rho_2^4 = T_{\delta_{g-1}}^{-1}$.When we embed $L_1, L_2, \ldots, L_{(g+2)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\phi_1, \ldots, \phi_{(g+2)/3}$ and a representative of ρ_2. When ϕ denotes the homotopy class of the homeomorphism of Σ_g, $\phi \in \mathcal{M}_g$ has order 4. From the construction of ϕ we find that $\phi(\alpha_2) = x_1$ and $\phi(\gamma_2) = \alpha_3$.

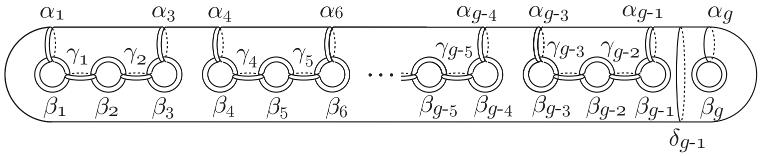

Fig. 26. Cutting the surface of genus $3m + 1$, I.

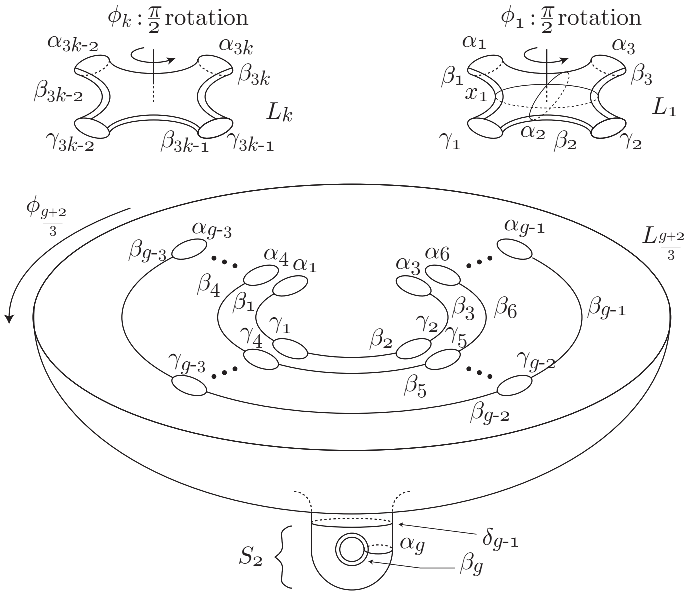

Fig. 27. $\mathbb{Z}_4$-symmetry of Σ_{3m+1}, 1.

We construct a second element ψ of order 4 for each $g = 3m + 1$. We define that the homeomorphism ψ_1 of L_1 is rotation by $\pi/2$ about the axis indicated in Figure 28 and that the homeomorphism ψ_k of L_k $(k = 1, \ldots, (g-1)/3)$ is ϕ_k. We define that the homeomorphism $\psi_{(g+2)/3}$ of $L_{(g+2)/3}$ is a 4th root of a twist in a neighbourhood of δ_{g-1} in S_2 as shown in Figure 28. Note that $(\psi_{(g+2)/3}|_{\delta_{g-1}})^4$ is homotopic to $T_{\delta_{g-1}}$.When we embed L_1, L_2,..., $L_{(g+2)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\psi_1, \ldots, \psi_{(g+2)/3}$ and a representative of ρ_2. When ψ denotes the homotopy class of the homeomorphism of Σ_g, $\psi \in \mathcal{M}_g$ has order 4. From the construction of ψ we find that $\psi(x_1) = x_2$, $\psi(\alpha_3) = \gamma_1$. Moreover, since $\psi|_{S_2} = \rho_2$, we see that $\psi(\alpha_g) = \beta_g$.

We construct a third element ω of order 4 for each $g = 3m+1$. We cut Σ_g along the curves α_{3j-1}, γ_{3j-1}, γ_{3j} and α_{3j+1} $(j = 1, \ldots, (g-1)/3)$ to obtain $(g+2)/3$ surfaces L'_1, L'_2,..., $L'_{(g-1)/3}$ and $L'_{(g+2)/3}$ as shown in Figure 29. $L'_{(g+2)/3}$ is a torus with $4(g-1)/3$ boundary components, and L_j $(i = 1, \cdots, (g-1)/3)$ is a sphere bounded by α_{3j-1}, γ_{3j-1}, γ_{3j} and α_{3j+1}.

We define that the homeomorphism ω_j of L'_j $(j = 1, \ldots, (g-1)/3)$ is rotation by $\pi/2$ about the axis indicated in Figure 30. Let δ_1 be the separating curve on Σ_g as shown in Figure 29 and let S_3 be a torus bounded by δ_1. Then, there exists a 4th root $\omega_{(g+2)/3}$: $L'_{(g+2)/3} \rightarrow L'_{(g+2)/3}$ of a twist in a neighbourhood of δ_1 in S_3 as shown in Figure 30.

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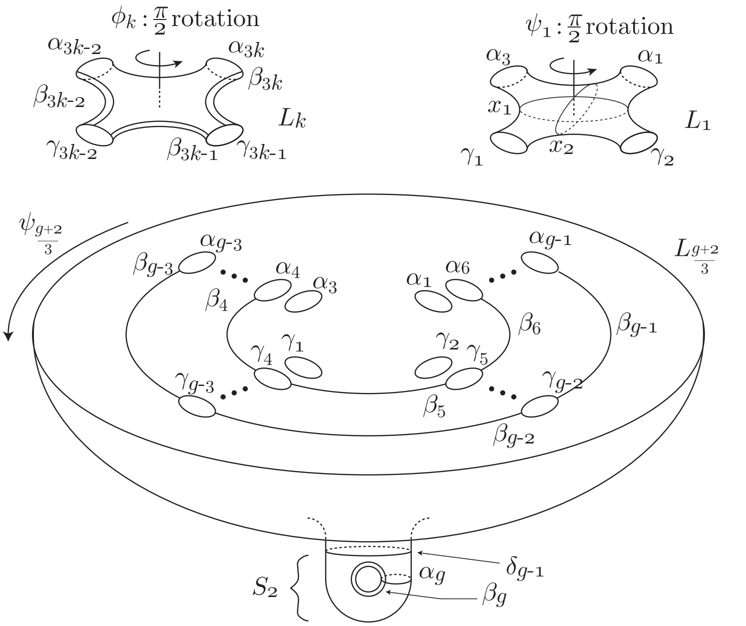

Fig. 28. $\mathbb{Z}_4$-symmetry of Σ_{3m+1}, 2.

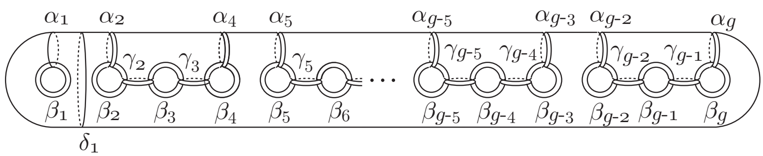

Fig. 29. Cutting the surface of genus $3m+1$, II.

We define $\rho_3 = (T_{\alpha_1} T_{\beta_1} T_{\alpha_1})^{-1}$. Note that $(\omega_{(g+2)/3}|_{\delta_1})^4$ is homotopic to T_{δ_1}, and that $\rho_3^4 = T_{\delta_1}^{-1}$.When we embed L'_1, L'_2,..., $L'_{(g+2)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\omega_1, \ldots, \omega_{(g+2)/3}$ and a representative of ρ_3. When ω denotes the homotopy class of the homeomorphism of Σ_g, $\omega \in \mathcal{M}_g$ has order 4. From the construction of ω we find that $\omega(\alpha_2) = \gamma_2$.

By the constructions of ϕ,ψ and ω we can send α_1 to all γ_i and β_i by ϕ, ψ and ω as shown in Figure 31. Moreover, we can send α_1 to α_2 by ϕ, ψ and ω.

4·1·4. *The genus is* $3m+2$

We assume that $g = 3m+2$ ($m \geqslant 2$).

We construct an element ϕ of order 4 for each $g = 3m+2$. We cut Σ_g along the curves α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k} ($k = 1, \ldots, (g-2)/3$) to obtain $(g+1)/3$ surfaces L_1, L_2,..., $L_{(g+1)/3}$ as shown in Figure 32. $L_{(g+2)/3}$ is a torus with $4(g-2)/3$ boundary components, and L_k ($k = 1, \cdots, (g-2)/3$) is a sphere bounded by α_{3k-2}, γ_{3k-2}, γ_{3k-1} and α_{3k}.

We define that the homeomorphism ϕ_k of L_k ($k = 1, \ldots, (g-1)/3$) is rotation by $\pi/2$ about the axis indicated in Figure 33. Let S_4 be a surface of genus 2 bounded by δ_{g-2}. Then, there exists a 4th root $\phi_{(g+1)/3} : L_{(g+1)/3} \to L_{(g+1)/3}$ of a twist in a neighbourhood of δ_{g-2} in S_4 as shown in Figure 33. We define $\rho_4 = (T_{\alpha_{g-1}} T_{\beta_{g-1}} T_{\alpha_{g-1}})^{-1}(T_{\alpha_g} T_{\beta_g} T_{\alpha_g})$. By the chain

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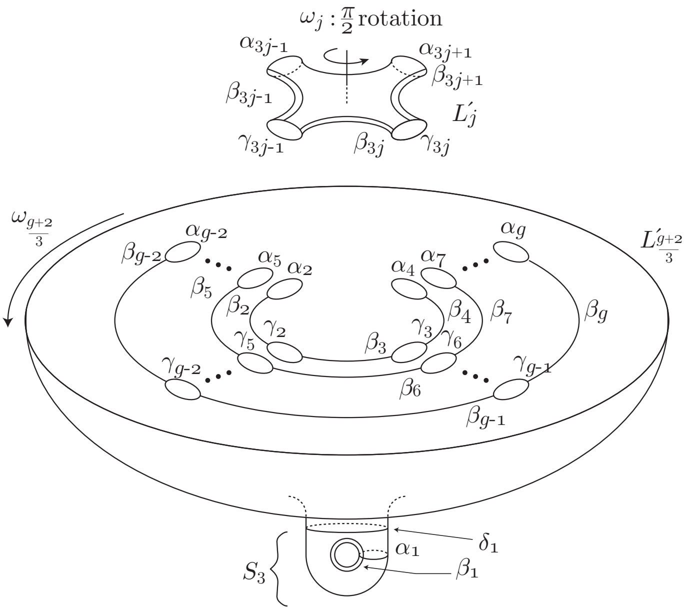

Fig. 30. $\mathbb{Z}_4$-symmetry of Σ_{3m+1}, 3.

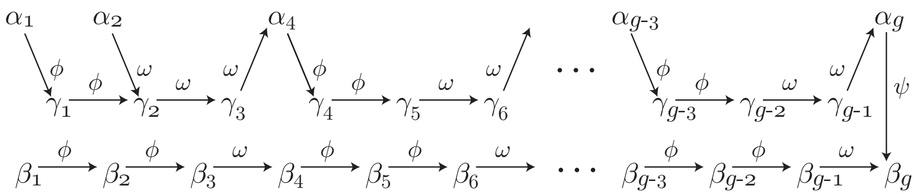

Fig. 31.

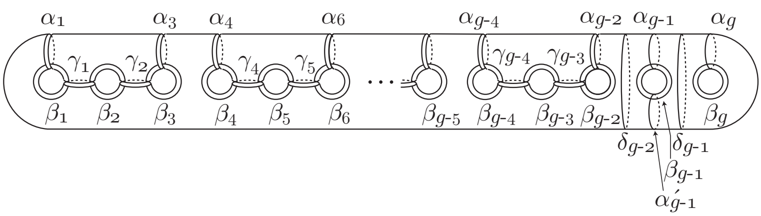

Fig. 32. Cutting the surface of genus $3m+2$, I.

relation and Lemma 3 we have

$$\rho_4^4 = (T_{\alpha_{g-1}} T_{\beta_{g-1}} T_{\alpha_{g-1}})^{-4} (T_{\alpha_g} T_{\beta_g} T_{\alpha_g})^4$$
$$= (T_{\delta_{g-2}}^{-1} T_{\delta_{g-1}}^{-1})(T_{\delta_{g-1}}) = T_{\delta_{g-2}}^{-1}.$$

Moreover, we find that $\rho_4(\alpha_g) = \beta_g$. Note that $(\phi_{(g+1)/3}|_{\delta_{g-2}})^4$ is homotopic to $T_{\delta_{g-2}}$. When we embed L_1, L_2,..., $L_{(g+1)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms $\phi_1, \ldots, \phi_{(g+1)/3}$ and a representative of ρ_4. When ϕ denotes the

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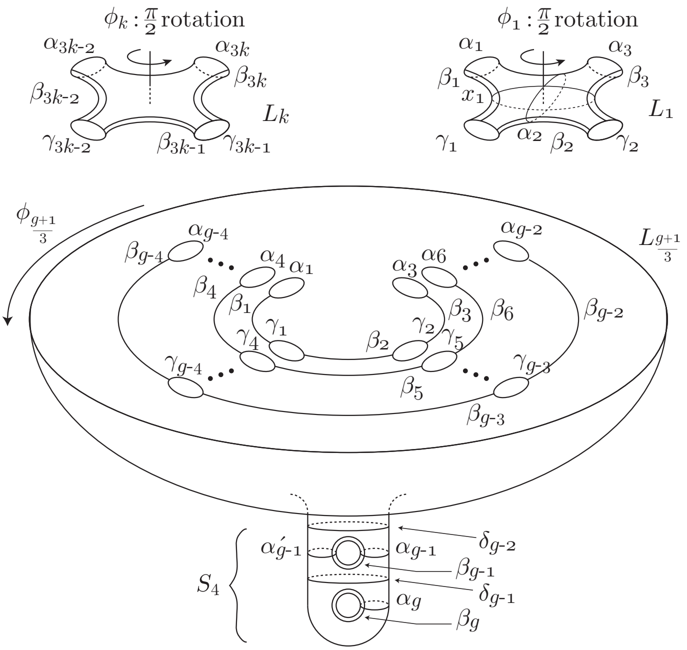

Fig. 33. $\mathbb{Z}_4$-symmetry of Σ_{3m+2}, 1.

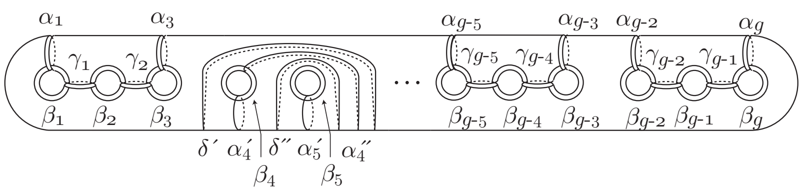

Fig. 34. Cutting the surface of genus $3m+2$, II.

homotopy class of the homeomorphism of Σ_g, $\phi \in \mathcal{M}_g$ has order 4. From the construction of ϕ we find that $\phi(\alpha_2) = x_1$, $\phi(\gamma_2) = \alpha_3$ and $\phi(\alpha_g) = \beta_g$.

We construct a second element ψ of order 4 for each $g = 3m+2$. We cut Σ_g along the curves α_1, γ_1, γ_2, α_3, α_{3i}, γ_{3i}, γ_{3i+1} and α_{3i+2} $(i = 2, \ldots, (g-2)/3)$ to obtain $(g+1)/3$ surfaces L''_1, $L''_2, \ldots, L''_{(g+1)/3}$ as shown in Figure 34. $L''_{(g+2)/3}$ is a torus with $4(g-2)/3$ boundary components, and L''_1 and L''_i $(i = 2, \cdots, (g-2)/3)$ are spheres bounded by α_1, γ_1, γ_2, α_3 and α_{3i}, γ_{3i}, γ_{3i+1}, α_{3i+2}, respectively.

We define that the homeomorphism ψ_i of L''_i $(i = 1, \ldots, (g-2)/3)$ is rotation by $\pi/2$ about the axis indicated in Figure 35. Let δ' and δ'' (resp. α'_4 and α'_5) be the separating (resp. the nonseparating) curves as shown in Figure 34. We denote by S_5 a surface of genus 2 bounded by δ''. Then, there exists a 4th root $\psi_{(g+1)/3} : L''_{(g+1)/3} \to L''_{(g+1)/3}$ of a twist in a neighbourhood of δ'' in S_5 as shown in Figure 35. We define $\rho_5 = (T_{\alpha_4} T_{\beta_4} T_{\alpha'_4})^{-1} (T_{\alpha_5} T_{\beta_5} T_{\alpha_5})$. By the chain relation and Lemma 3 we have

$$\begin{aligned}\rho_5^4 &= (T_{\alpha_4} T_{\beta_4} T_{\alpha'_4})^{-4} (T_{\alpha_5} T_{\beta_5} T_{\alpha_5})^4 \\ &= (T_{\delta'}^{-1} T_{\delta''}^{-1})(T_{\delta''}) = T_{\delta'}^{-1}.\end{aligned}$$

Note that $(\psi_{(g+1)/3}|_{\delta'})^4$ is homotopic to $T_{\delta'}$. When we embed L''_1, $L''_2, \ldots, L''_{(g+1)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms

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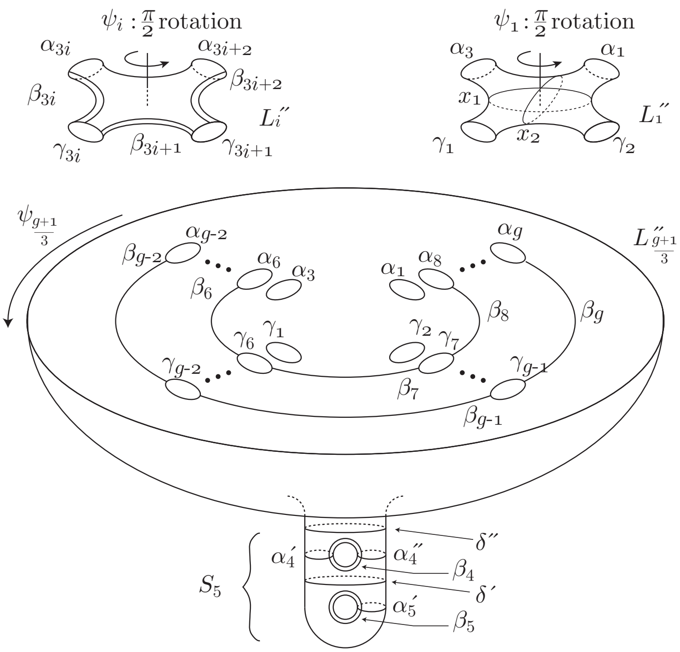

Fig. 35. $\mathbb{Z}_4$-symmetry of Σ_{3m+2}, 2.

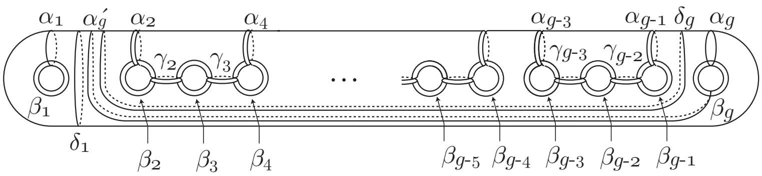

Fig. 36. Cutting the surface of genus $3m + 2$, III.

$\psi_1, \ldots, \psi_{(g+1)/3}$ and a representative of ρ_5. When ψ denotes the homotopy class of the homeomorphism of Σ_g, $\psi \in \mathcal{M}_g$ has order 4. From the construction of ψ we find that $\psi(x_1) = x_2,\ \psi(\alpha_3) = \gamma_1$.

We construct a third element ω of order 4 for each $g = 3m+2$. We cut Σ_g along the curves $\alpha_{3j-1}, \gamma_{3j-1}, \gamma_{3j}$ and α_{3j+1} $(j = 1, \ldots, (g-2)/3)$ to obtain $(g+1)/3$ surfaces $L_1', L_2', \ldots, L'_{(g+1)/3}$ as shown in Figure 36. $L'_{(g+2)/3}$ is a torus with $4(g-2)/3$ boundary components, and L_j $(j = 1, \cdots, (g-2)/3)$ is a sphere bounded by $\alpha_{3j-1}, \gamma_{3j-1}, \gamma_{3j}$ and α_{3j+1}.

We define that the homeomorphism ω_j of L_j $(j = 1, \ldots, (g-1)/3)$ is rotation by $\pi/2$ about the axis indicated in Figure 37. Let δ_g and α_g' be the separating and nonseparating curves as shown in Figure 36. We denote by S_6 a surface of genus 2 bounded by δ_g. Then, there exists a 4th root $\omega_{(g+1)/3} : L'_{(g+1)/3} \to L'_{(g+1)/3}$ of a twist in a neighbourhood of δ_g in S_6 as shown in Figure 37. We define $\rho_6 = (T_{\alpha_g} T_{\beta_g} T_{\alpha_g'})^{-1}(T_{\alpha_1} T_{\beta_1} T_{\alpha_1})$. By the chain relation and Lemma 3 we have

$$\begin{aligned}\rho_6^4 &= (T_{\alpha_g} T_{\beta_g} T_{\alpha_g'})^{-4}(T_{\alpha_1} T_{\beta_1} T_{\alpha_1})^4 \\ &= (T_{\delta_g}^{-1} T_{\delta_1}^{-1})(T_{\delta_1}) = T_{\delta_g}^{-1}.\end{aligned}$$

Note that $(\omega_{(g+1)/3}|_{\delta_g})^4$ is homotopic to T_{δ_g}. When we embed $L_1', L_2', \ldots, L'_{(g+1)/3}$ in Σ_g, we can define a homeomorphism of Σ_g by gluing together the homeomorphisms

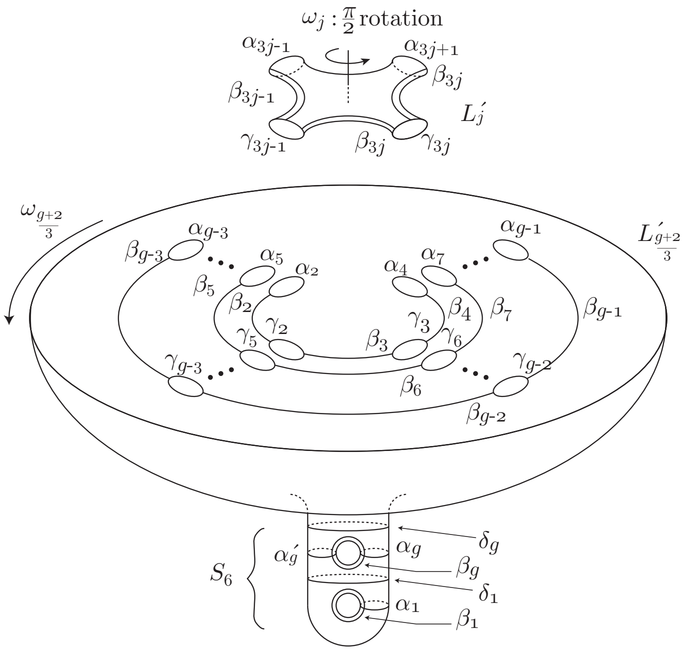

Fig. 37. $\mathbb{Z}_4$-symmetry of Σ_{3m+2}, 3.

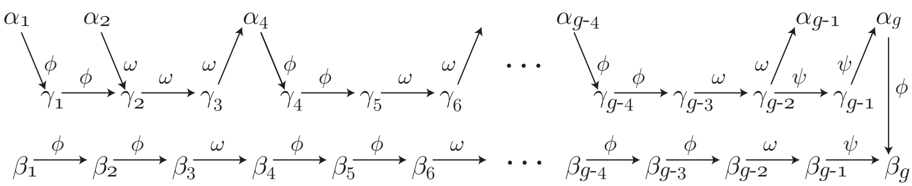

Fig. 38.

$\omega_1, \ldots, \omega_{(g+1)/3}$ and a representative of ρ_6. When ω denotes the homotopy class of the homeomorphism of Σ_g, $\omega \in \mathcal{M}_g$ has order 4. From the construction of ω we find that $\omega(\alpha_2) = \gamma_2$.

By the constructions of ϕ, ψ and ω we can send α_1 to all γ_i and β_i by ϕ, ψ and ω as shown in Figure 38. Moreover, we can send α_1 to α_2 by ϕ, ψ and ω.

4·1·5. *The genus is* 5

We assume that $g = 5$.

We construct an element ϕ of order 4. Let α_4' be the nonseparating curve on Σ_5 as shown in Figure 39. We define

$$\phi = (T_{\gamma_3} T_{\beta_3} T_{\gamma_2} T_{\beta_2} T_{\gamma_1} T_{\beta_1} T_{\alpha_1})^2 (T_{\alpha_5} T_{\beta_5} T_{\gamma_4})^{-1}.$$

By the chain relation and Lemma 3 we have

$$\begin{aligned}\phi^4 &= (T_{\gamma_3} T_{\beta_3} T_{\gamma_2} T_{\beta_2} T_{\gamma_1} T_{\beta_1} T_{\alpha_1})^8 (T_{\alpha_5} T_{\beta_5} T_{\gamma_4})^{-4}.\\ &= (T_{\alpha_4} T_{\alpha_4'})(T_{\alpha_4}^{-1} T_{\alpha_4'}^{-1}) = 1.\end{aligned}$$

Hence, ϕ has order 4. We note that $\phi^{-1}(\alpha_2) = x_1$, $\phi^{-1}(\gamma_2) = \gamma_1$ and $\phi(\beta_5) = \gamma_4$.

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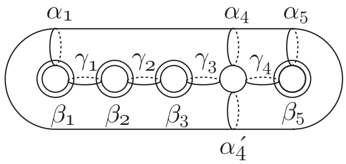

Fig. 39. $\mathbb{Z}_4$-symmetry of Σ_5, 1.

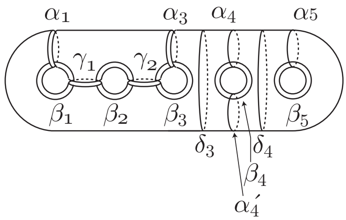

Fig. 40. Cutting the surface of genus 5.

We construct a second element ψ of order 4. We cut Σ_5 along the curves α_1, γ_1, γ_2 and α_3 to obtain two surfaces L_1, L_2 as shown in Figure 40. L_2 is a surface of genus 2 with 4 boundary components, and L_1 is a sphere bounded by α_1, γ_1, γ_2 and α_3.

We define that the homeomorphism ψ_1 of L_1 is rotation by $\pi/2$ about the axis indicated in Figure 41. Let δ_3, δ_4 be the separating curves as shown in Figure 40, and let α_4' be nonseparating curves as shown in Figure 40. We denote by S_7 a surface of genus 2 bounded by δ_3. Then, there exists a 4th root $\psi_2 : L_2 \to L_2$ of a twist in a neighbourhood of δ_3 in S_7 as shown in Figure 41. We define $\rho_7 = (T_{\alpha_4} T_{\beta_4} T_{\alpha_4'})^{-1}(T_{\alpha_5} T_{\beta_5} T_{\alpha_5})$. By the chain relation and Lemma 3 we have

$$\begin{aligned}\rho_7^4 &= (T_{\alpha_4} T_{\beta_4} T_{\alpha_4'})^{-4}(T_{\alpha_5} T_{\beta_5} T_{\alpha_5})^4\\ &= (T_{\delta_3}^{-1} T_{\delta_4}^{-1})(T_{\delta_4}) = T_{\delta_3}^{-1}.\end{aligned}$$

Note that $(\psi_2|_{\delta_3})^4$ is homotopic to T_{δ_3}. When we embed L_1, L_2 in Σ_5, we can define a homeomorphism of Σ_5 by gluing together the homeomorphisms ψ_1, ψ_2 and a representative of ρ_7. When ψ denotes the homotopy class of the homeomorphism of Σ_5, $\psi \in \mathcal{M}_5$ has order 4. From the construction of ψ we find that $\psi^{-1}(x_1) = x_2$ and $\psi^{-1}(\gamma_1) = \alpha_3$.

We construct a third element ω of order 4. Let α_5' and ϵ be the nonseparating curves on Σ_5 as shown in Figure 42. We define

$$\omega = (T_{\gamma_4} T_{\beta_3} T_{\gamma_3} T_{\beta_2} T_{\gamma_2} T_{\beta_2} T_{\alpha_2})^2 (T_\epsilon T_{\beta_1} T_{\alpha_1})^{-1}.$$

By the chain relation and Lemma 3 we have

$$\begin{aligned}\phi^4 &= (T_{\gamma_4} T_{\beta_3} T_{\gamma_3} T_{\beta_2} T_{\gamma_2} T_{\beta_2} T_{\alpha_2})^8 (T_\epsilon T_{\beta_1} T_{\alpha_1})^{-4}.\\ &= (T_{\alpha_5} T_{\alpha_5'})(T_{\alpha_5}^{-1} T_{\alpha_5'}^{-1}) = 1.\end{aligned}$$

Hence, ω has order 4. We note that $\omega^{-1}(\alpha_2) = \gamma_2$.

By the constructions of ϕ and ω we can send α_1 to all γ_i by ϕ and ω as shown in Figure 43. Moreover, as shown on Figure 43, we find that α_1 can be send to all β_i by ϕ and ω.

4·2. *Generating the Dehn twist by* 4 *elements of order* 4

By using the lantern relation we generate the Dehn twist by 4 elements of order 4.

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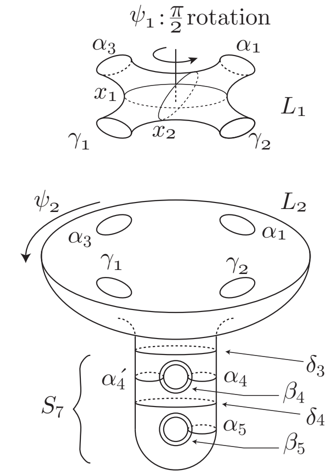

Fig. 41. $\mathbb{Z}_4$-symmetry of Σ_5, 2.

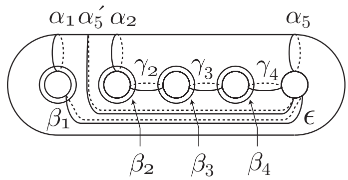

Fig. 42. $\mathbb{Z}_4$-symmetry of Σ_5, 3.

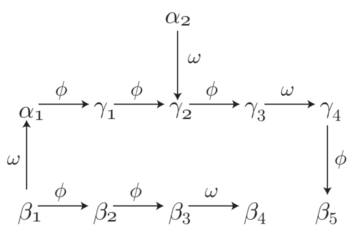

Fig. 43.

We assume $g \neq 3, 5$. From the lantern relation we can rewrite the relation as

$$T_{\alpha_1} = (T_{x_1} T_{\alpha_3}^{-1})(T_{x_2} T_{\gamma_1}^{-1})(T_{\alpha_2} T_{\gamma_2}^{-1}).$$

Since $\phi(\alpha_2) = x_1$, $\phi(\gamma_2) = \alpha_3$, $\psi(x_1) = x_2$ and $\psi(\alpha_3) = \gamma_1$, we have

$$T_{x_1} T_{\alpha_3}^{-1} = \phi T_{\alpha_2} T_{\gamma_2}^{-1} \phi^{-1},$$
$$T_{x_2} T_{\gamma_1}^{-1} = \psi T_{x_1} T_{\alpha_3}^{-1} \psi^{-1} = \psi \phi T_{\alpha_2} T_{\gamma_2}^{-1} \phi^{-1} \psi^{-1}.$$

Moreover, since $\omega(\alpha_2) = \gamma_2$, we see that

$$T_{\alpha_2} T_{\gamma_2}^{-1} = T_{\alpha_2} \omega T_{\alpha_2}^{-1} \omega^{-1} = (T_{\alpha_2} \omega T_{\alpha_2}^{-1}) \omega^{-1}.$$

Let $\tilde{\omega}$ denote $T_{\alpha_2} \omega T_{\alpha_2}^{-1}$. Then, we have $T_{\alpha_2} T_{\gamma_2}^{-1} = \tilde{\omega} \omega^{-1}$. Hence, we have

$$T_{\alpha_1} = (\phi \tilde{\omega} \omega^{-1} \phi^{-1})(\psi \phi \tilde{\omega} \omega^{-1} \phi^{-1} \psi^{-1})(\tilde{\omega} \omega^{-1}). \qquad (4{\cdot}1)$$

Therefore, T_{α_1} is generated by ϕ, ψ, ω and $\tilde{\omega}$.

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We assume $g = 3$. From the lantern relation we can rewrite the relation as

$$T_{\alpha_1} = (T_{x_1}T_{\alpha_3}^{-1})(T_{x_2}T_{\gamma_1}^{-1})(T_{\alpha_2}T_{\gamma_2}^{-1}).$$

Since $\phi(\alpha_2) = x_1$, $\phi(\gamma_2) = \alpha_3$, $\psi(x_1) = x_2$ and $\psi(\alpha_3) = \gamma_1$, we have

$$T_{x_1}T_{\alpha_3}^{-1} = \phi T_{\alpha_2}T_{\gamma_2}^{-1}\phi^{-1},$$
$$T_{x_2}T_{\gamma_1}^{-1} = \psi T_{x_1}T_{\alpha_3}^{-1}\psi^{-1} = \psi\phi T_{\alpha_2}T_{\gamma_2}^{-1}\phi^{-1}\psi^{-1}.$$

Moreover, since $\omega^2(\alpha_2) = \gamma_2$, we see that

$$T_{\alpha_2}T_{\gamma_2}^{-1} = T_{\alpha_2}\omega^2T_{\alpha_2}^{-1}\omega^{-2} = (T_{\alpha_2}\omega T_{\alpha_2}^{-1})^2\omega^{-2}.$$

As before, let $\tilde{\omega}$ denote $T_{\alpha_2}\omega T_{\alpha_2}^{-1}$. Then, we have $T_{\alpha_2}T_{\gamma_2}^{-1} = \tilde{\omega}^2\omega^{-2}$. Hence, we have

$$T_{\alpha_1} = (\phi\tilde{\omega}^2\omega^{-2}\phi^{-1})(\psi\phi\tilde{\omega}^2\omega^{-2}\phi^{-1}\psi^{-1})(\tilde{\omega}^2\omega^{-2}). \tag{4·2}$$

Therefore, T_{α_1} is generated by ϕ, ψ, ω and $\tilde{\omega}$.

We assume $g = 5$. From the lantern relation we can rewrite the relation as

$$T_{\alpha_1} = (T_{x_1}T_{\gamma_1}^{-1})(T_{x_2}T_{\alpha_3}^{-1})(T_{\alpha_2}T_{\gamma_2}^{-1}).$$

Since $\phi^{-1}(\alpha_2) = x_1$, $\phi^{-1}(\gamma_2) = \gamma_1$, $\psi^{-1}(x_1) = x_2$ and $\psi^{-1}(\gamma_1) = \alpha_3$, we have

$$T_{x_1}T_{\gamma_1}^{-1} = \phi^{-1}T_{\alpha_2}T_{\gamma_2}^{-1}\phi,$$
$$T_{x_2}T_{\alpha_3}^{-1} = \psi^{-1}T_{x_1}T_{\gamma_1}^{-1}\psi = \psi^{-1}\phi^{-1}T_{\alpha_2}T_{\gamma_2}^{-1}\phi\psi.$$

Moreover, since $\omega^{-1}(\alpha_2) = \gamma_2$, we see that

$$T_{\alpha_2}T_{\gamma_2}^{-1} = T_{\alpha_2}\omega^{-1}T_{\alpha_2}^{-1}\omega = (T_{\alpha_2}\omega T_{\alpha_2}^{-1})^{-1}\omega.$$

Let $\tilde{\omega}$ denote $T_{\alpha_2}\omega T_{\alpha_2}^{-1}$. Then, we have $T_{\alpha_2}T_{\gamma_2}^{-1} = \tilde{\omega}^{-1}\omega$. Hence, we have

$$T_{\alpha_1} = (\phi^{-1}\tilde{\omega}^{-1}\omega\phi)(\psi^{-1}\phi^{-1}\tilde{\omega}^{-1}\omega\phi\psi)(\tilde{\omega}^{-1}\omega). \tag{4·3}$$

Therefore, T_{α_1} is generated by ϕ, ψ, ω and $\tilde{\omega}$.

Proof of Theorem 1 *(*ii*).* We show that $\mathcal{M}_g$ is generated by ϕ, ψ, ω and $\tilde{\omega}$.

Let G denote the group generated by ϕ, ψ, ω and $\tilde{\omega}$. From the equations (4·1), (4·2) and (4·3) we have $T_{\alpha_1} \in G$. Since we can send α_1 to all γ_i and β_i by ϕ, ψ and ω (see Figures 21, 25, 31, 38 and 43), by Lemma 2, T_{γ_i} and $T_{\beta_i} \in G$ for all i. Similarly, we have $T_{\alpha_2} \in G$. Therefore, since we have shown that all Humphries's generators are in G, G is equal to $\mathcal{M}_g$.

Remark 6. It seems that for $g = 3$ we can not construct elements of order 5 by our method. In fact, is well-known that $\mathcal{M}_3$ has no elements of order 5.

5. *Remarks*

5·1. *Low genus*

By using the argument of McCarthy and Papadopulos [**13**] and the work of Hirose [**5**], we find that $\mathcal{M}_g$ can not be generated by elements of same order for $g = 1, 2$.

Hirose gave presentations of finite order elements by Dehn twists up to conjugacy for $g = 1, \ldots, 4$. We introduce the presentation of finite order elements in the case of $g = 1, 2$. The list is as follows:

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McCarthy and Papadopulos proved that $\mathcal{M}_2$ can not be generated by elements of order 2. The argument of McCarthy and Papadopulos is as follows:

genus	elements	order
1	$T_{\beta_1}T_{\alpha_1}$	6
	$T_{\alpha_1}T_{\beta_1}T_{\alpha_1}$	4
2	$T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1}$	10
	$T_{\beta_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1}$	8
	$T_{\alpha_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1}$	6
	$(T_{\alpha_1}T_{\beta_1}T_{\gamma_1}T_{\beta_2}T_{\alpha_2})(T_{\alpha_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1})^3$	6

Let c be a nonseparating simple closed curve and p be the abelianization map given by Powell's result [**14**]:

$$
\begin{array}{ccc}
p : \mathcal{M}_2 & \longrightarrow & \mathbb{Z}_{10} \\
\Cup & & \Cup \\
T_c & \longmapsto & 1.
\end{array}
$$

We can find that

$$
\begin{aligned}
p((T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1})^5) &= p((T_{\beta_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1})^4) \\
&= p(((T_{\alpha_1}T_{\beta_1}T_{\gamma_1}T_{\beta_2}T_{\alpha_2})(T_{\alpha_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1})^3)^3) \\
&= 0 \\
p((T_{\alpha_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1})^3) &= 5.
\end{aligned}
$$

Since $\mathbb{Z}_{10}$ can not be generated by 0 and 5, we see that $\mathcal{M}_2$ can not be generated by elements of order 2.

By the similar proof, we can see that $\mathcal{M}_1$ and $\mathcal{M}_2$ can not be generated by elements of same order.

Remark 7. *$\mathcal{M}_1$ and $\mathcal{M}_2$ can be generated by elements of different order. For example, $\mathcal{M}_1$ can be generated by $T_{\beta_1}T_{\alpha_1}$ and $T_{\alpha_1}T_{\beta_1}T_{\alpha_1}$, and $\mathcal{M}_2$ can be generated by $T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1}$ and $T_{\alpha_2}T_{\beta_2}T_{\gamma_1}T_{\beta_1}T_{\alpha_1}$.*

5·2. *Lower bound*

The order of $\mathcal{M}_g$ is not finite. Therefore, it is clear that a lower bound of the number of generators whose order are 3 (resp. 4) is 2. The author has the following question:

Question 1. What is the minimal number of elements of order 3 (resp. 4) required to generate $\mathcal{M}_g$?

Kassabov [**8**] proved that for $g \geqslant 7$ $\mathcal{M}_g$ is generated by three involutions. Since $\mathcal{M}_g$ does not have a finite index cyclic subgroup, it is not generated by 2 involutions. The following problem remains open.

Problem 1 ([**3**], [**8**]). For $g \geqslant 7$, determine whether or not $\mathcal{M}_g$ can be generated by three involutions.

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