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COMBINATORIAL APPROACH TO COMPUTING COMPONENT IMPORTANCE INDEXES IN COHERENT SYSTEMS

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We consider binary coherent systems with independent binary components having equal failure probability q. The system *DOWN* probability is expressed via its signature's combinatorial analogue, the so-called D-spectrum. Using the definition of the Birnbaum importance measure (BIM), we introduce for each component a new combinatorial parameter, so-called BIM-spectrum, and develop a simple formula expressing component BIM via the component BIM-spectrum. Further extension of this approach allows obtaining a combinatorial representation for the joint reliability importance (JRI) of two components. To estimate component BIMs and JRIs, there is no need to know the analytic formula for system reliability. We demonstrate how our method works using the Monte Carlo approach. We present several examples of estimating component importance measures in a network when the *DOWN* state is defined as the loss of terminal connectivity.

1. INTRODUCTION

It is well known that the so-called component important measures play crucial role in the optimal reliability design of coherent systems. The use of Birnbaum importance

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measure (BIM) (see Barlow and Proschan [1] and Birnbaum [2]) and the joint reliability importance (JRI) introduced by Hong and Lie in [5] (see also Gao, Cui and Li [4]) is severely limited by the necessity to have an analytic expression of the system reliability function via its component reliability values. It turns out that for the case of the equal component reliability $p_i \equiv p$, it is possible to develop a Monte Carlo approximations to component BIMs and JRIs using the system combinatorial parameter related to the so-called *structural signature* (Spizzichino and Navarro [7]) and so-called *C-spectrum* (Gertsbakh and Shpungin, [3, Chap. 10]). In this article we present a systematic exposition of the combinatorial approach to finding out the BIMs and JRIs. The exposition is as follows.

In Section 2 we present the system *DOWN* probability using Samaniego's signature [6] and its combinatorial analogue, the so-called cumulative D-spectrum (Gertsbakh and Shpungin [3]). In Section 3 we analyze the BIM and its representation via the pivotal decomposition, and we develop a combinatorial representation of the BIMs using the so-called component BIM-spectra. We also formulate sufficient conditions guaranteeing that the BIM of component i dominates the BIM of component j for all q values.

In Section 4 we extend the approach of Section 3 to the combinatorial representation of JRI and investigate the JRI for a highly reliable system. Section 5 presents the Taylor series second-order approximation to system reliability via component BIMs and JRIs.

Section 6 explains how to obtain Monte Carlo approximations to the D-spectra, the BIMs and JRIs. Finally, Section 7 presents numerical examples.

2. STRUCTURE FUNCTION, SIGNATURE, AND FORMULA FOR $P(DOWN)$

Following Barlow and Proschan [1, Chapt. 1], we will consider binary coherent systems consisting of binary components. It will be assumed that we are given system structure function [1]

$$\varphi(\mathbf{x}) \to \{0, 1\}, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is system state vector, and the x_i are binary variables.

It follows from (1) that we know all system *path* sets $K_1, K_2, \ldots, K_a$ and all system *cut* sets $C_1, C_2, \ldots, C_b$. Denote by $C(y)$ the number of cut sets of size y, $y = 1, \ldots, n$, and by $K(y)$ the number of path sets of size y, $y = 1, \ldots, n$.

The following is the standard definition of *signature* by Samaniego [6, p. 21]. Assume that the lifetimes of system's n components are independent and identically distributed (i.i.d.) according to the continuous distribution F. The signature $\mathbf{s}$ of the system is an n-dimensional probability vector whose ith element s_i is equal to the probability that the ith component failure causes the system to fail. In brief, $s_i = P(T = X_{i:n})$, where T is the failure time of the system and $X_{i:n}$ is the ith-order statistic of the n component failure times.

From here it follows Samaniego's principal formula for the cumulative distribution function (CDF) $G(t)$ of system lifetime T [6, p. 27]:

$$G(t) = P(T \leq t) = \sum_{i=1}^{n} s_i P(X_{i:n} \leq t), \tag{2}$$

where $P(X_{i:n} \leq t) = F_{(i)}(t)$ is the CDF of the ith order statistic.

After substituting the expression for $F_{(i)}(t)$ into (2), changing the order of summation and using little algebra, (2) takes the form

$$G(t) = \sum_{i=1}^{n} S_i \cdot (F(t))^i (1 - F(t))^{(n-i)} n!/(i!(n-i)!), \tag{3}$$

where

$$S_i = \sum_{j=1}^{i} s_j$$

is so called *cumulative signature*.

By its definition, $G(t)$ is the probability that the system is *DOWN* at time t, and $F(t)$ is the probability that component is *DOWN* at time t. Denote $F(t) = q$ and

$$S_i \cdot n!/(i!(n-i)!) = A(i). \tag{4}$$

Then (3) takes the form

$$Q = P(DOWN) = \sum_{x=1}^{n} A(x) q^x (1-q)^{(n-x)}. \tag{5}$$

However, from (5) it follows that $A(x)$ must be equal $C(x)$ because the event "the system is *DOWN*" means that the system is in one of its failure states, and the measure of one failure set with x components *down* and $(n-x)$ *up* equals $q^x(1-q)^{n-x}$. It follows from (4) that the cumulative signature (and the signature itself) is expressed via the number of cut sets $C(x)$ of size x, which, in turn, is defined by the system structure function.

In light of the above, it seems logical to use an alternative definition of signature called *structural signature* by Spizzichino and Navarro [7], which is equivalent to the notion of a D-spectrum introduced by Gertsbakh and Shpungin [3]. The D-spectrum is defined as follows.

Imagine a *random* permutation $\pi = (x_{i_1}, x_{i_2}, \ldots, x_{i_n})$ of component numbers and assume that all components are *up*. Start moving along π from left to right and turn *down* one component after another. Let f_j be the probability that the system will get *DOWN* on the jth step of this process (i.e., on turning *down* the component i_j). The discrete density

$$\mathbf{f} = (f_1, f_2, \ldots, f_n)$$

is called the D-spectrum (D stands for "destruction"). The space of all $n!$ permutations is supplied by a uniform measure, and each particular π has probability $1/n!$. It is easy

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to prove that $f_i = s_i$. Similarly to the cumulative signature S_j, we define the cumulative D-spectrum as

$$F(x) = f_1 + \cdots + f_x.$$

Let Y be the number of components needed to be turned from *up* to *down* to cause the system get *DOWN*. Then, obviously, $f_r = P(Y = r)$ and

$$F(x) = P(Y \leq x).$$

3. BIRNBAUM IMPORTANCE MEASURE AND ITS COMBINATORIAL REPRESENTATION

In this section we present a combinatorial interpretation of the so-called BIM [2]. Suppose that a system consists of n independent components and that component i has reliability p_i, $i = 1, \ldots, n$. Let the system reliability be

$$R = \Psi(p_1, p_2, \ldots, p_n).$$

Component j's BIM has been originally defined by Birnbaum [2] as

$$\text{BIM}_j = \frac{\partial \Psi(p_1, p_2, \ldots, p_n)}{\partial p_j}. \tag{6}$$

Taking into account the so-called pivotal decomposition (Barlow and Proschan [1]),

$$R = \Psi(p_1, p_2, \ldots, p_n) = p_j \cdot \Psi(p_1, p_2, \ldots, 1_j, \ldots, p_n) + (1 - p_j)\Psi(p_1, p_2, \ldots, 0_j, \ldots, p_n),$$

$$\text{BIM}_j = \Psi(p_1, p_2, \ldots, 1_j, \ldots, p_n) - \Psi(p_1, p_2, \ldots, 0_j, \ldots, p_n). \tag{7}$$

In (7), the first term is the reliability of the original system in which component j is replaced by an absolutely reliable one, and the second term is the reliability of the original system in which the component j is permanently *down*. Note also that both reliability functions $\Psi(1_j; \cdot)$ and $\Psi(0_j; \cdot)$ *do not* depend on p_j. Therefore, the expression for BIM_j remains the same for the case that is of interest to us, namely for the case of equal component reliability $p_j \equiv p$.

It will be more convenient to use the *DOWN* probability

$$G(p_1, p_2, \ldots, p_n) = 1 - \Psi(p_1, p_2, \ldots, p_n)$$

and to rewrite (7) as

$$\text{BIM}_j = G(p_1, p_2, \ldots, 0_j, \ldots, p_n) - G(p_1, p_2, \ldots, 1_j, \ldots, p_n). \tag{8}$$

Note that for the case of $q_j = 1 - p_j \equiv q = 1 - p$, the pivotal formula for $G = 1 - R = P(DOWN)$ takes the form

$$G(p) = Q = G(p, p, \ldots, p) = (1 - p) \cdot G(p, \ldots, 0_j, \ldots, p) + p \cdot G(p, \ldots, 1_j, \ldots, p).$$

Let us now rewrite (5) in accordance with this pivotal decomposition. For this purpose we note that

$$A(x) = A(x; 0_j) + A(x; 1_j),$$

where $A(x; 0_j)$ is the number of failure sets (cut sets) that

(i) have exactly x *down* components

(ii) component j is among the *down* components

and $A(x; 1_j)$ is the number of failure sets (cut sets) that

(i) have exactly x *down* components and

(ii) component j is *not* among the *down* components.

Now we can express the *DOWN* probability as

$$Q = q \cdot \sum_{x=1}^{n} A(x; 0_j) q^{x-1}(1-q)^{(n-x)} + (1-q) \sum_{x=1}^{n} A(x; 1_j) q^{x}(1-q)^{(n-x-1)}. \quad \textbf{(9)}$$

In the second sum, the summation in fact goes up to $x = n - 1$ since $A(n; 1_j) = 0$.

The crucial observation is that the first sum is nothing but $G(p, \ldots, 0_j, \ldots, p)$ because the probabilistic measure of the cut set $A(x; 0_j)$ equals $q^{(x-1)}p^{(n-x)}$. By a similar reason, the second sum equals $G(p, \ldots, 1_j, \ldots, p)$.

Let us now connect $A(x; 0_j)$ and $A(x; 1_j)$ with the cumulative spectrum. Recalling the definition of random variable Y in the end of previous section, we see that the probability that system is *DOWN* when x components are *down* splits into two probabilities

$$F(x; 0_j) = P((Y \leq x) \bigcap (\text{component } j \text{ is } down)) \quad \textbf{(10)}$$

and

$$F(x; 1_j) = F(x) - F(x; 0_j) = P((Y \leq x) \bigcap (\text{component } j \text{ is } up)). \quad \textbf{(11)}$$

Therefore,

$$A(x; 0_j) = F(x; 0_j) \frac{n!}{x!(n-x)!}, \qquad A(x; 1_j) = F(x; 1_j) \frac{n!}{x!(n-x)!}. \quad \textbf{(12)}$$

Collecting all together, we arrive at the expression for BIM_j.

Claim 1:

$$\text{BIM}_j = n! \Bigg[\sum_{x=1}^{n} F(x; 0_j) q^{x-1}(1-q)^{(n-x)}/(x!(n-x)!) - \sum_{x=1}^{n} (F(x) - F(x; 1_j)) q^{x}(1-q)^{(n-x-1)} n!/(x!(n-x)!) \Bigg]. \quad \textbf{(13)}$$

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We will call the collection $F(x; 0_j)$, $x = 1, 2, \dots, n$, the *BIM-spectrum* of component j. We note that the BIM of a component depends on its BIM-spectrum *and* on the q value.

Comparing (13) for i and $j \neq i$, we arrive at the following claim.

Claim 2 [3, p. 145]: If

$$F(x; 0_i) \geq F(x; 0_j) \quad \text{for all } x = 1, 2, \dots, n,$$

then for all $q \in (0, 1)$,

$$\mathrm{BIM}_i \geq \mathrm{BIM}_j. \tag{14}$$

In words—if the BIM-spectrum of component i *dominates* the BIM-spectrum of component j, i is more important than j, regardless of the values of q.

4. JOINT RELIABILITY IMPORTANCE

Joint reliability importance (JRI) for two components have been introduced by Hong and Lie [5] as a measure of how components interact in determining system reliability. JRI for components i and j is defined as

$$\mathrm{JRI}_{(i,j)} = \frac{\partial^2 \Psi(\mathbf{p})}{\partial p_i \, \partial p_j}.$$

Using the pivotal formula [1] and the expression (8) for BIM_j, it is easy to obtain the expression

$$\mathrm{JRI}_{(i,j)} = G(1_i, 0_j; \mathbf{p}) + G(0_i, 1_j; \mathbf{p}) - G(0_i, 0_j; \mathbf{p}) - G(1_i, 1_j; \mathbf{p}), \tag{15}$$

where $G(\delta_i, \delta_j; \mathbf{p})$ is the probability that the system is *DOWN* when its component i is in the state δ_i *and* component j is in the state δ_j; $\delta_{(\cdot)} = 1(0)$ if the corresponding component is in the *up* or *down* state, respectively. A similar formula has been derived by Gao et al. in [4].

Now, we follow the main steps that led to Claim 1. First, we turn to (5) and present $A(x)$ in the following form:

$$A(x) = A(x; 1_i, 0_j) + A(x; 0_i, 1_j) + A(x; 0_i, 0_j) + A(x; 1_i, 1_j). \tag{16}$$

Here $A(x; 1_i, 0_j)$ is the number of failure sets (cut sets) that

(i) have exactly x *down* components
(ii) component i is *not* among the *down* components
(iii) component j is among the *down* components.

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(ii) component i is among the *down* components
(iii) component j is *not* among the *down* components.

$A(x; 0_i, 0_j)$ is the number of failure sets (cut sets) that

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(iii) component j is among the *down* components.

$A(x; 1_i, 1_j)$ is the number of failure sets (cut sets) that

(i) have exactly x *down* components
(ii) component i is *not* among the *down* components
(iii) component j is *not* among the *down* components.

Now, rewrite (5) in accordance with the above representation of $A(x)$:

$$Q = qp\sum_{x=1}^{n} A(x; 1_i, 0_j)q^{x-1}p^{(n-x-1)} + qp\sum_{x=1}^{n} A(x; 0_i, 1_j)q^{x-1}p^{(n-x-1)} + q^2\sum_{x=1}^{n} A(x; 0_i, 0_j)q^{x-2}p^{(n-x)} + p^2\sum_{x=1}^{n} A(x; 1_i, 1_j)q^{x}p^{(n-x-2)}. \quad \textbf{(17)}$$

For the sake of symmetry we left x running from 1 to n in all four sums, but we keep in mind that $A(n; 1_i, 0_j) = A(n; 0_i, 1_j) = A(1; 0_i, 0_j) = 0$, and $A(n; 1_i, 1_j) = A(n-1; 1_i, 1_j) = 0$.

Similarly to the reasoning following (9), we observe that the first sum in (17) equals $G(p, \ldots, p, 1_i, \ldots, 0_j, \ldots, p)$, the second sum equals $G(p, \ldots, p, 0_i, \ldots, 1_j, \ldots, p)$, the third sum equals $G(p, \ldots, p, 0_i, \ldots, 0_j, \ldots, p)$, and the last sum is $G(p, \ldots, p, 1_i, \ldots, 1_j, \ldots, p)$.

Now, similarly to the previous decomposition of $F(x)$ (10) and (11), we define

$$F(x; 1_i, 0_j) = P((Y \le x) \bigcap (\text{component } i \text{ is } up) \bigcap (\text{component } j \text{ is } down)),$$

$$F(x; 0_i, 1_j) = P((Y \le x) \bigcap (\text{component } i \text{ is } down) \bigcap (\text{component } j \text{ is } up)),$$

$$F(x; 0_i, 0_j) = P((Y \le x) \bigcap (\text{component } i \text{ is } down) \bigcap (\text{component } j \text{ is } down)),$$

and

$$F(x; 1_i, 1_j) = P((Y \le x) \bigcap (\text{component } i \text{ is } up) \bigcap (\text{component } j \text{ is } up)).$$

Now, we can represent

$$F(x) = F(x; 1_i, 0_j) + F(x; 0_i, 1_j) + F(x; 0_i, 0_j) + F(x; 1_i, 1_j). \tag{18}$$

Finally, we conclude that

$$\begin{aligned} A(x; 1_i, 0_j) &= F(x; 1_i, 0_j)\frac{n!}{x!(n-x)!}; \qquad A(x; 0_i, 1_j) = F(x; 0_i, 1_j)\frac{n!}{x!(n-x)!}; \\ A(x; 0_i, 0_j) &= F(x; 0_i, 0_j)\frac{n!}{x!(n-x)!}; \qquad A(x; 1_i, 1_j) = F(x; 1_i, 1_j)\frac{n!}{x!(n-x)!}. \end{aligned} \tag{19}$$

Note that, for example, $A(x; 0_i, 0_j) \cdot (n!/(x!(n-x)!))^{-1}$ is the probability that a randomly chosen set of x components from n components is a failure set having x components *and* components i and j present in this set, exactly in accord with the definition of $F(x; 0_i, 0_j)$.

Collecting all together, we arrive at the following claim.

Claim 3:

$$\begin{aligned} \mathrm{JRI}_{(i,j)} = n! \Bigg[&\sum_{x=1}^{n} F(x; 1_i, 0_j) q^{x-1} p^{(n-x-1)}/(x!(n-x)!) \\ &+ \sum_{x=1}^{n} F(x; 0_i, 1_j) q^{x-1} p^{(n-x-1)}/(x!(n-x)!) \\ &- \sum_{x=1}^{n} F(x; 0_i, 0_j) q^{x-2} p^{(n-x)}/(x!(n-x)!) \\ &- \sum_{x=1}^{n} F(x; 1_i, 1_j) q^{x} p^{(n-x-2)}/(x!(n-x)!) \Bigg]. \end{aligned} \tag{20}$$

Unfortunately, rather a complicated form of the JRI makes it not possible to find in a general form a simple sufficient condition of dominance of $\mathrm{JRI}_{(i,j)}$, over $\mathrm{JRI}_{(i,s)}$, $s \neq j$, uniformly with respect to q. There is, however, a possibility to investigate the behavior of the JRI for small values of q (i.e., for highly reliable systems).

Assume that $q \to 0$. Then $p^m = (1-q)^m = 1 - qm + o(q)$. Let $x_{\min} \geq 2$ be the size of the minimal min-cut set in the system. The main contribution to $\mathrm{JRI}_{(i,j)}$ will be made by the terms with the lowest degrees of q (i.e., by the first terms in the sums of formula (20)). We arrive, therefore, at Claim 4.

Claim 4: The main term in the asymptotic representation of $\mathrm{JRI}_{(i,j)}$ as $q \to 0$ is

$$\begin{aligned} JRI_{(i,j)} = \{&[A(x_{\min}; 1_i, 0_j) + A(x_{\min}; 0_i, 1_j)] q^{(x_{\min}-1)} \\ &+ A(x_{\min}+1; 0_i, 0_j) q^{(x_{\min}-1)}\}(1 + o(1)). \end{aligned} \tag{21}$$

Let us illustrate this claim by an example considered in Hong and Lie [5]. The article studies an *s-t* network with 12 edges and 8 nodes. Consider its $\mathrm{JRI}_{(3,12)}$. The network has one min-cut of minimal size $x_{\min} = 2$ of edges $(11,12)$ and two cuts of size 3 containing edge 12, namely $(3,9,12)$ and $(3,11,12)$. By (21),

$$\mathrm{JRI}_{(3,12)} = 1 \cdot q - 2 \cdot q + o(q) \approx -q.$$

It exactly coincides with the value of $JRI_{(3,12)}$ given in [5] (p. 21, last column of Table 2), which has been computed by an exact enumeration.

5. SECOND-ORDER APPROXIMATION TO SYSTEM RELIABILITY

Let us return to the reliability function $\Psi(p_1, p_2, \ldots, p_n) = R(\mathbf{p})$ of a coherent system with independent components. Suppose that we have the possibility to increase the reliability of components 1 and 2 by Δp_1 and Δp_2, respectively. Let us present the second-order approximation to the reliability of the "reinforced" system via the Taylor series. The standard expression well known from calculus can be considerably simplified if we take into account that all second derivatives of type

$$\frac{\partial^2 R}{\partial^2 p_i} = 0.$$

It follows directly from (7). Taking this into account and the definitions of BIM_j and $\mathrm{JRI}_{(i,j)}$, we can write the Taylor expansion as

$$R(p_1 + \Delta p_1, p_2 + \Delta p_2; p_3, \ldots, p_n) \approx R(p_1, p_2, \ldots, p_n) + \mathrm{BIM}_1 \cdot \Delta p_1 + \mathrm{BIM}_2 \cdot \Delta p_2 + JRI_{(1,2)} \cdot \Delta p_1 \cdot \Delta p_2. \quad \mathbf{(22)}$$

Several simple recommendations follow from this formula.

1. If a single component is reinforced, the best result is achieved by replacing component r which has the largest value of $\mathrm{BIM}_r \cdot \Delta p_r$.
2. If two components i and j are reinforced, all components have the same BIMs, and $\Delta p_i, i = 1, 2, \ldots, n$ are the same, then the best choice of the (i, j) pair is determined by the maximal value of their JRI.

We will illustrate the use of these recommendations in Example 2.

6. MONTE CARLO APPROXIMATION TO THE BIM-SPECTRA AND JRI

In [3] we have described a Monte Carlo (MC) algorithm for approximating the system D-spectrum (signature). It works as follows. To estimate $F(x)$, we simulated M random permutations $\pi = (i_1, i_2, \ldots, i_n)$ of component numbers and imitated a sequential destruction of components by moving along a permutation from left to right and by

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counting the number N_i of such permutations that the system went *DOWN* on the ith step of the destruction process. Afterward, as an MC estimate of $F(x)$ was taken the quantity

$$\widehat{F}(x) = (N_1 + \cdots + N_x)/M.$$

To obtain an MC estimate of $F(x; 1_j)$, we modify the above described procedure and split the sum $\sum_{i=1}^{x} N_i = M(x)$ as $M(x; 0_j) + M(x; 1_j)$, where the first term is the number of permutations such that the system went *DOWN* during the first x failures *and* component j was among these x components, and the second term equals the number of permutations where j was missing in the first x components. Then

$$\widehat{F}(x; 0_j) = M(x; 0_j)/M, \qquad \widehat{F}(x; 1_j) = M(x; 1_j)/M.$$

The MC procedure for approximating the JRI is quite similar. $M(x)$ is split into four terms:

$$M(x) = M(x; 0_i, 1_j) + M(x; 1_i, 0_j) + M(x; 0_i, 0_j) + M(x; 1_i, 1_j)$$

and

$$\widehat{F}(x; 1_i, 0_j) = M(x; 1_i, 0_j)/M, \qquad \widehat{F}(x; 0_i, 1_j) = M(x; 0_i, 1_j)/M,$$
$$\widehat{F}(x; 0_i, 0_j) = M(x; 0_i, 0_j)/M, \qquad \widehat{F}(x; 1_i, 1_j) = M(x; 1_i, 1_j)/M.$$

7. EXAMPLES

Example 1 (*Terminal connectivity of a cubic network*): Consider the hypercube H_3 network shown in Figure 1. Elements subject to failure are the edges. Nodes 1, 3, and

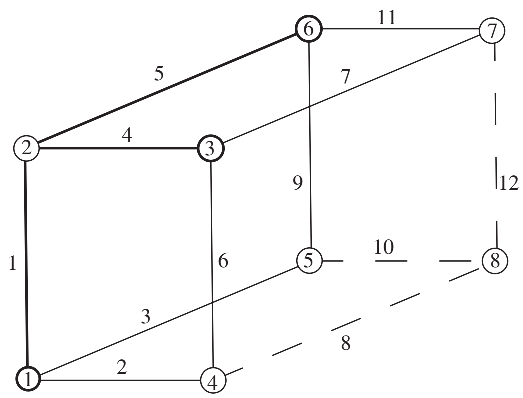

FIGURE 1. H_3 network. Nodes 1, 6, and 3 are terminals. First group edges is shown by bold lines; second group edges is shown by thin lines and third group edges are shown by dotted lines.

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TABLE 1. Simulated BIM-spectra for Nodes 1, 7, and 10. $M = 10,000$ replications

x	$F(x; 0_1)$	$F(x; 0_7)$	$F(x; 0_{10})$
3	0.0041	0.0040	0.0000
4	0.0322	0.0282	0.0112
5	0.1316	0.1209	0.0788
6	0.3476	0.3287	0.2684
7	0.5288	0.5047	0.4652
8	0.6572	0.6387	0.6285
9	0.7463	0.7480	0.7413
10	0.8319	0.8355	0.8344
11	0.9140	0.9185	0.9195
12	1	1	1

6 are terminals. Network failure is defined as the loss of terminal connectivity. The *UP* state is therefore the situation when all three terminals are connected to each other.

As follows from Claim 2, for ranking elements by their BIMs, it is sufficient to compare their BIM-spectra. Table 1 presents the estimated BIM-spectra for three edges 1, 7, and 10, based on $M = 10,000$ Monte Carlo replications. It is seen from Table 1 that for all x, except for for $x = 10, 11$, $F(x; 0_1) > F(x; 0_7) > F(x; 0_{10})$. The statistical error for $x = 10$ in the case of $M = 10,000$ replications is of magnitude ± 0.010. So, it may be assumed that the violation of domination for $x = 10$, 11 by 0.004 is due to a random errors.

Therefore, comparing these three edges, we can conclude that edge 1 is the "most important," edge 7 is the "second important," and edge 10 is the less important. We write it as

$$1 \succ 7 \succ 10.$$

Analyzing the BIM-spectra of all 12 edges of the network, we arrive at the conclusion that there are 3 groups of edges ranked by their importance. The first group consists of equally important edges 1, 4, and 5. In the second group, there are six equally important edges 2, 3, 6, 7, 9, and 11, and in the third group, the three edges 8, 10, and 12 of equal importance:

$$\{1, 4, 5\} \succ \{2, 3, 6, 7, 9, 11\} \succ \{8, 10, 12\}.$$

This ranking has a clear intuitive explanation. Indeed, the edges from the first group have the common property: One node of each edge is a terminal and the second is on the distance 1 from two other terminals. In the second group, one node of each edge is some terminal and the second is on the distance 1 from one terminal node. All other edges are in the third group. If two components can be reinforced, then it seems plausible to choose them from the first group and follow the above recommendations about second derivatives; see Section 4.

For $q = 0.35$, the probability of terminal connectivity equals $R_0 = 0.7670$. Suppose that we decide to reinforce edges 1 and 4 (first group) by 0.1; that is, their

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reliability will be raised to 0.75 from $1 - q = 0.65$. Using the data on BIM-spectra for edges 1 and 4 and formula (13), we find that for $q = 0.35$, $\mathrm{BIM}_1 = 0.204 \approx \mathrm{BIM}_4 = 0.202$. The calculations using (20) show that $\mathrm{JRI}_{(1,4)} = -0.185$. Then using the Taylor series approximation (22), we find out that the probability R_0 will increase by

$$\Delta R \approx \mathrm{BIM}_1 \cdot 0.1 + \mathrm{BIM}_4 \cdot 0.1 + \mathrm{JRI}_{(1,4)} \cdot 0.1^2 \approx 0.0388 \approx 0.04,$$

which is a 4% increase. This is slightly more accurate result that would have been obtained without taking into account the $\mathrm{JRI}_{(1,4)}$.

Example 2 (*a bridge structure*): Consider a bridge structure. It has five edges: $1 = (s, a), 2 = (s, b), 3 = (a, b), 4 = (a, t)$, and $5 = (b, t)$. Edges are subject to failure and system failure is the loss of the $s - t$ connection.

The reliability of the bridge is given by a well-known formula (see, e.g., Barlow and Proschan [1]):

$$R = p_1p_3p_5 + p_2p_3p_4 + p_2p_5 + p_1p_4 - \sum_{r=1}^{5} \left(\prod_{1}^{5} p_i \right) / p_r + 2 \prod_{1}^{5} p_i.$$

Assume that all $p_i = p$. It is easy to check that components (1, 2, 4, 5) have the same BIM values and that $(\mathrm{BIM})_1 > (\mathrm{BIM})_3$. Let us take any subset of three components (e.g., the edges 1, 2, and 5). It is easy to compare their JRIs:

$$\mathrm{JRI}_{(2,5)} = 1 - 3p^2 + 2p^3 > \mathrm{JRI}_{(1,5)} = p - 3p^2 + 2p^3 > \mathrm{JRI}_{(1,2)} = -3p^2 + 2p^3.$$

From here it follows that if one component is reinforced, it is to be taken (if all Δp_i are the same) from the group (2, 4, 5). The best choice of a *pair* of components is (2, 5).

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