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THE MONOTONICITY AND LOG-BEHAVIOUR OF SOME FUNCTIONS RELATED TO THE EULER GAMMA FUNCTION

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Abstract The aim of this paper is to develop analytic techniques to deal with the monotonicity of certain combinatorial sequences. On the one hand, a criterion for the monotonicity of the function $\sqrt[x]{f(x)}$ is given, which is a continuous analogue of a result of Wang and Zhu. On the other hand, the log-behaviour of the functions

$$\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)} \quad \text{and} \quad F(x) = \sqrt[x]{\frac{\Gamma(ax+b+1)}{\Gamma(cx+d+1)\Gamma(ex+f+1)}}$$

is considered, where $\zeta(x)$ and $\Gamma(x)$ are the Riemann zeta function and the Euler Gamma function, respectively. Consequently, the strict log-concavities of the function $\theta(x)$ (a conjecture of Chen *et al.*) and $\{\sqrt[n]{z_n}\}$ for some combinatorial sequences (including the Bernoulli numbers, the tangent numbers, the Catalan numbers, the Fuss–Catalan numbers, and the binomial coefficients $\binom{2n}{n}$, $\binom{3n}{n}$, $\binom{4n}{n}$, $\binom{5n}{n}$, $\binom{5n}{2n}$) are demonstrated. In particular, this contains some results of Chen *et al.*, and Luca and Stănică. Finally, by researching the logarithmically complete monotonicity of some functions, the infinite log-monotonicity of the sequence

$$\left\{\frac{(n_0+ia)!}{(k_0+ib)!(\overline{k_0}+i\bar{b})!}\right\}_{i\geqslant 0}$$

is proved. This generalizes two results of Chen *et al.* that both the Catalan numbers $(1/(n+1))\binom{2n}{n}$ and the central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic, and strengthens one result of Su and Wang that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. In addition, the asymptotically infinite log-monotonicity of derangement numbers is showed. In order to research the stronger properties of the above functions $\theta(x)$ and $F(x)$, the logarithmically complete monotonicity of functions

$$\frac{1}{\sqrt[x]{a\zeta(x+b)\Gamma(x+c)}} \quad \text{and} \quad \sqrt[x]{\rho\prod_{i=1}^{n}\frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}}$$

is also obtained, which generalizes the results of Lee and Tepedelenlioğlu, and Qi and Li.

Keywords: monotonicity; log-convexity; log-concavity; completely monotonic functions; infinite log-monotonicity

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1. Introduction

Let $\{z_n\}_{n\geqslant 0}$ be a sequence of positive numbers. It is called *log-concave* (respectively, *log-convex*) if $z_{n-1}z_{n+1} \leqslant z_n^2$ (respectively, $z_{n-1}z_{n+1} \geqslant z_n^2$) for all $n \geqslant 1$. Clearly, the sequence $\{z_n\}_{n\geqslant 0}$ is log-concave (respectively, log-convex) if and only if the sequence $\{z_{n+1}/z_n\}_{n\geqslant 0}$ is decreasing (respectively, increasing). Generally speaking, a sequence will have good behaviour (for example, distribution properties, bounds by inequalities) if it is log-concave or log-convex. In addition, sequences with log-behaviour arise often in combinatorics, algebra, geometry, analysis, probability and statistics and have been extensively investigated (see, for example, [**3**, **10**, **15**, **20**, **23**]).

Motivated by a series of conjectures of Sun [**17**] about the monotonicity of sequences of the forms $\{\sqrt[n]{z_n}\}$, and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, where $\{z_n\}_{n\geqslant 0}$ is a familiar number-theoretic or combinatorial sequence, for example, the Bernoulli numbers, the Fibonacci numbers, the derangement numbers, the tangent numbers, the Euler numbers, the Schröder numbers, the Motzkin numbers, the Domb numbers, and so on. These conjectures have recently been investigated by some researchers (see [**4**, **5**, **8**, **11**, **21**]). The main aim of this paper is to develop some analytic techniques to deal with the monotonicity of $\{\sqrt[n]{z_n}\}$ and $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ (note that the monotonicity of $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$ is equivalent to the log-behaviour of $\{\sqrt[n]{z_n}\}$).

Recently, Wang and Zhu [**21**] observed sufficient conditions that the log-behaviour of $\{z_n\}_{n\geqslant 0}$ implies the monotonicity of $\{\sqrt[n]{z_n}\}_{n\geqslant 1}$. For example, for a positive log-convex sequence $\{z_n\}_{n\geqslant 0}$, if $z_0 \leqslant 1$, then the sequence $\{\sqrt[n]{z_n}\}_{n\geqslant 1}$ is increasing. Using the analytic approach of Chen *et al*. [**5**], the following continuous analogue can be proved, whose proof is given in § 2.

Theorem 1.1. *Let N be a positive number. If $f(x)$ is a positive increasing log-convex function for $x \geqslant N$ and $f(N) \leqslant 1$, then $\sqrt[x]{f(x)}$ is strictly increasing on (N, ∞).*

Remark 1.2. Theorem 1.1 can be applied to the monotonicity of $\{\sqrt[n]{z_n}\}_{n\geqslant 1}$ for some combinatorial sequences $\{z_n\}_{n\geqslant 0}$. Some further examples and applications related to Theorem 1.1 can be found in [**5**].

Thus, one may ask whether there are some analytic techniques to deal with the log-behaviour of $\{\sqrt[n]{z_n}\}_{n\geqslant 1}$. This is another motivation of this paper. In particular, Conjecture 1.4 in the following following example is still open.

Example 1.3. Recall that the classical Bernoulli numbers are defined by

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0, \quad n = 1, 2, \ldots.$$

It is well known that $B_{2n+1} = 0$, $(-1)^{n-1}B_{2n} > 0$ for $n \geqslant 1$ and

$$(-1)^{n-1}B_{2n} = \frac{2(2n)!\zeta(2n)}{(2\pi)^{2n}}$$

(see, for example, [**6**, (6.89)]). In order to show that $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}$ is increasing, Chen *et al*. [**5**] introduced the function $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$, where

$$\zeta(x) = \sum_{n\geqslant 1} \frac{1}{n^x}$$

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is the Riemann zeta function and $\Gamma(x)$ is the Euler Gamma function. Thus,

$$\sqrt[n]{(-1)^{n-1}B_{2n}} = \theta^2(2n)/4\pi^2.$$

They proved that $\theta(x)$ is increasing on $(6,\infty)$. In addition, in order to get the log-concavity of $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geqslant 1}$, they furthermore made the following conjecture.

Conjecture 1.4 (Chen *et al.* [5]). *The function* $\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$ *is log-concave on* $(6,\infty)$.

Using some inequalities of the Riemann zeta function and the Euler Gamma function, Conjecture 1.4 will almost be confirmed in § 3 (see Theorem 3.3). As applications, the results of Luca and Stănică [**11**] on strict log-concavities of $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geqslant 1}$ and $\{\sqrt[n]{T(n)}\}_{n\geqslant 1}$ can be verified, where $T(n)$ are the tangent numbers.

In addition, motivated by the strict log-concavities of

$$\sqrt[n]{\binom{2n}{n}} \quad \text{and} \quad \sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}$$

(see [**4**]), the log-behaviour of the function

$$F(x) = \sqrt[x]{\frac{\Gamma(ax+b+1)}{\Gamma(cx+d+1)\Gamma(ex+f+1)}}$$

is considered (see Theorem 3.6). As consequences, for any positive integers $p \geqslant 2$ and $a > c$, the strict log-concavities of

$$\left\{\sqrt[n]{\frac{1}{(p-1)n+1}\binom{pn}{n}}\right\}_{n\geqslant 2} \quad \text{and} \quad \left\{\sqrt[n]{\binom{an}{cn}}\right\}_{n\geqslant 30}$$

are obtained (see Corollary 3.7). For more examples, the sequences

$$\left\{\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}}\right\}_{n\geqslant 1}, \quad \left\{\sqrt[n]{\binom{2n}{n}}\right\}_{n\geqslant 1}, \quad \left\{\sqrt[n]{\binom{3n}{n}}\right\}_{n\geqslant 1},$$
$$\left\{\sqrt[n]{\binom{4n}{n}}\right\}_{n\geqslant 1}, \quad \left\{\sqrt[n]{\binom{5n}{n}}\right\}_{n\geqslant 1}, \quad \left\{\sqrt[n]{\binom{5n}{2n}}\right\}_{n\geqslant 1}$$

are strictly log-concave.

To study the conjectures of Sun on the monotonicity of $\{\sqrt[n+1]{z_{n+1}}/\sqrt[n]{z_n}\}$, Chen *et al.* [**4**] found a connection between the log-behaviour of $\{\sqrt[n]{z_n}\}_{n\geqslant 1}$ and that of $\{z_{n+1}/z_n\}_{n\geqslant 0}$. Moreover, they introduced a stronger concept as follows: define an operator R on a sequence $\{z_n\}_{n\geqslant 0}$ by

$$R\{z_n\}_{n\geqslant 0} = \{x_n\}_{n\geqslant 0},$$

where $x_n = z_{n+1}/z_n$. The sequence $\{z_n\}_{n\geqslant 0}$ is called *infinitely log-monotonic* if the sequence $R^r\{z_n\}_{n\geqslant 0}$ is log-concave for all positive odd r and is log-convex for all non-negative even r. In fact, the infinite log-monotonicity is related to the logarithmically completely monotonic function.

Recall that a function $f(x)$ is said to be *completely monotonic* on an interval I if $f(x)$ has derivatives of all orders on I that alternate successively in sign, that is,

$$(-1)^n f^{(n)}(x) \geqslant 0$$

for all $x \in I$ and for all $n \geqslant 0$. If these inequalities are strict for all $x \in I$ and for all $n \geqslant 0$, then $f(x)$ is said to be strictly completely monotonic. A positive function $f(x)$ is said to be *logarithmically completely monotonic* on an interval I if $\log f(x)$ satisfies

$$(-1)^n [\log f(x)]^n \geqslant 0$$

for all $x \in I$ and for all $n \geqslant 1$. A logarithmically completely monotonic function is completely monotonic, but the reverse is not necessarily true (see Berg [**2**]). The reader is referred to [**22**] for the properties of completely monotonic functions and to [**13**] for a survey of logarithmically completely monotonic functions. In [**4**], Chen *et al*. found the link between logarithmically completely monotonic functions and the infinite log-monotonicity of combinatorial sequences. Thus, in § 4, the logarithmically complete monotonicity of some functions related to combinatorial sequences will be considered. As applications, for non-negative integers n_0, k_0, $\overline{k_0}$ and positive integers a, b, $\bar{b}$, if $a \geqslant b + \bar{b}$ and $-1 \leqslant k_0 - (n_0 + 1)b/a \leqslant 0$, then the sequence

$$\left\{ \frac{(n_0 + ia)!}{(k_0 + ib)!(\overline{k_0} + i\bar{b})!} \right\}_{i \geqslant 0}$$

is infinitely log-monotonic. This generalizes two results of Chen *et al*. [**4**] that both the Catalan numbers $(1/(n+1))\binom{2n}{n}$ and the central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic, and strengthens one result of Su and Wang [**16**] that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. In addition, the asymptotically infinite log-monotonicity of derangement numbers is also demonstrated.

In order to research the stronger properties of the above functions $\theta(x)$ and $F(x)$, the logarithmically complete monotonicity of the functions

$$\frac{1}{\sqrt[x]{a\zeta(x+b)\Gamma(x+c)}} \quad \text{and} \quad \sqrt[x]{\rho \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}}$$

is also given, which generalizes one result of Lee and Tepedelenlioğlu about the logarithmically complete monotonicity of

$$\sqrt[x]{\frac{2\sqrt{\pi}\Gamma(x+1)}{\Gamma(x+1/2)}},$$

and one result of Qi and Li about the logarithmically complete monotonicity of

$$\sqrt[x]{\frac{a\Gamma(x+b)}{\Gamma(x+c)}}.$$

2. Analytic results for the monotonicity of the sequence $\sqrt[n]{z_n}$

This section is devoted to the proof of the analytic result Theorem 1.1.

Proof. Let $y = \sqrt[x]{f(x)}$. Then one can get

$$y' = \frac{y}{x}\left(\frac{f'(x)}{f(x)} - \frac{\log f(x)}{x}\right).$$

In order to show that $\sqrt[x]{f(x)}$ is strictly increasing, it suffices to prove that

$$\frac{f'(x)}{f(x)} - \frac{\log f(x)}{x} > 0 \tag{2.1}$$

for $x \geqslant N$. Since $f(N) \leqslant 1$ and $f(x)$ is increasing, one can derive that

$$\frac{\log f(x)}{x} \leqslant \frac{\log f(x) - \log f(N)}{x} < \frac{\log f(x) - \log f(N)}{x - N} \tag{2.2}$$

for $x \geqslant N$.

By the mean value theorem, one can obtain

$$\frac{\log f(x) - \log f(N)}{x - N} = \frac{f'(\xi)}{f(\xi)}, \tag{2.3}$$

where $N \leqslant \xi \leqslant x$. On the other hand, it follows from log-convexity of the function $f(x)$ that

$$(\log f(x))'' = \left(\frac{f'(x)}{f(x)}\right)' = \frac{f''(x)f(x) - f'(x)^2}{f^2(x)} \geqslant 0, \tag{2.4}$$

which implies that $f'(x)/f(x)$ is increasing. Thus, it follows that

$$\frac{f'(\xi)}{f(\xi)} \leqslant \frac{f'(x)}{f(x)} \tag{2.5}$$

for $x \geqslant \xi$. Combining (2.2), (2.3) and (2.5), one can obtain (2.1). So $\sqrt[x]{f(x)}$ is increasing. □

3. Analytic results for the log-behaviour of the sequence $\sqrt[n]{z_n}$

In order to deal with the log-behaviour of the sequence $\sqrt[n]{z_n}$, some analytic methods will be developed in this section. There are two main results in this section, one being the proof of Conjecture 1.4 and the other being the log-behaviour of the function $F(x)$.

In the proofs the following known facts are needed. It follows from [**1**, Theorem 8] that the function

$$G_0(x) = -\log \Gamma(x) + (x - 1/2)\log x - x + \log\sqrt{2\pi} + \frac{1}{12x}$$

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is strictly completely monotonic on $(0,\infty)$. This implies that

$$\log \Gamma(x) < (x-1/2)\log x - x + \log\sqrt{2\pi} + \frac{1}{12x}, \tag{3.1}$$

$$(\log \Gamma(x))' > \log x - \frac{1}{2x} - \frac{1}{12x^2}, \tag{3.2}$$

$$(\log \Gamma(x))'' < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}. \tag{3.3}$$

On the other hand, [**1**, Theorem 8] also says that the function

$$F_0(x) = \log \Gamma(x) - (x-1/2)\log x + x - \log\sqrt{2\pi}$$

is strictly completely monotonic on $(0,\infty)$. So

$$\log \Gamma(x) > (x-1/2)\log x - x + \log\sqrt{2\pi}, \tag{3.4}$$

$$(\log \Gamma(x))' < \log x - \frac{1}{2x}, \tag{3.5}$$

$$(\log \Gamma(x))'' > \frac{1}{x} + \frac{1}{2x^2}. \tag{3.6}$$

Thus, by combining these inequalities, one can get the next result, which will be used repeatedly in the proofs.

Lemma 3.1. *Let $a>0$. Assume that $h(x)=\log\Gamma(x)$. If $b\geqslant -1$ and $ax+b\geqslant 0$, then*

$$x^3\left(\frac{h(ax+b+1)}{x}\right)'' \leqslant -ax + (2b+1)\log(ax+b+1) - 3b - \tfrac{3}{2} + \log 2\pi + \frac{b^2+b+1/2}{ax+b+1},$$

$$x^3\left(\frac{h(ax+b+1)}{x}\right)'' \geqslant -ax + (2b+1)\log(ax+b+1) - 3b - 3 + \log 2\pi.$$

Proof. By $h(x)=\log\Gamma(x)$, it is not hard to deduce that

$$\left(\frac{h(ax+b+1)}{x}\right)'' = \frac{a^2x^2h''(ax+b+1) - 2axh'(ax+b+1) + 2h(ax+b+1)}{x^3}.$$

By (3.1)–(3.3), it follows that

$$\begin{aligned} &a^2x^2h''(ax+b+1) - 2axh'(ax+b+1) + 2h(ax+b+1)\\ &\qquad \leqslant -ax + (2b+1)\log(ax+b+1) - 3b - \tfrac{3}{2} + \log 2\pi + \frac{b^2+b+1/2}{ax+b+1}. \end{aligned}$$

In addition, by (3.4)–(3.6), one can also obtain that

$$\begin{aligned} &a^2x^2h''(ax+b+1) - 2axh'(ax+b+1) + 2h(ax+b+1)\\ &\qquad \geqslant -ax + (2b+1)\log(ax+b+1) - 3b - 3 + \log 2\pi. \end{aligned}$$

This completes the proof. □

In order to prove Conjecture 1.4, the next result will be used.

Lemma 3.2. *Let* $\zeta(x) = \sum_{n\geqslant 1} 1/n^x$ *be the Riemann zeta function. Define a function* $\eta(x) = \zeta(x) - 1$*. The bound* $\eta(x) \leqslant 3/2^x$ *holds for all* $x \geqslant 4$*.*

Proof. Since

$$\begin{aligned}\eta(x) &= \frac{1}{2^x}\left(1 + \frac{1}{1.5^x} + \frac{1}{2^x} + \cdots\right)\\ &\leqslant \frac{1}{2^x}\left(1 + \frac{1}{1.5^x} + 2(\zeta(x) - 1)\right)\\ &\leqslant \frac{1}{2^x}(1 + \tfrac{1}{2} + 2\eta(x))\end{aligned}$$

for $x \geqslant 4$, one can get $\eta(x) \leqslant 3/2^x$. □

We can obtain a partial result for Conjecture 1.4 as follows.

Theorem 3.3. *The function*

$$\theta(x) = \sqrt[x]{2\zeta(x)\Gamma(x+1)}$$

is log-concave on $(7.1, \infty)$*.*

Proof. In order to show that $\theta(x)$ is log-concave on $(7.1, \infty)$, it suffices to prove that

$$\begin{aligned}(\log\theta(x))'' &= \left(\frac{\log 2}{x}\right)'' + \left(\frac{\log\zeta(x)}{x}\right)'' + \left(\frac{\log\Gamma(x+1)}{x}\right)''\\ &= \frac{2\log 2}{x^3} + \left(\frac{\log\zeta(x)}{x}\right)'' + \left(\frac{\log\Gamma(x+1)}{x}\right)'' \qquad (3.7)\\ &< 0.\end{aligned}$$

Noting that $\log x < \sqrt{x}$ for $x \geqslant 2$, one has $\zeta''(x) < \eta(x-1)$ and $|\zeta'(x)| < \eta(x-0.5)$. In addition, it follows from $\log(x+1) \leqslant x$ for $x > 0$ that $\log(1+\eta(x)) \leqslant \eta(x) \leqslant 3/2^x$ by Lemma 3.2. Thus, for $x \geqslant 7.1$, it follows that

$$\begin{aligned}x^3\left(\frac{\log\zeta(x)}{x}\right)'' &= x^2\left(\frac{\zeta(x)\zeta''(x) - \zeta'(x)^2}{\zeta(x)^2}\right) - 2x\frac{\zeta'(x)}{\zeta(x)} + 2\log\zeta(x)\\ &< \frac{x^2\zeta''(x)}{\zeta(x)} - \frac{2x\zeta'(x)}{\zeta(x)} + 2\log\zeta(x)\\ &< 2.67, \qquad (3.8)\end{aligned}$$

where the final inequality can be obtained by considering the monotonicity of the right-hand function.

On the other hand, by Lemma 3.1, one can get

$$\begin{aligned}x^3\left(\frac{\log\Gamma(x+1)}{x}\right)'' &\leqslant -x + \log(x+1) - 1 + \log 2\pi + \frac{1}{2(x+1)}\\ &< -4.1 \qquad (3.9)\end{aligned}$$

for $x \geqslant 7.1$.

Thus, combining (3.7)–(3.9), one can conclude that

$$(\log \theta(x))'' = \frac{2\log 2}{x^3} + \left(\frac{\log \zeta(x)}{x}\right)'' + \left(\frac{\log \Gamma(x+1)}{x}\right)'' < 0,$$

as desired. This completes the proof. □

Notice that

$$\sqrt[n]{(-1)^{n-1}B_{2n}} = \frac{\theta^2(2n)}{4\pi^2}.$$

Thus, it follows from the strict log-concavity of $\{\theta(2n)\}_{n\geqslant 4}$ that $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geqslant 4}$ is strictly log-concave. In addition, it is easy to check that $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geqslant 0}$ is strictly log-concave for $1 \leqslant n \leqslant 4$. Thus, the following result is immediate, which was conjectured by Sun [**17**, Conjecture 2.15] and has been verified by Luca and Stănică [**11**] and Chen *et al.* [**4**] by different methods.

Corollary 3.4. *The sequence* $\{\sqrt[n]{(-1)^{n-1}B_{2n}}\}_{n\geqslant 1}$ *is strictly log-concave.*

Now consider the tangent numbers [**14**, A000182]

$$\{T(n)\}_{n\geqslant 0} = \{1, 2, 16, 272, 7936, 353\,792, \dots\},$$

which are defined by

$$\tan x = \sum_{n\geqslant 1} T(n)\frac{x^{2n-1}}{(2n-1)!}$$

and are closely related to the Bernoulli numbers:

$$T(n) = (-1)^{n-1}B_{2n}\frac{(4^n-1)}{2n}4^n$$

(see, for example, [**6**, (6.93)]). So

$$\sqrt[n]{T(n)} = 4\sqrt[n]{(-1)^{n-1}B_{2n}}\,\sqrt[n]{4^n-1}\,\sqrt[n]{\frac{1}{2n}}.$$

It is not difficult to verify that both $\sqrt[n]{4^n-1}$ and $\sqrt[n]{1/2n}$ are log-concave in n (we leave the details to the reader). The product of log-concave sequences is still log-concave. Thus the next result, which was conjectured by Sun [**17**, Conjecture 3.5] and was verified by Luca and Stănică [**11**] by a discrete method, is immediate.

Corollary 3.5. *The sequence* $\{\sqrt[n]{T(n)}\}_{n\geqslant 1}$ *is strictly log-concave.*

In order to develop analytic techniques to deal with the log-behaviour of $\{\sqrt[n]{z_n}\}$, in the following the log-behaviour of a function $F(x)$ related to the Euler Gamma function will be considered, which can be applied to some interesting binomial coefficients.

Theorem 3.6. *Given real numbers b, d, f and non-negative real numbers a, c, e, define the function*

$$F(x) = \sqrt[x]{\frac{\Gamma(ax+b+1)}{\Gamma(cx+d+1)\Gamma(ex+f+1)}}.$$

(i) *If $a > c+e$, then $F(x)$ is an asymptotically log-concave function.*

(ii) *Assume that $a = c+e$. If $c \geqslant e > 0$ and $b < d+f+1/2$, then $F(x)$ is an asymptotically log-concave function. In particular, if $c \geqslant 1$ and $b = d = f = 0$, then we have that $F(x)$ is a log-concave function for $x \geqslant 30$; if $c \geqslant 1$, $b = d = 0$ and $f \geqslant 1$, then $F(x)$ is a log-concave function for $x \geqslant 2$.*

(iii) *Assume that $a = c+e$. If $c > e = 0$ and $b < d$, then $F(x)$ is an asymptotically log-concave function.*

(iv) *If $a < c+e$, then $F(x)$ is an asymptotically log-convex function.*

Proof. Let $h(x) = \log \Gamma(x)$. By Lemma 3.1, one has

$$\begin{aligned}(\log F(x))'' &= \left(\frac{h(ax+b+1)}{x}\right)'' - \left(\frac{h(cx+d+1)}{x}\right)'' - \left(\frac{h(ex+f+1)}{x}\right)'' \\ &< (c+e-a)x + \log\frac{(ax+b+1)^{(2b+1)}}{(cx+d+1)^{(2d+1)}(ex+f+1)^{(2f+1)}} + 3(d+f-b) \\ &\quad + \tfrac{9}{2} - \log 2\pi + \frac{b^2+b+1/2}{ax+b+1}. \end{aligned} \tag{3.10}$$

It is easy to prove for $a > c+e$ that

$$\lim_{x\to+\infty}(c+e-a)x + \log\frac{(ax+b+1)^{(2b+1)}}{(cx+d+1)^{(2d+1)}(ex+f+1)^{(2f+1)}} = -\infty,$$

and for $a = c+e$ that

$$\lim_{x\to+\infty}\log\frac{(ax+b+1)^{(2b+1)}}{(cx+d+1)^{(2d+1)}(ex+f+1)^{(2f+1)}} = -\infty$$

if $c \geqslant e > 0$ and $b < d+f+1/2$ or $c > e = 0$ and $b < d$. Thus, under conditions (i)–(iii), by (3.10) one can get

$$\lim_{x\to+\infty}(\log F(x))'' = -\infty,$$

implying that $F(x)$ is an asymptotically log-concave function.

Assume that $a = c+e$ and $c \geqslant e \geqslant 1$. If $b = d = f = 0$, then, by (3.10),

$$\begin{aligned}(\log F(x))'' &< \log\frac{(ax+1)}{(cx+1)(ex+1)} + \tfrac{9}{2} - \log 2\pi + \frac{1}{2(ax+1)} \\ &< -0.04\end{aligned}$$

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for $x \geqslant 30$. If $b = d = 0$ and $f \geqslant 1$, then, by (3.10),

$$(\log F(x))'' < \log \frac{(ax+1)}{(cx+1)(ex+2)^3} + \tfrac{9}{2} - \log 2\pi + \frac{1}{2(ax+1)} < -0.37$$

for $x \geqslant 2$.

Finally, since the proof of (iv) is similar to that of (i), it is omitted for brevity. This completes the proof. □

By Theorem 3.6, the next result is immediate.

Proposition 3.7. *Let integers a, b, c, d, f satisfy $a > c > 0$ and $b < d + f + 1/2$. Then the sequence*

$$\left\{ \sqrt[n]{\frac{\Gamma(an+b+1)}{\Gamma(cn+d+1)\Gamma((a-c)n+f+1)}} \right\}_{n\geqslant 1}$$

is asymptotically log-concave. In particular, $\{\sqrt[n]{\binom{an}{cn}}\}_{n\geqslant 30}$ and

$$\left\{ \sqrt[n]{\frac{\Gamma(an+1)}{\Gamma(cn+1)\Gamma((a-c)n+f+1)}} \right\}_{n\geqslant 2}$$

are strictly log-concave for $f \geqslant 1$.

For integer $p \geqslant 2$, Fuss–Catalan numbers [**7**] are given by the formula

$$C_p(n) = \frac{1}{(p-1)n+1}\binom{pn}{n} = \frac{\Gamma(pn+1)}{\Gamma(n+1)\Gamma((p-1)n+2)}.$$

It is well known that the Fuss–Catalan numbers count the number of paths in the integer lattice $\mathbb{Z} \times \mathbb{Z}$ (with directed vertices from (i, j) to either $(i, j+1)$ or $(i+1, j)$) from the origin $(0, 0)$ to $(n, (p-1)n)$ that never go above the diagonal $(p-1)x = y$. Su and Wang [**16**] showed that $\{\binom{an}{bn}\}_{n\geqslant 0}$ is log-convex for positive integers $a > b$. Thus, it is easy to see that $\{C_p(n)\}_{n\geqslant 0}$ is log-convex. Chen *et al*. [**4**] proved that

$$\sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}} \quad \text{and} \quad \sqrt[n]{\binom{2n}{n}}$$

are strictly log-concave. By verifying the first few terms, one can get the following corollary using Corollary 3.7.

Corollary 3.8. *The sequences*

$$\left\{ \sqrt[n]{\frac{1}{2n+1}\binom{2n}{n}} \right\}_{n\geqslant 1}, \quad \left\{ \sqrt[n]{\binom{2n}{n}} \right\}_{n\geqslant 1}, \quad \left\{ \sqrt[n]{\binom{3n}{n}} \right\}_{n\geqslant 1},$$

$$\left\{ \sqrt[n]{\binom{4n}{n}} \right\}_{n\geqslant 1}, \quad \left\{ \sqrt[n]{\binom{5n}{n}} \right\}_{n\geqslant 1}, \quad \left\{ \sqrt[n]{\binom{5n}{2n}} \right\}_{n\geqslant 1}, \quad \left\{ \sqrt[n]{C_p(n)} \right\}_{n\geqslant 2}$$

are strictly log-concave for any positive integer $p \geqslant 2$.

4. Logarithmically completely monotonic functions

Since logarithmically completely monotonic functions have many applications, it is important to know which functions have such a property. In particular, Chen *et al*. [**4**] found the connection between logarithmically completely monotonic functions and infinite log-monotonicity of combinatorial sequences as follows.

Theorem 4.1 (Chen *et al*. [4]). *Assume that a function $f(x)$ is such that $[\log f(x)]''$ is completely monotonic for $x \geqslant 1$, and $a_n = f(n)$ for $n \geqslant 1$. Then the sequence $\{a_n\}_{n\geqslant 1}$ is infinitely log-monotonic.*

Thus, it is very interesting to research the logarithmically complete monotonicity of some functions related to combinatorial sequences, which is the aim of this section.

Many sequences of binomial coefficients share various log-behaviour properties (see, for example, [**16**,**18**,**19**]). In particular, Su and Wang proved that $\binom{dn}{\delta n}$ is log-convex in n for positive integers $d > \delta$. Recently, Chen *et al*. [**4**] proved that both the Catalan numbers $(1/(n+1))\binom{2n}{n}$ and central binomial coefficients $\binom{2n}{n}$ are infinitely log-monotonic. Motivated by these results, a generalization can be stated as follows.

Theorem 4.2. *Let n_0, k_0, $\overline{k_0}$ be non-negative integers and let a, b, $\bar{b}$ be positive integers. Define the function*

$$G(x) = \frac{\Gamma(n_0 + ax + 1)}{\Gamma(k_0 + bx + 1)\Gamma(\overline{k_0} + x\bar{b} + 1)}.$$

If $a \geqslant b + \bar{b}$ and $-1 \leqslant k_0 - (n_0+1)b/a \leqslant 0$, then $(\log G(x))''$ is a completely monotonic function for $x \geqslant 0$. In particular,

$$\left\{ \frac{(n_0 + ia)!}{(k_0 + ib)!(\overline{k_0} + i\bar{b})!} \right\}_{i\geqslant 0}$$

is infinitely log-monotonic.

Proof. By Theorem 4.1, it suffices to show that $(\log G(x))''$ is a completely monotonic function for $x \geqslant 0$. Let $g(x) = \log G(x)$. So

$$\begin{aligned}
[g(x)]^{(n)} &= [\log \Gamma(n_0 + ax + 1)]^{(n)} - [\log \Gamma(k_0 + bx + 1)]^{(n)} - [\log \Gamma(\overline{k_0} + x\bar{b} + 1)]^{(n)} \\
&= (-1)^n \int_0^\infty \frac{t^{n-1}}{1 - \mathrm{e}^{-t}} [a^n \mathrm{e}^{-t(n_0+ax+1)} - b^n \mathrm{e}^{-t(k_0+bx+1)} - \bar{b}^n \mathrm{e}^{-t(\overline{k_0}+x\bar{b}+1)}]\, \mathrm{d}t \\
&= (-1)^n \int_0^\infty a^n t^{n-1} \mathrm{e}^{-tax} \left[\frac{\mathrm{e}^{-(n_0+1)t}}{1 - \mathrm{e}^{-t}} - \frac{\mathrm{e}^{-ta(k_0+1)/b}}{1 - \mathrm{e}^{-at/b}} - \frac{\mathrm{e}^{-ta(\overline{k_0}+1)/\bar{b}}}{1 - \mathrm{e}^{-at/\bar{b}}} \right] \mathrm{d}t \qquad (4.1)
\end{aligned}$$

since

$$[\log \Gamma(x)]^{(n)} = (-1)^n \int_0^\infty \frac{t^{n-1}\mathrm{e}^{-tx}}{1 - \mathrm{e}^{-t}}\, \mathrm{d}t$$

for $x > 0$ and $n \geqslant 2$ (see, for example, [**12**, p. 16]).

Therefore it follows from $a > b > 0$ that for further simplification one can define $u = k_0 - (n_0+1)b/a$, $p = a/b$, and $q = a/\bar{b}$. Clearly, $1/p + 1/q \leqslant 1$. So one can deduce that

$$(-1)^n[g(x)]^{(n)} = \int_0^\infty a^n t^{n-1} \mathrm{e}^{-t(n_0+ax+1)} h(t,u)\,\mathrm{d}t, \tag{4.2}$$

where

$$h(t,u) = \frac{1}{1-\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-tp(u+1)}}{1-\mathrm{e}^{-pt}} - \frac{\mathrm{e}^{uqt}}{1-\mathrm{e}^{-qt}}.$$

Furthermore, one can obtain the next claim for $-1 \leqslant k_0 - (n_0+1)b/a \leqslant 0$.

Claim 4.3. *If* $-1 \leqslant u \leqslant 0$, *then* $h(t,u) > 0$.

Proof of Claim 4.3. It is obvious that $h(t,u)$ is concave in u. Thus, it suffices to show that $h(t,u) > 0$ for $u = -1$ and $u = 0$. Setting $u = 0$, since the case $u = -1$ can be obtained by switching the roles of p and q, one has

$$h(t,0) = \frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-tp}}{1-\mathrm{e}^{-pt}} - \frac{\mathrm{e}^{-qt}}{1-\mathrm{e}^{-qt}}.$$

Noting for $s > 0$ that the function

$$f(s) = \frac{s\mathrm{e}^{-s}}{1-\mathrm{e}^{-s}}$$

strictly decreases in s and $1/p + 1/q \leqslant 1$, one gets that

$$\begin{aligned} h(t,0) &\geqslant \left(\frac{1}{p} + \frac{1}{q}\right)\frac{\mathrm{e}^{-t}}{1-\mathrm{e}^{-t}} - \frac{\mathrm{e}^{-tp}}{1-\mathrm{e}^{-pt}} - \frac{\mathrm{e}^{-qt}}{1-\mathrm{e}^{-qt}} \\ &= \frac{f(t) - f(tp)}{tp} + \frac{f(t) - f(tq)}{tq} \\ &\geqslant 0. \end{aligned}$$

This completes the proof of Claim 4.3. □

Thus, by (4.2) and Claim 4.3, one has $(-1)^n[g(x)]^{(n)} > 0$, which implies that $(\log G(x))''$ is a completely monotonic function. This completes the proof. □

By Theorem 4.2, the following two corollaries are immediate.

Corollary 4.4. *Let* n_0, k_0, d, δ *be four non-negative integers. Define the sequence*

$$C_i = \binom{n_0 + id}{k_0 + i\delta}, \quad i = 0, 1, 2, \ldots.$$

If $d > \delta > 0$ *and* $-1 \leqslant k_0 - (n_0+1)\delta/d \leqslant 0$, *then the sequence* $\{C_n\}_{n \geqslant 0}$ *is infinitely log-monotonic.*

Corollary 4.5. *The Fuss–Catalan sequence* $\{C_p(n)\}_{n\geqslant 0}$ *is infinitely log-monotonic, where* $p \geqslant 2$ *and*

$$C_p(n) = \frac{1}{(p-1)n+1}\binom{pn}{n}.$$

The derangement number d_n is a classical combinatorial number. It is log-convex and ratio log-concave: see [**10**] and [**5**], respectively. Note that $\{\Gamma(n)\}_{n\geqslant 1}$ is strictly infinitely log-monotonic (see [**4**]) and

$$\left| d_n - \frac{n!}{\mathrm{e}} \right| \leqslant \frac{1}{2} \tag{4.3}$$

for $n \geqslant 3$ (see [**8**]), from which the following interesting result can be demonstrated.

Theorem 4.6. *The sequence of the derangement numbers* $\{d_n\}_{n\geqslant 3}$ *is asymptotically infinitely log-monotonic.*

Proof. From (4.3), one can deduce that

$$\frac{n!}{\mathrm{e}} - \frac{1}{2} \leqslant d_n \leqslant \frac{n!}{\mathrm{e}} + \frac{1}{2},$$

which implies that

$$\Gamma(n+1) - \tfrac{3}{2} \leqslant \mathrm{e}d_n \leqslant \Gamma(n+1) + \tfrac{3}{2}.$$

Thus,

$$\begin{aligned} \mathrm{e}^2(d_{n+1}d_{n-1} - d_n^2) &\geqslant [\Gamma(n+2) - 1.5][\Gamma(n) - 1.5] - [\Gamma(n+1) + 1.5]^2 \\ &> 0 \end{aligned}$$

for $n \geqslant 4$, which implies that $\{d_n\}_{n\geqslant 4}$ is log-convex. Note that

$$\begin{aligned} &\mathrm{e}^4(d_{n+1}^3 d_{n-1} - d_n^3 d_{n+2}) \\ &\qquad \geqslant [\Gamma(n+2) - 1.5]^3[\Gamma(n) - 1.5] - [\Gamma(n+1) + 1.5]^3[\Gamma(n+3) + 1.5] \\ &\qquad > 0 \end{aligned}$$

for $n \geqslant 8$, which implies that $R\{d_n\}_{n\geqslant 8}$ is log-concave. Because $\{\Gamma(n)\}_{n\geqslant 1}$ is strictly infinitely log-monotonic, similarly, it can be proceeded to the higher-order log-monotonicity. Thus, for any positive integer k, by the sign-preserving property of limits, one can obtain that there exists a positive N such that the sequence $R^r\{d_n\}_{n\geqslant N}$ is log-concave for positive odd r and is log-convex for positive even r. Thus, the sequence of the derangement numbers $\{d_n\}_{n\geqslant 3}$ is asymptotically infinitely log-monotonic. □

In the following we will continue to give two kinds of logarithmically completely monotonic functions. In order to consider a stronger result for Theorem 3.3, given a, b, $c > 0$, define the function

$$\theta_{a,b,c}(x) = \sqrt[x]{a\zeta(x+b)\Gamma(x+c)}.$$

It is known that the Riemann zeta function $\zeta(x)$ is logarithmically completely monotonic on $(1, +\infty)$ and the function $[\log \Gamma(x)]''$ is completely monotonic on $(0, +\infty)$ (see [**4**]). Based on these results, one can demonstrate the next theorem.

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Theorem 4.7. *Let a, b, c be positive real numbers, where $b \geqslant 1$. If $a\zeta(b)\Gamma(c) \leqslant 1$, then the reciprocal of the function $\theta_{a,b,c}(x)$ is logarithmically completely monotonic on $(1,\infty)$.*

Proof. Since

$$\log \theta^{-1}_{a,b,c}(x) = -\frac{\log(a\zeta(x+b)\Gamma(x+c))}{x} = -\frac{\log a + \log\zeta(x+b) + \log\Gamma(x+c)}{x},$$

in order to show that $\theta^{-1}_{a,b,c}(x)$ is logarithmically completely monotonic on $(1,\infty)$, it suffices to prove that

$$(-1)^n \log^{(n)} \theta^{-1}_{a,b,c}(x) \geqslant 0$$

for all $n \geqslant 1$. Note that a known formula is as follows:

$$\left(\frac{g(x)}{x}\right)^{(n)} = \frac{(-1)^n g(0) n!}{x^{n+1}} + x^{-n-1}\int_0^x t^n g^{(n+1)}(x)\,\mathrm{d}t, \tag{4.4}$$

which can be easily proved by induction. Thus, one can deduce for $n \geqslant 1$ and $x > 1$ that

$$\begin{aligned}(-1)^n \log^{(n)} \theta^{-1}_{a,b,c}(x) &= \frac{-n!\log a\zeta(b)\Gamma(c)}{x^{n+1}} \\ &\quad + x^{-n-1}\int_0^x t^n(-1)^{n+1}[(\log\zeta(x+b))^{(n+1)} + (\log\Gamma(x+c))^{(n+1)}]\,\mathrm{d}t \\ &\geqslant 0\end{aligned}$$

since

$$\log a\zeta(b)\Gamma(c) \leqslant 0, \quad (-1)^{n+1}(\log\zeta(x+b))^{(n+1)} \geqslant 0, \quad (-1)^{n+1}(\log\Gamma(x+c))^{(n+1)} \geqslant 0.$$

This completes the proof. □

The next result was proved by Alzer [**1**].

Theorem 4.8 (Alzer [1]). *Let there be non-negative sequences $0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_n$ and $0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant \cdots \leqslant b_n$. If $\sum_{i=1}^k a_i \leqslant \sum_{i=1}^k b_i$ for $k = 1, 2, \ldots, n$, then the function*

$$\prod_{i=1}^n \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}$$

is completely monotonic on $(0,\infty)$.

On the other hand, Lee and Tepedelenlioğlu [**9**] proved that the function

$$\sqrt[x]{\frac{2\sqrt{\pi}\Gamma(x+1)}{\Gamma(x+1/2)}}$$

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originating from the coding gain is logarithmically completely monotonic on $(0,\infty)$. In addition, Qi and Li [**13**] considered the logarithmically complete monotonicity of

$$\sqrt[x]{\frac{a\Gamma(x+b)}{\Gamma(x+c)}}.$$

In what follows a general result for a kind of logarithmically completely monotonic function is obtained.

Theorem 4.9. *Let $0 \leqslant a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_n$, let $0 \leqslant b_1 \leqslant b_2 \leqslant b_3 \leqslant \cdots \leqslant b_n$, let $\rho > 0$, and define the function*

$$\chi(x) = \sqrt[x]{\rho \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}}.$$

(i) *If*

$$\rho \prod_{i=1}^{n} \frac{\Gamma(a_i)}{\Gamma(b_i)} \geqslant 1 \quad \text{and} \quad \sum_{i=1}^{k} a_i \geqslant \sum_{i=1}^{k} b_i$$

for $k = 1, 2, \ldots, n$, then the function $\chi(x)$ is logarithmically completely monotonic on $(0,\infty)$.

(ii) *If*

$$\rho \prod_{i=1}^{n} \frac{\Gamma(a_i)}{\Gamma(b_i)} \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i$$

for $k = 1, 2, \ldots, n$, then the reciprocal of the function $\chi(x)$ is logarithmically completely monotonic on $(0,\infty)$.

Proof. Because (ii) can be obtained in a similar way, we only need to prove (i). Define the function $h(x) = \sum_{i=1}^{n} \log \Gamma(x+a_i) - \log \Gamma(x+b_i)$. Then

$$\log \sqrt[x]{\rho \prod_{i=1}^{n} \frac{\Gamma(x+a_i)}{\Gamma(x+b_i)}} = \frac{\log \rho \prod_{i=1}^{n} \Gamma(x+a_i)/\Gamma(x+b_i)}{x} = \frac{\log \rho + h(x)}{x}.$$

So it is not hard to get

$$(-1)^k [\log \chi(x)]^{(k)} = \frac{k!(\log \rho + h(0))}{x^{k+1}} + x^{-k-1} \int_0^x t^k (-1)^k h^{(k+1)}(x)\, \mathrm{d}t. \tag{4.5}$$

If $\rho \prod_{i=1}^{n} \Gamma(a_i)/\Gamma(b_i) \geqslant 1$, then it is clear that

$$\log \rho + h(0) \geqslant 0.$$

In addition, Alzer [**1**] proved that $(-1)^k h^{(k+1)}(x) \geqslant 0$ for $k \geqslant 0$ and $x \geqslant 0$. Thus,

$$(-1)^k [\log \chi(x)]^{(k)} \geqslant 0,$$

that is, $\chi(x)$ is logarithmically completely monotonic on $(0,\infty)$. This completes the proof. □

Remark 4.10. If $\rho = 2\sqrt{\pi}$, $a_1 = 1$ and $b_1 = 1/2$, then $2\sqrt{\pi}\Gamma(1)/\Gamma(1/2) = 2 > 1$. So the function $\sqrt[x]{2\sqrt{\pi}\Gamma(x+1)/\Gamma(x+1/2)}$ is logarithmically completely monotonic on $(0,\infty)$ (see [**9**]). In addition, the case in which $n = 1$ in Theorem 4.9 was proved by Qi and Li [**13**]. Thus, the result in Theorem 4.9 is a generalization.

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References

1. H. Alzer, On some inequalities for the gamma and psi functions, *Math. Comp.* **66** (1997), 373–389.
2. C. Berg, Integral representation of some functions related to the gamma function, *Med. J. Math.* **1** (2004), 433–439.
3. F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, *Jerusalem combinatorics '93*, Contemporary Mathematics, Volume 178, pp. 71–89 (American Mathematical Society, Providence, RI, 1994).
4. W. Y. C. Chen, J. J. F. Guo and L. X. W. Wang, Infinitely log-monotonic combinatorial sequences, *Adv. Appl. Math.* **52** (2014), 99–120.
5. W. Y. C. Chen, J. J. F. Guo and L. X. W. Wang, Zeta functions and the log-behavior of combinatorial sequences, *Proc. Edinb. Math. Soc.* **58**(3) (2015), 637–651.
6. R. L. Graham, D. E. Knuth and O. Patashnik, *Concrete mathematics: a foundation for computer science*, 2nd edn (Addison-Wesley, 1994).
7. P. Hilton and J. Pedersen, Catalan numbers, their generalizations, and their uses, *Math. Intelligencer* **13** (1991), 64–75.
8. Q.-H. Hou, Z.-W. Sun and H.-M. Wen, On monotonicity of some combinatorial sequences, Preprint (arXiv.org/abs/1208.3903; 2014).
9. J. Lee and C. Tepedelenlioğlu, Space-time coding over fading channels with stable noise, *IEEE Trans. Veh. Technol.* **60**(7) (2011), 3169–3177.
10. L. L. Liu and Y. Wang, On the log-convexity of combinatorial sequences, *Adv. Appl. Math.* **39** (2007), 453–476.
11. F. Luca and P. Stănică, On some conjectures on the monotonicity of some arithmetical sequences, *J. Combin. Number Theory* **4** (2012), 1–10.
12. W. Magnus, F. Oberhettinger and R. P. Soni, *Formulas and theorems for the special functions of mathematical physics* (Springer, 1966).
13. F. Qi and W.-H. Li, A logarithmically completely monotonic function involving the ratio of two Gamma functions, *J. Appl. Analysis Comp.* **5**(4) (2015), 626–634.
14. N. J. A. Sloane, *The on-line encyclopedia of integer sequences*, available at http://oeis.org (1964).
15. R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, *Annals New York Acad. Sci.* **576** (1989), 500–534.
16. X.-T. Su and Y. Wang, On the unimodality problems in Pascal's triangle, *Electron. J. Combin.* **15** (2008), R113.
17. Z.-W. Sun, Conjectures involving arithmetical sequences, *Numbers theory: arithmetic in Shangri-La* (ed. S. Kanemitsu, H. Li and J. Liu), Series on Number Theory and Its Applications, Volume 8, pp. 244–258 (World Scientific, 2013).

18. S. Tanny and M. Zuker, On a unimodal sequence of binomial coefficients, *Discr. Math.* **9** (1974), 79–89.
19. S. Tanny and M. Zuker, On a unimodal sequence of binomial coefficients II, *J. Combin. Inf. Syst. Sci.* **1** (1976), 81–91.
20. Y. Wang and Y.-N. Yeh, Log-concavity and LC-positivity, *J. Combin. Theory* A **114** (2007), 195–210.
21. Y. Wang and B.-X. Zhu, Proofs of some conjectures on monotonicity of number-theoretic and combinatorial sequences, *Sci. China Math.* **57**(11) (2014), 2429–2435.
22. D. V. Widder, *The Laplace transform* (Princeton University Press, 1946).
23. B.-X. Zhu, Log-convexity and strong q-log-convexity for some triangular arrays, *Adv. Appl. Math.* **50**(4) (2013), 595–606.