ON A CLASS OF INEQUALITIES

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1. Introduction

First consider some familiar results, the inequality of the arithmetic and geometric means is:

$$\int_0^1 f(x)\,\mathrm{d}x \geqq \exp \int_0^1 \log f(x)\,\mathrm{d}x \quad \text{for all } f > 0.$$

Kantorovich's inequality (reference [1]) asserts that if $0 < A \leqq f(x) \leqq B$ then:

$$\int_0^1 f\,\mathrm{d}x \int_0^1 1/f\,\mathrm{d}x \leqq (A+B)^2/(4AB)$$

The Cauchy-Schwarz inequality is:

$$\left|\int f(x)g(x)\,\mathrm{d}x\right| \leqq \left(\int f^2\,\mathrm{d}x\right)^{\frac{1}{2}} \left(\int g^2\,\mathrm{d}x\right)^{\frac{1}{2}}$$

This paper discusses a certain class of inequalities which includes the three above.

Three theorems are proved which apply to any inequality of this class; then follow some examples. They are mainly to show how the general theory helps in the finding of inequalities, but the result of Example 1 seems worth reporting for its own sake.

Roughly, the conclusion is that the problem of finding the best possible inequalities connecting any k such integrals is the problem of finding a convex hull in k-dimensional space.

2. General results

Let $f_1, f_2, \cdots f_r$ be measurable functions on a set E, which is either the unit interval $(0, 1)$ or the real line $(-\infty, \infty)$, their values being restricted to sets $S_1, S_2, \cdots S_r$ respectively. For each $i = 1, 2, \cdots k$ let ϕ_i be a real continuous function on the product space $S_1 \times S_2 \times \cdots \times S_r$, and let $u_i = \int_E \phi_i(f_1, f_2, \cdots f_r)\,\mathrm{d}x$. We restrict ourselves to functions such that each u_i is finite.

For each family of functions $f_1, \cdots f_r$, there is a point $(u_1, \cdots u_k)$ in k-dimensional space. Let C be the set of all such points.

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THEOREM 1. *The set C is convex.*

PROOF. Take any two points $\boldsymbol{u} = (u_1, \cdots u_k)$ and $\boldsymbol{u}^* = (u_1^*, \cdots u_k^*)$ in C, and let λ and μ be positive numbers with sum equal to 1.

Let $f_1, \cdots f_r$ be a family of functions that gives rise to the point $\boldsymbol{u}$ in C, and similarly take any $f_1^*, \cdots f_r^*$. Define a new family of functions as follows, if n is an integer and $0 \leqq \theta < 1$ then:

$$f_i^{**}(n+\lambda\theta) = f_i(n+\theta)$$
$$f_i^{**}(n+\lambda+\mu\theta) = f_i^*(n+\theta)$$

Corresponding to this family of functions is the point given by $u_s^{**} = \int \varphi_s(f_1^{**}, \cdots f_r^{**})dx$, and it is clear that $u_s^{**} = \lambda u_s + \mu u_s^*$ so that the point $\lambda\boldsymbol{u} + \mu\boldsymbol{u}^*$ is in C, which proves convexity.

The set C is not in general closed (see Example 3 below) but its closure $\bar{C}$ is convex. In the next theorem we use the fact that a closed convex set in finite-dimensional space is the intersection of all the closed half spaces that contain it (see for example Bonnesen and Fenchel (2) page 5).

THEOREM 2 (*a*) *If the interval of integration, E, is* (0, 1) *then $\bar{C}$ is the intersection of all the half spaces*:

$$\{\boldsymbol{u};\, a_1 u_1 + \cdots + a_k u_k \geqq a_0\}$$

that have coefficients such that:

$$\sum a_s \varphi_s(x_1, \cdots x_r) \geqq a_0 \quad \textit{for all} \quad \boldsymbol{x} \textit{ in } S_1 \times \cdots \times S_r.$$

(*b*) *If E is $(-\infty, \infty)$ then the same result holds except that a_0 is to be put equal to zero.*

PROOF OF (*a*) First it follows by integration that every point of C is in the intersection of these half-spaces, and therefore so is every point of $\bar{C}$ (because the intersection is closed).

To prove the converse, take any point $(y_1, \cdots y_k)$ not in $\bar{C}$; since $\bar{C}$ is a closed convex set there must be coefficients $b_0, b_1, \cdots b_k$ such that both:

$$b_1 y_1 + \cdots b_k y_k < b_0$$

and for all $\boldsymbol{z}$ in $\bar{C}$ and a fortiori for all $\boldsymbol{z}$ in C:

$$b_1 z_1 + \cdots + b_k z_k \geqq b_0.$$

From this last inequality, taking the case of functions $f_1, \cdots f_r$, each constant, it follows for all $h_1, \cdots h_r$ in $S_1 \times \cdots \times S_r$ that

$$\sum b_s \varphi_s(h_1, \cdots h_r) \geqq b_0$$

Therefore the half-space defined by these coefficients is one of those

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mentioned in the statement of the theorem. This proves that the point $y_1, \cdots y_k$ (which was just an arbitrary point not in $\bar{C}$) is not in the intersection of the set of half-spaces, so that part (a) is established.

PROOF OF *(b)* In this case the set C is a cone, in the sense that with any point $\boldsymbol{u}$ it also contains $t\boldsymbol{u}$ for any $t > 0$, because

$$\int_0^\infty \varphi(f_1(x/t), \cdots f_r(x/t))\mathrm{d}x = t\int_0^\infty \varphi(f_1(x) \cdots f_r(x))\mathrm{d}x$$

Therefore $\bar{C}$ is also a cone. The proof of case (a) may then be taken over with two modifications, that the coefficients a_0 and b_0 are zero and that we use the theorem that a closed convex cone with vertex at the origin is the intersection of all the closed homogeneous half-spaces that contain it.

LEMMA. *Given any sequence $f_1, f_2 \cdots$ of measurable functions on* $(0, 1)$ *with values in a compact set S, there exists a subsequence $f^{(1)}, f^{(2)}, \cdots$ and a function f (also on* $(0, 1)$ *with values in S) such that if φ is continuous then*

$$\int_0^1 \varphi(f^{(n)}(x))\,\mathrm{d}x \to \int_0^1 \varphi(f(x))\,\mathrm{d}x.$$

PROOF. To any function f on $(0, 1)$ with values in S corresponds another:

$$f^*(x) = \inf\{y; \mu\{t; f(t) < y\} > x\}$$

which is increasing and equimeasurable, so that $\int \varphi(f)\,\mathrm{d}x = \int \varphi(f^*)\,\mathrm{d}x$, (see chapter 10 of [4]).

Therefore we may assume without loss of generality that the given functions are all increasing.

Let $r(1), r(2), \cdots$ be the rationals in the unit interval. From the given sequence of functions choose a subsequence such that the values at $r(1)$ converge. Denote the first member by $f^{(1)}$, and from the others choose a sub-sub-sequence for which the values at $r(2)$ converge. Denote the first member by $f^{(2)}$, and so on. The function $f(x)$ is defined for rational x by $f(x) = \lim f^{(n)}(x)$ and for almost all irrational x by the condition that it is monotonic. It remains to prove convergence of the integrals; first we note that for a uniformly bounded sequence of increasing functions, convergence at the rational points implies convergence p.p. and therefore convergence of the integrals. Since any continuous φ can be uniformly approximated by the difference of two increasing functions, the lemma is established.

The following result then becomes clear.

THEOREM 3. *If $r = 1$, the set E is* $(0, 1)$, *the functions $\varphi_1, \cdots \varphi_k$ are continuous and the set S of permitted values for the functions f is compact, then the set C is closed.*

It is sometimes of interest to obtain strict inequalities, that is to distinguish the interior points from the boundary points of C. If a point

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$(u_1^*, \cdots u_k^*)$ of $\bar{C}$ is on the boundary then it is on the boundary of some half-space *containing* C, *let it be that defined by:*

$$a_1 u_1 + \cdots + a_k u_k \geqq a_0.$$

Then the function $\sum a_s \varphi_s(f_1(x), \cdots f_r(x)) - a_0$ must be $\geqq 0$ on E, and the families of functions, if any, that give the point $(u_1^*, \cdots u_k^*)$ are those whose values are the zeros of $\sum a_s \varphi_s - a_0$ (as a function of the r variables $f_1, \cdots f_r$).

These considerations generally make it easy to find the cases where the inequalities that we seek cannot be improved to strict inequalities; also they enable us to find what points of $\bar{C}$ are not in C.

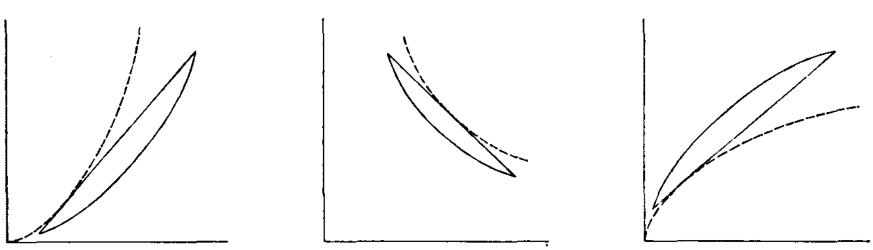
3. Some illustrations

Example 1. Take real numbers $r < s$ and $0 < A < B$. For any measurable function $f(x)$ that satisfies $A \leqq f(x) \leqq B$ in the interval $(0, 1)$, let:

$$U_r(f) = \int_0^1 (f(x))^r \mathrm{d}x \quad \textit{and} \quad U_s(f) = \int_0^1 (f(x))^s \mathrm{d}x.$$

The problem is to find the inequalities that connect U_r with U_s.

Let E be the plane set of all points with coordinates (f^r, f^s) for f between A and B. It is an arc of a curve, including the two end-points, (A^r, A^s) and (B^r, B^s). The theorems above show that the set C of all (U_r, U_s) is a closed convex set and is the intersection of all the closed half-spaces that contain E, that is to say C is the convex hull of E (the convex hull of E is closed because E is compact).



$0 < r < s$ $\qquad$ $r < 0 < s$ $\qquad$ $r < s < 0$

The three illustrations are typical of the three possible combinations of sign of r and s. They show the arc E and the chord joining its end-points, which together enclose the set C. The dotted curves will be explained below. In each case the arc is entirely on one side of the chord, and the curvature of the arc does not change sign.

Therefore the relevant set of linear inequalities will consist of a single

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one corresponding to the chord, and a one-parameter family corresponding to the tangents to the curve.

An analytic, rather than geometrical, treatment of the problem is as follows. It covers each of the three possible combinations of sign of r and s.

$$\frac{f^r}{B^r-A^r}-\frac{f^s}{B^s-A^s}-\frac{A^rB^s-A^sB^r}{(B^r-A^r)(B^s-A^s)}$$

as a function of the strictly positive real variable f, has just one stationary value, is zero at A and B, and is positive between A and B. By regarding f as a function of x and integrating, we obtain the inequality:

$$\frac{U_r}{B^r-A^r}-\frac{U_s}{B^s-A^s}\geqq\frac{A^rB^s-A^sB^r}{(B^r-A^r)(B^s-A^s)}$$

which gives the chord bounding the set C.

Secondly we see that if $t>0$ then

$$(f/t)^r/r-(f/t)^s/s-1/r+1/s$$

as a function of f on $(0, \infty)$ is strictly negative except for a zero at t. Therefore by integration we have the inequalities

$$t^{-r}U_r/r-t^{-s}U_s/s\leqq 1/r-1/s$$

which correspond to the tangents to the curve E. To find the intersection of these half-planes we take advantage of the fact that for each one we know the point of contact of the tangent with the curve, in fact it is the case of equality, given by $f(x)=t$, so that we may put $t=U_r^{1/r}$, which gives $U_s^{1/s}\geqq U_r^{1/r}$.

The two inequalities that determine C are therefore:

$$\frac{U_r}{B^r-A^r}-\frac{U_s}{B^s-A^s}\geqq\frac{A^rB^s-A^sB^r}{(B^r-A^r)(B^s-A^s)}$$

and $U_r^{1/r}\leqq U_s^{1/s}$.

The generalisation of Kantorovich's inequality by G. T. Cargo and O. Shisha is that the lower bound for $U_r^{1/r}U_s^{-1/s}$ is

$$\left(\frac{A^rB^r-A^sB^r}{s-r}\right)^{1/r-1/s}\left(\frac{B^r-A^r}{r}\right)^{1/s}\left(\frac{B^s-A^s}{s}\right)^{-1/r}.$$

Their result may be deduced from the first of the two inequalities above, for we have $U_r^{1/r}\geqq a$ certain function of U_s, and the minimum may be found by differentiation. The result of Cargo and Shisha may be illustrated in the diagrams above as being that the set C is on the convex side of the dotted curve, which touches the chord of the arc E.

Of the two inqualities above that determine the set C, the first can be

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strengthened to a strict inequality unless f takes p.p. only the values A and B, the second can be so strengthened unless f is constant p.p.

Example 2. For functions on $(-\infty, \infty)$ let $U_n = \int |f|^n dx$. To find the possible values of (U_1, U_2, U_4), we may take $f \geqq 0$. From the inequality:

$$f(f+2t)(f-t)^2 \geqq 0 \quad (\text{all } t > 0)$$

we obtain $U_4 \geqq 3t^2U_2-2t^3U_1$ in which there is equality if f takes only the values 0 and t, in which case $t = U_2/U_1$. Therefore:

$$U_4U_1^2 \geqq U_2^3$$

and it is clear that the set C is the set that is specified by this inequality. (Theorem 17 of reference [4]).

Example 3. Modifying the previous example by taking the interval of integration E to be $(0, 1)$, the inequalities specifying $\bar{C}$ are found to be:

$$U_4U_1^2 \geqq U_2^3 \quad \text{and} \quad U_2 \geqq U_1^2.$$

The point $U_1 = U_2 = 1$, $U_4 = 2$ is in $\bar{C}$ but not in C. To show this, take a sequence of functions of which the nth has the value n on an interval of length $1/n^4$ and the value 1 elsewhere, which gives a sequence of points of C converging to $(1, 1, 2)$; but any function for which $U_1 = U_2 = 1$ must be equal to 1 p.p., so that $U_4 = 1$, and $(1, 1, 2)$ therefore is not in C.

Example 4. Modify the previous example by taking $S = [0, B]$. A further inequality is obtained by integrating

$$(f-t)^2(f-B)(f+2t+B) \leqq 0 \quad (t \text{ in } S).$$

The set C is found to be given by:

$$U_2^3 \leqq U_4U_1^2$$
$$(B^2-U_2)(BU_1-U_2)^2 \leqq (B^2U_2-U_4)(B-U_1)^2$$
$$U_2/B \leqq U_1 \leqq B$$

In fact the boundary of C consists of portions of the cubic surfaces given by the first and second inequalities above, each portion being bounded on one side by the line $BU_1 = U_2 = U_4/B^2$ and on the other by the curve $U_1 = U_2^{1/2} = U_4^{1/4}$.

The third inequality above serves the purpose of excluding another region where the first and second inequalities are satisfied; for the two cubic surfaces also intersect in another curve, and if $B = 3$ the point $(4, 8, 42)$ satisfies the first two inequalities but is not in C.

Example 5. For functions $f > 0$ on $(0, 1)$ put $\varphi_1(f) = f$ and $\varphi_2(f) = \log f$.

Using $\log f \leqq \log t - 1 + f/t$, which holds for all $t > 0$ with equality when $f = t$, we obtain the inequality of the arithmetic and geometric

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means:

$$\exp \int_0^1 \log f \, dx \leqq \int_0^1 f \, dx.$$

This example is like Theorem 204 of [4] in deriving the inequality from the convexity of minus the logarithm function.

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