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Alternative results for option pricing and implied volatility in jump-diffusion models using Mellin transforms

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In this article, we use Mellin transforms to derive alternative results for option pricing and implied volatility estimation when the underlying asset price is governed by jump-diffusion dynamics. The current well known results are restrictive since the jump is assumed to follow a predetermined distribution (e.g., lognormal or double exponential). However, the results we present are general since we do not specify a particular jump-diffusion model within the derivations. In particular, we construct and derive an exact solution to the option pricing problem in a general jump-diffusion framework via Mellin transforms. This approach of Mellin transforms is further extended to derive a Dupire-like partial integro-differential equation, which ultimately yields an implied volatility estimator for assets subjected to instantaneous jumps in the price. Numerical simulations are provided to show the accuracy of the estimator.

Key words: Mellin transform, Black–Scholes partial differential equation, jump-diffusion model, implied volatility estimation, Dupire equation.

1 Introduction

1.1 Option pricing theory

An option is a contract between two parties (known as the *holder* and the *writer*) that gives the holder the right, but not the obligation, to buy/sell an underlying asset from/to the writer at a mutually agreed price (known as the *exercise* or *strike* price) on or before a specified future date (known as the *expiry* date). On or before the expiry date, the holder may "exercise" the option. The right to buy is called a *call* option whereas the right to sell is called a *put* option. Furthermore, a *European* option can only be exercised at expiry, whereas an *American* option can be exercised before or on the expiry date. A well-known result for determining the European option value is known as the Black–Scholes formula [6].

It was verified by Merton [45] that one of the fundamental assumptions for the Black–Scholes model to hold is that the asset price follows a continuous-time, diffusion process with a continuous sample path. In [47], Merton considered a "jump" stochastic process for the asset price that allows for the probability for it to change at large magnitudes irrespective of the time interval between successive observations. The jumps in the asset price can be accommodated for by appending an additional source of uncertainty into the asset price dynamics that models the discontinuity. Moreover, subsequent empirical studies (e.g., Rosenfeld [55], Jarrow & Rosenfeld [35], Ball & Torous [4], and Brown & Dybvig [10]) asserted that the asset price process is best resembled by a stochastic process with a discontinuous sample path. This phenomenon implies that the asset price dynamics follow a *jump-diffusion model*.

Merton [47] derived a partial integro-differential equation (or PIDE) to represent a modified Black–Scholes system that accounts for the inclusion of jumps. A solution was also reported, which can be viewed as an explicit European option pricing formula in terms of an infinite series of Black–Scholes prices multiplied by a factor that encapsulates the behaviour of the jump. Essentially, the Merton model adds the Poisson process to the Wiener process that governs the asset price. The result is a continuous-time, stochastic process with stationary increments independent of one another, known as a Lévy process [50].

The importance of developing such a system extends beyond attempting to capture the option market's behaviour at any given point. The need lies within being able to deliver fundamental explanations to why certain phenomena occur. For example, when one wishes to estimate the implied volatility surfaces to calibrate the standard Black–Scholes option values to actual market quotes, the Black–Scholes model where the underlying asset follows a standard diffusion process assumes the implied volatility surface to be flat. That is, a constant value during the option's lifetime and for varying values of the strike price (options are commonly listed as a function of their strike price). But empirical observations have shown that these implied volatility surfaces are heavily dependent on both the strike price and the expiry date (in particular, refer to Heynen [32], Dumas *et al.* [25], Rebonato [51], and Cont & Fonseca [18, 19]). As a result, these surfaces actually form either a "smile" or "skew" depending on the values of the strike and time to expiry. Dupire [26] developed a technique for computing the local implied volatility surfaces and showed that the standard Black–Scholes model with an asset under diffusion dynamics can embody all the distinguishing features of this "smile problem". However, it only gives us a tool needed to ensure we recover the required option values. It does not explain for why these smiles and skews occur. A jump-diffusion model, however, is able to encapsulate both a justification for these smiles and skews, their increased occurrences after the 1987 crash (see Andersen and Andreasen [2]), and how the jumps in the asset price bear some psychological parallel to the potential tentative demeanour of the market participants [20].

In terms of option valuation in jump-diffusion models, the literature is quite rich (e.g., see Amin [3], Kou [39], Kou & Wang [40], Hilliard & Schwartz [34], Carr & Mayo [11], Feng & Linetsky [28], Cheang & Chiarella [15], and Frontczak [29]) with many resourceful texts (e.g., see Rogers [54], Kijima [37], Cont & Tankov [20], and Vercer [57]).

Amin [3] developed one of the earliest numerical schemes for pricing options in a jump-diffusion framework by adapting the binomial model proposed by Cox *et al.* [22]. The extension is achieved by allowing multiple movements in the asset price at every discrete time step to simulate the discontinuous jumps, whereas the standard binomial model allows for only one discrete movement in the asset price at every discrete point in time. This discrete approach is then compared numerically against the closed-form solution provided by Merton [47], with the resultant options values having little differences between one another.

Further empirical investigations by Kou [39] led to the proposal of a double exponential jump-diffusion model where the jump intensities are double exponentially distributed. The author's empirical studies contradicted the previous assumptions that the underlying asset's jump-diffusion model was lognormal. Specifically, the findings showed that the return distribution of the asset possessed features uncharacteristic of a normal distribution (i.e., higher peak and heavier asymmetric tails than that of a normal distribution), and the "volatility smile" observed in the option markets. Despite the normal distribution being a central mechanism in simulating the asset price process, Kou provided in-depth explanations for the aforementioned empirical analysis and introduced an updated model. This model assumed that the jumps in the asset price follow a double exponential distribution. Analytical solutions for pricing of European call/put options and path-dependent options in a double exponential jump-diffusion model were derived in [40] co-authored with Wang. However, limitations of the model were noted by Kou [39] in regards to hedging difficulties and assumed dependence of the jump increments.

In terms of other numerical implementations, Hilliard and Schwartz [34] introduced a bivariate tree approach for pricing both European and American derivatives with jumps, where one factor represents a discrete-time version of the standard continuous asset price path whilst the second factor models a discrete-time version of the jumps arriving as a Poisson process. Feng and Linetsky [28] also provided a computational alternative to pricing options with jumps by introducing a high-order time discretisation scheme to solve the PIDE in Merton's article [47]. The authors demonstrated that their method provides rapid convergence to the solution in comparison to standard implicit-explicit time discretisation methods, using Kou's model as a comparative example.

Carr and Mayo [11] also reported a novel numerical implementation for calculating option prices when the asset is subjected to jump-diffusion dynamics. The authors devised a method that involves converting the integral term in the PIDE derived by Merton [47] to a correlation integral. They stated that in many instances this correlation integral is a solution to an ordinary differential equation (ODE) or partial differential equation (PDE). Carr and Mayo also argued that solving these associated ODEs and PDEs substantially reduces computational effort since it effectively bypasses numerical evaluation of the aforementioned integral. They illustrated their concept by examining both Merton's lognormal model and Kou's double exponential model.

Cheang and Chiarella [15] advocated for amendments to be made to Merton's original jump-diffusion model. They argued that the Merton model makes assumptions that lead to the jump-risk [31] being unpriced and force the distribution of the Poisson jumps to remain unchanged under a change of measure. The authors stressed the significance of this since a realistic market that contains assets with jumps is incomplete. Additionally, when the market price of the jump-risk is accounted for, there exist many equivalent martingale measures that ultimately produce different prices for options. Hence, they introduced a Radon–Nikodým derivative process which translates the market measure to an equivalent martingale measure (EMM) for option valuation. However, the EMM is non-unique in the presence of jumps; one must choose parameters in the Radon–Nikodým derivative to establish an EMM to price options. Furthermore, Cheang and Chiarella derived a PIDE and thus a general pricing formula which reduces to Merton's solution [47] as a special case.

Frontczak [29] adopted a method of solving the PIDE seen in [47] using Mellin transforms. He proceeded to re-derive Merton's solution for a European put option via direct inversion. Frontczak's approach of directly evaluating of the inverse Mellin integral (i.e., a complex integral) is where the approach could be improved. Moreover, this process needs to be repeated for different payoffs, making this procedure computationally expensive and tedious.

1.2 Implied volatility

For estimating the volatility σ in the standard diffusion model (2.1), there exist two primary methods. The first scheme involves estimating σ from previous asset price movements. That is, suppose a model for the behaviour of the asset involving σ is known and the asset prices for all times up until the present are accessible. Then, σ can be fitted to this observed data. This method is dubbed *historical volatility* as σ is approximated using data of previous asset prices. The second is using all the known parameters, treating the option value as a function of σ (the option price is one of the known parameter values) and solving for σ. This approach determines σ implicitly from the Black–Scholes formula using the option price and the observed parameters, and is referred to as *implied volatility*. Aside from option pricing in a jump-diffusion framework, another aim of this article is to present a novel implied volatility scheme using Mellin transforms.

One of the earliest methods for implied volatility estimation was proposed by Latané and Rendleman [42], where σ is computed using a technique called weighted implied standard deviation (WISD). Their idea consisted of obtaining a set of option prices, approximating the implied volatility using the Black–Scholes formula and calculating a WISD using a "weight" against the Black–Scholes derived implied volatility. The crux of the method was to reduce any sampling error. Latané and Rendleman concluded the WISD approach was superior in comparison to corresponding historical volatility estimations. Furthermore, the weighting scheme selected provided more weight to options at-the-money and possessing a longer time to expiry.

Cox and Rubinstein [23] further analysed the weighting scheme proposed by Latané and Rendleman and stressed the importance of employing data from at-the-money options. Their justification was because at-the-money options are the most actively and frequently traded options; thus, the implied volatility obtained using at-the-money option values would yield a credible estimation as the data used closely simulates actual trading conditions.

As data from at-the-money options were becoming increasingly appealing to incorporate in implied volatility estimation, Brenner and Subrahmanyam [8] introduced a simplified formula for calculating σ. Their article focussed on reducing the complexity of the Black–Scholes pricing formula by assuming the option was at-the-money and close to expiry. These assumptions, coupled with using an asymptotic approximation for the cumulative distribution function (CDF) for a standard normal, resulted in an approximate option valuation formula where σ could be evaluated explicitly as a time-constant value. This process allowed one to forego the need to use an iterative procedure to calculate the implied volatility (e.g., the Newton–Raphson method), which was a common practice at the time. The article highlighted that for options close to at-the-money, the value of the option is comparatively proportional to the value of σ. Furthermore, Brenner and Subrahmanyam stated that their approximation formula may also be implemented as a good initial guess for numerical algorithms like the Newton–Raphson method since the starting seed is essential for improving the likelihood and speed of convergence [44]. The result of Feinstein [27] is nearly identical to Brenner and Subrahmanyan; however, it was developed independently. Curtis and Carriker [24] also introduced a closed-form solution for implied volatility estimation for at-the-money options. It can be shown that under certain circumstances, the result by Brenner and Subrahmanyam is a special case of Curtis and Carriker's formula for σ (see "Final remarks" in [14]).

Despite the resemblance conveyed by at-the-money implied volatility calculations to true trading circumstances, the aforementioned estimations were ill-suited for evaluating implied volatility for option moneyness that is not at-the-money. Studies have been conducted to develop approximations that account for times when the underlying asset price differs from the exercise price (i.e., in-the-money or out-of-the-money options). A notable result was published by Corrado and Miller [21], where their approximation for σ reduces to the Brenner–Subrahmanyam formula for options at-the-money. Their motivation was primarily to improve the accuracy range of implied volatility estimations to a wider scope of option moneyness not necessarily at-the-money. The derivation presented by Corrado and Miller illustrates similarities to that of Brenner and Subrahamyam's approach due to both articles incorporating an asymptotic expansion of the CDF for a standard normal random variable as a gateway to producing simplified approximations. The numeric generated by the authors' result exhibited good agreement with the actual implied volatility via the Black–Scholes formula for options close to and at-the-money. In addition, their numerical output also demonstrated and confirmed that the use of the Brenner–Subrahmanyam result was only accurate for at-the-money options.

Chance [13] developed an implied volatility approximation that extended the result by Brenner and Subrahmanyam. The author's motivation mimicked that of Corrado and Miller as they derived an expression σ to accommodate for the strike price bias. Chance's formula involved assuming all parameters are known for an at-the-money option, then first deriving an initial guess for σ using the Brenner–Subrahmanyan formula (i.e., implied volatility for an at-the-money option). He then demonstrated that the value of an option not at-the-money is simply an at-the-money option perturbed by a value Δv, which could be the result of differences in strike price and σ values. The perturbation Δv is then obtained by second-order Taylor expansions resulting in an equation that is quadratic in $\Delta\sigma$. Upon computing $\Delta\sigma$ via the quadratic formula, the final σ value for an option not at-the-money is the addition of both σ at-the-money plus $\Delta\sigma$. Chance numerically verified the result and illustrated its effectiveness for options near at-the-money (no more than 20% in- or out-of-the-money) and options far from expiry. The significance of this was also asserted as long-term options were becoming increasingly popular in practice; however, the author also noted the accuracy decay when the option is closer to expiry. Furthermore, the model requires extra information including an at-the-money option value and its associated Greeks (specifically, vega and the partial derivative with respect to the strike price).

Bharadia *et al.* [5] also reported a result that claimed to be a highly simplified volatility estimation formula, where the primary advantages of the approximation are its simplicity in form and the fact that it does not require the option to be at-the-money.

Amidst these optimistic results, Chambers and Nawalkha [12] comparatively examined the implied volatility estimation formulas of Bharadia *et al.*, Corrado and Miller, and Chance. Chambers and Nawalkha praised the result from Bharadia *et al.* for being very condense in form, but also pointed out the inaccuracy (possessing the highest weighted approximation error amongst the three aforementioned estimates). Chambers and Nawalkha commended the Corrado–Miller formula in that an at-the-money option value was not a prerequisite, yet highlighted that the limitation was the square root term (which could be negative). Furthermore, whether the formula would produce a complex solution is unknown *a priori*; however, the likelihood is minimised substantially for reasonable parameter values. Chambers and Nawalkha accommodated for the possibility of a negative argument for the square root term by setting the term to be zero if the case occurred. It was commented that Corrado and Miller's model is extremely accurate for options near at-the-money, but substantial errors are prevalent for options very far from at-the-money. Corrado and Miller's formula for σ possessed the second highest weighted approximation error.

Special attention was devoted to Chance's estimate in [12] as it produced the lowest weighted approximation error amongst the three models. The assessment of Chance's approximation gave positive mention of accuracy and ease of understanding/implementation. Similar to Corrado and Miller, Chance's formula yields the highest accuracy for near at-the-money options, but deteriorates for options significantly far from at-the-money. This consequently provided the mathematical structure for Chambers and Nawalkha's result, developing a simplified extension to Chance's formula that dramatically improved the accuracy.

Chambers and Nawalkha attempted to improve the accuracy of Chance's implied volatility model for options relatively far from at-the-money (where all three formulas suffered in accuracy). Recall that Chance employed a second-order Taylor series expansion in two variables as there was justification for both the strike price and volatility to contribute to the change in option prices. Chambers and Nawalkha adopted a similar approach by performing a second-order Taylor Series expansion around Δv, but only with respect to σ. The result was a much simpler quadratic equation in $\Delta\sigma$ and, similar to Chance's formula, required an initial guess for σ at-the-money (which is computed via Brenner and Subrahmanyam's approximation also). Chambers and Nawalkha asserted that the effect of strike price differences can be encapsulated in the Brenner–Subrahmanyam formula for σ, thus, requiring only the partial derivative with respect to volatility in the Taylor series expansion. The weighted approximation error is ultimately the lowest in comparison to the three models by Chance, Corrado and Miller, and Bharadia *et al.* Despite this, Chambers and Nawalkha's method shares the same detriment to Chance's formula in that an option value at-the-money is required to estimate a starting σ value.

From all the schemes presented above, the common hindrance is either the need for additional data (e.g., at-the-money option value) or the deterioration of accuracy for options very far from at-the-money. Li [43] attempted to rectify the need for extra information and improved reliability for options deep in- or out-of-the-money. By incorporating a substitution of variables and Taylor series expansion, Li derived two separate formulas for σ depending on whether the option was at-the-money or not. The numerical results provided in Li's paper demonstrate greater accuracy for σ for both at-the-money and not-at-the-money scenarios. Interestingly, Li's approximation for σ not at-the-money reduces to the Brenner–Subrahmanyan formula under special conditions. Several other approaches have also been developed over recent years (see Park *et al.* [49], Choi *et al.* [17], Zhang & Man [59], Chen & Xu [16]). For the majority of the aforementioned cited works, the primary motivation was to develop an analytical approximation for the implied volatility that possessed benefits over their predecessors. Although many of the methods were derived from seemingly ad hoc methodologies, the validity of these analytical results is still valuable as it provides us with a means to evaluate and analyse the sensitivity of the implied volatility to the other financial parameters. If one took a standard iterative (e.g., Newton–Raphson) numerical approach to compute the implied volatility, it may be difficult to gauge how the behaviour this obtained σ value varies with the other parameters.

1.3 Outline of the article

In this article, we will present alternative results to pricing options and evaluating implied volatility in jump-diffusion models. Our approach implements Mellin transforms similar to [29] to derive the necessary results. The structure will be as follows. Section 2 will provide the preliminary information necessary for the remainder of the article. In Section 3, we provide the main results (i.e., the derivations). Sections 4–7 will demonstrate specific cases to the main results, associated verifications and other pertinent analogous relations. The implied volatility content commences in Section 8, where we begin by deriving a Dupre-like PIDE. Section 9 will yield the implied volatility formula required. Also in Section 9, numeric will be analysed and investigated to assess the potential application of the aforementioned results. Finally, we will present a discussion of the findings followed by a conclusion with tentative future directions.

2 Preliminaries

2.1 Black–Scholes option pricing framework

For the Black–Scholes framework [6], we assume that the option price depends on the asset price under any risk-neutral probability measure given by the following stochastic differential equation (SDE):

$$\frac{\mathrm{d}S_t}{S_t} = (r(t) - q(t))\,\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t, \tag{2.1}$$

where $\{S_t : t \in [0, T]\}$ is the asset price process, $\{W_t : t \in [0, T]\}$ is a Wiener process with respect to the risk-neutral measure, $T > 0$ is the expiry, $r(t) > 0$ is the risk-free interest rate, $q(t) \geqslant 0$ is the dividend yield, and $\sigma(t) > 0$ is the volatility. Here, we assume that the parameters r, q, and σ are continuous functions of time. It is well known that the option value is given by $V_t = v(S_t, t)$, where $v = v(x, t)$ is a function that satisfies the following terminal value problem:

$$\frac{\partial v}{\partial t}(x,t) + \frac{1}{2}\sigma(t)^2 x^2 \frac{\partial^2 v}{\partial x^2}(x,t) + (r(t) - q(t))x\frac{\partial v}{\partial x}(x,t) - r(t)v(x,t) = 0, \tag{2.2}$$

$$v(x,T) = \phi(x). \tag{2.3}$$

Equation (2.2) is commonly known as the Black–Scholes PDE and equation (2.3) is the payoff (i.e., loss or profit at expiry), where $\phi : [0,\infty) \to [0,\infty)$. For the European call and put options, $\phi(x) = \max(x - K, 0)$ and $\phi(x) = \max(K - x, 0)$, respectively, where K is the strike price. Note that at expiry, $v(S_T, T) = \phi(S_T)$. It is well known [52, 58] that the value of the European call option is given by

$$v^{\text{call}}(x,t) = xe^{-\int_t^T q(\tau)\,d\tau} N\left(z_1\left(\frac{x}{K}, t, T\right)\right) - Ke^{-\int_t^T r(\tau)\,d\tau} N\left(z_2\left(\frac{x}{K}, t, T\right)\right), \tag{2.4}$$

and the European put option is

$$v^{\text{put}}(x,t) = Ke^{-\int_t^T r(\tau)\,d\tau} N\left(-z_2\left(\frac{x}{K}, t, T\right)\right) - xe^{-\int_t^T q(\tau)\,d\tau} N\left(-z_1\left(\frac{x}{K}, t, T\right)\right), \tag{2.5}$$

where

$$z_1(x,t,u) = \frac{\log x + \int_t^u (r(\tau) - q(\tau) + \sigma(\tau)^2/2)\,d\tau}{\left(\int_t^u \sigma(\tau)^2\,d\tau\right)^{1/2}}, \tag{2.6}$$

$$z_2(x,t,u) = \frac{\log x + \int_t^u (r(\tau) - q(\tau) - \sigma(\tau)^2/2)\,d\tau}{\left(\int_t^u \sigma(\tau)^2\,d\tau\right)^{1/2}}, \tag{2.7}$$

and N is the CDF of a standard normal random variable.

2.2 The Black–Scholes kernel and its properties

We will also require the Black–Scholes kernel first introduced by Rodrigo and Mamon in [52] and then extended upon in [53]. This is defined by

$$\mathcal{K}(x,t,u) = \frac{e^{-\int_t^u r(\tau)\,d\tau}}{\left(\int_t^u \sigma(\tau)^2\,d\tau\right)^{1/2}} N'(z_2(x,t,u)), \tag{2.8}$$

and an alternative form given as

$$\mathcal{K}(x,t,u) = \frac{xe^{-\int_t^u q(\tau)\,d\tau}}{\left(\int_t^u \sigma(\tau)^2\,d\tau\right)^{1/2}} N'(z_1(x,t,u)). \tag{2.9}$$

Furthermore, it was shown [52] that for an arbitrary payoff function ϕ, the European option price can be formulated as

$$v(x,t) = \int_0^\infty \frac{1}{z}\mathcal{K}\left(\frac{x}{z}, t, T\right)\phi(z)\,dz. \tag{2.10}$$

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It can be seen that the option price (2.10) is expressible as a convolution of the Black–Scholes kernel and the payoff (see the convolution definition in Section 2.5).

2.3 Option pricing under a jump-diffusion model

The asset price process in (2.1) needs to be adjusted when accounting for the possibility of instantaneous jumps. Under the assumption that the discontinuous jumps arrive as a Poisson process, the risk-neutral asset price dynamics are given by

$$\frac{\mathrm{d}S_t}{S_t} = (r(t) - q(t) - \kappa\lambda)\mathrm{d}t + \sigma(t)\,\mathrm{d}W_t + (Y - 1)\,\mathrm{d}N_t, \tag{2.11}$$

where $\{W_t : t \in [0, T]\}$ is the standard Wiener process as defined earlier, Y is a nonnegative random variable with $Y - 1$ denoting the impulse change in the asset price from S_t to $Y S_t$ as a consequence of the jump, $\kappa = \mathbb{E}[Y - 1]$ with $\mathbb{E}[\cdot]$ as the expectation operator, $\{N_t : t \in [0, T]\}$ is the aforementioned Poisson process with intensity λ, and

$$\mathrm{d}N_t = \begin{cases} 1 & \text{with probability } \lambda\,\mathrm{d}t, \\ 0 & \text{with probability } (1 - \lambda\,\mathrm{d}t). \end{cases} \tag{2.12}$$

Additionally, W_t, N_t, and samples $\{Y_1, Y_2, \ldots\}$ from Y are assumed to be independent.

In [47], Merton extended (2.2) to ensure the behaviour of the jumps is properly encapsulated. The result is a PIDE system:

$$\begin{aligned} &\frac{\partial v}{\partial t}(x,t) + (r(t) - q(t) - \kappa\lambda)x\frac{\partial v}{\partial x}(x,t) - r(t)v(x,t) + \frac{1}{2}\sigma(t)^2x^2\frac{\partial^2 v}{\partial x^2}(x,t) \\ &\quad + \lambda\int_0^\infty (v(xy,t) - v(x,t))f(y)\,\mathrm{d}y = 0, \end{aligned} \tag{2.13}$$

$$v(x, T) = \phi(x), \tag{2.14}$$

where f is the probability density function (PDF) of Y such that $\int_0^\infty f(y)\,\mathrm{d}y = 1$. A special case of Merton's infinite series solution mentioned in Section 1.1 is when Y is lognormally distributed (i.e., $Y \sim \mathrm{LN}(\mu_Y, \sigma_Y^2)$). The European option pricing formula was shown to be

$$v_{\mathrm{M}}(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(1+\kappa)(T-t))^n}{n!} e^{-\lambda(1+\kappa)(T-t)} v_n(x,t), \tag{2.15}$$

where

$$v_n(x,t) = v(x,t;r,q,\sigma)\big|_{r=r_n(t),q=q,\sigma=\sigma_n(t)},$$

with

$$r_n(t) = r - \kappa\lambda + \frac{n\log(1+\kappa)}{T-t}, \quad \sigma_n(t)^2 = \sigma^2 + \frac{n\sigma_Y^2}{T-t}. \tag{2.16}$$

That is, v is the European option price due to the Black–Scholes formula with constant coefficients, and v_n is the result of directly substituting $r_n(t)$ and $\sigma_n(t)$ for r and σ, respectively, into v.

2.4 Useful properties of the CDF of a standard normal variable

Several properties of the cumulative normal distribution N and its derivative N' will also be required [36, pp. 235–239], namely

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, \mathrm{d}y, \quad N(-x) = 1 - N(x), \quad N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \\ N(\infty) = 1, \quad N(-\infty) = 0. \tag{2.17}$$

We will also need the following lemmas.

Lemma 1 *For $a_1, a_2 > 0$ and $b_1, b_2 \in \mathbb{R}$, we have*

$$\int_0^\infty \frac{1}{z} N'(a_1 \log z + b_1) N'\left(a_2 \log\left(1/z\right) + b_2\right)\, \mathrm{d}z = \frac{1}{\sqrt{a_1^2 + a_2^2}} N'\left(\frac{a_1 b_2 + a_2 b_1}{\sqrt{a_1^2 + a_2^2}}\right).$$

Proof See Appendix A. □

Lemma 2 *Assume $a_1, a_2 > 0$ and $b_1, b_2, a, b \in \mathbb{R}$. Then,*

$$\int_a^b \frac{1}{y} N'(a_1 \log(1/y) + b_1)\, \mathrm{d}y = \frac{1}{a_1}\left(N'(a_1 \log(1/a) + b_1) - N'(a_1 \log(1/b) + b_1)\right),$$
$$\int_a^b N'(a_1 \log(1/y) + b_1)\, \mathrm{d}y = \frac{e^{b_1/a_1 + 1/(2a_1^2)}}{a_1}\Big(N(a_1 \log(1/a) + b_1 + 1/a_1) \\ - N(a_1 \log(1/b) + b_1 + 1/a_1)\Big).$$

Proof See Appendix B. □

2.5 Mellin transform

Suppose that $f : [0, \infty) \to \mathbb{R}$ is such that $f = f(x)$. The *Mellin transform* $\hat{f}$ of f at $\xi \in \mathbb{R}$ is defined as

$$\hat{f}(\xi) = \mathscr{M}\{f\}(\xi) = \int_0^\infty x^{\xi - 1} f(x)\, \mathrm{d}x,$$

provided the integral converges at ξ. Now, we denote the function id by $\mathrm{id}(x) = x$. Then, for each $x \in [0, \infty)$, define the functions $(\mathrm{id} \cdot f')$ and $(\mathrm{id}^2 \cdot f'')$ by

$$(\mathrm{id} \cdot f')(x) = x f'(x), \quad (\mathrm{id}^2 \cdot f'')(x) = x^2 f''(x),$$

respectively. It can be shown that [48, pp. 362–363]

$$\mathscr{M}\{x f'(x)\} = \widehat{x f'}(\xi) = -\xi \hat{f}(\xi), \quad \mathscr{M}\left\{x^2 f''(x)\right\} = \widehat{x^2 f''}(\xi) = \xi(\xi + 1)\hat{f}(\xi), \tag{2.18}$$

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assuming that f satisfies

$$x^{\xi} f(x)\big|_0^{\infty} = 0, \quad x^{\xi+1} f'(x)\big|_0^{\infty} = 0.$$

We also recall

$$\mathscr{M}\{(f * g)(x)\} = (\widehat{f * g})(\xi) = \hat{f}(\xi)\hat{g}(\xi),$$

where $f * g$ is the convolution of f and g defined to be

$$(f * g)(x) = \int_0^{\infty} \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y \quad \text{for all } x \geqslant 0.$$

In addition to the Mellin transform and its properties, it was shown in [52] that the Mellin transform of (2.10) is

$$\hat{v}(\xi, t) = \hat{\mathscr{K}}(\xi, t, T)\hat{\phi}(\xi), \tag{2.19}$$

where $\hat{\mathscr{K}}(\xi, t, T) = e^{-\int_t^T p(\xi,\tau)\,\mathrm{d}\tau}$ with

$$p(\xi, \tau) = r(\tau) + \left(r(\tau) - q(\tau) - \frac{1}{2}\sigma(\tau)^2\right)\xi - \frac{1}{2}\sigma(\tau)^2\xi^2. \tag{2.20}$$

The following lemma will be useful in what follows.

Lemma 3 *For $a > 0$ and $b \in \mathbb{R}$, we have*

$$\mathscr{M}^{-1}\left\{e^{-b\xi/a} e^{\xi^2/(2a^2)}\right\} = aN'(a \log x + b).$$

Proof See Appendix C. □

3 Main results

3.1 Alternative option pricing formula where the underlying asset is subjected to jump-diffusion dynamics

Analogous to [47] and [29], we want to solve the problem (2.13), (2.14). Applying the Mellin transform of v with respect to x to (2.13), (2.14), we get

$$\begin{aligned}\frac{\partial \hat{v}}{\partial t}(\xi, t) &- (r(t) - q(t) - \kappa\lambda)\xi\hat{v}(\xi, t) - r(t)\hat{v}(\xi, t) + \frac{1}{2}\sigma(t)^2\xi(\xi+1)\hat{v}(\xi, t)\\ &+ \lambda\int_0^{\infty} x^{\xi-1}\left(\int_0^{\infty}(v(xy, t) - v(x, t))f(y)\,\mathrm{d}y\right)\mathrm{d}x = 0, \quad \hat{v}(\xi, T) = \hat{\phi}(\xi).\end{aligned} \tag{3.1}$$

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For the integral term, reversing the order of integration and using $z = xy$, we obtain

$$\begin{aligned}
&\int_0^\infty x^{\xi-1}\left(\int_0^\infty (v(xy,t)-v(x,t))f(y)\,\mathrm{d}y\right)\mathrm{d}x \\
&\quad= \int_0^\infty f(y)\left(\int_0^\infty x^{\xi-1}v(xy,t)\,\mathrm{d}x\right)\mathrm{d}y \;-\int_0^\infty f(y)\left(\int_0^\infty x^{\xi-1}v(x,t)\,\mathrm{d}x\right)\mathrm{d}y \\
&\quad= \int_0^\infty y^{-\xi}f(y)\left(\int_0^\infty z^{\xi-1}v(z,t)\,\mathrm{d}z\right)\mathrm{d}y \;-\int_0^\infty f(y)\hat{v}(\xi,t)\,\mathrm{d}y \\
&\quad= \hat{v}(\xi,t)(\mathbb{E}[Y^{-\xi}]-1).
\end{aligned}$$

Therefore, (3.1) simplifies to

$$\frac{\partial \hat{v}}{\partial t} = \left(p_\lambda(\xi,t)-\lambda\mathbb{E}[Y^{-\xi}]\right)\hat{v}(\xi,t), \qquad \hat{v}(\xi,T)=\hat{\phi}(\xi), \tag{3.2}$$

where p_λ is defined to be

$$p_\lambda(\xi,t) = r(t)+\lambda+\left(r(t)-q(t)-\kappa\lambda-\frac{1}{2}\sigma(t)^2\right)\xi-\frac{1}{2}\sigma(t)^2\xi^2. \tag{3.3}$$

Note that when $\lambda = 0$, equation (3.3) simplifies to $p_0(\xi,t) = p(\xi,t)$, where $p(\xi,t)$ is given in (2.20). The solution to (3.2) is

$$\hat{v}(\xi,t) = e^{\lambda(T-t)\mathbb{E}[Y^{-\xi}]}e^{-\int_t^T p_\lambda(\xi,\tau)\,\mathrm{d}\tau}\hat{\phi}(\xi). \tag{3.4}$$

To proceed, we let $v_\lambda = v_\lambda(x,t)$ be the solution to the Black–Scholes system (2.2), (2.3) with shifted parameters $r(t)\to r(t)+\lambda$ and $q(t)\to q(t)+\lambda+\kappa\lambda$. The payoff function ϕ remains unchanged. Using (2.10), we can deduce the analogous formula:

$$v_\lambda(x,t) = \int_0^\infty \frac{1}{z}\mathscr{K}_\lambda\left(\frac{x}{z},t,T\right)\phi(z)\,\mathrm{d}z, \tag{3.5}$$

with $\mathscr{K}_\lambda$ being the shifted Black–Scholes kernel given by

$$\mathscr{K}_\lambda(x,t,u) = \frac{e^{-\int_t^u(r(\tau)+\lambda)\,\mathrm{d}\tau}}{(\int_t^u\sigma(\tau)^2\,\mathrm{d}\tau)^{1/2}}N'(z_{2\lambda}(x,t,u)) = \frac{xe^{-\int_t^u(q(\tau)+\lambda+\kappa\lambda)\,\mathrm{d}\tau}}{(\int_t^u\sigma(\tau)^2\,\mathrm{d}\tau)^{1/2}}N'(z_{1\lambda}(x,t,u)), \tag{3.6}$$

where

$$z_{1\lambda}(x,t,u) = \frac{\log x+\int_t^u(r(\tau)-q(\tau)-\kappa\lambda+\sigma(\tau)^2/2)\,\mathrm{d}\tau}{\left(\int_t^u\sigma(\tau)^2\,\mathrm{d}\tau\right)^{1/2}}, \tag{3.7}$$

$$z_{2\lambda}(x,t,u) = \frac{\log x+\int_t^u(r(\tau)-q(\tau)-\kappa\lambda-\sigma(\tau)^2/2)\,\mathrm{d}\tau}{\left(\int_t^u\sigma(\tau)^2\,\mathrm{d}\tau\right)^{1/2}}. \tag{3.8}$$

Thus, using (2.19) in (3.4), we get

$$\hat{v}(\xi,t) = e^{\lambda(T-t)\mathbb{E}[Y^{-\xi}]}\hat{v}_\lambda(\xi,t). \tag{3.9}$$

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Now, let $\mathscr{J} = \mathscr{J}(x,t)$ be a function whose Mellin transform is

$$\hat{\mathscr{J}}(\xi,t) = e^{\lambda(T-t)\mathbb{E}[Y^{-\xi}]}. \tag{3.10}$$

Then, we can write (3.9) as $\hat{v}(\xi,t) = \hat{v}_\lambda(\xi,t)\hat{\mathscr{J}}(\xi,t)$, and from the convolution property we obtain

$$v(x,t) = (v_\lambda(\cdot,t) * \mathscr{J}(\cdot,t))(x) = \int_0^\infty \frac{1}{z} v_\lambda\left(\frac{x}{z},t\right)\mathscr{J}(z,t)\,\mathrm{d}z. \tag{3.11}$$

3.2 The jump term $\mathscr{J}$

To find $\mathscr{J}$, we can actually bypass the complex integral required for an inverse Mellin transform. From (3.10), we have

$$\hat{\mathscr{J}}(\xi,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t)\mathbb{E}[Y^{-\xi}])^n}{n!},$$

and as only the factor that depends on ξ is the one with the expectation, we invert $\hat{\mathscr{J}}$ to get

$$\mathscr{J}(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!}\mathscr{M}^{-1}\{\mathbb{E}[Y^{-\xi}]^n\}. \tag{3.12}$$

We now let $F_n = F_n(x)$ be a function such that $\hat{F}_n(\xi) = \mathbb{E}[Y^{-\xi}]^n$, where $n = 0, 1, \ldots$. We can rewrite $\hat{F}_n$ as

$$\hat{F}_n(\xi) = \mathbb{E}[Y^{-\xi}]\mathbb{E}[Y^{-\xi}]^{n-1} \quad (n = 1, 2, \ldots),$$

and from the convolution property, $\hat{F}_n$ can be inverted to yield

$$\begin{aligned} F_n(x) = \mathscr{M}^{-1}\{\mathbb{E}[Y^{-\xi}]\} * \mathscr{M}^{-1}\{\mathbb{E}[Y^{-\xi}]^{n-1}\} &= \mathscr{M}^{-1}\{\hat{F}_1(\xi)\} * \mathscr{M}^{-1}\{\hat{F}_{n-1}(\xi)\} \\ &= \int_0^\infty \frac{1}{z} F_1(z) F_{n-1}\left(\frac{x}{z}\right)\,\mathrm{d}z. \end{aligned}$$

Since F_n is recursive, we need the base cases F_0 and F_1. To find F_0, we refer to its Mellin transform and find that $\hat{F}_0(\xi) = \mathbb{E}[Y^{-\xi}]^0 = 1$. This can be inverted to give

$$F_0(x) = \mathscr{M}^{-1}\{1\} = \delta(x-1),$$

where δ is the Dirac delta function.[1] To find F_1, we know that $\mathscr{M}\{F_1(x)\} = \mathbb{E}[Y^{-\xi}]$. From the definition of the expectation, we see that

$$\mathscr{M}\{F_1(x)\} = \int_0^\infty y^{-\xi} f(y)\,\mathrm{d}y,$$

[1] To see how $\mathscr{M}^{-1}\{1\} = \delta(x-1)$, we simply take the Mellin transform of $\delta(x-1)$ to give $\mathscr{M}\{\delta(x-1)\} = \int_0^\infty x^{\xi-1}\delta(x-1)\,\mathrm{d}x$. Then, using the property that $\int_0^\infty f(x)\delta(x-a)\,\mathrm{d}x = f(a)$ for $a > 0$, where $f(x) = x^{\xi-1}$, we obtain $\mathscr{M}\{\delta(x-1)\} = 1$.

where f is the PDF of Y. The substitution $x = 1/y$ gives

$$\mathscr{M}\{F_1(x)\} = \int_0^\infty x^{\xi-1}\frac{1}{x}f\left(\frac{1}{x}\right)\,\mathrm{d}x = \mathscr{M}\left\{\frac{1}{x}f\left(\frac{1}{x}\right)\right\};$$

hence, we get

$$F_1(x) = \frac{1}{x}f\left(\frac{1}{x}\right).$$

We can then express (3.12) as

$$\mathscr{J}(x,t) = \sum_{n=0}^\infty \frac{(\lambda(T-t))^n}{n!}F_n(x), \tag{3.13}$$

with

$$F_n(x) = \begin{cases} \delta(x-1) & n=0, \\ \dfrac{1}{x}f\left(\dfrac{1}{x}\right) & n=1, \\ \displaystyle\int_0^\infty \frac{1}{z}F_1(z)F_{n-1}\left(\frac{x}{z}\right)\,\mathrm{d}z & n \geqslant 2. \end{cases} \tag{3.14}$$

Formula (3.13) for $\mathscr{J}$ can then be substituted into (3.11) for computation. Equation (3.11) gives us the European option pricing formula with a general payoff where the underlying asset has jumps. The key attributes of (3.11) are as follows:

(1) The formula can be applied to any payoff and any jump (cf. [39,40,47]).

(2) The option price can be expressed as the convolution of a standard European option with shifted parameters and a separate function that encapsulates the behaviour of the jump.

(3) No complex integrals are required to be computed (cf. [29]).[2]

Now, we give an alternative expression for (3.11). We have

$$v(x,t) = \int_0^\infty \frac{1}{z}v_\lambda\left(\frac{x}{z},t\right)\sum_{n=0}^\infty \frac{(\lambda(T-t))^n}{n!}F_n(z)\,\mathrm{d}z.$$

Interchanging the summation and integral and using (3.5) for v_λ, we obtain

$$v(x,t) = \sum_{n=0}^\infty \frac{(\lambda(T-t))^n}{n!}\int_0^\infty \frac{F_n(z)}{z}\int_0^\infty \frac{1}{y}\mathscr{K}_\lambda\left(\frac{x}{yz},t,T\right)\phi(y)\,\mathrm{d}y\,\mathrm{d}z.$$

[2] In [29], the option price was given as the Mellin inverse of an integrand that depends on the payoff. Consequently, this inversion process has to be done every time the payoff is changed. On the other hand, in our approach, since we are using the Mellin convolution theorem, it is only necessary to invert the Black–Scholes kernel once, which we have done (refer to (3.15)). Thus, our expression for the option price only involves real integrals.

Swapping the order of integration yields

$$v(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \int_0^{\infty} \frac{\phi(y)}{y} \int_0^{\infty} \frac{1}{z} \mathscr{K}_\lambda\left(\frac{x}{yz}, t, T\right) F_n(z)\, \mathrm{d}z\, \mathrm{d}y.$$

The innermost integral with respect to z resembles an option whose payoff function is F_n in accordance to (3.5) and (2.10). We will label this function w_n, i.e., let

$$w_n\left(\frac{x}{y}, t\right) = \int_0^{\infty} \frac{1}{z} \mathscr{K}_\lambda\left(\frac{x}{yz}, t, T\right) F_n(z)\, \mathrm{d}z,$$

and get

$$\begin{aligned} v(x,t) &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \int_0^{\infty} \frac{\phi(y)}{y} w_n\left(\frac{x}{y}, t\right) \mathrm{d}y \\ &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \left(w_n(\cdot, t) * \phi\right)(x) \qquad (3.15) \\ &= \left(\left(\sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} w_n(\cdot, t)\right) * \phi\right)(x), \end{aligned}$$

where the Mellin convolution theorem was used in the second equality and its distributivity property in the third line. Note that (2.10) can be expressed as a convolution of the Black–Scholes kernel and the payoff, namely

$$v(x,t) = (\mathscr{K}(\cdot, t, T) * \phi)(x). \qquad (3.16)$$

This leads to an interpretation for the summation in (3.15) as an analogue of the Black–Scholes kernel in the case of jump-diffusion dynamics. When there are no jumps (i.e., $\lambda = 0$), equation (3.15) reduces to (3.16). This idea of representing the option price as an iterated integral and swapping the order will be useful in Section 4.3 when we show equality between our solution and Merton's solution in the case of lognormal jumps.

4 Example: lognormally distributed jumps

We will now derive a specific formula for v when Y is lognormal (i.e., $Y \sim \mathrm{LN}(\mu_Y, \sigma_Y)$). It is known [47] that

$$f(y) = \frac{1}{y\sqrt{2\pi\sigma_Y^2}} e^{-(\log y - \mu_Y)^2/(2\sigma_Y^2)}, \qquad \mathbb{E}[Y^{-\xi}] = e^{-\mu_Y \xi + \sigma_Y^2 \xi^2/2}. \qquad (4.1)$$

To proceed, we present two ways of deriving the explicit formula: one by the general recursion formula and the other by using a direct Mellin approach.

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4.1 Result via the general recursion formula

Using (3.14), we obtain F_1 for lognormal jumps as

$$F_1(x) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-(\log x+\mu_Y)^2/(2\sigma_Y^2)} = \frac{1}{\sqrt{\sigma_Y^2}} N'\left(\frac{\log x + \mu_Y}{\sqrt{\sigma_Y^2}}\right),$$

using (2.17) to change the exponential to N'. Similarly, F_2 is given by

$$F_2(x) = \int_0^\infty \frac{1}{z} F_1(z) F_1\left(\frac{x}{z}\right) \mathrm{d}z = \frac{1}{\sigma_Y^2} \int_0^\infty \frac{1}{z} N'\left(\frac{\log z + \mu_Y}{\sqrt{\sigma_Y^2}}\right) N'\left(\frac{\log(x/z) + \mu_Y}{\sqrt{\sigma_Y^2}}\right) \mathrm{d}z.$$

Using Lemma 1, we choose $a_1 = a_2 = 1/\sigma_Y$ and $b_1 = b_2 = \mu_Y/\sigma_Y$ to simplify the integral and yield

$$F_2(x) = \frac{1}{\sqrt{2\sigma_Y^2}} N'\left(\frac{\log x + 2\mu_Y}{\sqrt{2\sigma_Y^2}}\right).$$

Hence, using an induction argument, we can deduce that

$$F_n(x) = \frac{1}{\sqrt{n\sigma_Y^2}} N'\left(\frac{\log x + n\mu_Y}{\sqrt{n\sigma_Y^2}}\right).$$

The resulting formula for the jump is

$$\mathscr{J}(x,t) = \delta(x-1) + \sum_{n=1}^{\infty} \frac{(\lambda(T-t))^n}{n!} \frac{1}{\sqrt{n\sigma_Y^2}} N'\left(\frac{\log x + n\mu_Y}{\sqrt{n\sigma_Y^2}}\right), \tag{4.2}$$

recalling the definition of F_0 from (3.14). Therefore, v is

$$v(x,t) = v_\lambda(x,t) + \sum_{n=1}^{\infty} \frac{(\lambda(T-t))^n}{\sigma_Y n! \sqrt{n}} \int_0^\infty \frac{1}{z} N'\left(\frac{\log z + n\mu_Y}{\sigma_Y \sqrt{n}}\right) v_\lambda\left(\frac{x}{z}, t\right) \mathrm{d}z, \tag{4.3}$$

using a standard property of the Dirac delta function. Note that we can also express (4.3) as a summation from $n = 0$, where the 0th term corresponds to $v_\lambda(x,t)$ as can be seen due to the properties of the Dirac delta function.

4.2 Result using Mellin identities

Substituting the second equation of (4.1) into (3.10), we get

$$\hat{\mathscr{J}}(\xi,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} e^{-n\mu_Y \xi + n\sigma_Y^2 \xi^2/2}.$$

Inverting $\hat{\mathscr{J}}$ gives

$$\mathscr{J}(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \mathscr{M}^{-1}\left\{e^{-n\mu_Y \xi + n\sigma_Y^2 \xi^2/2}\right\}.$$

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Using Lemma 3, choosing $a = 1/(\sigma_Y\sqrt{n})$ and $b = \mu_Y\sqrt{n}/\sigma_Y$, we see that

$$\mathscr{M}^{-1}\left\{e^{-n\mu_Y\xi+n\sigma_Y^2\xi^2/2}\right\} = \frac{1}{\sqrt{n\sigma_Y^2}}N'\left(\frac{\log x + n\mu_Y}{\sqrt{n\sigma_Y^2}}\right).$$

Then, (3.12) for lognormally distributed jumps is given by

$$\mathscr{J}(x,t) = \delta(x-1) + \sum_{n=1}^{\infty}\frac{(\lambda(T-t))^n}{n!}\frac{1}{\sqrt{n\sigma_Y^2}}N'\left(\frac{\log x + n\mu_Y}{\sqrt{n\sigma_Y^2}}\right), \tag{4.4}$$

which is identical to (4.2). Hence, (3.11) for lognormally distributed jumps is identical to (4.3), as expected.

4.3 Verification of equality to Merton's solution

We will now verify that (4.3) is identical to Merton's option pricing formula in (2.15) for lognormal jumps and an arbitrary payoff function ϕ. Note that Merton assumed that r, q, and σ are constant, so we too will make that assumption. The goal is to show that

$$v(x,t) = v_M(x,t), \tag{4.5}$$

for constant r, q, and σ. We will start with the left-hand side using (4.3). We first convert both v_λ terms with its integral form (3.5) to get

$$\begin{aligned}
v(x,t) &= \int_0^\infty \frac{1}{y}\mathscr{K}_\lambda\left(\frac{x}{y},t,T\right)\phi(y)\,\mathrm{d}y \\
&\quad + \sum_{n=1}^{\infty}\frac{(\lambda(T-t))^n}{\sigma_Y n!\sqrt{n}}\int_0^\infty\int_0^\infty \frac{1}{z}N'\left(\frac{\log z + n\mu_Y}{\sigma_Y\sqrt{n}}\right)\frac{1}{y}\mathscr{K}_\lambda\left(\frac{x}{yz},t,T\right)\phi(y)\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_0^\infty \frac{1}{y}\mathscr{K}_\lambda\left(\frac{x}{y},t,T\right)\phi(y)\,\mathrm{d}y \\
&\quad + \sum_{n=1}^{\infty}\frac{(\lambda(T-t))^n}{\sigma_Y n!\sqrt{n}}\int_0^\infty \frac{\phi(y)}{y}\int_0^\infty \frac{1}{z}N'\left(\frac{\log z + n\mu_Y}{\sigma_Y\sqrt{n}}\right)\mathscr{K}_\lambda\left(\frac{x}{yz},t,T\right)\mathrm{d}z\,\mathrm{d}y.
\end{aligned}$$

We then want to evaluate

$$I = \int_0^\infty \frac{1}{z}N'\left(\frac{\log z + n\mu_Y}{\sigma_Y\sqrt{n}}\right)N'\left(z_{2\lambda}\left(\frac{x}{yz},t,T\right)\right)\mathrm{d}z.$$

To do this, we substitute the first expression in (3.6) for $\mathscr{K}_\lambda$. Recalling the form for $z_{2\lambda}$ using (3.8), we apply Lemma 1 and we choose

$$a_1 = \frac{1}{\sigma_Y\sqrt{n}},\quad b_1 = \frac{n\mu_Y}{\sigma_Y\sqrt{n}},\quad a_2 = \frac{1}{\sigma\sqrt{T-t}},\quad b_2 = \frac{\log(x/y) + (r-q-\kappa\lambda-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

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This simplifies I to be

$$I = \frac{\sigma_Y \sqrt{n}\, \sigma \sqrt{T-t}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} N'(z_n),$$

where

$$z_n = \frac{\log(x/y) + n\mu_Y + (r - q - \kappa\lambda - \sigma^2/2)(T-t)}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}}.$$

So far we have

$$v(x,t) = \frac{e^{-(r+\lambda)(T-t)}}{\sigma\sqrt{T-t}} \int_0^\infty \frac{1}{y} N'(z_0)\, \phi(y)\, \mathrm{d}y \\ + \sum_{n=1}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(z_n)\phi(y)\, \mathrm{d}y,$$

where we expand the first $\mathscr{K}_\lambda$ using (3.6) assuming constant r, q, and σ with

$$z_0 = \frac{\log(x/y) + (r - q - \kappa\lambda - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} = z_{2\lambda}\left(\frac{x}{y}, t, T\right).$$

This can actually be contracted to

$$v(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(z_n)\phi(y)\, \mathrm{d}y$$

by recognising the relation between z_0 and z_n along with $\sigma\sqrt{T-t}$ and $(n\sigma_Y^2 + \sigma^2(T-t))^{1/2}$. The integral can also be simplified if we briefly recall from Merton's solution (2.15) that

$$v_n(x,t) = v(x,t;r,q,\sigma)|_{r=r_n(t),\ q=q,\ \sigma=\sigma_n(t)}.$$

From (2.10), this gives

$$v_n(x,t) = \left[\int_0^\infty \frac{1}{y}\mathscr{K}\left(\frac{x}{y}, t, T\right)\phi(y)\, \mathrm{d}y\right]\Bigg|_{r=r_n(t),\, q=q,\, \sigma=\sigma_n(t)}.$$

Now, recalling the definition for $r_n(t)$ and $\sigma_n(t)$ from (2.16), we choose (2.8) and get

$$v_n(x,t) = \frac{e^{-(r-\kappa\lambda)(T-t) - n\log(1+\kappa)}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(d_n)\phi(y)\, \mathrm{d}y,$$

where

$$d_n = \frac{\log(x/y) + [r - \kappa\lambda + n\log(1+\kappa)/(T-t) - q - \sigma^2/2 - n\sigma_Y^2/(2(T-t))](T-t)}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}}.$$

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To proceed, we now turn to v_M on the right-hand side of (4.5). We first change v_n into its kernel form using (2.8) and the definition of v_n, $r_n(t)$, $\sigma_n(t)$ from Section 2.1 to give

$$
\begin{aligned}
v_M(x,t) &= \sum_{n=0}^{\infty} \frac{(\lambda(1+\kappa)(T-t))^n}{n!} \cdot e^{-\lambda(1+\kappa)(T-t)} \cdot \frac{e^{-(r-\kappa\lambda)(T-t)-n\log(1+\kappa)}}{\left(n\sigma_Y^2+\sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(d_n)\phi(y)\,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2+\sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(d_n)\phi(y)\,\mathrm{d}y.
\end{aligned}
$$

For a lognormal distribution, we have $\kappa = e^{\mu_Y+\sigma_Y^2/2}-1$ which reduces d_n to

$$
d_n = \frac{\log(x/y) + n\mu_Y + (r-q-\kappa\lambda-\sigma^2/2)(T-t)}{\left(n\sigma_Y^2+\sigma^2(T-t)\right)^{1/2}} = z_n.
$$

Therefore,

$$
v_M(x,t) = \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2+\sigma^2(T-t)\right)^{1/2}} \int_0^\infty \frac{1}{y} N'(z_n)\phi(y)\,\mathrm{d}y = v(x,t),
$$

hence, showing equality between (4.3) and (2.15). The integrals containing $N'(z_n)$ can be evaluated using Lemma 2 once the payoff function ϕ is defined. In practice, many financial payoffs can be expressed as finite linear combinations of

$$
x \mapsto \mathbb{1}_I(x), \quad x \mapsto x\mathbb{1}_I(x),
$$

with $\mathbb{1}_I$ is the indicator function defined as

$$
\mathbb{1}_I(x) = \begin{cases} 1, & x \in I, \\ 0, & x \notin I, \end{cases}
$$

where I is an arbitrary interval with endpoints a and b with $a < b$. This specified interval can be open, half-closed, or closed. For example, a call option has payoff $\phi(x) = \max(x-K, 0)$ which can be formulated as

$$
\max(x-K,0) = x\mathbb{1}_{[K,\infty)}(x) - K\mathbb{1}_{[K,\infty)}(x),
$$

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where I is the interval $[K, \infty)$. So, the expression would be

$$
\begin{aligned}
v(x,t) &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} \int_0^{\infty} \frac{1}{y} N'(z_n) \max(y-K,0)\,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \frac{e^{-(r+\lambda)(T-t)}}{\left(n\sigma_Y^2 + \sigma^2(T-t)\right)^{1/2}} \int_K^{\infty} \frac{1}{y} N'(z_n)(y-K,0)\,\mathrm{d}y \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n}{n!} \cdot \Bigg(x e^{-(q+\lambda(1+\kappa))(T-t)+n\mu_Y+n\sigma_Y^2/2} \cdot \\
&\quad N\left(\frac{\log(x/K) + n\mu_Y + n\sigma_Y^2 + (r-q-\kappa\lambda+\sigma^2/2)(T-t)}{(n\sigma_Y^2+\sigma^2(T-t))^{1/2}}\right) \\
&\quad - K e^{-(r+\lambda)(T-t)} N\left(\frac{\log(x/K) + n\mu_Y + (r-q-\kappa\lambda-\sigma^2/2)(T-t)}{(n\sigma_Y^2+\sigma^2(T-t))^{1/2}}\right)\Bigg).
\end{aligned}
$$

Therefore, the two expressions in Lemma 2 will account for any potential payoff one may encounter in options pricing.

4.4 Comparison of the jump-diffusion and Black–Scholes models

For completeness, we will present some elementary numerical comparisons between the option values for when the asset price is governed by a jump-diffusion model and when it follows the standard diffusion model. We will assume the jumps are lognormally distributed. The chosen parameters are $r = 0.05$, $q = 0.0$, $\sigma = 0.3$, $T - t = 0.5$, $K = 100$, $\lambda = 0.5$, $\mu_Y = -0.90, \sigma_Y = 0.45$. We will use a European call option and vary S_0 between 50 and 500 to investigate the behaviour both in-the-money and out-of-the-money. Comparing both plots in Figure 1, we see that options in a jump-diffusion framework possess a higher value than those of the standard diffusion model. This is expected as there is an extra component of uncertainty governed by the SDE in (2.11).

5 Example: double exponentially distributed jumps

We will also demonstrate how to derive a recursive formula for double exponentially distributed jumps. A pricing formula does exist [39, 40] for a double exponential jump-diffusion model, but it is expressed in a way that showing equality to the recursive form (3.14) is very difficult. Thus, only F_1 will be determined since it is all that is required to generate the other terms.

Suppose $Y > 0$ is drawn from a double exponential distribution with parameters $\omega_1 > 0$, $\omega_2 > 0$, and $p, q \geqslant 0$ such that $p + q = 1$. Frontczak [29] gives the corresponding PDF and expectation as

$$
f(y) = p\omega_1 y^{-\omega_1 - 1} \mathbb{1}_{\{y \geqslant 1\}} + q\omega_2 y^{\omega_2 - 1} \mathbb{1}_{\{0<y<1\}}, \quad \mathbb{E}[Y^{-\xi}] = \frac{p\omega_1}{\omega_1 + \xi} + \frac{q\omega_2}{\omega_2 - \xi},
$$

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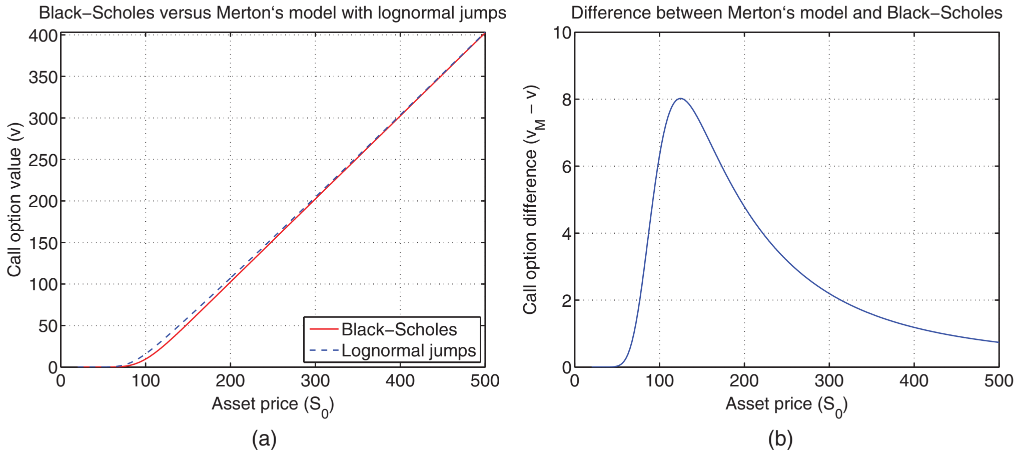

FIGURE 1. Call option profiles using (2.4) and (2.15). The financial parameters are $r = 0.05$, $q = 0.00$, $\sigma = 0.3$, $T - t = 0.5$, $K = 100$, and $S_0 \in [50, 500]$. The lognormal jump parameters are $\lambda = 0.5$, $\mu_Y = -0.90$, and $\sigma_Y = 0.45$. (a) Option profiles for Merton's solution (2.15) with lognormal jumps and standard Black–Scholes call option (2.4). (b) Difference between (2.15) and (2.4).

where $\mathbb{1}_I$ is the indicator function of the interval I. Using (3.14), we get

$$\begin{aligned} F_1(x) &= \frac{1}{x}\left(p\omega_1\left(\frac{1}{x}\right)^{-\omega_1-1}\mathbb{1}_{\{1/x\geqslant 1\}} + q\omega_2\left(\frac{1}{x}\right)^{\omega_2-1}\mathbb{1}_{\{0<1/x<1\}}\right) \\ &= p\omega_1 x^{\omega_1}\mathbb{1}_{\{x\leqslant 1\}} + q\omega_2 x^{-\omega_2}\mathbb{1}_{\{x>1\}}, \end{aligned} \tag{5.1}$$

and from here we can obtain F_n recursively from (3.14). Using this, we can substitute into (3.13) and then (3.11) to find the option price.

6 Example: gamma distributed jumps

Whilst a pricing formula for lognormal jumps and double exponential jumps has been derived in previously, none exists for gamma distributed jumps. We will show a recursive solution that is still exact and analytic.

Suppose $Y \sim \text{Gamma}(\alpha_Y, \beta_Y)$, where $\alpha_Y > 0$ affects the distribution shape and $\beta_Y > 0$ determines the scale (i.e., how far spread out the distribution is). The associated PDF of Y is given by [29]

$$f(y) = \frac{1}{\Gamma(\alpha_Y)\beta_Y^{\alpha_Y}} y^{\alpha_Y - 1} e^{-y/\beta_Y}, \qquad \mathbb{E}[Y^{-\xi}] = \frac{\beta_Y^{-\xi}\Gamma(\alpha_Y - \xi)}{\Gamma(\alpha_Y)}.$$

Then, using (3.13), we have

$$F_1(x) = \frac{1}{x}\frac{1}{\Gamma(\alpha_Y)\beta_Y^{\alpha_Y}}\left(\frac{1}{x}\right)^{\alpha_Y - 1} e^{-1/(x\beta_Y)} = \frac{(x\beta_Y)^{-\alpha_Y}}{\Gamma(\alpha_Y)} e^{-1/(x\beta_Y)}, \tag{6.1}$$

which can then be employed recursively to compute F_n. Similarly, F_n can be substituted into (3.13) and then (3.11) to find the corresponding option price.

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7 A PIDE analogue of Dupire's equation

In this section, we will derive a PIDE that is the analogue of Dupire's equation as seen in [30]. The Dupire-like PIDE will serve as the platform for computing the implied volatility of options with jump-diffusion asset dynamics.

7.1 Homogeneity of the solution

First, we assume that the payoff function ϕ now depends on a parameter $x' > 0$ (i.e., $\phi = \phi(x; x')$). The motivation for this is that in the case of a European put or call, x' represents the strike price. Furthermore, we assume that ϕ is homogeneous of degree one in x and x'. Note that the put and call payoffs satisfy this assumption. We show that the option price function v is homogeneous of degree one in x and x'. That is, we want to show for $v = v(x, t; x')$ that

$$v(\beta x, t; \beta x') = \beta v(x, t; x') \tag{7.1}$$

for all $\beta > 0$. This equality can be proven via a uniqueness argument as follows. We first express (2.13) as $\mathscr{L} v = 0$. Now let $w = w(x, t; x')$ solve the following final value problem:

$$\mathscr{L} w = 0, \qquad w(x, T; x') = \beta\phi(x; x'). \tag{7.2}$$

Next, we define the function $v_1(x, t; x') = \beta v(x, t; x')$, where v is a solution to (2.13), (2.14). Then,

$$\mathscr{L} v_1 = \beta \mathscr{L} v = 0.$$

For the terminal condition, since $v(x, T; x') = \phi(x; x')$, this implies that

$$v_1(x, T; x') = \beta v(x, T; x') = \beta\phi(x; x').$$

Therefore, v_1 satisfies the final value problem (7.2). On the other hand, we now let $v_2(x, t, x') = v(\beta x, t; \beta x')$. Computing the derivatives gives

$$\frac{\partial v_2}{\partial x} = \beta D_1 v, \quad x^2 \frac{\partial^2 v_2}{\partial x^2} = \beta^2 D_{11} v,$$

where D_1 and D_{11} represent the first and second partial derivatives with respect to the first argument, respectively. Substituting these into (7.2), we get

$$\mathscr{L} v_2 = \mathscr{L} v = 0,$$

and by the homogeneity of ϕ, the terminal condition is

$$v_2(x, T; x') = v\left(\beta x, T; \beta x'\right) = \phi(\beta x; \beta x') = \beta\phi(x; x').$$

Hence, v_2 also satisfies the final value problem (7.2). By uniqueness, we have

$$v\left(\beta x, t; \beta x'\right) = v_2(x, t; x') = v_1(x, t; x') = \beta v(x, t; x'),$$

thus, proving the homogeneity property for v and any general payoff ϕ that is homogeneous of degree one in x and x'.

7.2 Derivation of a Dupire-like PIDE via Euler's theorem on homogeneous functions

The partial derivatives of the Dupire equation [30] are in terms of the strike price K. Thus, to derive a Dupire-like PIDE, we will require partial derivatives in terms of x' (the analogous variable for K). This can be done by invoking Euler's theorem for homogeneous functions [38, pp. 317] to v and we get

$$x\frac{\partial v}{\partial x} + x'\frac{\partial v}{\partial x'} = v,$$

since v has been shown to be homogeneous in x and x' of degree one. By differentiating the above equation with respect to x and x', we obtain

$$x\frac{\partial^2 v}{\partial x^2} = -x'\frac{\partial^2 v}{\partial x \partial x'}, \quad x'\frac{\partial^2 v}{\partial x'^2} = -x\frac{\partial^2 v}{\partial x' \partial x},$$

respectively. Hence, it follows that

$$x^2\frac{\partial^2 v}{\partial x^2} = x'^2\frac{\partial^2 v}{\partial x'^2}.$$

The only term left to account for is the integral in (2.13). Notice that the first integrand term depends on y; we want to transfer the dependency on y to the third argument (i.e., x'). This can be achieved by the homogeneity property in (7.1), and we obtain

$$v(xy, t; x') = v\left(xy, t; \frac{x'y}{y}\right) = yv\left(x, t; \frac{x'}{y}\right).$$

Thus, setting $u(x', t; x) = v(x', t; x)$ and replacing all the x derivatives with x' derivatives and substituting the above rearrangement for the integrand, we get

$$\begin{aligned}\frac{\partial u}{\partial t} - (q(t) + \kappa\lambda)u - (r(t) - q(t) - \kappa\lambda)x'\frac{\partial u}{\partial x'} + \frac{1}{2}\sigma(t)^2 x'^2\frac{\partial^2 u}{\partial x'^2}\\ + \lambda\int_0^\infty \left(yu\left(\frac{x'}{y}, t; x\right) - u(x', t; x)\right) f(y)\,\mathrm{d}y = 0\end{aligned} \tag{7.3}$$

with

$$u(x', T; x) = \phi(x'; x), \tag{7.4}$$

since u now depends on the variables x' and t with x as a parameter. Equations (7.3) and (7.4) together form the Dupire-like PIDE system for options in a jump-diffusion framework. Note that this reduces to the standard Dupire PDE as seen in [30] in the absence of jumps (i.e., $\lambda = 0$) when ϕ is either the call or put payoff.

8 Implied volatility formula

From (7.3) and (7.4), it is possible to now solve the inverse problem of implied volatility estimation. Throughout the remainder of this section, we will assume that r, q, and σ are constants. Suppose that we are given $u(x', 0; S_0)$ for all $x' > 0$. We wish to derive an explicit formula for σ in terms of certain integrals of v with respect to x'. The reason for this is that in practice one can observe different time-zero option prices $u_1, u_2, \ldots, u_m$ for varying strike prices $K_1, K_2, \ldots, K_m$, here corresponding to different values of x'. Once we can extrapolate u for extreme values of x', we would know the entire time-zero profile of u.

First, denote by $\hat{u}$ the Mellin transform of u with respect to x', i.e.,

$$\hat{u}(\xi, t) = \int_0^\infty (x')^{\xi-1} u(x', t; x)\,\mathrm{d}x'.$$

We take the Mellin transform of (7.3) and (7.4) with respect to x' to obtain

$$\frac{\partial \hat{u}}{\partial t} - G_\lambda(\xi)\hat{u}(\xi, t) = 0, \quad \hat{u}(\xi, T) = \hat{\phi}(\xi), \tag{8.1}$$

where

$$G_\lambda(\xi) = -\left(\frac{\sigma^2}{2}\xi(\xi+1) + (r - q - \kappa\lambda)\xi - (q + \kappa\lambda) + \lambda\mathbb{E}[Y^{\xi+1} - 1]\right). \tag{8.2}$$

We are left with an ODE in t. Solving (8.1) gives

$$\hat{u}(\xi, t) = e^{-G_\lambda(\xi)(T-t)}\hat{\phi}(\xi). \tag{8.3}$$

We can proceed to isolate σ^2 in (8.3) to yield

$$\sigma^2 = \frac{2}{\xi(\xi+1)}\left(\frac{\ln(\hat{u}(\xi, t)/\hat{\phi}(\xi))}{T - t} - (r - q - \kappa\lambda)\xi + (q + \kappa\lambda) - \lambda\mathbb{E}[Y^{\xi+1} - 1]\right), \tag{8.4}$$

where we have the flexibility to choose a value of ξ. Theoretically, σ^2 should be constant for any value of ξ and t, provided the Mellin transform of u exists. Furthermore, it should be emphasised that (8.4) can be applied to any type of payoff and jump. When $\lambda = 0$, (8.4) gives an explicit formula for the implied volatility in the usual diffusion framework.

9 Numerical simulations

This section will contain the numerical results obtained from the implied volatility formula (8.4) for lognormal jumps. To test the validity of the model, we will require an initial σ value to generate option prices before solving the inverse problem. The results will be divided into two sets: the first set will be implementing purely theoretical data; the second set will be generated using pseudo-market data that attempts to mimic observed market prices and values. We will now elaborate on how the option prices are obtained. For definiteness, we will consider a time-zero European call. That is, in (2.4) we set $t = 0$,

$x = S_0$, assume r, q, and σ are constant, and view this as a function of K given as

$$v^{\text{call}}(K) = S_0 e^{-qT} N\left(z_1\left(\frac{S_0}{K}, 0, T\right)\right) - Ke^{-rT} N\left(z_2\left(\frac{S_0}{K}, 0, T\right)\right), \tag{9.1}$$

where z_1 and z_2 are defined as they are in (2.6) and (2.7), respectively.

9.1 Theoretical data for option prices

The Mellin transform is valid in the domain $[0, \infty)$. Since this implied volatility scheme incorporates a Mellin transform with respect to the strike price $K = x'$ for a fixed $x = S_0$, we require time-zero option prices for varying $K \in [0, \infty)$. Numerically, we will use discrete 200 values of $K \in [1.0 \times 10^{-6}, 8S_0]$ evenly spaced to simulate continuity for the entire domain $K > 0$. This will yield 200 call prices. In practice, this is seldom applicable as many sources for financial data will only list discrete option prices for a finite set of K values (i.e., much less than 200) and for a fixed asset price S_0. Furthermore, it is often implausible to expect the domain of K to be uniformly spaced. This approach is only included to illustrate the accuracy of the model assuming a very smooth dataset.

9.2 Pseudo-market data for option prices

As mentioned before, the finite number of discrete option prices may prove insufficient in exhibiting a continuous behaviour in the option price profile. Hence, we require a method for approximating the data beyond the option prices provided. The following procedure will be demonstrated for a call option in the absence of jumps to simplify the calculations. However, these steps can be adapted when accounting for jumps in the asset dynamics.

We assume that we have a set of call prices $v_1 > v_2 > \cdots > v_{m-1} > v_m$ with corresponding strike prices $K_1 < K_2 < \cdots < K_{m-1} < K_m$. It is known from [7] that the best one-parameter logistic approximation of the standard normal CDF N for all $z \in \mathbb{R}$ is given by

$$N(z) \approx \frac{1}{1 + e^{-az}}, \quad a = 1.702,$$

where the maximum difference between the approximation and exact expression for N is less than 0.001 for $z \in [-4.5, 4.5]$. Now, for $z < 0$ we have

$$N(z) \approx \frac{e^{az}}{1 + e^{az}} = e^{az}(1 - e^{az} + e^{2az} - \cdots) = e^{az} - e^{2az} + e^{3az} - \cdots.$$

Hence, we can take $N(z) \approx e^{az}$ for $z \ll -1$. Using this logistic estimation in (9.1), this approximates to

$$v^{\text{call}}(K) \approx \frac{S_0 e^{-qT}}{1 + e^{-ad_1(S_0/K, 0, T)}} - \frac{Ke^{-rT}}{1 + e^{-ad_2(S_0/K, 0, T)}} = \frac{S_0 e^{-qT}}{1 + e^{-ad_1}} - \frac{Ke^{-rT}}{1 + e^{-ad_2}},$$

where d_1 and d_2 are defined as

$$d_1 = z_1\left(\frac{S_0}{K}, 0, T\right), \quad d_2 = z_2\left(\frac{S_0}{K}, 0, T\right)$$

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using (2.6) and (2.7), respectively, under the assumption of constant parameters. When $|K| \ll 1$ we see that $d_1 \gg 1$ and $d_2 \gg 1$; hence, $v^{\text{call}}(K) \approx S_0 e^{-qT} - K e^{-rT}$. Therefore, we assume that

$$v^{\text{call}}(K) = S_0 e^{-qT} - \beta K, \quad 0 < K \leqslant K_1$$

for some $\beta > 0$. Using K_1 to extrapolate, we see from $v^{\text{call}}(K_1) = v_1$ that we obtain

$$\beta = \frac{S_0 e^{-qT} - v_1}{K_1}.$$

Conversely, when $K \gg 0$ we have $-d_1 \gg 1$ and $-d_2 \gg 1$. Using $N(z) \approx e^{az}$ for $-z \gg 1$, we can simplify $N(d_1)$ and $N(d_2)$ and approximate (9.1) by

$$v^{\text{call}}(K) \approx S_0 e^{-qT} e^{ad_1} - K e^{-rT} e^{ad_2}.$$

As $d_1 = d_2 + \sigma\sqrt{T}$,

$$e^{ad_2} = e^{a\left(\log(S_0/K) + (r-q-\sigma^2/2)T\right)/\left(\sigma\sqrt{T}\right)} = \left(\frac{S_0}{K}\right)^{a/\left(\sigma\sqrt{T}\right)} e^{a(r-q-\sigma^2/2)\sqrt{T}/\sigma}.$$

Similarly,

$$e^{ad_1} = e^{a\sigma\sqrt{T}} e^{ad_2},$$

hence, we have

$$v^{\text{call}}(K) \approx \left(S_0 e^{-qT} e^{a\sigma\sqrt{T}} - K e^{-rT}\right) \left(\frac{S_0}{K}\right)^{a/\left(\sigma\sqrt{T}\right)} e^{a(r-q-\sigma^2/2)\sqrt{T}/\sigma}.$$

Therefore, we assume that

$$v^{\text{call}}(K) = \frac{\gamma_1}{K^\delta} + \frac{\gamma_2}{K^{\delta-1}}, \quad K \geqslant K_m$$

for some $\gamma_1, \gamma_2, \delta > 0$. We will need to use K_{m-1} and K_m to extrapolate, but we also require another data point. For the call option, $v^{\text{call}}(K) \to 0$ as $K \to \infty$, thus we let $K_L \gg K_m$ represent the strike price "near" infinity. We see from $v^{\text{call}}(K_{m-1}) = v_{m-1}$, $v^{\text{call}}(K_m) = v_m$, and $v^{\text{call}}(K_L) \approx 0$, and we deduce that

$$\delta = \frac{\log\left(v_{m-1}/v_m\right) + \log\left((K_L - K_m)/(K_L - K_{m-1})\right)}{\log(K_m/K_{m-1})},$$

$$\gamma_2 = \frac{v_m K_m^\delta}{K_m - K_L}, \quad \gamma_1 = -K_L \gamma_2, \quad K_L \gg K_m.$$

Thus, the call option function can be reformulated to become

$$v^{\text{call}}(K) = \begin{cases} S_0 e^{-qT} - \beta K & 0 < K \leqslant K_1, \\ v_j & K = K_j, \;\; j = 1, \ldots, m, \\ \frac{\gamma_1}{K^\delta} + \frac{\gamma_2}{K^{\delta-1}} & K \geqslant K_m, \end{cases} \tag{9.2}$$

where $v_1, \ldots, v_m$ are the observed call prices. A similar process can also be adopted for the

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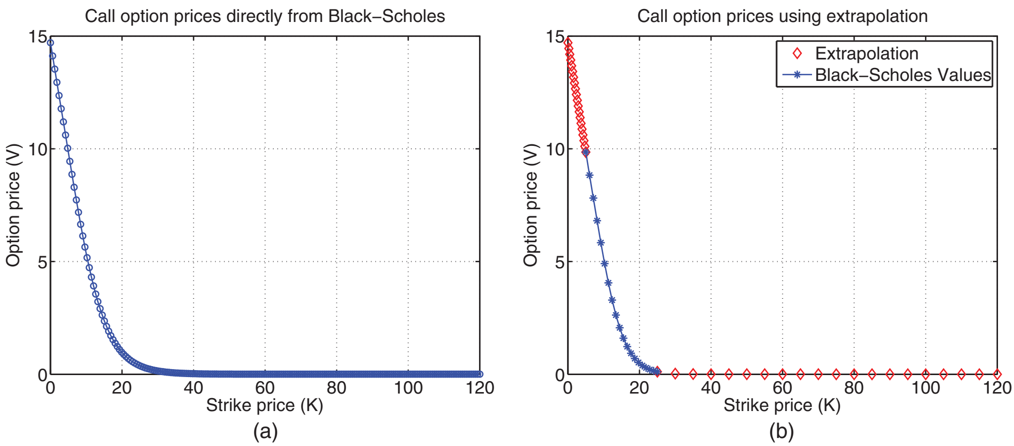

FIGURE 2. Call option profiles for $K > 0$. The parameter values are $S_0 = 15$, $T = 0.3$, $r = 0.03$, $q = 0.02$, and $\sigma = 0.3$. (a) Call prices computed using (9.1) with 200 equally spaced nodes for K between 10^{-6} and $8S_0$. (b) Call prices computed using (9.1) for pseudo-observed Black–Scholes values and (9.2) to extrapolate.

European call or put with jumps. Figure 2(a) and (b) shows the profile for the call option with both theoretical and pseudo-market data, respectively.

9.3 Algorithm

The algorithm for computing σ^2 for a call option is as follows:

(1) Obtain option data $v_1, v_2, \cdots, v_m$ for $K_1 < K_2 < \cdots < K_m$ either using theoretical, pseudo-market or actual market data.

(a) Theoretical data – use (9.1) or (2.15) (with appropriate adjustments to the notation) and ensure $K_1, K_2, \ldots, K_m$ are 200 evenly spaced nodes between 10^{-6} and $8S_0$ (adjust if $8S_0 < K_m$).

(b) Pseudo-real or market data – generate $v_1, v_2, \ldots, v_m$ using theoretical data or observed from the market, then use (9.2) (adapt for jumps if necessary) to create more data points for a smoother profile. For $K \approx 0$, use 1.0×10^{-6}; for $K \gg 0$, use $8S_0$ (adjust if $8S_0 < K_m$).

(2) Choose a value of ξ.

(3) Evaluate $\hat{v}^{\text{call}}(\xi) = \int_0^\infty K^{\xi-1} v^{\text{call}}(K)\,dK$ via numerical integration (e.g., Gauss–Lobatto or Gauss–Kronrod quadrature), where v^{call} is the entire time-zero option profile.

(4) Substitute the value for $\hat{v}^{\text{call}}(\xi)$ into (8.4) and compute σ^2.

9.4 Results

We will now report the implied volatility estimations for both theoretical option data and pseudo-market option prices via extrapolation. The parameter values used are $S_0 = 15$,

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Table 1. *Implied volatility estimations and errors for different* ξ *when* $\sigma = 0.15$ *using pure theoretical option data from* (4.3). *Average CPU time is given in seconds*

Implied volatility estimation for $\sigma = 0.15$ Avg. CPU time: 0.1 s		
ξ	Estimated σ	Absolute error
1.0	0.150001657453424	1.6×10^{-6}
1.25	0.149999916945745	8.3×10^{-8}
1.5	0.150004007140589	4.0×10^{-6}
1.75	0.150000231848765	2.3×10^{-7}
2.0	0.150000372512579	3.7×10^{-7}
2.25	0.150000519892163	5.1×10^{-7}
2.5	0.150000665164591	6.6×10^{-7}
2.75	0.150000808303685	8.0×10^{-7}
3.0	0.150000949189142	9.4×10^{-7}
3.25	0.150001087644665	1.0×10^{-6}
3.5	0.150001223391692	1.2×10^{-6}
3.75	0.150001355979046	1.3×10^{-6}
4.0	0.150001484661131	1.4×10^{-6}
4.25	0.150001608185628	1.6×10^{-6}
4.5	0.150001724425047	1.7×10^{-6}
4.75	0.150001829737164	1.8×10^{-6}
5.0	0.150001917853050	1.9×10^{-6}

$r = 0.05$, $q = 0.03$, and $T = 0.025$. We will use $\sigma = 0.15$ and $\sigma = 0.3$ as initial seeds to generate the corresponding option prices. All simulations will be performed in MATLAB using a European call option (with and without jumps). The Mellin transform will be computed using the adaptive Gauss–Kronrod quadrature scheme available in MATLAB.

9.4.1 *Theoretical data*

For the theoretical data, (4.3) is used to generate 200 European call option prices with lognormal jumps for $K \in [1.0 \times 10^{-6}, 8S_0]$. The associated lognormal parameters are chosen to be $\lambda = 0.10$, $\mu_Y = -0.90$, and $\sigma_Y = 0.45$. To illustrate the consistency of the algorithm, several ξ values are selected for the Mellin transform. The domain chosen is $\xi \in [1.0, 5.0]$ in discrete increments of 0.25. Tables 1 and 2 show the numerical approximations for σ against the true values.

For the theoretical option prices, the implied volatility estimations for $\sigma = 0.15$ and $\sigma = 0.3$ prove to be quite accurate with errors in the order of 10^{-7}–10^{-6}. The error remains relatively consistent for all ξ in the allocated domain, which further highlights the precision of the algorithm. It can be argued for $\sigma = 0.15$ that the absolute error is increasing as ξ increases; however, this is primarily linked to approximation errors since the Mellin transform is computed numerically.

Table 2. *Implied volatility estimations and errors for different* ξ *when* $\sigma = 0.3$ *using pure theoretical option data from* (4.3). *Average CPU time is given in seconds*

	Implied volatility estimation for $\sigma = 0.3$ Avg. CPU time: 0.1 s	
ξ	Estimated σ	Absolute error
1.0	0.300000812537166	8.1×10^{-7}
1.25	0.300000602264498	6.0×10^{-7}
1.5	0.300001642090901	1.6×10^{-6}
1.75	0.300000647148822	6.4×10^{-7}
2.0	0.300000660419985	6.6×10^{-7}
2.25	0.300000671406796	6.7×10^{-7}
2.5	0.300000680367558	6.8×10^{-7}
2.75	0.300000687348957	6.8×10^{-7}
3.0	0.300000692310426	6.9×10^{-7}
3.25	0.300000695116983	6.9×10^{-7}
3.5	0.300000695475283	6.9×10^{-7}
3.75	0.300000692830239	6.9×10^{-7}
4.0	0.300000686190937	6.8×10^{-7}
4.25	0.300000329275066	3.2×10^{-7}
4.5	0.300000337379160	3.3×10^{-7}
4.75	0.300000329776874	3.2×10^{-7}
5.0	0.300000298131713	2.9×10^{-7}

9.4.2 *Pseudo-market data*

The pseudo-market option prices are computed using (9.1) with 20 discrete values of $K \in [5, 25]$, and then incorporates (9.2) to extrapolate and provide continuity to the data. Although we are considering a scenario with no jumps (i.e., $\lambda = 0$), a similar procedure may be applied in the case of jumps as seen in the previous section using pure theoretical data. Note that the discrete domain for K will need to be adjusted accordingly if S_0 changes. Tables 3 and 4 list the results for the implied volatility estimation.

Once again, the results are quite satisfactory but the overall absolute error has increased in order of magnitude in comparison to the estimations yielded by the purely theoretical dataset. This is mainly attributed to the extrapolating functions in (9.2). Whilst it maintains the monotonicity of the option profile versus the strike price (e.g., monotonically decreasing for a European call against strike), the main source of error lies within the "tail" function (i.e., the approximation for the option price as $K \to \infty$). This will be elaborated upon in the discussion.

9.4.3 *Comparison to other methods*

We will now give a comparison of (8.4) against two other formulas for implied volatility estimation. We first denote v^{call} to be observed European call price that is required to

Table 3. *Implied volatility estimations and errors for different ξ when $\sigma = 0.15$ using* (9.1) *to generate pseudo-market data. Average CPU time is given in seconds*

Implied volatility estimation for $\sigma = 0.15$ Avg. CPU time: 0.002 s		
ξ	Estimated σ	Absolute error
1.0	0.149998728439811	1.2×10^{-6}
1.25	0.150027834813882	2.7×10^{-5}
1.5	0.150054557535829	5.4×10^{-5}
1.75	0.150080609531067	8.0×10^{-5}
2.0	0.150106032110318	1.0×10^{-4}
2.25	0.150130843920878	1.3×10^{-4}
2.5	0.150155063080464	1.5×10^{-4}
2.75	0.150178294968951	1.7×10^{-4}
3.0	0.150201382384214	2.0×10^{-4}
3.25	0.150223926634414	2.2×10^{-4}
3.5	0.150245946705591	2.4×10^{-4}
3.75	0.150267456617342	2.6×10^{-4}
4.0	0.150288471381866	2.8×10^{-4}
4.25	0.150309005551112	3.0×10^{-4}
4.5	0.150329021452815	3.2×10^{-4}
4.75	0.150348633294877	3.4×10^{-4}
5.0	0.150367805351298	3.6×10^{-4}

Table 4. *Implied volatility estimations and errors for different ξ when $\sigma = 0.3$ using* (9.1) *to generate pseudo-market data. Average CPU time is given in seconds*

Implied volatility estimation for $\sigma = 0.3$ Avg. CPU time: 0.002 s		
ξ	Estimated σ	Absolute error
1.0	0.300031895572929	3.1×10^{-5}
1.25	0.300045257605552	4.5×10^{-5}
1.5	0.300058090390309	5.8×10^{-5}
1.75	0.300071242016552	7.1×10^{-5}
2.0	0.300085040710682	8.5×10^{-5}
2.25	0.300099577530501	9.9×10^{-5}
2.5	0.300115017586394	1.1×10^{-4}
2.75	0.300131558128381	1.3×10^{-4}
3.0	0.300149410201347	1.4×10^{-4}
3.25	0.300168805884818	1.6×10^{-4}
3.5	0.300190000681277	1.9×10^{-4}
3.75	0.300213193710094	2.1×10^{-4}
4.0	0.300238860932383	2.3×10^{-4}
4.25	0.300267261118478	2.6×10^{-4}
4.5	0.300298771264214	2.9×10^{-4}
4.75	0.300333806724048	3.3×10^{-4}
5.0	0.300372824834949	3.7×10^{-4}

compute the implied volatility. We will use the result by Brenner and Subrahmanyam [8]

$$\sigma \approx \frac{v^{\text{call}}}{S_0}\sqrt{\frac{2\pi}{T}}, \tag{9.3}$$

Corrado and Miller [21]

$$\sigma \approx \frac{1}{S_0 + K}\sqrt{\frac{2\pi}{T}}\left(v^{\text{call}} - \frac{S_0 - K}{2} + \sqrt{\left(v^{\text{call}} - \frac{(S_0 - K)}{2}\right)^2 - \frac{(S_0 - K)^2}{\pi}}\right), \tag{9.4}$$

and a standard Newton's method approach [33]

$$\sigma_{n+1} = \sigma_n - \frac{F(\sigma_n)}{F'(\sigma_n)}, \tag{9.5}$$

where F is the difference between value of the European call (2.4) at $\sigma = \sigma_n$ and the observed price v^{call}, and F' is the *vega* of the European call: the partial derivative of (2.4) with respect to σ. The analysis will be conducted with 20 discrete strike values $K \in [5, 25]$ and the aforementioned parameters values used to compute the call prices using (9.1). Table 5 gives the approximations for $\sigma = 0.30$.

It is immediately clear that formulas (9.3) and (9.4) are heavily dependent on the value of K. Brenner and Subrahmanyam's formula yields plausible approximations when the option is at-the-money which is exemplified in Table 5. The Corrado–Miller formula appears to allow more flexibility in the option's moneyness; however, the Corrado–Miller formula possesses the possibility for complex solutions as seen by the numerical results. Both outcomes coincide with the details provided in the introduction; (9.3) is only valid for options at-the-money and (9.4) is not restricted to at-the-money options but may generate complex values depending on the moneyness or parameter values (see Chambers and Nawalkha [12]). Newton's method (9.5) proved to the most reliable of the three schemes. But the focus of the article is more on implied volatility estimation in the scenario of a jump-diffusion model where Newton's method (or a standard root-finding scheme) would not be a desirable approach.

10 Discussion and conclusion

The first key result presented in this article was the alternative pricing formula (3.11) for options in a jump-diffusion model for the underlying asset. There are several advantages to this new formula. First, (3.11) is applicable to any general payoff and type of jump. Merton's formula is applicable for when the jump is drawn from a particular distribution, namely, the lognormal distribution. On the other hand, Frontczak's formula is also applicable to any general payoff and type of jump as in (3.11), but a complex integral has to be evaluated in Frontczak's result and reduces to (2.15) for a given payoff and jump. However, the integrals in (3.11) are all real since the Mellin transform inversion has been performed unlike in [29] where the inversion was completed via a complex integral.

Equation (3.11) conveniently represents the standard European option value with shifted parameters and a function which mimics the discontinuous jumps. If multiple

Table 5. *Comparison of implied volatility formulas for* $\sigma = 0.3$

Implied volatility comparison for $\sigma = 0.30$

K	Equation (9.3) BS formula Avg. CPU time: 0.00014 s	Equation (9.4) CM formula Avg. CPU time: 0.00020 s	Equation (9.5) Newton's method Avg. CPU time: 0.00016 s
5.0	3.3255	1.2408–0.6786i	0.2998
6.0	2.9954	1.0653–0.5784i	0.3000
7.0	2.6653	0.9058–0.4873i	0.3000
8.0	2.3353	0.7601–0.4040i	0.3000
9.0	2.0053	0.6266–0.3276i	0.3000
10.0	1.6757	0.5041–0.2567i	0.3000
11.0	1.3492	0.3927–0.1874i	0.3000
12.0	1.0336	0.2957–0.1064i	0.3000
13.0	0.7440	0.3054	0.3000
14.0	0.4984	0.3123	0.3000
15.0	0.3093	0.3093	0.3000
16.0	0.1778	0.3066	0.3000
17.0	0.0949	0.2972	0.3000
18.0	0.0474	0.2494–0.0625i	0.3000
19.0	0.0222	0.3047–0.1337i	0.3000
20.0	0.0099	0.3623–0.1789i	0.3000
21.0	0.0042	0.4195–0.2150i	0.3000
22.0	0.0017	0.4749–0.2466i	0.3000
23.0	0.0007	0.5280–0.2753i	0.3000
24.0	0.0003	0.5785–0.3022i	0.3000
25.0	0.0001	0.6267–0.3275i	0.3000

types of options are to be priced or if the jump dynamics were changed, (3.11) is in a form whereby any alterations can be easily incorporated since the jump function is completely separated from any other component of the pricing formula. Additionally, the general pricing formula in [29] is expressed as a complex integral with the jump dynamics embedded across multiple terms. In practice, this would be unfavourable as computing complex integrals is relatively expensive when compared to real integrals.

Examples were given for when the jumps have distributions that are lognormal, double exponential, and gamma. For lognormal jumps, both [47] and [29] also derived similar results; Merton's classical formula (2.15) exploited the properties of expectations, whereas Frontczak's formula computed the Mellin inverse via algebraic manipulation and the Mellin convolution. Equation (4.3) was derived using convolution and direct inversion that bypasses the complex integral evaluation employed by Frontczak. One approach used (3.13) to compute the terms recursively whilst the other relied on the properties of the exponential function which simplified the algebra tremendously. It should be emphasised that in (4.3), having the jump term isolated from the remainder of the formula is convenient since it allows for the pricing process to be modular. That is, one can calculate the necessary jump term before determining the option price at the specified parameter values. Not only is the separation preferable for computation, it reiterates the notion of interchangeability: if the jump dynamics were to change, (3.11) together with (3.13) would be able to accommodate this efficiently. Although (3.13) is recursive in the general case, one may obtain some insight into what F_n is by carefully analysing the distribution of the jump. This could ultimately lead to easier calculations. Consequently, it is possible to derive pricing formulas for any types of jumps as shown with the double exponential distribution in (5.1) and the gamma distribution in (6.1). The key is being able to calculate each term in the sequence $F_1, F_2, \ldots$. If the integrals associated with F_n are too complicated to solve analytically, one may resort to numeric to yield approximate solutions to (3.13). For the double exponential and gamma distributions, we kept the jump terms in a recursive form to demonstrate the capability of (3.13). However, it is not clear how to obtain a non-recursive form for F_n for double exponentially and gamma distributed jumps as we did for the lognormal jumps. There is also interest in finding an exact solution for Kou's double exponentially distributed jumps [39, 40] using (3.11) and (3.13) since recent empirical studies suggest a better model for the asset process involves the jumps following a double exponential distribution. In particular, it may be of interest to see whether or not a Mellin transform route would generate a more elegant and simple solution in lieu of Kou's original solution, which involves the computation of quite complicated Hh functions (see [1, pp. 691]). There is also the possibility to extend (3.11) to price American options in jump-diffusion models; this is current work in progress.

The second main result of this article was the implied volatility formula (8.4) for options under a jump-diffusion framework. Many estimators already exist for the implied volatility, but none of these schemes accommodates the possibility of jumps in the asset price. It should be highlighted that (8.4) also works in the absence of jumps by setting $\lambda = 0$. Both sets of implied volatility results for theoretical and pseudo-market data produced accurate estimations for the true value of σ as shown in the numerical simulations.

For the theoretical data, the absolute errors remain in the order of 10^{-7}–10^{-6}. The low order of magnitude for the errors is not surprising as the options price profile exhibits nice continuity as all the values are evenly distributed between 10^{-6} and $8S_0$. It was mentioned that the error appeared to be marginally increasing for $\sigma = 0.15$ for larger values of ξ, but this is associated with the numeric as the Mellin transform was performed via numerical integration.

For the pseudo-market data, the implied volatility estimation possessed higher orders for the absolute error ranging from 10^{-5} to 10^{-4}. The cause is undoubtedly the extrapolation functions in (9.2). Both functions manage to capture the profile and monotonicity of the option prices, but the main problem is their failure to replicate how the standard normal CDF behaves. Although the logistic approximation in [7] is deemed to be one of the most accurate, further testing has shown that the approximation (9.2) as $K \to \infty$ does not actually decay at the same rate as it does in the Black–Scholes formula (9.1). Hence, it can be inferred that the relatively larger errors are attributed to this subtle artefact in the extrapolation.

A brief comparison against three other methods was used to gauge the validity and accuracy of (8.4) under the assumption of constant volatility. Both results by Brenner–Subrahmanyan (9.3) and Corrado–Miller (9.4) provided acceptable implied volatility estimations for particular strike prices, but continued to exemplify the drawbacks that are inherent to their respective models. Brenner and Subrahmanyan's formula is effectively feasible only when the option is at-the-money; Corrado and Miller's formula permits marginal freedom in the option's moneyness but suffers from the potential of complex solutions which can be unknown *a priori*. Newton's method proved to give the most favourable results out of these three numerical schemes, but these are only applicable for a standard diffusion case. A major advantage of (8.4) is its independence from the option moneyness condition, although we are assuming we possess different option prices for different strikes (which are readily available anyway). Although one could argue that (8.4) has a slower execution time compared to the other three methods we used to benchmark against, we justify our scheme's versatility at being able to counteract the flaws of (9.3) and (9.4), as well to illustrate the use of the extrapolating functions. We did not demonstrate the extrapolation procedure for the jump-diffusion case for simplicity, but the extension should be straightforward.

The implied volatility result could also be potentially modified for American options in both standard diffusion and jump-diffusion frameworks. The main challenge would be adapting to the moving boundary problem that exists in the asset price due to the possibility of exercising the option before the expiry date.

To summarise, we have devised and introduced a new scheme for option pricing when the asset possesses jump-diffusion dynamics. In particular, we were able to formulate this new model to fit any type of jump. The consequent result can be computed recursively within an infinite sum. This was achieved by implementing the properties of the Mellin transform and the Black–Scholes kernel. We also highlighted how the recursion is handled when the jump is extracted from a lognormal distribution and also provided some insight into how the recursion can be computed when the jump is double exponentially and gamma distributed. Additionally, we derived a Dupire-like PIDE for options in a jump-diffusion environment and ultimately an implied volatility formula within this framework. Numerical approximations for implied volatility with and without jumps rival in accuracy and robustness to two well-known implied volatility results. The analysis and approach once again incorporated the Mellin transform.

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Appendix A Proof of Lemma 1

To begin, we let I equal

$$I = \int_0^\infty \frac{1}{z} N'(a_1 \log z + b_1) N'\left(a_2 \log\left(\frac{1}{z}\right) + b_2\right) \mathrm{d}z.$$

Setting $\rho = a_2 \log\left(1/z\right) + b_2$, the integral becomes

$$I = \frac{1}{a_2} \int_{-\infty}^\infty N'\left(\frac{a_1 b_2}{a_2} + b_1 - \frac{a_1}{a_2}\rho\right) N'(\rho)\,\mathrm{d}\rho = \frac{1}{a_2}\int_{-\infty}^\infty N'\left(\gamma - \frac{a_1}{a_2}\rho\right) N'(\rho)\,\mathrm{d}\rho,$$

where we write $\gamma = a_1 b_2/a_2 + b_1$. Using (2.17) to replace the N' terms, we get

$$\begin{aligned} I &= \frac{1}{2\pi a_2}\int_{-\infty}^\infty e^{-((\gamma - a_1\rho/a_2)^2 + \rho^2)/2}\,\mathrm{d}\rho \\ &= \frac{e^{-\gamma^2/2} e^{(a_1\gamma)^2/(2(a_1^2+a_2^2))}}{2\pi a_2} \int_{-\infty}^\infty e^{-\left(\left((a_1^2+a_2^2)/a_2^2\right)\left(\rho - (a_1 a_2 \gamma)/(a_1^2+a_2^2)\right)^2\right)/2}\,\mathrm{d}\rho, \end{aligned}$$

Now setting

$$\omega = \sqrt{\frac{a_1^2 + a_2^2}{a_2^2}}\left(u - \frac{a_1 a_2 \gamma}{a_1^2 + a_2^2}\right),$$

we obtain

$$I = \frac{e^{-\gamma^2/2} e^{(a_1\gamma)^2/(2(a_1^2+a_2^2))}}{2\pi\sqrt{a_1^2+a_2^2}} \int_{-\infty}^\infty e^{-\omega^2/2}\,\mathrm{d}\omega.$$

To finish off, we use (2.17) to replace the exponential term with N', integrate, and arrive at

$$I = \frac{1}{\sqrt{a_1^2+a_2^2}} N'\left(\frac{\gamma a_2}{\sqrt{a_1^2+a_2^2}}\right) = \frac{1}{\sqrt{a_1^2+a_2^2}} N'\left(\frac{a_1 b_2 + a_2 b_1}{\sqrt{a_1^2+a_2^2}}\right),$$

where we substituted the expression for γ back in. This completes the proof.

Appendix B Proof of Lemma 2

The proof of the first expression is simple. We denote the left-hand side integral to be I. By setting $u = a_1 \log(1/y) + b_1$, we get

$$I = \frac{1}{a_1}\int_{a_1 \log(1/b) + b_1}^{a_1 \log(1/a) + b_1} N'(u)\,\mathrm{d}u = \frac{1}{a_1}\Big(N(a_1 \log(1/a) + b_1) - N(a_1 \log(1/b) + b_1)\Big).$$

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For the second expression, we perform the exact same step as we did for the first expression to begin with. We first obtain

$$I = \frac{e^{b_1/a_1}}{a_1} \int_{a_1 \log(1/b)+b_1}^{a_1 \log(1/a)+b_1} N'(u) e^{-u/a_1} \, \mathrm{d}u = \frac{e^{b_1/a_1}}{a_1\sqrt{2\pi}} \int_{a_1 \log(1/b)+b_1}^{a_1 \log(1/a)+b_1} e^{-u^2/2-u/a_1} \, \mathrm{d}u,$$

where we used (2.17) to convert N' to its integral form. Looking at the power of the exponential term, we complete the square and arrive at

$$I = \frac{e^{b_1/a_1+1/(2a_1^2)}}{a_1\sqrt{2\pi}} \int_{a_1 \log(1/b)+b_1}^{a_1 \log(1/a)+b_1} e^{-(u+1/a_1)^2/2} \, \mathrm{d}u = \frac{e^{b_1/a_1+1/(2a_1^2)}}{a_1} \int_{a_1 \log(1/b)+b_1}^{a_1 \log(1/a)+b_1} N'\left(u+1/a_1\right) \, \mathrm{d}u$$
$$= \frac{e^{b_1/a_1+1/(2a_1^2)}}{a_1} \bigg(N(a_1 \log(1/a) + b_1 + 1/a_1) - N(a_1 \log(1/b) + b_1 + 1/a_1) \bigg).$$

This equals the right-hand side of the second expression and concludes the proof.

Appendix C Proof of Lemma 3

To prove the result, the Mellin transform will need to be evaluated directly. Using the definition of the Mellin transform and (2.17), the left-hand side becomes

$$\mathscr{M}\left\{N'(a \log x + b)\right\} = \int_0^\infty x^{\xi-1} N'(a \log x + b) \, \mathrm{d}x = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{\xi-1} e^{-(a \log x + b)^2/2} \, \mathrm{d}x.$$

Using $\omega = a \log x + b$, the integral becomes

$$\mathscr{M}\left\{N'(a \log x + b)\right\} = \frac{e^{-b\xi/a}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\omega^2/2+\omega\xi/a} \, \mathrm{d}\omega.$$

Upon completing the square inside the exponential with respect to ω, this simplifies to

$$\mathscr{M}\left\{N'(a \log x + b)\right\} = \frac{e^{-b\xi/a} e^{\xi^2/(2a^2)}}{a\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\omega-\xi/a)^2/2} \, \mathrm{d}\omega = \frac{e^{-b\xi/a} e^{\xi^2/(2a^2)}}{a},$$

where we used the standard identity $\int_{-\infty}^{\infty} e^{-(\omega-y)^2/2} \, \mathrm{d}\omega = \sqrt{2\pi}$ for $y \in \mathbb{R}$. This completes the proof.