J. Fluid Mech. (2004), *vol.* 517, *pp.* 55–70.
DOI: 10.1017/S0022112004000497 Printed in the United Kingdom

Bounds on horizontal convection

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(Received 5 December 2003 and in revised form 12 May 2004)

For a fluid layer heated and cooled differentially at its surface, we use a variational approach to place bounds on the viscous dissipation rate and a horizontal Nusselt measure based on the entropy production. With a general temperature distribution imposed at the top of the layer and a variety of thermal boundary conditions at the base of the layer, the horizontal Nusselt number is bounded by $cR_H^{1/3}$ as the horizontal Rayleigh number $R_H \rightarrow \infty$, for some constant c. The analysis suggests that the ultimate regime for this so-called 'horizontal convection' is one in which the temperature field develops a boundary layer of width $O(R_H^{-1/3})$ at the surface, but has no variation in the interior. Although this scenario resonates with results of dimensional scaling theory and numerical computations, the bounds differ in the dependence of the Nusselt measure on R_H. Numerical solutions for steady convection appear to confirm Rossby's result that the horizontal Nusselt number scales like $R_H^{1/5}$, suggesting either that the bound is not tight or that the numerics have yet to reach the asymptotic regime.

1. Introduction

A key feature of the ocean is its non-uniform surface heating over latitude by the sun. Since any variation in the surface temperature will set the ocean into motion (called 'horizontal convection' after Stern 1975), an important question is whether such forcing makes an impact on the general ocean circulation. Currently, this differential surface heating is not considered an effective mechanism for supplying energy to the ocean circulation (Defant 1961; Houghton 1986). However, this viewpoint is based merely upon a general thermodynamic argument by Sandström (1908), and the fact that previous numerical investigations have only ever found steady weak flow (Somerville 1967; Beardsley & Festa 1972; Rossby 1998). In fact, Sandström's argument makes no direct contact with the fluid's governing equations and is imprecisely stated (although his general conclusion seems reasonable, see Jeffreys 1925). Moreover, the regime accessed by the numerical work is well removed from the physical condition of the ocean. Nevertheless, some authors have gone as far as suggesting that the flow remains steady and stable regardless of how strongly the system is forced (e.g. Huang 1999; Wunsch 2000).

Paparella & Young (2002) have tried to formalize Sandström's argument by focusing attention on the associated energy dissipation rate of the flow. They obtain an upper bound on the dissipation rate that vanishes as the thermal and viscous diffusivities tend to zero (at fixed Prandtl number). By quoting a popular experimental law of turbulence (Frisch 1995), they are thereby led to an 'anti-turbulence' theorem for horizontal convection. Whilst this may be too stringent a criterion for precluding turbulence according to some tastes, the implication of the bound is clear: uneven surface heating cannot provide net energy to the fluid in the non-diffusive limit. Paparella & Young also present numerical computations at higher Rayleigh numbers than previously studied, which clearly show the existence of unsteady flows. Contrary to what has previously been assumed, this suggests the presence of further bifurcations and the concomitant emergence of increasingly energetic flows as the thermal driving becomes larger. Thus, an outstanding question concerns the form of the ultimate state of horizontal convection.

The purpose of the present paper is to constrain this ultimate regime by using a variational technique for securing upper bounds on certain physical measures that describe the flow. The key idea is to maximize the quantity of interest, subject to various constraints obtained from projections of the governing equations. Previous applications of this technique indicate that the bound obtained can possess the same scaling behaviour with the relevant controlling parameter as observed experimentally or numerically, e.g. Rayleigh–Bénard convection, where evidence is accumulating for a Nusselt number scaling like $Ra^{1/2}$ (where Ra is the Rayleigh number, Roche *et al.* 2001; Lohse & Toschi 2003), and in wall-bounded shear flows, the dissipation rate bound is $O(1)$, which is the observed behaviour except for logarithmic corrections in the Reynolds number (e.g. Zagarola & Smits 1998). In addition to the bound itself, the procedure may also predict salient time-averaged features of the realized solution, such as the presence and thickness of boundary layers.

For horizontal convection, the first problem is to identify a global flow measure that captures key features of the flow. In Rayleigh–Bénard convection, a popular choice is the Nusselt number, which is the factor by which the heat flux is enhanced by convection over that expected for pure conduction. Here, we follow suit and define a convenient *horizontal* Nusselt number, Nu_H, based on an approximation of the horizontal heat flux. We seek bounds on this quantity using the method of Doering & Constantin (1996), which is more easily adapted to the current problem than the older, more classical Euler–Lagrange approach (Howard 1963, 1972; Busse 1978); although both should ultimately offer the same result (Kerswell 1998).

2. Formulation

Following Stommel's (1962) idealization of horizontal convection, we consider a planar layer of Boussinesq fluid of constant depth. We assume periodicity (with period L) in the horizontal direction and impose a fixed temperature distribution on the top surface. We use L, the thermal diffusion time L^2/κ (where κ is the thermal diffusivity) and a typical temperature variation over the surface, ΔT, to non-dimensionalize the governing equations, so that they take the form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + \widehat{\boldsymbol{z}}\times f\boldsymbol{u} + \nabla p = \sigma R_H T\widehat{\boldsymbol{z}} + \sigma\nabla^2\boldsymbol{u}, \tag{2.1}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\nabla T = \nabla^2 T, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{2.3}$$

where $\boldsymbol{u}$, T and p are the non-dimensional velocity field, temperature and pressure respectively, $\sigma = \nu/\kappa$ is the Prandtl number, $R_H = L^3 g\alpha_T\Delta T/\kappa\nu$ is the horizontal

Rayleigh number, ν is the kinematic viscosity, g is the acceleration due to gravity and α_T is the thermal expansion coefficient. Rotation is included through the Coriolis term, $\widehat{\boldsymbol{z}} \times f\boldsymbol{u}$, within the edifice of the conventional f-plane; however, it plays no role in the bounds derived below since it does not contribute to the integral relations that are ultimately exploited.

The equations are solved on the domain, $0 \leqslant x \leqslant 1$, $-\infty \leqslant y \leqslant \infty$ and $0 \leqslant z \leqslant d$. At the top surface we impose stress-free velocity boundary conditions and $T = T_1(x)$. By altering the definition of p, we may assume $\int_0^1 T_1(x)\,\mathrm{d}x = 0$. At the bottom, we use no-slip velocity boundary conditions and a mixed thermal boundary condition, $T = T_0 + \lambda\,\mathrm{d}T/\mathrm{d}z$, where T_0 and λ are parameters that we may vary to gauge the sensitivity of the system on the thermal properties of the lower boundary. We shall vary λ over the range $[0, \infty]$, to pass from a perfectly conducting ($\lambda = 0$) to a perfectly insulating ($\lambda = \infty$) boundary. In addition, the constant flux problem ($T_z|_{z=0} = -F$) is accessible by keeping $F = T_0/\lambda \geqslant 0$ fixed as $\lambda \to \infty$. The oceanographic problem might be characterized by a constant flux condition involving the relatively slight geothermal heating. However, by adopting a general lower thermal boundary condition, we may avoid any discussion of the detailed physics of the ocean floor, whilst simultaneously positioning ourselves to compare the results with known bounds for either fixed-temperature or fixed-flux Rayleigh–Bénard convection. In any event, we shall establish that our results are largely insensitive to the particular choice of λ.

It is also worth remarking that this system does not possess a static solution with $\boldsymbol{u} = 0$, unlike the Rayleigh–Bénard problem: from (2.1), we must satisfy $\nabla p = \sigma R_H T \widehat{\boldsymbol{z}}$, but since T must depend non-trivially on x in order to satisfy the boundary conditions, $T\widehat{\boldsymbol{z}}$ cannot be a gradient (unless $T \equiv 0$). Furthermore, it has not been possible to construct any exact analytic solution to the system.

2.1. *Boundedness of temperature and velocity fields*

Before establishing bounds on any functionals of the temperature and velocity fields, we first prove that these fields are themselves bounded (in an L_2 sense) in time. In this subsection, we shall use an overbar to denote the horizontal average and angle brackets to denote the space average:

$$\overline{(\cdot)} = \lim_{y_0 \to \infty} \frac{1}{2y_0} \int_{-y_0}^{y_0} \int_0^1 (\cdot)\,\mathrm{d}x\,\mathrm{d}y, \qquad \langle(\cdot)\rangle = \frac{1}{d} \int_0^d \overline{(\cdot)}\,\mathrm{d}z.$$

We begin by proving that the temperature field is bounded. Consider the solution of (2.2) over the time interval $t \in [0, t_0]$, starting from a bounded initial temperature distribution at $t = 0$. Suppose first that the maximum value of T occurs at a point at which $z \neq 0, d$. There, $\nabla T = 0$, $\nabla^2 T \leqslant 0$ and so from (2.2), $\partial T/\partial t \leqslant 0$, demanding that the maximum of T is attained at $t = 0$. If the maximum occurs at $z = 0$, then $T_z \leqslant 0$ there, which implies, using the boundary condition, that $T \leqslant T_0$, and if it is at $z = d$, then $T \leqslant \max(T_1)$. A similar principle can be used to bound T from below and thus T is everywhere in the range

$$[\min(T_0, \inf(T|_{t=0})), \max(T_0, \sup(T|_{t=0}))].$$

If the system is allowed to relax for sufficiently long, then eventually

$$\min(\min(T_1), T_0) \leqslant T \leqslant \max(\max(T_1), T_0), \tag{2.4}$$

a result that will be exploited below.

To bound the velocity field, we take $\langle \boldsymbol{u}\cdot(2.1)\rangle$, rearrange the results and use the Cauchy–Schwartz inequality, to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle|\boldsymbol{u}|^2\rangle = \sigma R_H\langle wT\rangle - \sigma\langle|\nabla\boldsymbol{u}|^2\rangle \leqslant \sigma\left(R_H\sqrt{\langle|\boldsymbol{u}|^2\rangle\langle T^2\rangle} - \langle|\nabla\boldsymbol{u}|^2\rangle\right). \tag{2.5}$$

After dividing by $\sqrt{\langle|\boldsymbol{u}|^2\rangle}$ and using Poincaré's inequality, $\langle|\nabla\boldsymbol{u}|^2\rangle \geqslant \pi^2\langle|\boldsymbol{u}|^2\rangle/d^2$, we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\langle|\boldsymbol{u}|^2\rangle} \leqslant \sigma\left(R_H\sqrt{\langle T^2\rangle} - \frac{\langle|\nabla\boldsymbol{u}|^2\rangle}{\sqrt{\langle|\boldsymbol{u}|^2\rangle}}\right) \leqslant \sigma\left(R_H\sqrt{\langle T^2\rangle} - \frac{\pi^2}{d^2}\sqrt{\langle|\boldsymbol{u}|^2\rangle}\right), \tag{2.6}$$

so $\langle|\boldsymbol{u}|^2\rangle$ is bounded above by its initial value or by $R_H^2 d^4 \max(\max(T_1^2), T_0^2)/\pi^4$.

2.2. *Bound on the viscous dissipation rate*

Now we derive a simple generalization of the result by Paparella & Young (2002). Hereinafter, we extend the definitions of the 'averages' to include long time averaging, i.e.

$$\overline{(\cdot)} = \lim_{t_0, y_0\to\infty}\frac{1}{2t_0y_0}\int_0^{t_0}\int_{-y_0}^{y_0}\int_0^1 (\cdot)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t, \qquad \langle(\cdot)\rangle = \frac{1}{d}\int_0^d \overline{(\cdot)}\,\mathrm{d}z.$$

The dimensional viscous dissipation rate per unit volume is given by

$$\varepsilon = \frac{\nu\kappa^2}{L^4}\langle|\nabla\boldsymbol{u}|^2\rangle.$$

Taking $\langle \boldsymbol{u}\cdot(2.1)\rangle$ gives

$$\langle|\nabla\boldsymbol{u}|^2\rangle = R_H\langle wT\rangle,$$

since we know from the preceding section that the kinetic energy is bounded. Similarly, $\langle(d-z)(2.2)\rangle$ leads to

$$\langle wT\rangle = -\overline{T}_z|_0 - \frac{1}{d}\overline{T}|_0 = \frac{1}{\lambda}T_0 - \frac{d+\lambda}{d\lambda}\overline{T}|_0 \leqslant M, \tag{2.7}$$

using (2.4), where

$$M = \max\left[\frac{dT_0 - (d+\lambda)\min(T_1)}{d\lambda}, -\frac{T_0}{d}\right].$$

Thus

$$\varepsilon \leqslant \frac{\nu\kappa^2 R_H}{L^4}M \equiv \kappa\frac{g\alpha_T M\Delta T}{L}, \tag{2.8}$$

which indicates that the viscous dissipation rate must vanish in the limit $\nu, \kappa \to 0$ with $\sigma = \nu/\kappa$ and all other parameters held fixed. This extends the 'anti-turbulence theorem' of Paparella & Young (2002) to more general bottom boundary conditions; their result is recovered as $\lambda\to\infty$. Additionally, if $F = T_0/\lambda$ is held fixed as $\lambda\to\infty$, (2.8) also bounds a flow with constant thermal flux F at the bottom boundary. We cannot, however, establish a similar result for a fixed bottom temperature. In this case, $\lambda\to 0$ and we are unable to relate the flux through the bottom to the temperature there as in (2.7). Instead, we have used variational methods similar to those employed below, to establish the bound, $\varepsilon \lesssim g(T_0)\nu\kappa^2 R_H^{3/2}/L^4$ as $R_H \to \infty$ for some function g, indicating that ε is bounded by a non-zero constant as $\nu, \kappa \to 0$.

2.3. *Definition of the horizontal Nusselt number*

The horizontal heat flux is given by

$$\overline{\chi(x, y, t)T_z(x, y, d, t)},$$

where $\chi(x, y, t) = 1$ if $T_z(x, y, d, t) > 0$ (corresponding to places where there is flux in) and 0 otherwise (corresponding to flux out). Unfortunately, this quantity proves difficult to work with because of the non-continuous function χ, and because we cannot predict *a priori* where heat flows into and out of the domain. Indeed, numerical simulations (e.g. Rossby 1998; Paparella & Young 2002) indicate the presence of narrow cold plumes and hence a marked asymmetry between the heat inflow and outflow. Instead, we define a pseudo-flux, $\overline{TT_z}|_{z=d}$, which only approximates the heat flux, but is significantly easier to work with. In Rayleigh–Bénard convection, the non-dimensional heat flux can be associated exactly with the entropy production integral $\langle|\nabla T|^2\rangle$. Likewise, from (2.2) the pseudo-flux is distinguished by its connection to $\langle|\nabla T|^2\rangle$ via

$$\overline{TT_z}|_{z=d} = d\langle|\nabla T|^2\rangle + \overline{TT_z}|_{z=0}, \tag{2.9}$$

a relationship that will be used in the bounding procedure.

A horizontal Nusselt number can now be defined by normalizing the pseudo-flux with that corresponding to a 'conduction' temperature distribution, $T_c(x, z)$. Following Paparella & Young (2002), we define T_c to be the steady solution where the fluid motion is ignored. That is, T_c satisfies $\nabla^2 T_c = 0$ together with the thermal boundary conditions. Hence,

$$Nu_H = \frac{\overline{TT_z}|_d}{\overline{T_c T_{cz}}|_d} = \frac{d\langle|\nabla T|^2\rangle + \overline{TT_z}|_0}{\overline{T_c T_{cz}}|_d}. \tag{2.10}$$

For the zero-flux bottom boundary condition used by Paparella & Young, this quantity reduces to the functional, Φ, used to characterize their numerical solutions.

2.4. *Bound on Nu_H*

We now bound the horizontal Nusselt number, Nu_H, using the background method of Doering & Constantin (1996). We let $T(\boldsymbol{x}, t) = \tau(x, z) + \theta(\boldsymbol{x}, t)$, where τ satisfies the boundary conditions on T, and so θ satisfies the homogeneous conditions, $\theta = 0$ at $z = d$ and $\theta - \lambda\theta_z = 0$ at $z = 0$. It is important to notice that this decomposition is non-unique: the essence of the Doering–Constantin approach is to exploit this freedom by choosing τ conveniently. Note also that in contrast to Doering & Constantin's treatment for Rayleigh–Bénard convection, in which τ is a function of z only, here τ must also depend on x in order to satisfy the boundary conditions.

We construct a Lagrangian to bound the numerator of (2.10):

$$\mathscr{L} = \overline{TT_z}|_d - a\langle\boldsymbol{u}\cdot(2.1)\rangle - b\langle\theta(2.2)\rangle, \tag{2.11}$$

where $\boldsymbol{u}$ satisfies (2.3) everywhere, and a and b are Lagrange multipliers enforcing additional constraints that amount to 'power integrals' of the governing equations (Kerswell 2001). The ideal would be to require (2.1)–(2.3) to be satisfied at every point in the domain at all times, but this is too complicated since it is equivalent to solving those equations. Instead, by using the formulation (2.11) we only require that certain time-averaged projections are satisfied, hoping that this still captures the main dependence of the bound on R_H.

Expanding (2.11) gives

$$\mathscr{L} = \langle d|\nabla\tau|^2 - a\sigma|\nabla \boldsymbol{u}|^2 + (b-d)\theta\nabla^2\theta + (b-2d)\theta\nabla^2\tau + a\sigma R_H w(\tau+\theta) - b\theta \boldsymbol{u}\cdot\nabla\tau\rangle + \frac{1}{\lambda}\,\overline{(\tau-T_0)^2}\Big|_0 + \frac{T_0}{\lambda}(\overline{T}|_0 - T_0). \quad (2.12)$$

From (2.7), $\overline{T}|_0 = d(T_0 - \lambda\langle wT\rangle)/(d+\lambda)$ and so

$$\mathscr{L} = \langle d|\nabla\tau|^2 - a\sigma|\nabla \boldsymbol{u}|^2 + (b-d)\theta\nabla^2\theta + (b-2d)\theta\nabla^2\tau + \mu w(\tau+\theta) - b\theta \boldsymbol{u}\cdot\nabla\tau\rangle + \frac{1}{\lambda}\,\overline{(\tau-T_0)^2}\Big|_0 - \frac{T_0^2}{d+\lambda}, \quad (2.13)$$

where

$$\mu = a\sigma R_H - \frac{dT_0}{d+\lambda}$$

is assumed positive since R_H is large. If $a \geqslant 0$ and $b \geqslant d$, then all terms in (2.13) are either independent of θ and $\boldsymbol{u}$, depend linearly on one of them, or may be related to quadratic semi-definite terms, with the exceptions of $\langle \mu w\theta\rangle$ and $\langle -b\theta\boldsymbol{u}\cdot\nabla\tau\rangle$. For Rayleigh–Bénard convection (Doering & Constantin 1996; Kerswell 2001) these latter two terms must be dealt with together. Here, by contrast, the fact that the bottom temperature gradient is constrained by the lower boundary condition allows us to deal with the two terms separately and more efficiently, ultimately enabling us to obtain a tighter scaling as $R_H \to \infty$. More specifically, from (2.7), we find $\langle \mu w(\tau+\theta)\rangle = \mu\langle wT\rangle \leqslant \mu M$. Then, we need only to choose τ such that

$$\langle -b\theta\boldsymbol{u}\cdot\nabla\tau\rangle \leqslant \alpha\langle|\nabla\boldsymbol{u}|^2\rangle + \beta\langle|\nabla\theta|^2\rangle, \quad (2.14)$$

allowing the whole expression in (2.13) to be bounded above straightforwardly.

We choose the background field τ to minimize the worst estimate of $\langle -b\theta\boldsymbol{u}\cdot\nabla\tau\rangle$, which we accomplish by setting $\nabla\tau = \boldsymbol{0}$ over as much of the layer as possible. We let $\tau = \tau_0(z) + T_1(x)\tau_1(z)$, where

$$\tau_0 = \begin{cases} (\delta_0 - z)T_0/(\delta_0+\lambda), & 0<z<\delta_0,\\ 0, & \delta_0<z<d, \end{cases} \qquad \tau_1 = \begin{cases} 0, & 0<z<d-\delta_1,\\ (z-d+\delta_1)/\delta_1, & d-\delta_1<z<d, \end{cases} \quad (2.15)$$

thereby creating boundary layers of width δ_0 and δ_1 at the bottom and top of the box, respectively, to satisfy the boundary conditions. With this choice, and using the estimates (A 1), (A 3) and (A 4) laid out in Appendix A, we can show that (2.14) holds for all fields $\boldsymbol{u}$ and θ, where

$$\alpha = b\left(\frac{|T_0|}{\delta_0+\lambda}\frac{\delta_0^2 c_0}{2\pi^2} + \frac{\delta_1 c_1}{2\pi^2}\max|T_1|\left(1+\frac{\max|T_1'|}{\max|T_1|}\pi\sqrt{d\delta_1}\right)\right) > 0,$$

$$\beta = b\left(\frac{|T_0|}{\delta_0+\lambda}\frac{d\delta_0}{2c_0} + \frac{2\delta_1}{\pi^2 c_1}\max|T_1|\left(1+\frac{\max|T_1'|}{\max|T_1|}\pi\sqrt{d\delta_1}\right)\right) > 0,$$

where c_0 and c_1 are arbitrary positive constants. Incorporating these estimates into

(2.13) leads to

$$\mathscr{L} \leqslant \langle d|\nabla\tau|^2 - (a\sigma - \alpha)|\nabla \boldsymbol{u}|^2 + (b - d - \beta)\theta\nabla^2\theta + (b - 2d)\theta\nabla^2\tau\rangle + M\mu + \frac{1}{\lambda}\,\overline{(\tau - T_0)^2}\Big|_0 - \frac{T_0^2}{d+\lambda},$$

the right-hand side of which is easily minimized to find the smallest upper bound (the boundary term $-\beta\overline{\theta\theta_z}|_0/d$ arising from the integration by parts has been omitted since it is negative semi-definite). The Euler–Lagrange equations for minimizing the right-hand side over the fields $\boldsymbol{u}$ and θ are

$$\nabla p - 2(a\sigma - \alpha)\nabla^2\boldsymbol{u} = 0,$$
$$-2(b - d - \beta)\nabla^2\theta = (b - 2d)\nabla^2\tau,$$

where p is introduced to ensure that $\nabla\cdot\boldsymbol{u} = 0$ (and mimics a pressure field). The unique solution is

$$\boldsymbol{u}^* = 0, \quad \theta^* = -\frac{(b-2d)}{2(b-d-\beta)}(\tau - T_c), \tag{2.16}$$

which minimizes the functional as long as the 'spectral constraints', $a\sigma \geqslant \alpha$ and $b - d \geqslant \beta$, are satisfied. The extremal bound is

$$\mathscr{L} \leqslant d\langle|\nabla\tau|^2\rangle + \frac{(b-2d)^2}{4(b-d-\beta)}\left[\langle\nabla\tau\cdot\nabla(\tau - T_c)\rangle + \frac{1}{d\lambda}\,\overline{(\tau - T_c)(\tau - T_0)}\Big|_0\right] + M\mu + \frac{1}{\lambda}\,\overline{(\tau - T_0)^2}\Big|_0 - \frac{T_0^2}{d+\lambda}.$$

Choosing $b = 2d$ for simplicity, for $R_H \gg 1$ the tightest leading-order bound (subject to the spectral constraints) is obtained with the parameter selections,

$$\delta_0 = 0, \quad \delta_1 = \frac{1}{2}\left(\frac{\pi^4\overline{T_1^2}}{\max\left(T_1^2\right)dMR_H}\right)^{1/3},$$
$$a\sigma = d\left(\frac{\overline{T_1^2}\max|T_1|}{\pi^2 dMR_H}\right)^{2/3}, \quad c_1 = 2\left(\frac{\overline{T_1^2}\max|T_1|}{\pi^2 dMR_H}\right)^{1/3},$$

(with c_0 undefined). These values also ensure that $\mu \geqslant 0$, and in particular they secure the lowest possible exponent of R_H in the bound, which is $1/3$. Thence,

$$\mathscr{L} \leqslant 3\,\pi^{-4/3}\left(\overline{T_1^2}^{\,2}\max\left(T_1^2\right)dMR_H\right)^{1/3} + O(1),$$

and so

$$Nu_H \lesssim \frac{3\left(\overline{T_1^2}^{\,2}\max\left(T_1^2\right)dMR_H\right)^{1/3}}{\pi^{4/3}\;\overline{T_cT_{cz}}\big|_{z=d}}. \tag{2.17}$$

In the simple case of a cosine temperature distribution $T_1 = \cos 2\pi x$ on the top surface, the bound is

$$Nu_H \lesssim \frac{3(dMR_H)^{1/3}}{2^{2/3}\,\pi^{7/3}J_c}, \tag{2.18}$$

https://doi.org/10.1017/S0022112004000497 Published online by Cambridge University Press

where

$$J_c = \left(\frac{\cosh 2\pi d + 2\pi\lambda \sinh 2\pi d}{\sinh 2\pi d + 2\pi\lambda \cosh 2\pi d} \right).$$

2.5. *Sensitivity to the boundary conditions*

The distinguished limit of $\lambda \to \infty$ and $T_0/\lambda \to F > 0$ yields a constant flux F through the bottom (or a perfectly insulating bottom if $F = 0$). The bound (2.17) then simplifies to

$$Nu_H \lesssim \frac{3\left(\overline{T_1^2}^2 \max\left(T_1^2\right)(1 + Fd)R_H\right)^{1/3}}{\pi^{4/3}\overline{T_c T_{cz}}|_{z=d}}. \tag{2.19}$$

For large F, this bound scales with the one-third power of a *flux* Rayleigh number, FR_H, as found by Otero *et al.* (2002) who studied fixed-flux Rayleigh–Bénard convection.

There is a problem, however, lurking in the limit $\lambda \to 0$, for a perfectly conducting bottom. In this case, the prefactor of $M^{1/3}$ in (2.17) diverges for $T_0 > -1$. Furthermore, if $T_0 \leqslant -1$, the bound still diverges, but through another term that has been neglected in arriving at (2.17). Of course, the bound holds at any given $\lambda > 0$, but the fact that the limits $R_H \to \infty$ and $\lambda \to 0$ cannot be interchanged indicates a singular limit. There are two separate sticking points in the above analysis when $\lambda = 0$. The first is that we can no longer use the boundedness of the temperature field to control $\langle wT \rangle$ since the heat flux through the bottom boundary is now unknown. In fact, had we not made use of this boundedness in the above analysis, then the best bound possible would have been the more conservative $Nu_H \leqslant O(R_H^{2/5})$. The second sticking point is that a strong boundary layer is required at the bottom boundary to adjust the fluid temperature to the boundary condition at $\lambda = 0$, as opposed to the relatively weak boundary layer required with $\lambda > 0$, which made an insignificant contribution to the bound for asymptotically large R_H. Both of these features are essential in converting the current results to forms that can be compared with known bounds for the fixed-temperature Rayleigh–Bénard problem.

In Appendix B, we offer a separate analysis for a fixed temperature on the lower boundary, which yields the leading-order bound,

$$Nu_H \lesssim \frac{d^{1/2}}{4\pi^2} f R_H^{1/2}, \tag{2.20}$$

for some f depending on the lower boundary temperature T_0. This signifies that the exponent of R_H has increased from 1/3 to 1/2. Also, $f \sim T_0^{5/2}$ for $T_0 \gg 1$ and $f \sim (-T_0)^{-1/2}$ for $-O(R_H) < T_0 \ll -1$. If, instead, we consider bounds on the usual Nusselt number in terms of a *vertical* Rayleigh number, rather than R_H, we recover those found by Doering & Constantin (1996) for Rayleigh–Bénard convection (see Appendix B), at least up to the prefactor in front of the main Rayleigh-number scaling.

The bound is relatively insensitive to the choice of the other boundary conditions. For example, interchanging no-slip and stress-free boundaries on the top and bottom makes no difference to the leading-order bound. Neither does switching the horizontal periodic boundary conditions to insulating and stress-free sidewalls.

3. Discussion

The main results of this paper are the bounds expressed in (2.17), (2.19) and (2.20). We found that the bound on the horizontal Nusselt number scales with $R_H^{1/3}$ for any bottom boundary condition that involves the temperature gradient. However, in the singular case of a constant-temperature bottom boundary condition, the best bound available scales as $R_H^{1/2}$. A simple re-expression of the latter in terms of a vertical Rayleigh number recovers the known scaling of the bound for fixed-temperature Rayleigh–Bénard convection.

To gauge the physical significance of our results for ocean circulation, we estimate the bounds on ε and Nu_H using numbers suggested by the real ocean. We use a simple cosine distribution $T_1 = \cos 2\pi x$ on the top surface, and since the ocean floor is thought to be a nearly perfect insulator with a weak geothermal heat flux, we use (2.19) to estimate the Nu_H-bound. We take $L = 10^7$ m, $H = 10^3$ m (so $d = 10^{-4}$), $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$, $\kappa = 10^{-7}\,\mathrm{m^2\,s^{-1}}$, $\alpha_T = 10^{-5}\,\mathrm{K^{-1}}$, $\Delta T = 10$ K and $g = 10\,\mathrm{m\,s^{-2}}$ (so $R_H = 10^{31}$) and adopt the dimensional value of $F_E \approx 3 \times 10^{13}$ W for the total geothermal flux through the Earth's surface. Thence,

$$F = \frac{F_E/L^2}{\rho c \kappa \Delta T/L} \approx 10^6,$$

where $\rho = 1000\,\mathrm{kg\,m^{-3}}$ and $c = 4200\,\mathrm{J\,kg^{-1}\,K^{-1}}$ is the specific heat capacity of water. The numerator of (2.19) implies that the total horizontal heat flow rate (the dimensional 'pseudo-flux') satisfies

$$L^2 (c\rho\kappa\Delta T/L)\,\overline{\cos 2\pi x T_z}|_d \lesssim 10^{18}\,W$$

and the energy dissipation rate bound (from (2.8)) is

$$\varepsilon \lesssim 10^{-11}\,\mathrm{m^2\,s^{-3}}.$$

Munk & Wunsch (1998) argue that the actual planetary-scale ocean circulation provides an equator-to-pole heat flux of about 2×10^{15} W. Thus, our bound cannot exclude the possibility that horizontal convection due to surface heating is responsible, in contrast to commonplace views in the oceanographic community. An improvement of the prefactor in (2.17) might lead to a more telling result. The geothermal heat flux also contributes significantly to the estimate, although it is not clear whether this reflects more the quality of the bound than a real physical effect.

Associated with the bound (2.17) is the temperature field $\tau + \theta^*$, which has a boundary layer of thickness $O(R_H^{-1/3})$ at the top surface and vanishing interior temperature gradient. This certainly resonates qualitatively with what is seen numerically in a time-averaged sense (e.g. Paparella & Young 2002) although our boundary layer is thinner than that predicted by the dimensional scaling theory of Rossby (1965). In his picture, there is a surface boundary layer of thickness δ, in which vertical derivatives are $O(\delta^{-1})$, and horizontal ones are $O(1)$. Balancing advection terms with diffusion in (2.2), a consistent scaling is obtained with $\boldsymbol{u} = O(\delta^{-2}, \delta^{-2}, \delta^{-1})$, and balancing pressure, buoyancy and viscous dissipation terms in (2.1) gives $\delta = O(R_H^{-1/5})$ and $Nu_H \sim \langle|\nabla T^2|\rangle = O(R_H^{1/5})$. In fact, we can trace the difference between Rossby's scalings and the bound to our treatment of the term $\int_{d-\delta_1}^{d} |w\theta|\,\mathrm{d}z$. In particular, we use the estimate,

$$\int_{d-\delta_1}^{d} w_z^2\,\mathrm{d}z \leqslant \frac{1}{4}\int_{d-\delta_1}^{d} |\nabla \boldsymbol{u}|^2\,\mathrm{d}z,$$

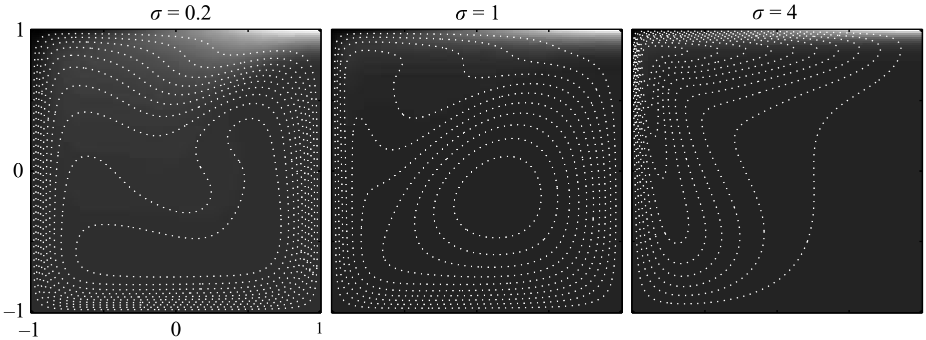

FIGURE 1. Numerical solutions for steady horizontal convection. The boundaries are stress free and insulating, except for the top surface on which we fix the profile $T = \sin(\pi x/2)$. The aspect ratio of the fluid layer is unity and $R_H = 10^6$. Three solutions with different Prandtl numbers are shown. The dotted lines display contours of constant streamfunction, and the shading indicates temperature.

whereas Rossby's scalings suggest that

$$\int_{d-\delta_1}^{d} w_z^2 \, \mathrm{d}z \sim \delta_1^2 \int_{d-\delta_1}^{d} |\nabla \boldsymbol{u}|^2 \, \mathrm{d}z,$$

which imply that our bound is needlessly conservative.

Rossby's scaling theory is not, however, free of criticism. In particular, he adopts a boundary-layer structure in the streamfunction representation of the velocity field which is open to question with the stress-free boundary conditions he also imposes. This is seen particularly clearly in figure 1, which shows numerical solutions for steady horizontal convection, and illustrates detailed flow structure in the fluid interior, despite the confinement of the temperature gradients to a boundary layer; the interior flow is largely a deep inertial response to the overlying boundary currents. Although these computations correspond to a slightly different physical set-up from that for which we have formulated the bounding problem, the bound (2.19) with $F = 0$ still holds. Moreover, the Nusselt number for steady Rayleigh–Bénard convection with stress-free boundaries is commonly thought to scale with the 1/3 power of the vertical Rayleigh number (Roberts 1979), rather than 1/2. Nevertheless, although Rossby's scalings can be criticized, the numerical computations also appear to vindicate them, provided we focus on the thermal boundary layer (which is where our pseudo-flux is determined) and not the fluid interior (see figure 2) The Nusselt number, $Nu_H \sim R_H^{1/3}$, emerges, and the fluid velocities take Rossby's scaling inside the boundary layer (which, expressed in terms of a maximum streamfunction is $\Psi_{BL} \sim R_H^{1/5}$; by contrast, the maximum streamfunction over the whole domain has a stronger, but less clear dependence on R_H). In other words, our bound does not seem to be tight.

If we adopt the implied relationship, $Nu_H = cR_H^{1/5}$ where $c \approx 1$, the data presented in figure 2 can be extrapolated up to oceanographic Rayleigh numbers. This suggests a heat flux of $O(10^{11})$ W implying that steady horizontal convection is too weak to explain observations. Nevertheless, as suggested by the computations of Paparella & Young, unsteady forms of horizontal convection can exist and become preferred as R_H increases. Thus, the discord between our bound and the numerical computations

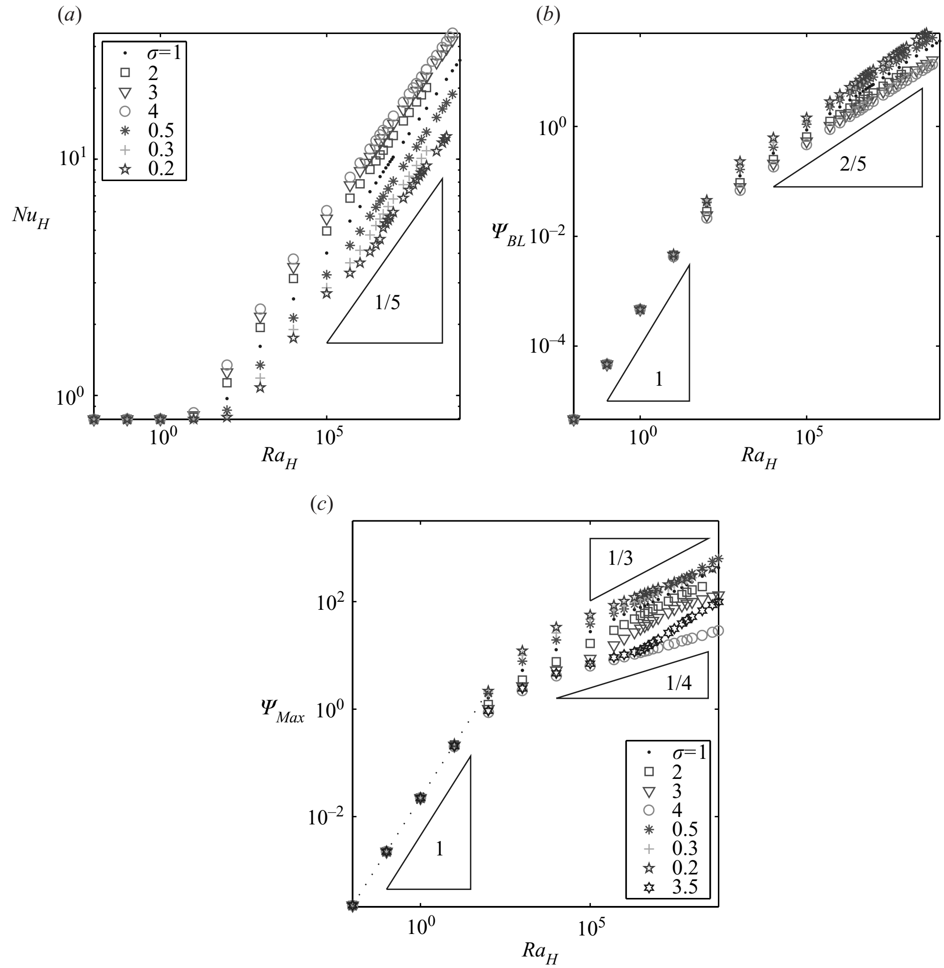

FIGURE 2. Scaling data for numerical computations of steady horizontal convection. The aspect ratio of the fluid layer is unity; solutions with various Prandtl numbers are shown. (*a*) The scaling of the horizontal Nusselt number with R_H. The remaining panels show the maximum value of the streamfunction (*b*) in the boundary layer and (*c*) over the fluid layer. In (*c*), the dotted line shows the maximum streamfunction of an asymptotic solution valid for $R_H \ll 1$.

may simply reflect that the ultimate Nusselt-number scaling of horizontal convection has not yet been reached numerically.

The extremal solution for $\boldsymbol{u}^*$ (2.16) also delivers no information about the velocity field, which is another consequence of our relatively conservative analysis. Thus, we are unable to offer any insight into the scaling of the velocity field with R_H (figure 2 displays our efforts in this direction). To reap the full rewards of the Doering–Constantin approach, we must tackle the complete Euler–Lagrange equations head on. Here, we have attempted to harvest the main scaling of the bound quickly by sidestepping some aspects of these equations and using conservative functional analytic estimation. In doing so, information about the velocity field has been lost, although we expect its structure to be secondary to that of the temperature field.

This work was carried out during the Geophysical Fluid Dynamics 2002 Summer School at the Woods Hole Oceanographic Institution. The authors would like to thank the Director Jack Whitehead and all the staff for providing such a stimulating research environment. We especially thank Charles Doering, Bill Young and Michael Proctor for insights into the problem. N. J. B. was supported by the National Science Foundation under the Collaborations in Mathematical Geosciences initiative (grant number ATM0222109). The computations shown in figures 1 and 2 were performed using a code kindly provided by Glenn Ierley.

Appendix A. Estimates of boundary-layer integrals

In this section, we estimate the maximum possible size of some integrals that are required to bound the sign-indeterminate quadratic terms. The required integrals are

$$\int_{d-\delta_1}^{d} \overline{|w\theta|}\,\mathrm{d}z, \qquad \int_{d-\delta_1}^{d} \overline{|u\theta|}\,\mathrm{d}z, \qquad \int_{0}^{\delta_0} \overline{|w\theta|}\,\mathrm{d}z,$$

where $u=w=\theta=0$ at $z=d$, and at $z=0$, we have $u_z=w=0$ and either $\theta=0$ or $\theta-\lambda\theta_z=0$ where $\lambda>0$.

First, we note that if the functions f and g are both zero on the plane $z=z_0$, then using the Cauchy–Schwartz inequality and Young's inequality ($\sqrt{ab}\leqslant(ca+b/c)/2$ for any $c>0$),

$$\int_{z_0}^{z_0+\delta} \overline{|fg|}\,\mathrm{d}z \leqslant \overline{\left(\int_{z_0}^{z_0+\delta} f^2\,\mathrm{d}z \int_{z_0}^{z_0+\delta} g^2\,\mathrm{d}z\right)^{1/2}} \leqslant \frac{1}{2}\overline{\left(c\int_{z_0}^{z_0+\delta} f^2\,\mathrm{d}z + \frac{1}{c}\int_{z_0}^{z_0+\delta} g^2\,\mathrm{d}z\right)}.$$

Since

$$\min_{h(z_0)=0} \frac{\displaystyle\int_{z_0}^{z_0+\delta} h_z^2\,\mathrm{d}z}{\displaystyle\int_{z_0}^{z_0+\delta} h^2\,\mathrm{d}z} = \frac{\pi^2}{4\delta^2},$$

this means that

$$\int_{z_0}^{z_0+\delta} \overline{|fg|}\,\mathrm{d}z \leqslant \frac{2\delta^2}{\pi^2}\int_{z_0}^{z_0+\delta}\left(c\overline{f_z^2}+\frac{1}{c}\overline{g_z^2}\right)\mathrm{d}z,$$

for any $c>0$.

Thus, we have

$$\int_{d-\delta_1}^{d} \overline{|w\theta|}\,\mathrm{d}z \leqslant \frac{2\delta_1^2}{\pi^2}\int_{d-\delta_1}^{d}\left(c\overline{w_z^2}+\frac{\overline{\theta_z^2}}{c}\right)\mathrm{d}z \leqslant \frac{2\delta_1^2 d}{\pi^2}\left(\frac{c}{4}\langle|\nabla \boldsymbol{u}|^2\rangle + \frac{1}{c}\langle|\nabla\theta|^2\rangle\right), \qquad \text{(A 1)}$$

and similarly, if $\theta=0$ at $z=0$ then

$$\int_{0}^{\delta_0} \overline{|w\theta|}\,\mathrm{d}z \leqslant \frac{2\delta_0^2 d}{\pi^2}\left(\frac{c}{4}\langle|\nabla \boldsymbol{u}|^2\rangle + \frac{1}{c}\langle|\nabla\theta|^2\rangle\right). \qquad \text{(A 2)}$$

However, if $\theta - \lambda\theta_z = 0$ at $z = 0$, there is a weaker bound:

$$\int_0^{\delta_0} \overline{|w\theta|}\,\mathrm{d}z \leqslant \overline{\left(\int_0^{\delta_0} w^2\,\mathrm{d}z \int_0^{\delta_0} \theta^2\,\mathrm{d}z\right)^{1/2}}, \text{ (Cauchy–Schwartz)},$$
$$\leqslant \frac{1}{2}\left(c\int_0^{\delta_0} \overline{w^2}\,\mathrm{d}z + \frac{1}{c}\int_0^{\delta_0} \overline{\theta^2}\,\mathrm{d}z\right), \text{ (Young's inequality)},$$
$$\leqslant \frac{1}{2}\left(\frac{4\delta_0^2 c}{\pi^2}\int_0^{\delta_0} \overline{w_z^2}\,\mathrm{d}z + \frac{1}{c}\int_0^{\delta_0} \overline{\left(\int_z^d \theta_{z'}\,\mathrm{d}z'\right)^2}\,\mathrm{d}z\right),$$
$$\leqslant \frac{2\delta_0^2 c}{\pi^2}\int_0^{\delta_0} \overline{w_z^2}\,\mathrm{d}z + \frac{1}{2c}\int_0^{\delta_0} \overline{\left((d-z)\int_z^d \theta_{z'}^2\,\mathrm{d}z'\right)}\,\mathrm{d}z,$$
(Cauchy–Schwartz),
$$\leqslant \frac{2\delta_0^2 c}{\pi^2}\int_0^{\delta_0} \overline{w_z^2}\,\mathrm{d}z + \frac{d^2\delta_0}{2c}\langle|\nabla\theta|^2\rangle \leqslant \frac{d\delta_0^2 c}{2\pi^2}\langle|\nabla \boldsymbol{u}|^2\rangle + \frac{d^2\delta_0}{2c}\langle|\nabla\theta|^2\rangle, \tag{A 3}$$

and by a similar calculation,

$$\int_{d-\delta_1}^{d} \overline{|u\theta|}\,\mathrm{d}z \leqslant \frac{d^2\delta_1 c}{2}\langle|\nabla \boldsymbol{u}|^2\rangle + \frac{2d\delta_1^2}{\pi^2 c}\langle|\nabla\theta|^2\rangle. \tag{A 4}$$

In (A 1)–(A 4), the inequality holds for any $c > 0$. We have also used the fact that $\langle w_z^2\rangle \leqslant \langle|\nabla \boldsymbol{u}|^2\rangle/4$ (see equation (5.16) of Doering & Constantin 1996).

Appendix B. Bound on Nu_H for constant bottom temperature

In this section, we consider the boundary condition, $T = T_0$ at $z = 0$. We proceed as in the main text by letting $T = \tau(x, z) + \theta(\boldsymbol{x}, t)$, and constructing the functional

$$\mathscr{L} = \overline{TT_z}\big|_d - a\langle \boldsymbol{u}\cdot(2.1)\rangle - b\langle\theta\,(2.2)\rangle,$$
$$= \langle d|\nabla\tau|^2 - a\sigma|\nabla \boldsymbol{u}|^2 - (b-d)|\nabla\theta|^2 - (b-2d)\nabla\tau\cdot\nabla\theta + \mu w(\tau+\theta) - b\theta\boldsymbol{u}\cdot\nabla\tau\rangle - T_0^2/d,$$

where μ is now equal to $a\sigma R_H - T_0$. As before $\langle w(\tau+\theta)\rangle = -\overline{T}_z|_0 - T_0/d$, but in this case we cannot estimate $\overline{T}_z|_0$ directly. Instead, we choose the background field to minimize the worst estimate of the sign indeterminate quadratic terms, that is $\langle \mu w\theta - b\theta\boldsymbol{u}\cdot\nabla\tau\rangle$. We let $\tau = \tau_0(z) + T_1(x)\tau_1(z)$, and choose τ_0 and τ_1 so that the integrand is zero over the bulk of the layer. It is not immediately clear whether or not it is best to have just a single boundary layer at the bottom for τ_0 or to have boundary layers top and bottom. In both cases, however, we obtain the same scaling on the bound, and only the prefactor changes, so for simplicity we have a boundary layer just at the bottom and take

$$\tau_0 = \begin{cases} (T_0(\delta_0 - z) - \mu(d-\delta_0)z/b)/\delta_0 & \text{for } 0 < z < \delta_0, \\ -\mu(d-z)/b & \text{for } \delta_0 < z < d. \end{cases} \tag{B 1}$$

Again, we take τ_1 to be given by (2.15).

Proceeding as for the mixed lower thermal boundary condition case, and solving the Euler–Lagrange equations, we find that the extremalizing velocity field satisfies

$$\nabla p = 2(a\sigma - \alpha)\nabla^2\boldsymbol{u} + \mu\tau\widehat{\boldsymbol{z}}, \tag{B 2}$$

whilst the extremalizing temperature is given by (2.16), where

$$\alpha = \frac{\delta_0 c_0 |d\mu + bT_0|}{2\pi^2} + \frac{b\delta_1 c_1}{2\pi^2} \max|T_1| \left(1 + \frac{\max|T_1'|}{\max|T_1|}\pi\sqrt{d\delta_1}\right),$$
$$\beta = \frac{2\delta_0 |d\mu + bT_0|}{\pi^2 c_0} + \frac{2b\delta_1}{\pi^2 c_1} \max|T_1| \left(1 + \frac{\max|T_1'|}{\max|T_1|}\pi\sqrt{d\delta_1}\right).$$

Since w must be zero on the top boundary, from (B 2) we can see that near this boundary $w = O(\mu(d-z)/(a\sigma - \alpha))$, and thus $\langle w^* T\rangle$ is at most $O(\mu\|T_1\|\delta_1^2/(a\sigma-\alpha))$. We obtain the bound

$$\mathscr{L} \leqslant d\langle|\nabla\tau|^2\rangle + \frac{(b-2d)^2}{4(b-d-\beta)}\langle\nabla\tau\cdot\nabla(\tau - T_c)\rangle - \frac{T_0^2}{d} + \frac{\mu^2\|T_1\|}{a\sigma-\alpha}O(\delta_1^2),$$

and making the simplifying assumption $b = 2d$ yields

$$\mathscr{L} \leqslant \frac{(a\sigma R_H + T_0)^2}{4\delta_0} + \frac{\overline{T_1^2}}{\delta_1} - \frac{(a\sigma R_H + T_0)^2}{4d} + \tfrac{1}{3}\overline{T_1'^2}\delta_1 + \frac{(a\sigma R_H - T_0)^2\|T_1\|}{a\sigma - \alpha}O(\delta_1^2).$$

Upon making the substitutions

$$\delta_0 = \frac{\pi^2 \Delta_0}{\sqrt{dR_H}}, \quad \delta_1 = \frac{\pi^2\Delta_1}{2\sqrt{dR_H}\,\max|T_1|}, \quad c_0 = \frac{2C_0}{\sqrt{dR_H}}, \quad c_1 = \frac{2C_1}{\sqrt{dR_H}}, \quad a\sigma = \frac{A}{R_H},$$

this becomes

$$\mathscr{L} \lesssim \frac{\sqrt{dR_H}}{4\pi^2} f + O(1), \text{ where } f = \frac{(A+T_0)^2}{\Delta_0} + \frac{4\overline{T_1^2}\max|T_1|}{\Delta_1}, \tag{B 3}$$

subject to the spectral constraints

$$A \gtrsim \Delta_0 C_0|A + T_0| + \Delta_1 C_1, \quad 1 \gtrsim \frac{\Delta_0}{C_0}|A+T_0| + \frac{\Delta_1}{C_1}. \tag{B 4}$$

The minimization of f subject to the constraints gives $\Delta_0 = 1/6\sqrt{A}$, $\Delta_1 = (5A - T_0)/6\sqrt{A}$, $C_0 = \sqrt{A}$, $C_1 = \sqrt{A}$, where A satisfies $(A + T_0)(5A - T_0)^2 = 4\overline{T_1^2}\max|T_1|$. The corresponding extremal value of f is $36A^{3/2}(A + T_0)$.

When $T_0 \gg 1$, then $f \sim 216(T_0/5)^{5/2}$, and the bound becomes

$$Nu_H \lesssim \frac{54(T_0/5)^{5/2}}{\pi^2\,\overline{T_c T_{cz}}\big|_{z=d}}\sqrt{dR_H},$$

and when $T_0 \ll -1$, then $f \sim 4(-T_0)^{-1/2}$, so the bound is

$$Nu_H \lesssim \frac{\overline{T_1^2}\max|T_1|(-T_0)^{-1/2}}{\pi^2\,\overline{T_c T_{cz}}\big|_{z=d}}\sqrt{dR_H}.$$

However, if $T_0 \lesssim -2d^3R_H/\pi^4$, this solution breaks down as it would have $\delta_1 > d$.

For very large T_0, we expect the motion to be dominated by the large vertical temperature gradient and look like Rayleigh–Bénard convection. Thus, we expect the vertical Nusselt number,

$$Nu = \frac{d\langle|\nabla T|^2\rangle}{d\langle|\nabla T_c|^2\rangle},$$

to be bounded by $Ra^{1/2}$ multiplied by some prefactor, as found by Doering & Constantin (1996), where $Ra = T_0 R_H$ is the usual Rayleigh number for Rayleigh–Bénard convection. Finding an upper bound on $d\langle|\nabla T|^2\rangle$ in a similar way gives (B 3) and (B 4), but with A replaced by $A + T_0$. When $|T_0| \gg 1$ then $T_c = T_0(1 - z/d) + O(1)$, meaning that

$$d\langle|\nabla T_c|^2\rangle \sim T_0^2/d,$$

which is much larger than $\overline{T_c T_{cz}}|_d$. If $T_0 \gg 1$, then we obtain

$$Nu \lesssim \frac{54d^{3/2}}{25\sqrt{5}\,\pi^2}(T_0 R_H)^{1/2} = \frac{54d^{3/2}}{25\sqrt{5}\,\pi^2} Ra^{1/2}$$

to leading order, which is the Doering–Constantin result (though our prefactor is not optimal since we only used a bottom boundary layer and not a top one, and we have not optimized the choice of the constants a, b, δ_0 and δ_1).

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