Forum of Mathematics, Sigma (2025), **Vol. 13**:e18 1–25
doi:10.1017/fms.2024.158

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Canonical heights for abelian group actions of maximal dynamical rank*

Fei Hu [1] and Guolei Zhong [2]

[1] School of Mathematics, Nanjing University, Nanjing, 210093, China; E-mail: fhu@nju.edu.cn.
[2] Center for Complex Geometry, Institute for Basic Science, Daejeon, 34126, Republic of Korea; E-mail: zhongguolei@u.nus.edu (corresponding author).

Received: 16 December 2023; **Revised:** 11 October 2024; **Accepted:** 20 December 2024

2020 Mathematics Subject Classification: *Primary* – 14J50; *Secondary* – 11G50, 37P15, 37P30

In memory of Professor Nessim Sibony

Abstract
Let X be a smooth projective variety of dimension $n \geq 2$ and $G \cong \mathbf{Z}^{n-1}$ a free abelian group of automorphisms of X over $\overline{\mathbf{Q}}$. Suppose that G is of positive entropy. We construct a canonical height function $\widehat{h}_G$ associated with G, corresponding to a nef and big $\mathbf{R}$-divisor, satisfying the Northcott property. By characterizing the zero locus of $\widehat{h}_G$, we prove the Kawaguchi–Silverman conjecture for each element of G. As for other applications, we determine the height counting function for non-periodic points and show that X satisfies potential density.

Contents

1. Introduction

Given a surjective holomorphic self-map f of a compact Kähler manifold M of dimension n, the topological entropy $h_{\mathrm{top}}(f)$ of f is a key dynamical invariant to measure the divergence of the orbits. A fundamental result due to Gromov [Gro03] and Yomdin [Yom87] establishes its equivalence to the

*The article has been updated since original publication. A notice detailing the change has also been published.

algebraic entropy $h_{\mathrm{alg}}(f)$, defined as the logarithm of the spectral radius of the linear pullback operation f^* acting on the cohomology group $\bigoplus_{k=0}^{n} H^{k,k}(M, \mathbf{C})$:

$$h_{\mathrm{alg}}(f) := \log \max_{0\le k\le n} \rho(f^*|_{H^{k,k}(M,\mathbf{C})}).$$

For each k, the quantity $\rho(f^*|_{H^{k,k}(M,\mathbf{C})})$ is called the *k-th dynamical degree* of f and denoted by $\lambda_k(f)$; see Definition 2.12 for an equivalent definition for projective M. We say that f is of *positive entropy* (resp. *null entropy*) if $h_{\mathrm{alg}}(f) > 0$ (resp. $= 0$) or, equivalently, if its first dynamical degree satisfies $\lambda_1(f) > 1$ (resp. $= 1$). Denote the full automorphism group of M by $\mathrm{Aut}(M)$. A subgroup $G \le \mathrm{Aut}(M)$ is of *positive entropy* if all the elements of $G\backslash\{\mathrm{id}\}$ are of positive entropy.

In their innovative work [DS04], Dinh and Sibony proved that for any abelian subgroup $G \le \mathrm{Aut}(M)$, the subset N of G consisting of automorphisms of null entropy is a group; moreover, there exists a free abelian subgroup P of G, of positive entropy and of rank $r \le n-1$, such that $G \cong P \times N$. The number r is also called the *dynamical rank* $\mathrm{dr}(G)$ of the abelian group G. We say that G is of *maximal dynamical rank* if $\mathrm{dr}(G) = n-1$. Subsequently, Zhang [Zha09] established a theorem of Tits type for $\mathrm{Aut}(M)$ and extended [DS04] to the solvable group case. Since then, algebraic dynamics on systems with maximal dynamical rank has been intensively studied (see, for example, [Din12, Zha13, CWZ14, DHZ15, Zha16, Les18, Hu20, Zho22, Zho23]).

On the other hand, topological dynamics on higher rank abelian group actions has also been investigated for decades, and various rigidity theorems hold on such systems (see, for example, [KS94, KK01, KKRH14]). However, arithmetic dynamics on these higher rank abelian group actions does not seem to attract as much attention. In this paper, we aim to explore the system of maximal dynamical rank from this perspective. Specifically, we construct a canonical height associated with a $G \cong \mathbf{Z}^{n-1}$-action of positive entropy on an n-dimensional smooth projective variety X, extending the works in [Sil91, Kaw08] from surfaces to higher dimensions.

Throughout the paper, unless otherwise stated, we will work over the field $\overline{\mathbf{Q}}$ of algebraic numbers. Below is our main result; see Theorem 4.2 for its more precise form and §2.3 for a brief review of Weil's height theory.

Theorem 1.1 (cf. Theorem 4.2). *Let X be a smooth projective variety of dimension $n \ge 2$ over $\overline{\mathbf{Q}}$. Let $G \cong \mathbf{Z}^{n-1}$ be a free abelian group of automorphisms of X such that any nontrivial element of G has positive entropy. Then there exist a function $\widehat{h}_G$ on $X(\overline{\mathbf{Q}})$ with respect to a set of generators of G and a G-invariant Zariski closed proper subset Z of X such that*

(1) *$\widehat{h}_G$ is a Weil height function corresponding to a nef and big $\mathbf{R}$-divisor on X;*
(2) *$\widehat{h}_G$ satisfies the Northcott property on $(X\backslash Z)(\overline{\mathbf{Q}})$;*
(3) *for any $x \in (X\backslash Z)(\overline{\mathbf{Q}})$, one has $\widehat{h}_G(x) = 0$ if and only if x is g-periodic for any $g \in G$.*

We refer to the function $\widehat{h}_G$ in Theorem 1.1 as *a canonical height function* associated with the abelian group G (cf. Theorem 2.10 for the one associated with a single endomorphism). Canonical height theory was initially developed by Néron and Tate for abelian varieties in the 1960s, and it has proved to be a crucial concept in arithmetic geometry. In his pioneering work [Sil91], Silverman constructed canonical height functions on the so-called Wehler K3 surfaces that are defined by the smooth complete intersection of two divisors of type $(1,1)$ and $(2,2)$ in $\mathbf{P}^2 \times \mathbf{P}^2$. Since then, there has been an extensive body of work on canonical height in arithmetic dynamics (see [BIJ+19, §16] and the references therein).

Before deriving into the applications of our Theorem 1.1, let us provide a brief sketch of the strategy behind its proof.

Remark 1.2. The idea of our proof of Theorem 1.1 draws significant inspiration from the innovative works of Dinh–Sibony [DS04] and Call–Silverman [CS93]. To be specific, we first follow the approach in [DS04] to construct n distinguished automorphisms g_i in G and n common nef $\mathbf{R}$-eigendivisors D_i on X. Each g_i^* expands D_i (up to $\mathbf{R}$-linear equivalence) and shrinks D_j with $j \ne i$ (up to numerical equivalence). We refer to Theorem 3.6 for details.

Furthermore, as established in [CS93], for each pair (g_i, D_i) as described above, there exists a unique nef canonical height function $\widehat{h}_{D_i,g_i}$. As the sum $\sum_i D_i$ forms a nef and big **R**-divisor, the corresponding sum $\sum_i \widehat{h}_{D_i,g_i}$ of nef canonical height functions turns out to be a Weil height function (denoted by $\widehat{h}_G$) satisfying the Northcott property. See Theorem 4.2 for details.

As the first application of our canonical height, we provide a positive answer to the Kawaguchi–Silverman conjecture under the assumption of maximal dynamical rank. In general, for an arithmetic dynamical system (X, f) over $\overline{\mathbf{Q}}$ and a point $x \in X(\overline{\mathbf{Q}})$, we use an ample height function to measure the arithmetic complexity of the orbit $\mathcal{O}_f(x)$. The guiding principle is that 'geometry governs arithmetic' in the sense that the height growth rate along the orbit is controlled by the first dynamical degree. The conjecture predicts that if x is sufficiently complicated (e.g., has a Zariski dense orbit), then the height growth achieves the maximum. For precise definitions and the statement of the conjecture, we refer the reader to §2.4; see also [DGH+22] for a higher-dimensional analog.

Corollary 1.3. *Under the assumption of Theorem 1.1, for each $g \in G$, the arithmetic degree $\alpha_g(x)$ of g at any $x \in X(\overline{\mathbf{Q}})$ equals the first dynamical degree $\lambda_1(g)$ of g, whenever the forward g-orbit $\mathcal{O}_g(x) := \{g^m(x) : m \in \mathbf{Z}_{\geq 0}\}$ of x is Zariski dense in X.*

Remark 1.4. The Kawaguchi–Silverman conjecture (i.e., Conjecture 2.14) has been successfully proved in many cases, leveraging canonical height theory. Notable contributions include works by [Sil91, CS93, Kaw06a, Kaw08, KS14, KS16a, Shi19, LS21], among others. Our approach to proving Corollary 1.3 closely follows this well-established strategy. In comparison, an alternative geometric approach to address Conjecture 2.14 involves the Equivariant Minimal Model Program, as developed by Meng and Zhang. See their survey paper [MZ24a] and the references therein. For the reader interested in this geometric perspective, we recommend exploring [MSS18, Mat20a, MZ22, MY22, MZ23, MZ24b]. For a comprehensive overview of the current state of Conjecture 2.14, we refer to a recent survey [Mat24].

Our second application involves an investigation of the bounded height property for periodic points, as well as an exploration of the height counting function for non-periodic points.

Corollary 1.5 (cf. [Kaw08, Theorem D] and [KS16b, Proposition 3]). *Under the assumption of Theorem 1.1 and with the notation therein, the following assertions hold for any $g \in G\backslash\{\mathrm{id}\}$.*

(1) *The subset below*

$$\{x \in (X\backslash Z)(\overline{\mathbf{Q}}) : x \text{ is } g\text{-periodic}\}$$

is of bounded height (see Definition 2.9).

(2) *For any ample divisor H_X on X and any non-g-periodic $x \in (X\backslash Z)(\overline{\mathbf{Q}})$, one has*

$$\lim_{T\to+\infty} \frac{\#\{m \in \mathbf{Z}_{\geq 0} : h_{H_X}(g^m(x)) \leq T\}}{\log T} = \frac{1}{\log \lambda_1(g)}.$$

As mentioned in Remark 1.2, one of the crucial ingredients to establish Theorem 1.1 is to construct a special nef and big **R**-divisor D, which is the sum of G-common nef eigendivisors (see Theorem 3.6). As a by-product of this construction, we obtain the existence of Zariski dense G-orbits, which also implies that our X satisfies potential density. We refer the reader to [Has03, Cam04, Wit18] and the references therein for the information on potential density.

Corollary 1.6. *Under the assumption of Theorem 1.1, there exists a rational point $x \in X(\overline{\mathbf{Q}})$ such that the G-orbit $\mathcal{O}_G(x) := \{g(x) : g \in G\}$ of x is Zariski dense in X. Moreover, if we assume that $X = Y_{\overline{\mathbf{Q}}} := Y \times_{\operatorname{Spec} K} \operatorname{Spec} \overline{\mathbf{Q}}$ for some Y defined over a number field K, then there exists a finite field extension L/K such that $Y_L(L)$ is Zariski dense in Y_L.*

For the sake of completeness, we recall a few examples satisfying the maximal dynamical rank assumption in Theorem 1.1.

Example 1.7 (cf. [DS04, Exemple 4.5] and [Ogu14, §5.2]). Let E be an elliptic curve over $\overline{\mathbf{Q}}$ and E^n the product variety of E. There is a natural faithful action of $\mathrm{SL}_n(\mathbf{Z})$ on E^n. It is known that $\mathrm{SL}_n(\mathbf{Z})$ admits a free abelian subgroup G of rank $n-1$ which is diagonalizable; any nontrivial element of G has an eigenvalue with modulus greater than 1 (see, for example, [Din12, Example 1.4]). It is not hard to see that for any $g \in G$, the first dynamical degree $\lambda_1(g)$ of g as an automorphism of E^n equals the square of the spectral radius of g as a matrix in $\mathrm{SL}_n(\mathbf{Z})$. So, $G \le \mathrm{Aut}(E^n)$ is of maximal dynamical rank. In this case, as E^n is an abelian variety, the nef and big $\mathbf{R}$-divisor D on E^n obtained from Theorem 3.6(2) is indeed ample. Consequently, the G-invariant Zariski closed proper subset Z in Theorem 1.1 is empty.

Further, thanks to Ueno [Uen75] and Campana [Cam11], one can also construct examples of Calabi–Yau varieties and rationally connected varieties. For instance, let E be an elliptic curve over $\overline{\mathbf{Q}}$ such that $\mathrm{Aut}(E)$ is cyclic of order 4, and σ the generator of $\mathrm{Aut}(E)$. Let $G \le \mathrm{SL}_n(\mathbf{Z})$ be as above. Since the diagonal action by σ is inside the centralizer of G in $\mathrm{Aut}(E^n)$, there is an induced faithful G-action on the quotient $E^n/\langle\sigma\rangle$. Let $X_{n,4}$ be a G-equivariant resolution of singularities of $E^n/\langle\sigma\rangle$. Then it is well known that the Kodaira dimension of $X_{n,4}$ satisfies

$$\kappa(X_{n,4}) = \begin{cases} 0 & \text{if } n \ge 4, \\ -\infty & \text{if } n \le 3. \end{cases}$$

Since dynamical degrees are invariant under generically finite rational maps (see Definition 2.12), $G \le \mathrm{Aut}(X_{n,4})$ is of maximal dynamical rank. In particular, the G-invariant Zariski closed proper subset Z in Theorem 1.1 coincides with the exceptional locus of the equivariant resolution (cf. Lemma 3.16). There is a parallel construction for any elliptic curve E with $\#\,\mathrm{Aut}(E) = 6$. See [Ogu14, §5.2] and the references therein for more details.

This paper is organized as follows. In Section 2, we provide a review of essential concepts and results, including weak numerical equivalence, stable base loci, Weil's height theory and dynamical/arithmetic degrees. In Section 3, we proceed to construct n distinguished automorphisms g_i in G and a nef and big $\mathbf{R}$-divisor as a sum of common nef eigendivisors D_i, possessing favorable properties from an arithmetic perspective (see Theorem 3.6 and Remark 3.7); then we prove Corollary 1.6 as a direct application. Following this, in Section 4, we define a canonical height function associated with the abelian group G of maximal dynamical rank. Finally, we present the proofs of Theorem 1.1 and Corollaries 1.3 and 1.5.

We conclude the introduction with the following remark.

Remark 1.8 (About the generalization of Theorem 1.1). It is noteworthy that extending our main result to normal projective varieties over $\overline{\mathbf{Q}}$ poses no essential difficulties, using intersection theory and Weil's height theory on $\mathbf{R}$-Cartier divisors. Furthermore, the extension of our main Theorem 1.1 to global fields in positive characteristic is also possible using the language in [Tru20, Dan20, Hu20], along with the result from [Hu24]. However, given that the primary focus of this paper is to present a distinctive perspective for studying abelian group actions of maximal dynamical rank, we choose to concentrate on smooth projective varieties over $\overline{\mathbf{Q}}$. This decision is made in the interest of maintaining clarity and simplicity in our exposition. We anticipate that this alternative viewpoint may provide valuable insights into the classification problem of abelian group actions with lower dynamical rank (see [Din12, Problem 1.5]).

2. Preliminaries

We start with notation and terminology. Let X be a smooth projective variety of dimension n over $\overline{\mathbf{Q}}$. The symbols $\sim$, $\approx$ and $\equiv$ stand for rational equivalence, algebraic equivalence and numerical equivalence for algebraic cycles, respectively. Abusing the notation, we also denote $\mathbf{Z}$-, $\mathbf{Q}$-, $\mathbf{R}$-linear equivalence for divisors by $\sim$, $\sim_{\mathbf{Q}}$, $\sim_{\mathbf{R}}$.

For any $0 \le i \le n$, denote by $\mathsf{CH}^i(X)$ the Chow group of algebraic cycles of codimension i on X modulo rational equivalence. It is well known in intersection theory that $\mathsf{CH}(X) := \bigoplus_{i=0}^n \mathsf{CH}^i(X)$ is a graded commutative ring with respect to the intersection product, called the Chow ring. When working with a coefficient field $\mathbf{K} = \mathbf{Q}$ or $\mathbf{R}$, we write

$$\mathsf{CH}^i(X)_{\mathbf{K}} := \mathsf{CH}^i(X) \otimes_{\mathbf{Z}} \mathbf{K}.$$

In particular, when $i = 1$, the Chow group $\mathsf{CH}^1(X)$ coincides with the Picard group $\mathsf{Pic}(X)$ of X. Denote by $\mathsf{Pic}^0(X)$ the subgroup of $\mathsf{Pic}(X)$ consisting of all integral divisors on X algebraically equivalent to zero (modulo linear equivalence); it has a structure of an abelian variety. The quotient group

$$\mathsf{NS}(X) := \mathsf{Pic}(X)/\mathsf{Pic}^0(X)$$

is called the *Néron–Severi group* of X, which is a finitely generated abelian group.

Let $\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$ denote the Picard variety of X over $\overline{\mathbf{Q}}$, i.e., the neutral connected component of the Picard group scheme $\mathbf{Pic}_{X/\overline{\mathbf{Q}}}$ of X over $\overline{\mathbf{Q}}$; it is also the dual abelian variety of the Albanese variety $\mathrm{Alb}(X)$ of X (see [Kle05, Theorem 9.5.4 and Remark 9.5.25]). Note that the group of the $\overline{\mathbf{Q}}$-points of the Picard variety $\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$ is exactly $\mathsf{Pic}^0(X)$.

For any $0 \le i \le n$, denote by $\mathsf{N}^i(X)$ the finitely generated free abelian group of algebraic cycles of codimension i on X modulo numerical equivalence (cf. [Ful98, Definition 19.1]), i.e.,

$$\mathsf{N}^i(X) := \mathsf{CH}^i(X)/\equiv .$$

For $\mathbf{K} = \mathbf{Q}$ or $\mathbf{R}$, denote by $\mathsf{N}^i(X)_{\mathbf{K}} := \mathsf{N}^i(X) \otimes_{\mathbf{Z}} \mathbf{K}$ the associated finite-dimensional $\mathbf{K}$-vector space. It is also well known that when $i = 1$,

$$\mathsf{N}^1(X)_{\mathbf{K}} = \mathsf{NS}(X)_{\mathbf{K}} := \mathsf{NS}(X) \otimes_{\mathbf{Z}} \mathbf{K};$$

see, for example, [Kle05, Theorem 9.6.3]. The *Picard number* $\rho(X)$ of X is defined as the rank of $\mathsf{N}^1(X)$ or $\dim_{\mathbf{R}} \mathsf{N}^1(X)_{\mathbf{R}}$. We henceforth endow $\mathsf{N}^1(X)_{\mathbf{R}}$ with the standard Euclidean topology and fix a norm $\|\cdot\|$ on it.

A divisor D on X is *nef* if the intersection number $D \cdot C$ is nonnegative for any curve C on X. The cone of all nef $\mathbf{R}$-divisors in $\mathsf{N}^1(X)_{\mathbf{R}}$ is called the *nef cone* $\mathsf{Nef}(X)$ of X, which is a salient closed convex cone of full dimension. Its interior is called the *ample cone* $\mathsf{Amp}(X)$ of X. An integral divisor D on X is *big* if the linear system $|mD|$ of some multiple of D induces a birational map $\Phi_{|mD|}$ from X onto its image. An $\mathbf{R}$-divisor D is *big* if it is a positive combination of integral big divisors. It is well known that for any nef $\mathbf{R}$-divisor D on X, it is big if and only if $D^n > 0$.

2.1. Weak numerical equivalence

It turns out to be convenient to consider the following notion (implicitly) introduced by Dinh–Sibony [DS04] and Zhang [Zha09].

Definition 2.1 (Weak numerical equivalence). Let X be a smooth projective variety of dimension n over $\overline{\mathbf{Q}}$. An algebraic cycle Z of codimension i on X is called *w*eakly numerically trivial and denoted by $Z \equiv_{\mathsf{w}} 0$ if

$$Z \cdot H_1 \cdots H_{n-i} = 0$$

for all ample (and hence for all) $\mathbf{R}$-divisors $H_1, \ldots, H_{n-i}$ on X.

Clearly, weak numerical equivalence is coarser than numerical equivalence. An important property of weak numerical equivalence is the following result due to Dinh and Sibony [DS04]. It essentially comes from the Hodge–Riemann bilinear relations and is crucial for deducing the nonvanishing of intersection numbers of divisors.

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

Lemma 2.2 (cf. [DS04, Lemme 4.4]). *Let $D_1, \ldots, D_j, D'_j$ be nef* **R***-divisors on X with $1 \le j \le n-1$ such that $D_1 \cdots D_{j-1} \cdot D_j \not\equiv_{\mathsf{w}} 0$ and $D_1 \cdots D_{j-1} \cdot D'_j \not\equiv_{\mathsf{w}} 0$. Let f be an automorphism of X such that*

$$f^*(D_1 \cdots D_{j-1} \cdot D_j) \equiv_{\mathsf{w}} \lambda D_1 \cdots D_{j-1} \cdot D_j \ \ \textit{and}$$
$$f^*(D_1 \cdots D_{j-1} \cdot D'_j) \equiv_{\mathsf{w}} \lambda' D_1 \cdots D_{j-1} \cdot D'_j$$

with positive real numbers $\lambda \neq \lambda'$. Then $D_1 \cdots D_{j-1} \cdot D_j \cdot D'_j \not\equiv_{\mathsf{w}} 0$.

2.2. Stable and augmented base loci

In the course of our construction of a canonical height function associated with an abelian group G of automorphisms of X, we actually first construct a nef and big **R**-divisor on X. In dealing with the height functions of **R**-divisors, it becomes necessary to consider the so-called augmented base loci of **R**-divisors.

Definition 2.3 (Augmented base loci). Let X be a smooth projective variety over $\overline{\mathbf{Q}}$. The *stable base locus* $\mathbf{B}(D)$ of a **Q**-divisor D is the Zariski closed subset of X defined by

$$\mathbf{B}(D) \coloneqq \bigcap_{m\geq 1,\, mD \text{ is Cartier}} \mathrm{Bs}(mD),$$

where $\mathrm{Bs}(mD)$ denotes the base locus of the linear system $|mD|$. It is an elementary fact that there is an $M \geq 1$ such that $\mathbf{B}(D) = \mathrm{Bs}(MD)$.

The *augmented base locus* $\mathbf{B}_+(D)$ of an **R**-divisor D is the Zariski closed subset of X defined by

$$\mathbf{B}_+(D) \coloneqq \bigcap_A \mathbf{B}(D - A),$$

where the intersection is taken over all ample **R**-divisors A such that $D - A$ is a **Q**-divisor.

For a detailed study of augmented base loci, we direct the reader to [ELM+06] and references therein. Here, we only state a few of them which will be utilized in the proofs of Theorem 1.1 and its corollaries.

Proposition 2.4 (cf. [ELM+06, Propositions 1.4 and 1.5, Example 1.7]). *Let X be a smooth projective variety over $\overline{\mathbf{Q}}$. Then the following assertions hold.*

(1) *For any* **R***-divisor D on X, it is big if and only if $\mathbf{B}_+(D) \neq X$.*
(2) *If D_1 and D_2 are numerically equivalent* **R***-divisors on X, then $\mathbf{B}_+(D_1) = \mathbf{B}_+(D_2)$.*
(3) *For any* **R***-divisor D on X, there is a positive number ε such that for any ample* **R***-divisor A with $\|A\| \leq \varepsilon$ and such that $D - A$ is a* **Q***-divisor, $\mathbf{B}_+(D) = \mathbf{B}(D - A)$.*

Lesieutre and Satriano [LS21] observed that for two nef **R**-divisors D_1, D_2 on X, the augmented base locus of $\mathbf{B}_+(a_1D_1 + a_2D_2)$ is independent of the positive coefficients a_1 and a_2. By induction, we can easily deduce the following.

Lemma 2.5 (cf. [LS21, Lemma 2.16]). *Let $D_1, \ldots, D_m$ be nef* **R***-divisors on X. Then for any $a_1, \ldots, a_m > 0$, one has*

$$\mathbf{B}_+(a_1D_1 + \cdots + a_mD_m) = \mathbf{B}_+(D_1 + \cdots + D_m).$$

2.3. Weil height and canonical height

We refer to [HS00, Part B] for an introduction to Weil's height theory. Among others, we collect some important facts from there.

Theorem 2.6 (cf. [HS00, Theorems B.3.2, B.3.6, and B.5.9] and [Kaw06b, Theorem 1.1.1]). *Let X be a smooth projective variety over $\overline{\mathbf{Q}}$. Then there exists a unique homomorphism*

$$h_X\colon \mathsf{Pic}(X)_{\mathbf{R}} \to \{\text{functions } X(\overline{\mathbf{Q}}) \to \mathbf{R}\}\big/\{\text{bounded functions } X(\overline{\mathbf{Q}}) \to \mathbf{R}\}$$

satisfying the following properties.

(i) *(Normalization) Let D be a very ample divisor on X and $\phi_D\colon X \hookrightarrow \mathbf{P}^N$ the associated embedding. Then we have*

$$h_{X,D} = h \circ \phi_D + O(1),$$

where h is the absolute logarithmic height on $\mathbf{P}^N$ (see [HS00, Definition, Page 176]).

(ii) *(Functoriality) Let $\pi\colon X \to Y$ be a morphism of smooth projective varieties and $D_Y \in \mathsf{Pic}(Y)_{\mathbf{R}}$. Then we have*

$$h_{X,\pi^*D_Y} = h_{Y,D_Y} \circ \pi + O(1).$$

(iii) *(Additivity) Let D_1, D_2 be $\mathbf{R}$-divisors on X. Then we have*

$$h_{X,D_1+D_2} = h_{X,D_1} + h_{X,D_2} + O(1).$$

(iv) *(Positivity) Let D be an effective integral divisor on X. Then $h_{X,D} \geq O(1)$ outside the base locus* $\mathrm{Bs}(D)$ *of D.*

(v) *(Algebraic equivalence) Let $H_X, D \in \mathsf{Pic}(X)_{\mathbf{R}}$ be $\mathbf{R}$-divisors with H_X ample and D algebraically equivalent to zero. Then there is a constant $C > 0$ such that*

$$h_{X,D} \leq C\sqrt{h^+_{X,H_X}},$$

where $h^+_{X,H_X} := \max(1, h_{X,H_X})$.

Remark 2.7. (1) It is worth mentioning that the terms $O(1)$ only depend on varieties, divisors and morphisms but are independent of rational points of varieties. This is why we omit the points $x \in X(\overline{\mathbf{Q}})$ in various height equations. See [HS00, Remarks B.3.2.1(ii)].

(2) When the ambient variety X is clear, we often use h_D to stand for $h_{X,D}$ for simplicity.

The following finiteness property, originally established by Northcott for integral ample divisors, becomes a fundamental tool in Weil's height theory (see [HS00, Theorem B.3.2(g)] and [Kaw06b, Theorem 1.1.2]). Lesieutre and Satriano proved a version for big $\mathbf{R}$-divisors (see [LS21, Lemma 2.26]). For the reader's convenience, we restate it here.

Theorem 2.8 (Northcott finiteness property). *Let X be a smooth projective variety over a number field K and D a big $\mathbf{R}$-divisor on X. Then for any $d \in \mathbf{Z}_{>0}$ and $T \in \mathbf{R}$, the set*

$$\left\{x \in (X\backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}}) : [K(x) : K] \leq d,\ h_D(x) \leq T\right\}$$

is finite, where $\mathbf{B}_+(D)$ denotes the augmented base locus of D. In particular, if D is an ample $\mathbf{R}$-divisor, then $\mathbf{B}_+(D) = \varnothing$, and this is the usual Northcott finiteness property.

Definition 2.9 (Bounded height). Fix an ample divisor H_X on a smooth projective variety X over $\overline{\mathbf{Q}}$. A subset $S \subseteq X(\overline{\mathbf{Q}})$ is called a set of *bounded height* if there is a constant C such that $h_{H_X}(s) \leq C$ for all $s \in S$. This property is independent of the choice of the ample divisor H_X.

The Northcott finiteness property Theorem 2.8 implies that if X is a smooth projective variety over a number field K and $S \subseteq X(\overline{\mathbf{Q}})$ is a set of bounded height, then $\{s \in S : [K(s) : K] \leq d\}$ is finite for any positive integer d.

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

In what follows, we recall a classical result on the canonical height associated with a single endomorphism due to Call and Silverman [CS93, Theorem 1.1], though in the book [HS00] the authors still attributed it to Néron and Tate.

Theorem 2.10 (cf. [HS00, Theorem B.4.1]). *Let $f\colon X \to X$ be an endomorphism of a smooth projective variety X over $\overline{\mathbf{Q}}$. Let D be an $\mathbf{R}$-divisor on X such that $f^*D \sim_{\mathbf{R}} \alpha D$ for some $\alpha > 1$. Then there is a unique function $\widehat{h}_{D,f} : X(\overline{\mathbf{Q}}) \to \mathbf{R}$, called the canonical height on X with respect to f and D, satisfying the following properties:*

(1) $\widehat{h}_{D,f} = h_D + O(1)$;
(2) $\widehat{h}_{D,f} \circ f = \alpha\, \widehat{h}_{D,f}$.

Moreover, the function $\widehat{h}_{D,f}$ is constructed by the limit

$$\widehat{h}_{D,f}(x) \coloneqq \lim_{m\to\infty} \frac{h_D(f^m(x))}{\alpha^m}.$$

Remark 2.11. In the above Theorem 2.10, if we replace $\mathbf{R}$-linear equivalence with algebraic or numerical equivalence – often more practical to verify – Kawaguchi and Silverman introduced a 'canonical height' in a similar manner. Specifically, they showed that given $f^*D' \equiv \beta D'$ for some $\mathbf{R}$-divisor D' and some $\beta > \sqrt{\lambda_1(f)}$, the limit

$$\widehat{h}_{D',f}(x) \coloneqq \lim_{m\to\infty} \frac{h_{D'}(f^m(x))}{\beta^m}$$

exists and satisfies the following properties:

(1) $\widehat{h}_{D',f} = h_{D'} + O\big(\sqrt{h_H^+}\big)$;
(2) $\widehat{h}_{D',f} \circ f = \beta\, \widehat{h}_{D',f}$.

See [KS16b, Theorem 5]. Its proof relies on their height estimate [KS16b, Theorem 24]. Yet, the correct proof of this height estimate is due to Matsuzawa (see [Mat20b, Theorem 1.4]). Further, we remark that if one needs to control the term $O(\sqrt{h_H^+})$, this height estimate is inevitable (see, for example, Lemmas 3.13 and 4.1).

2.4. Dynamical degrees and arithmetic degrees

In this subsection, we let $f\colon X \to X$ be a surjective endomorphism of a smooth projective variety X of dimension n over $\overline{\mathbf{Q}}$ and H_X an ample divisor on X. We recall the definitions of dynamical degrees and arithmetic degrees of this algebraic dynamical system (X, f), as well as the Kawaguchi–Silverman conjecture which reveals the relationship between these two dynamical invariants.

Definition 2.12 (Dynamical degrees). For each $0 \le k \le n$, the *k-th dynamical degree* of f is defined by

$$\lambda_k(f) \coloneqq \lim_{m\to\infty} ((f^m)^* H_X^k \cdot H_X^{n-k})^{1/m} \in \mathbf{R}_{\ge 1}.$$

It is well known that this definition is equivalent to the one given in the introduction. Dynamical degrees are invariant under generically finite rational maps and independent of the choice of the ample H_X. They also satisfy the log-concavity property as follows:

$$\lambda_k(f)^2 \ge \lambda_{k-1}(f)\lambda_{k+1}(f) \text{ for } 1 \le k \le n-1.$$

We refer to [DS05, Tru20, Dan20, HT21] for more detailed discussions on dynamical degrees. The *algebraic entropy* of f is defined by

$$h_{\mathrm{alg}}(f) := \log \max_{0 \le k \le n} \lambda_k(f).$$

Definition 2.13 (Arithmetic degrees). Let h_{H_X} be an absolute logarithmic Weil height function associated with H_X. Set $h^+_{H_X} = \max(1, h_{H_X})$. For each $x \in X(\overline{\mathbf{Q}})$, we define the *arithmetic degree* of f at x by

$$\alpha_f(x) := \lim_{m\to\infty} h^+_{H_X}(f^m(x))^{1/m} \in \mathbf{R}_{\ge 1}.$$

It is known that the limit always exists and is also independent of the choice of the ample divisor H_X (see [KS16a, Theorem 3] and [KS16b, Proposition 12], respectively).

The following conjecture due to Kawaguchi and Silverman asserts the properties of arithmetic degrees. Throughout this paper, we shall only consider this conjecture for automorphisms. We refer to [KS16b, Conjecture 6] for a general version of dominant rational self-maps.

Conjecture 2.14 (cf. [KS16b, Conjecture 6]). *Let $f\colon X \to X$ be a surjective endomorphism of a smooth projective variety X over $\overline{\mathbf{Q}}$. Then for any point $x \in X(\overline{\mathbf{Q}})$, if the forward f-orbit $\mathcal{O}_f(x) := \{f^m(x) : m \in \mathbf{Z}_{\ge 0}\}$ of x is Zariski dense in X, then*

$$\alpha_f(x) = \lambda_1(f).$$

3. Construction of distinguished automorphisms and divisors

This section is devoted to the construction of n distinguished automorphisms in the rank $n-1$ free abelian subgroup $G \le \mathrm{Aut}(X)$ of positive entropy and n common nef $\mathbf{R}$-divisors whose sum is a big $\mathbf{R}$-divisor (see Theorem 3.6). Hinging on them, we shall define a canonical height function in Section 4. As a by-product, we prove Corollary 1.6 at the end of this section.

3.1. Commuting families of linear maps preserving cones

Throughout this subsection, V is a finite-dimensional topological $\mathbf{R}$-vector space, and C is a *salient* closed convex cone in V of *full dimension* (i.e., $C \cap (-C) = \{0\}$ and C spans V, respectively). We recall a few facts on them from linear algebra which are crucial to the construction in the next §3.2. The following is Garrett Birkhoff's generalization of the classical Perron–Frobenius theorem.

Theorem 3.1 (cf. [Bir67]). *Let $\varphi \in \mathrm{End}(V)$ be an $\mathbf{R}$-linear endomorphism of V such that C is φ-invariant (i.e., $\varphi(C) \subseteq C$). Then the spectral radius $\rho(\varphi)$ is an eigenvalue of φ, and there is an eigenvector $v_\varphi \in C$ of φ associated with $\rho(\varphi)$.*

It is well known that a commuting family $\mathcal{F} \subseteq \mathrm{M}_n(\mathbf{C})$ of complex matrices possesses a nonzero common eigenvector $v \in \mathbf{C}^n$. Below is its analog when $\mathcal{F}$ is a family of real matrices preserving a salient closed convex cone of full dimension. It is essentially due to Dinh and Sibony (see [DS04, Proposition 4.1]).

Proposition 3.2. *Let $\mathcal{F} \subseteq \mathrm{End}(V)$ be a commuting family of $\mathbf{R}$-linear endomorphisms of V such that C is $\mathcal{F}$-invariant (i.e., $\psi(C) \subseteq C$ for any $\psi \in \mathcal{F}$). Then for any $\varphi \in \mathcal{F}$, there exists a nonzero vector $v_\varphi \in C$ such that*

(1) *for any $\psi \in \mathcal{F}$, $\psi(v_\varphi) \in \mathbf{R}_{\ge 0} \cdot v_\varphi$ (i.e., v_φ is a common eigenvector for all $\psi \in \mathcal{F}$ associated with some nonnegative eigenvalues); and moreover,*
(2) $\varphi(v_\varphi) = \rho(\varphi) v_\varphi$.

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

Proof. Let $\varphi \in \mathcal{F}$ be fixed. By Birkhoff's Theorem 3.1, the spectral radius $\rho(\varphi)$ of φ is an eigenvalue of φ, and the corresponding eigenvector can be chosen to lie in C. In particular,

$$C_\varphi := \{v \in C : \varphi(v) = \rho(\varphi)v\}$$

is a nonzero salient closed convex cone in V. It is easy to see that C_φ is $\mathcal{F}$-invariant. Indeed, for any $\psi \in \mathcal{F}$ and any $v \in C_\varphi \subseteq C$, by assumption that C is $\mathcal{F}$-invariant, we have $\psi(v) \in C$. Then it follows from the commutativity of $\mathcal{F}$ that

$$\varphi(\psi(v)) = \psi(\varphi(v)) = \psi(\rho(\varphi)v) = \rho(\varphi)\psi(v).$$

Hence $\psi(v) \in C_\varphi$ by the definition of C_φ.

Denote by V_φ the $\mathbf{R}$-vector subspace of V spanned by C_φ. Clearly, V_φ is nonzero, contained in the eigenspace of φ associated with the eigenvalue $\rho(\varphi)$, and $\mathcal{F}$-invariant, since so is C_φ. It suffices to show that there exists a common eigenvector $v_\varphi \in C_\varphi$ for any $\psi \in \mathcal{F}$. In other words, let ψ be arbitrary in $\mathcal{F}$ and denote

$$\widetilde{C}_\psi := \{v \in C_\varphi : \psi(v) = \chi_\psi v,\ \chi_\psi \in \mathbf{R}_{\geq 0}\},$$

which is a nonzero salient closed (possibly nonconvex) cone in V_φ. Then it remains to show that

$$\bigcap_{\psi \in \mathcal{F}} \widetilde{C}_\psi \neq \{0\}, \tag{3.1}$$

or equivalently, in the quotient space $\mathbf{P}^+(V_\varphi) := (V_\varphi \backslash \{0\})/\mathbf{R}_{>0}$ (think of $\mathbf{R}_{>0}$ as the multiplicative subgroup of $\mathbf{R}^* := \mathbf{R}\backslash\{0\}$),

$$\bigcap_{\psi \in \mathcal{F}} \mathbf{P}^+(\widetilde{C}_\psi) \neq \varnothing, \tag{3.2}$$

where each $\mathbf{P}^+(\widetilde{C}_\psi)$ denotes the image of $\widetilde{C}_\psi \backslash \{0\}$ under the natural quotient map

$$\pi\colon V_\varphi \backslash \{0\} \to \mathbf{P}^+(V_\varphi).$$

Note that $\mathbf{P}^+(V_\varphi)$ endowed with the quotient topology is homeomorphic with the $(\dim V_\varphi - 1)$-sphere and hence compact. Moreover, $\mathbf{P}^+(\widetilde{C}_\psi)$ is closed in $\mathbf{P}^+(V_\varphi)$ since so is $\widetilde{C}_\psi \backslash \{0\}$ in $V_\varphi \backslash \{0\}$. We are thus reduced to show Equation (3.2) or Equation (3.1) for any finite $\mathcal{F}' \subseteq \mathcal{F}$.

Suppose now that $\mathcal{F}' = \{\psi_0, \psi_1, \ldots, \psi_m\}$ is any fixed finite subset of $\mathcal{F}$. By adding the fixed endomorphism φ to $\mathcal{F}'$, if necessary, we may assume that $\psi_0 = \varphi$. Let $V_0 := V_\varphi$ and $C_0 := C_\varphi$. We shall inductively construct pairs (V_k, C_k), $0 \leq k \leq m$, satisfying the following properties:

(i) $V_0 \supseteq V_1 \supseteq \cdots \supseteq V_m \neq \{0\}$ is a decreasing sequence of nonzero $\mathcal{F}$-invariant $\mathbf{R}$-vector subspaces of V;
(ii) $C_0 \supseteq C_1 \supseteq \cdots \supseteq C_m \neq \{0\}$ is a decreasing sequence of nonzero $\mathcal{F}$-invariant salient closed convex cones in V;
(iii) for each $0 \leq k \leq m$, C_k spans V_k and

$$C_k \subseteq \bigcap_{i=0}^{k} \widetilde{C}_{\psi_i}.$$

As an immediate consequence of this construction, one gets $\bigcap_{i=0}^m \widetilde{C}_{\psi_i} \neq \{0\}$ since it contains the nonzero cone C_m, which completes the proof of Proposition 3.2.

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

By the definition of $\widetilde{C}_\varphi$, one has $C_\varphi = \widetilde{C}_\varphi = \widetilde{C}_{\psi_0}$. The assertion for (V_0, C_0) is hence true. By the inductive hypothesis, suppose that we have constructed pairs (V_i, C_i) for all $0 \le i \le k-1$ with $1 \le k \le m$. We then construct (V_k, C_k) satisfying all properties i to iii. Note that V_{k-1} and the spanning cone C_{k-1} are both $\mathcal{F}$-invariant and hence ψ_k-invariant. It follows from Theorem 3.1, applied to the triplet $(V_{k-1}, \psi_k|_{V_{k-1}}, C_{k-1})$, that

$$C_k := \{v \in C_{k-1} : \psi_k(v) = \rho(\psi_k|_{V_{k-1}})v\}$$

is a nonzero salient closed convex cone in V_{k-1}. Again by the commutativity of $\mathcal{F}$ and the $\mathcal{F}$-invariance of C_{k-1}, we see that C_k is $\mathcal{F}$-invariant. Let V_k denote the $\mathbf{R}$-vector subspace of V_{k-1} spanned by C_k. Then V_k is also $\mathcal{F}$-invariant. Moreover, as $C_k \subseteq C_{k-1} \cap \widetilde{C}_{\psi_k}$ by construction, the property iii for C_k follows by inductive hypothesis. □

Remark 3.3. Note that in the above Proposition 3.2, if we replace $\mathcal{F} \subseteq \mathrm{End}(V)$ with an abelian subgroup $\mathcal{G}$ of the $\mathbf{R}$-linear automorphism group $\mathrm{GL}(V)$ of V, then for any $\varphi \in \mathcal{G}$, there is a common eigenvector $v_\varphi \in C$ such that for any $\psi \in \mathcal{G}$, $\psi(v_\varphi) \in \mathbf{R}_{>0} \cdot v_\varphi$ and $\varphi(v_\varphi) = \rho(\varphi)v_\varphi$. It gives rise to a multiplicative group character $\chi_\varphi \colon \mathcal{G} \longrightarrow (\mathbf{R}_{>0}, \times)$ defined by $\psi(v_\varphi) = \chi_\varphi(\psi)v_\varphi$ for any $\psi \in \mathcal{G}$; the character χ_φ also satisfies that $\chi_\varphi(\varphi) = \rho(\varphi)$.

A priori, it is still unknown that for different φ and φ', the above characters χ_φ and $\chi_{\varphi'}$ are distinct, neither the uniqueness of χ_φ for each φ. Nonetheless, the following lemma shows that there are at most $\dim_{\mathbf{R}} V$ distinct characters of $\mathcal{G}$ constructed via common eigenvectors in C.

Lemma 3.4. *Let $\mathcal{G}$ be an abelian subgroup of $\mathrm{GL}(V)$ such that C is $\mathcal{G}$-invariant. As in Remark 3.3, let $\{\chi_1, \ldots, \chi_m\}$ denote the set of all distinct multiplicative group characters of $\mathcal{G}$, where each χ_i is associated with some (mutually noncolinear) common eigenvectors $v_i \in C$; that is, for any $\psi \in \mathcal{G}$ and any $1 \le i \le m$,*

$$\psi(v_i) = \chi_i(\psi)v_i.$$

Then the above $v_1, \ldots, v_m$ are linearly independent; in particular, $m \le \dim_{\mathbf{R}} V$ holds. Moreover, for any $\varphi \in \mathcal{G}$, there exists some $1 \le i \le m$ such that $\chi_i(\varphi) = \rho(\varphi)$, i.e., $\varphi(v_i) = \rho(\varphi)v_i$.

Proof. We prove the first half by induction. Suppose by inductive hypothesis that $v_1, \ldots, v_k$ are linearly independent for $1 \le k \le m-1$. Suppose that we have

$$\sum_{i=1}^{k+1} a_i v_i = 0 \tag{3.3}$$

for some $a_i \in \mathbf{R}$. Fix an arbitrary index j with $1 \le j \le k$. Since $\chi_{k+1} \ne \chi_j$, there is some $\psi_\circ \in \mathcal{G}$ such that $\chi_{k+1}(\psi_\circ) \ne \chi_j(\psi_\circ)$. Note that for any $1 \le i \le k+1$, one has $\psi_\circ(v_i) = \chi_i(\psi_\circ)v_i$ by definition. Hence, applying the above $\psi_\circ$ to Equation (3.3) yields that

$$\sum_{i=1}^{k+1} a_i \chi_i(\psi_\circ) v_i = 0. \tag{3.4}$$

Using the above two Equations (3.3) and (3.4) to cancel the coefficient of v_{k+1}, we obtain that

$$\sum_{i=1}^{k} a_i(\chi_{k+1}(\psi_\circ) - \chi_i(\psi_\circ))v_i = 0.$$

Since $v_1, \ldots, v_k$ are linearly independent, $a_i(\chi_{k+1}(\psi_\circ) - \chi_i(\psi_\circ)) = 0$ for all $1 \le i \le k$. In particular, $a_j(\chi_{k+1}(\psi_\circ) - \chi_j(\psi_\circ)) = 0$, and hence, $a_j = 0$. We thus prove the linear independence of $v_1, \ldots, v_m$ by induction.

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

For the second half, suppose to the contrary that there exists some $\varphi_\circ \in \mathcal{G}$ such that $\chi_i(\varphi_\circ) \neq \rho(\varphi_\circ)$ for all $1 \leq i \leq m$. Then by Proposition 3.2, there is a group character $\chi_{\varphi_\circ}$ of $\mathcal{G}$ associated with some common eigenvector $v_{\varphi_\circ} \in C$ such that $\chi_{\varphi_\circ}(\varphi_\circ) = \rho(\varphi_\circ)$, i.e., $\varphi_\circ(v_{\varphi_\circ}) = \rho(\varphi_\circ)v_{\varphi_\circ}$. Clearly, this new character $\chi_{\varphi_\circ}$ is different from any other χ_i, a contradiction. □

At the end of this subsection, we provide an auxiliary module-theoretic result that will be used in the proof of Lemma 3.13 to strengthen eigenequations of **R**-divisors modulo numerical equivalence to equations modulo **R**-linear equivalence.

Lemma 3.5. *Let $R \subseteq S$ be integral domains, M an (unnecessarily finitely generated) R-module and $\varphi\colon M \longrightarrow M$ an R-linear map such that $P(\varphi) = 0$ for some polynomial $P(t) \in R[t]$. Denote the field of fractions of S by K. Let $M_K := M \otimes_R K$ be the vector space over K and $\varphi_K := \varphi \otimes_R \mathrm{id}_K$ the induced K-linear map on M_K. Let $s \in S$ such that $P(s) \neq 0$ in S. Then $\varphi_K - s\,\mathrm{id}_{M_K}$ is an isomorphism of M_K.*

Proof. We first prove that $\varphi_K - s\,\mathrm{id}_{M_K}$ is injective. Let $v \in M_K$ such that $(\varphi_K - s\,\mathrm{id}_{M_K})(v) = 0$, i.e., $\varphi_K(v) = sv$. It is easy to verify that $P(\varphi_K) = (P(\varphi))_K = 0$. Then we have

$$0 = P(\varphi_K)(v) = P(s)v.$$

As $P(s) \neq 0$ is invertible in K, it follows that $v = 0$.

We next show that $\varphi_K - s\,\mathrm{id}_{M_K}$ is also surjective. Denote the degree of the polynomial $P(t)$ by n. Let $w \in M_K$ be arbitrary. As $P(\varphi_K)(w) = 0$, there exist $c_0, \ldots, c_{n-1} \in K$ (or rather, in the field of fractions of R), such that

$$\varphi_K^n(w) = c_{n-1}\varphi_K^{n-1}(w) + \cdots + c_1\varphi_K(w) + c_0 w.$$

In other words, the vector space B_w generated by $\{w, \varphi_K(w), \varphi_K^2(w), \ldots\}$ over K is a finite-dimensional φ_K-invariant subspace of M_K. It follows that the restriction map $(\varphi_K - s\,\mathrm{id}_{M_K})|_{B_w}$ of $\varphi_K - s\,\mathrm{id}_{M_K}$ on B_w is also injective. As B_w is of finite dimension, $(\varphi_K - s\,\mathrm{id}_{M_K})|_{B_w}$ is surjective. Since $w \in M_K$ is arbitrary, we see that $\varphi_K - s\,\mathrm{id}_{M_K}$ itself is surjective. □

3.2. Automorphisms and divisors associated with abelian groups of maximal dynamical rank

Below is the main result of this section. Given an abelian subgroup $\mathbf{Z}^{n-1} \cong G \leq \mathrm{Aut}(X)$ of positive entropy, we construct n distinguished automorphisms in G and a nef and big **R**-divisor associated with G. This construction forms a crucial ingredient for defining a canonical height in Section 4 (see Theorem 4.2).

Theorem 3.6. *Let X be a smooth projective variety of dimension $n \geq 2$ defined over $\overline{\mathbf{Q}}$ and $G \cong \mathbf{Z}^{n-1}$ a free abelian group of automorphisms of X of positive entropy. Then the following assertions hold.*

(1) *The set*

$$\left\{\chi\colon G \longrightarrow (\mathbf{R}_{>0}, \times) \;\middle|\; \begin{array}{l} \chi \text{ is a group homomorphism and there is a nef } \mathbf{R}\text{-divisor} \\ D \not\equiv 0 \text{ such that } g^*D \equiv \chi(g)D \text{ for any } g \in G \end{array}\right\}$$

has n elements χ_i with $1 \leq i \leq n$, where each χ_i is associated with some common nef eigendivisor D_i (i.e., $g^*D_i \equiv \chi_i(g)D_i$ for any $g \in G$).

(2) *$D_1 \cdots D_n \in \mathbf{R}_{>0}$; in particular, $D := \sum_{i=1}^n D_i$ is a nef and big **R**-divisor on X.*

(3) *For any $g \in G$, there is some $1 \leq i \leq n$ such that $\chi_i(g) = \lambda_1(g)$.*

(4) *There exist n automorphisms $g_1, \ldots, g_n \in G$ such that for any $1 \leq i \neq j \leq n$, we have $\chi_j(g_i) < 1$, and moreover,*

$$g_i^*D_i \sim_{\mathbf{R}} \chi_i(g_i)D_i = \lambda_1(g_i)D_i.$$

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

Remark 3.7. The construction of the common nef eigendivisors D_i is essentially attributed to Dinh and Sibony, who initially constructed $n-1$ of them and then separately constructed the last one (see the last paragraph of the proof of [DS04, Théorème 4.4]). This separateness is a crucial aspect, preventing a direct use of their construction to define a canonical height in an appropriate way. Upon revisiting [DS04] and examining each character $\chi_i\colon G \longrightarrow (\mathbf{R}_{>0}, \times)$, we demonstrate that there exist n distinguished automorphisms $g_1, \ldots, g_n$ in the rank $n-1$ abelian group G such that $\chi_i(g_i) = \lambda_1(g_i)$ and $\chi_j(g_i) < 1$ for all $j \neq i$. Consequently, all n nef eigendivisors D_i share the same status as in Theorem 3.6. It is worth mentioning that the automorphisms $g_1, \ldots, g_n$ we constructed may not necessarily be the generators of G. Instead, any $n-1$ of them generate a finite index subgroup of G (see Remark 3.15).

Before proving the above Theorem 3.6 at the end of this subsection, we prepare all necessary ingredients. Recall that $\mathsf{N}^1(X)_{\mathbf{R}}$ is the real Néron–Severi space of $\mathbf{R}$-divisors on X modulo numerical equivalence $\equiv$; its real dimension $\dim_{\mathbf{R}} \mathsf{N}^1(X)_{\mathbf{R}}$ is called the Picard number of X, denoted by $\rho(X)$. The nef cone $\mathsf{Nef}(X)$, consisting of the classes of all nef $\mathbf{R}$-divisors on X, is a salient closed convex cone in $\mathsf{N}^1(X)_{\mathbf{R}}$ of full dimension. The pullback action of automorphisms on $\mathsf{N}^1(X)_{\mathbf{R}}$ induces a natural representation of $\mathrm{Aut}(X)$:

$$\mathrm{Aut}(X) \longrightarrow \mathrm{GL}(\mathsf{N}^1(X)_{\mathbf{R}}), \quad g \mapsto g^*|_{\mathsf{N}^1(X)_{\mathbf{R}}}.$$

Note that any automorphism preserves the nef cone $\mathsf{Nef}(X) \subseteq \mathsf{N}^1(X)_{\mathbf{R}}$. For any subgroup G of $\mathrm{Aut}(X)$, denote by $G|_{\mathsf{N}^1(X)_{\mathbf{R}}} \leq \mathrm{GL}(\mathsf{N}^1(X)_{\mathbf{R}})$ the image of the above representation.

First of all, as a straightforward application of the previous discussion in §3.1 to the triplet $(\mathsf{N}^1(X)_{\mathbf{R}}, G|_{\mathsf{N}^1(X)_{\mathbf{R}}}, \mathsf{Nef}(X))$, we get some nonzero common nef eigendivisors D_i that naturally define multiplicative group characters χ_i of $G|_{\mathsf{N}^1(X)_{\mathbf{R}}}$. Composing them with the group homomorphism $G \longrightarrow G|_{\mathsf{N}^1(X)_{\mathbf{R}}}$ yields group characters of G itself, still denoted by χ_i. In summary, we obtain the following.

Proposition 3.8. *Let X be a smooth projective variety of dimension $n \geq 2$ defined over $\overline{\mathbf{Q}}$ and G an abelian subgroup of $\mathrm{Aut}(X)$. Let m be the number of all distinct multiplicative group characters $\chi_i\colon G \longrightarrow (\mathbf{R}_{>0}, \times)$ $(1 \leq i \leq m)$, where χ_i is associated with some common nef eigendivisor D_i (i.e., $g^*D_i \equiv \chi_i(g)D_i$ for any $g \in G$). Then $1 \leq m \leq \rho(X)$ holds. Further, for any $g \in G$, there is some $1 \leq i \leq m$ such that $\chi_i(g) = \rho(g^*|_{\mathsf{N}^1(X)_{\mathbf{R}}}) = \lambda_1(g)$.*

Proof. It follows readily from Proposition 3.2, Remark 3.3 and Lemma 3.4. □

By the above Proposition 3.8, following [DS04], we define a group homomorphism

$$\begin{aligned} \pi\colon G &\longrightarrow (\mathbf{R}^m, +) \\ g &\mapsto (\log \chi_1(g), \ldots, \log \chi_m(g)). \end{aligned}$$

Lemma 3.9 (cf. [DS04, Corollaire 2.2]). *Let X be a smooth projective variety of dimension $n \geq 2$ defined over $\overline{\mathbf{Q}}$. Then the set of the first dynamical degrees of surjective endomorphisms of X is discrete in $[1, +\infty)$.*

Proof. It suffices to show that for any $M > 1$, the following set

$$S_M := \{\lambda_1(f) : f \text{ a surjective endomorphism of} X \text{ with } \lambda_1(f) \leq M\}$$

is finite. We note that the first dynamical degree $\lambda_1(f)$ of a surjective endomorphism $f\colon X \longrightarrow X$ is the spectral radius of $f^*|_{\mathsf{N}^1(X)_{\mathbf{R}}}$, which is induced from $f^*|_{\mathsf{N}^1(X)}$. Since $\mathsf{N}^1(X)$ is a free abelian group of rank $\rho := \rho(X)$, all the eigenvalues of $f^*|_{\mathsf{N}^1(X)_{\mathbf{R}}}$ are algebraic integers. That is, every $\lambda_1(f)$ is the maximal modulus of the roots of a monic polynomial of degree ρ with integer coefficients. Let $P(t) = t^\rho + c_1 t^{\rho-1} + \cdots + c_\rho \in \mathbf{Z}[t]$ be such a polynomial that the maximal modulus of all roots $\alpha_1, \ldots, \alpha_\rho$, counting with multiplicities, is no more than M. It is well known that each $c_i = (-1)^i s_i(\alpha_1, \ldots, \alpha_\rho)$,

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

where s_i is the i-th elementary symmetric polynomial. Since the $|\alpha_i|$ are bounded and ρ is fixed, the $|c_i|$ are also bounded. In particular, $\#S_M$ is finite. We finish the proof of the lemma. $\square$

From now on, we consider abelian groups G of automorphisms of positive entropy. Recall that a *lattice* Γ in an $\mathbf{R}$-vector space V is a (possibly non-cocompact) discrete free subgroup. It is called *complete* if its rank equals the dimension of V.

Proposition 3.10 (cf. [DS04, Proposition 4.2]). *Let $G \le \mathrm{Aut}(X)$ be an abelian group of automorphisms of positive entropy. Then π is injective, and its image $\pi(G)$ is discrete in $\mathbf{R}^m$. In particular, G is free abelian, and $\pi(G)$ is a lattice in $\mathbf{R}^m$ of rank $r \le m$.*

Proof. For any $g \in G\backslash\{\mathrm{id}\}$, it follows from Proposition 3.8 that one of the coordinates of $\pi(g)$ coincides with $\log \lambda_1(g)$ which is positive; hence, π is injective. As π is a group homomorphism, to show that the image $\pi(G)$ is discrete, it is sufficient to show that $\pi(\mathrm{id}) = 0$ is an isolated point in the image $\pi(G)$. Applying Lemma 3.9, we see for each $g \in G\backslash\{\mathrm{id}\}$ that $\log \lambda_1(g)$ has a uniform lower bound (which is independent of g) and hence $\{0\}$ is an isolated point in $\pi(G)$. Note that the image $\pi(G)$ is an additive subgroup which is also discrete in $\mathbf{R}^m$. Hence, $\pi(G)$ is a lattice in $\mathbf{R}^m$. Because π is injective, G is free abelian of rank $r \le m$. $\square$

The lemma below provides a more accurate range of the number m of all distinct multiplicative group characters χ_i of G (cf. Proposition 3.8).

Lemma 3.11. *Let $G \le \mathrm{Aut}(X)$ be a free abelian group of rank r of positive entropy. Let $\chi_1, \ldots, \chi_m$ and $D_1, \ldots, D_m$ be in Proposition 3.8. Then we have*

$$D_1 \cdots D_m \not\equiv_{\mathrm{w}} 0 \textit{ and } r + 1 \le m \le \min(n, \rho(X)).$$

Proof. We first prove that $D_{i_1} \cdots D_{i_k} \not\equiv_{\mathrm{w}} 0$ for any multi-index $1 \le i_1 < \cdots < i_k \le \min(m, n)$ by induction. By the higher-dimensional Hodge index theorem (see, for example, [Zha16, Lemma 3.2]), we have $D_i \not\equiv_{\mathrm{w}} 0$ for each $1 \le i \le m$. Suppose that the intersection product of any $j \le k-1$ different divisors choosing from $D_1, \ldots, D_m$ is not weakly numerically trivial. Fix a multi-index $1 \le i_1 < \cdots < i_j < i_{j+1} \le \min(m, n)$. By inductive hypothesis, we have

$$D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_j} \not\equiv_{\mathrm{w}} 0 \text{ and } D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_{j+1}} \not\equiv_{\mathrm{w}} 0.$$

Since the χ_i are distinct, one has $\chi_{i_j} \neq \chi_{i_{j+1}}$, i.e., there is some $g_\circ \in G$ such that $\chi_{i_j}(g_\circ) \neq \chi_{i_{j+1}}(g_\circ)$. It follows from G-invariance of the D_i that

$$g_\circ^*(D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_j}) \equiv_{\mathrm{w}} \chi_{i_1}(g_\circ) \cdots \chi_{i_{j-1}}(g_\circ)\, \chi_{i_j}(g_\circ)(D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_j}) \text{ and}$$
$$g_\circ^*(D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_{j+1}}) \equiv_{\mathrm{w}} \chi_{i_1}(g_\circ) \cdots \chi_{i_{j-1}}(g_\circ)\, \chi_{i_{j+1}}(g_\circ)(D_{i_1} \cdots D_{i_{j-1}} \cdot D_{i_{j+1}}).$$

Noting that $j \le k-1 \le n-1$ by assumption, one has $D_{i_1} \cdots D_{i_j} \cdot D_{i_{j+1}} \not\equiv_{\mathrm{w}} 0$, thanks to Lemma 2.2. In other words, we have proved by induction that the product of any $k \le \min(m, n)$ different divisors from $D_1, \ldots, D_m$ is not weakly numerically trivial.

By Proposition 3.10 and Lemma 3.4, we already know that $r \le m \le \rho(X)$. Suppose to the contrary that there are $m \ge n+1$ distinct group characters $\chi_1, \ldots, \chi_{n+1}, \ldots, \chi_m$ of G associated with some common nef eigendivisors $D_1, \ldots, D_m$. By what we just proved, one has $D_1 \cdots D_n > 0$ and $D_2 \cdots D_{n+1} > 0$. Then by the projection formula, we have for any $g \in G$,

$$0 \neq D_1 \cdots D_n = \deg(g)(D_1 \cdots D_n) = g^*D_1 \cdots g^*D_n = \chi_1(g) \cdots \chi_n(g)(D_1 \cdots D_n),$$

and hence, $\chi_1(g) \cdots \chi_n(g) = 1$. Similarly, we also have $\chi_2(g) \cdots \chi_{n+1}(g) = 1$, so that $\chi_1(g) = \chi_{n+1}(g)$, a contradiction. So we get $m \le n$, and hence, $D_1 \cdots D_m \not\equiv_{\mathrm{w}} 0$, as desired.

It remains to show that $m \ge r+1$. Suppose to the contrary that $m = r$. Then $\pi(G)$ is a complete lattice in $\mathbf{R}^m$ (see Proposition 3.10). Namely, $\pi(G)$ spans $\mathbf{R}^m$. Therefore, there is some $g_\diamond \in G\backslash\{\mathrm{id}\}$

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

such that all m coordinates of $\pi(g_\diamond)$ are negative (i.e., $\chi_i(g_\diamond) < 1$ for all $1 \le i \le m$). On the other hand, Proposition 3.8 asserts that for such $g_\diamond$, there is some $1 \le i_\diamond \le m$ such that $\chi_{i_\diamond}(g_\diamond) = \lambda_1(g_\diamond) > 1$, since $g_\diamond$ is of positive entropy. This is a contradiction. □

Remark 3.12. Note that all these common nef eigendivisors D_i in Proposition 3.8 are constructed numerically. Namely, they only satisfy eigenequations modulo numerical equivalence. Though it is enough to define nef canonical height functions in the sense of [KS16b, Theorem 5(a)], the difference may not be bounded (see [KS16b, Theorem 5(b)]). In dimension two, Kawaguchi [Kaw08, Lemma 3.8] managed to improve them to eigenequations modulo $\mathbf{R}$-linear equivalence so that the difference is indeed bounded. Such an eigenequation modulo linear equivalence also appears in [Zha06, Proposition 1.1.3], [NZ10, Lemma 2.3] and [MMS+22, Theorem 6.4(1)].

Below is a higher-dimensional analog of [Kaw08, Lemma 3.8], which will be used in the proof of Theorem 3.6(4) shortly.

Lemma 3.13. *Let X be a smooth projective variety of dimension n over $\overline{\mathbf{Q}}$ and f a surjective endomorphism of X of positive entropy (i.e., $\lambda_1(f) > 1$). Then the following assertions hold:*

(1) *there is a nef $\mathbf{R}$-divisor D_f on X such that $f^*D_f \equiv \lambda_1(f)D_f$; further,*
(2) *for any D_f in the assertion (1), there is a unique nef $\mathbf{R}$-divisor D'_f on X, up to $\mathbf{R}$-linear equivalence, such that $D'_f \equiv D_f$ and $f^*D'_f \sim_{\mathbf{R}} \lambda_1(f)D'_f$.*

Proof. It is well known that the assertion (1) follows from Birkhoff's Theorem 3.1. Fix such a nef $\mathbf{R}$-divisor $D_f \in \mathsf{Nef}(X)$ such that $f^*D_f \equiv \lambda_1(f)D_f$. Let $\mathsf{Pic}^0(X)$ denote the subgroup of the Picard group $\mathsf{Pic}(X)$ consisting of integral divisors on X algebraically equivalent to zero (modulo linear equivalence), which has the structure of an abelian variety. Consider the exact sequence of $\mathbf{R}$-vector spaces:

$$0 \to \mathsf{Pic}^0(X)_{\mathbf{R}} \to \mathsf{Pic}(X)_{\mathbf{R}} \to \mathsf{NS}(X)_{\mathbf{R}} \cong \mathsf{N}^1(X)_{\mathbf{R}} \to 0.$$

If the irregularity $q(X) := h^1(X, \mathscr{O}_X) = 0$, then $\mathsf{N}^1(X)_{\mathbf{R}} \cong \mathsf{Pic}(X)_{\mathbf{R}}$, and hence, the assertion (2) follows. So let us consider the case $q(X) > 0$. Note that $f^*D_f - \lambda_1(f)D_f \in \mathsf{Pic}^0(X)_{\mathbf{R}}$.

Claim 3.14. The $\mathbf{R}$-linear map

$$f^* \otimes_{\mathbf{Z}} 1_{\mathbf{R}} - \lambda_1(f)\,\mathrm{id}\colon \mathsf{Pic}^0(X)_{\mathbf{R}} \to \mathsf{Pic}^0(X)_{\mathbf{R}}$$

on the (possibly infinite-dimensional) $\mathbf{R}$-vector space $\mathsf{Pic}^0(X)_{\mathbf{R}}$ is bijective.

Assuming Claim 3.14 for the time being, up to $\mathbf{R}$-linear equivalence, there is a unique $\mathbf{R}$-divisor $E \in \mathsf{Pic}^0(X)_{\mathbf{R}}$ such that $f^*E - \lambda_1(f)E \sim_{\mathbf{R}} f^*D_f - \lambda_1(f)D_f$, which yields that $f^*(D_f - E) \sim_{\mathbf{R}} \lambda_1(f)(D_f - E)$. Hence, $D'_f := D_f - E$ suffices to conclude the assertion (2). □

Proof of Claim 3.14. Let $\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$ denote the Picard variety of X. The pullback f^* of divisors on X induces an isogeny g of $\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$. Denote by $P_g(t) \in \mathbf{Z}[t]$ the characteristic polynomial of g, which has degree $2q(X)$ and satisfies that

$$\begin{aligned} P_g(t) &= \det(t\,\mathrm{id} - g^* \circlearrowleft H^1(\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}, \mathbf{Q})) \\ &= \det(t\,\mathrm{id} - f_* \circlearrowleft H^{2n-1}(X, \mathbf{Q})) \\ &= \det(t\,\mathrm{id} - f^* \circlearrowleft H^1(X, \mathbf{Q})), \end{aligned}$$

where $H^{2n-1}(X, \mathbf{Q})$ is canonically isomorphic to $H^1(\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}, \mathbf{Q})$ via the Poincaré divisor on $X \times \mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$ and the last equality follows from Poincaré duality. Thanks to [Din05, Proposition 5.8], all roots of $P_g(t)$ have moduli at most $\sqrt{\lambda_1(f)}$. In particular, $P_g(\lambda_1(f)) \neq 0$ since $\lambda_1(f) > 1$ by assumption. Besides, since the rational representation $\mathrm{End}(\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}) \to \mathrm{End}_{\mathbf{Q}}(H^1(\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}, \mathbf{Q}))$ is an injective homomorphism, $P_g(g) = 0$ as an endomorphism of $\mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}$. It follows that $P_g(f^*) = 0$ as a group homomorphism of $\mathsf{Pic}^0(X) = \mathbf{Pic}^0_{X/\overline{\mathbf{Q}}}(\overline{\mathbf{Q}})$. Now by applying Lemma 3.5 to the $\mathbf{Z}$-module $\mathsf{Pic}^0(X)$, Claim 3.14 follows. □

Now we are ready to prove Theorem 3.6.

Proof of Theorem 3.6. Assertions 1 to 3 follow readily from Proposition 3.8, Lemma 3.11 and the assumption that $\operatorname{rank} G = n - 1$. We are left to construct n automorphisms $g_1, \ldots, g_n \in G$ satisfying Assertion (4).

Let $1 \le i \le n$ be fixed. Let p_i denote the natural projection from $\mathbf{R}^n$ to $\mathbf{R}^{n-1}$ by omitting the i-th coordinate x_i. Recall that the group homomorphism $\pi\colon G \to (\mathbf{R}^n, +)$, defined by sending $g \in G$ to $(\log \chi_1(g), \ldots, \log \chi_n(g))$, is injective and $\pi(G)$ is a lattice in $\mathbf{R}^n$ of rank $n - 1$ (see Proposition 3.10). Besides, by the projection formula, we have for any $g \in G$,

$$D_1 \cdots D_n = \deg(g)(D_1 \cdots D_n) = g^*D_1 \cdots g^*D_n = \chi_1(g) \cdots \chi_n(g)(D_1 \cdots D_n).$$

Since $D_1 \cdots D_n > 0$ by Assertion (2), one has $\chi_1(g) \cdots \chi_n(g) = 1$. Hence, the image $\pi(G)$ of G is actually contained in the hyperplane $H := \{(x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{j=1}^n x_j = 0\}$.

Consider the following commutative diagram:

$$\begin{tikzcd} G \cong \mathbf{Z}^{n-1} \arrow[r, hook, "\pi"] \arrow[d, hook, "\tau"'] & \mathbf{R}^n \arrow[d, two heads, "p_i"] \\ H \arrow[ur, hook, "\iota"] \arrow[r, "p_i \circ \iota"] & \mathbf{R}^{n-1}. \end{tikzcd}$$

Clearly, $p_i \circ \iota\colon H \to \mathbf{R}^{n-1}$ is an isomorphism of $\mathbf{R}$-vector spaces. By the open mapping theorem, it is also an isomorphism of topological vector spaces. Denote $p_i \circ \pi$ by π_i. Since $\tau(G)$ is a lattice in H of rank $n - 1$ and $p_i \circ \iota$ is a topological isomorphism, $\pi_i(G) = (p_i \circ \iota)(\tau(G))$ is a lattice in $\mathbf{R}^{n-1}$ of rank $n - 1$. Therefore, there is some $g_i \in G\backslash\{\mathrm{id}\}$ such that all $n - 1$ coordinates of $\pi_i(g_i)$ are negative (i.e., $\chi_j(g_i) < 1$ for all $j \ne i$). Further, by Proposition 3.8, for such g_i, there is some $1 \le t_i \le n$ such that $\chi_{t_i}(g_i) = \lambda_1(g_i) > 1$. Clearly, t_i has to be i. We thus prove that $g_i^*D_i \equiv \chi_i(g_i)D_i = \lambda_1(g_i)D_i$. By Lemma 3.13(2), there is a nef $\mathbf{R}$-divisor $D_i' \equiv D_i$ such that $g_i^*D_i' \sim_{\mathbf{R}} \lambda_1(g_i)D_i'$.

In the end, we replace D_i with D_i' for all i. Since $D_i' \equiv D_i$, this does not affect Assertions (1) and (2). We thus complete the proof of Theorem 3.6. □

Remark 3.15. In the above Theorem 3.6, denote by M the matrix $(\log \chi_i(g_j))_{1\le i,j\le n}$. Clearly, the rank of M is at most $n-1$ since $\sum_{i=1}^n \log \chi_i = 0$. On the other hand, as $\log \chi_i(g_j) < 0$ for all $i \ne j$, one can see that any submatrix M_i of M obtained by deleting the i-th row and the i-th column is strictly diagonally dominant and hence nonsingular (see [HJ13, Theorem 6.1.10]). It follows that $\operatorname{rank} M = n - 1$. Hence, $g_1, \ldots, g_n$ generate a free abelian subgroup of G of full rank $n - 1$; moreover, by the same argument, so do any $n - 1$ automorphisms from $g_1, \ldots, g_n$.

At the end of this section, we quote a lemma of Zhang [Zha16], who essentially proved the existence of Zariski dense G-orbits (though he did not write it down explicitly). We will also use it later in the proof of Theorem 1.1 (or rather, proofs of Theorem 4.2 and its corollaries). For the sake of completeness, we present a proof. Recall that for a nef $\mathbf{R}$-divisor D on X, the *null locus* $\mathrm{Null}(D)$ of D is the union of all subvarieties V of X of positive dimension such that the restriction $D|_V$ is not big, i.e., $D^{\dim V} \cdot V = 0$.

Lemma 3.16 (cf. [Zha16, Lemma 3.9]). *With the notation as in Theorem 3.6, the augmented base locus $\mathbf{B}_+(D)$ of the nef and big $\mathbf{R}$-divisor D is a G-invariant Zariski closed proper subset of X equal to*

$$\bigcup_{G\text{-periodic } V \subsetneq X} V, \tag{3.5}$$

where V runs over all G-periodic proper subvarieties of X of positive dimension. In particular, $\mathbf{B}_+(D)$ is g-invariant for any $g \in G$, and hence, for any $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$, one has $\mathcal{O}_g(x) \cap \mathbf{B}_+(D) = \emptyset$ for any $g \in G$.

Proof. It is well known that $\mathbf{B}_+(D)$ is a Zariski closed proper subset of X, as D is big; see Proposition 2.4(1). We shall first show that $\mathbf{B}_+(D)$ is G-invariant. Let $g \in G$ be an arbitrary fixed automorphism of X. Thanks to an observation by Lesieutre and Satriano (see Lemma 2.5), for any positive numbers $a_1, \dots, a_n$, we have

$$\mathbf{B}_+(a_1 D_1 + \cdots + a_n D_n) = \mathbf{B}_+(D_1 + \cdots + D_n).$$

It thus follows from Proposition 2.4(2) that

$$g(\mathbf{B}_+(D)) = \mathbf{B}_+((g^{-1})^*(D_1 + \cdots + D_n)) = \mathbf{B}_+\left(\sum_{i=1}^n \chi_i(g)^{-1} D_i\right) = \mathbf{B}_+(D).$$

Therefore, $\mathbf{B}_+(D)$ is g-invariant and hence G-invariant.

We then prove that $\mathbf{B}_+(D)$ equals the union (3.5). First, by G-invariance of $\mathbf{B}_+(D)$, every irreducible component of the closed $\mathbf{B}_+(D)$ is G-periodic. It is also known that $\mathbf{B}_+(D)$ has no isolated points (see [ELM$^+$09, Proposition 1.1]). Hence, $\mathbf{B}_+(D)$ is contained in the union (3.5). On the other hand, thanks to [ELM$^+$09, Corollary 5.6], $\mathbf{B}_+(D)$ coincides with the null locus $\mathrm{Null}(D)$ of D. It remains to show that $\mathrm{Null}(D)$ contains the union (3.5). Suppose that $V \subsetneq X$ is a G-periodic proper subvariety of dimension k with $1 \le k < n$. We shall prove that $D|_V$ is not big or, equivalently, $D^k \cdot V = 0$. Since D is the sum of nef divisors D_i, it suffices to show that $D_{i_1} \cdots D_{i_k} \cdot V = 0$ for any multi-index $1 \le i_1 \le \cdots \le i_k \le n$. Fix such a multi-index $1 \le i_1 \le \cdots \le i_k \le n$. Then there always exists an $i_\circ$ with $1 \le i_\circ \le n$ different from all i_j with $1 \le j \le k$. Hence, by Theorem 3.6(4) , one has $\chi_{i_j}(g_{i_\circ}) < 1$ for any $1 \le j \le k$. Suppose that $g_{i_\circ}^e(V) = V$ for some $e \ge 1$ (depending on $g_{i_\circ}$). Then it follows from Theorem 3.6(1) and the projection formula that

$$D_{i_1} \cdots D_{i_k} \cdot V = (g_{i_\circ}^e)^* D_{i_1} \cdots (g_{i_\circ}^e)^* D_{i_k} \cdot (g_{i_\circ}^e)^* V = \prod_{j=1}^k \chi_{i_j}(g_{i_\circ})^e (D_{i_1} \cdots D_{i_k} \cdot V).$$

This implies that $D_{i_1} \cdots D_{i_k} \cdot V = 0$ and hence concludes the proof of Lemma 3.16. □

We are now in the position to prove the existence of Zariski dense G-orbits.

Proof of Corollary 1.6. According to Theorem 3.6 and Lemma 3.16, the augmented base locus $\mathbf{B}_+(D)$ of the nef and big $\mathbf{R}$-divisor D is a G-invariant Zariski closed proper subset of X. Thanks to [Ame11, Corollary 9 and the paragraph after it], there is a closed point $x \in X(\overline{\mathbf{Q}})$ away from $\mathbf{B}_+(D)$ such that the G-orbit $\mathcal{O}_G(x)$ of x is infinite. Let Z be the Zariski closure of $\mathcal{O}_G(x)$ in X. Then Z is a positive-dimensional G-invariant Zariski closed subset of X. Suppose that $Z \neq X$. Then by Lemma 3.16, Z is contained in $\mathbf{B}_+(D)$, contradicting the choice of x. Therefore, the G-orbit $\mathcal{O}_G(x)$ of x is Zariski dense in X. The first assertion is thus verified.

Let us prove the second assertion on the potential density of Y. By assumption, Y is defined over a number field K, and $X = Y \times_{\mathrm{Spec}\, K} \mathrm{Spec}\, \overline{\mathbf{Q}}$. By the first assertion, there is a closed point $x \in X(\overline{\mathbf{Q}})$ such that its G-orbit $\mathcal{O}_G(x)$ is Zariski dense in X. On the other hand, such a closed point $x \in X(\overline{\mathbf{Q}})$ is in fact defined over a number field L, which is a finite extension of the defining field K. Similarly, we denote by Y_L the base extension of Y and let $\varphi\colon X \longrightarrow Y_L$ be the natural projection. Then $\varphi(x)$ is an L-rational point of Y_L, and the φ-image $\varphi(\mathcal{O}_G(x))$ of the G-orbit $\mathcal{O}_G(x)$ is Zariski dense in Y_L by noting that φ is a finite surjective morphism. It follows that the set $Y_L(L)$ of all L-points of Y_L is also Zariski dense in Y_L, as we have a natural inclusion $\varphi(\mathcal{O}_G(x)) \subseteq Y_L(L)$. We thus finish the proof of our Corollary 1.6. □

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

4. Canonical heights for abelian group actions

Throughout this section, X is a smooth projective variety of dimension $n \geq 2$ defined over $\overline{\mathbf{Q}}$, and $G \cong \mathbf{Z}^{n-1}$ is a free abelian group of automorphisms of X of positive entropy. Thanks to Theorem 3.6, we can choose n automorphisms $g_1, \ldots, g_n \in G$ and n common nef $\mathbf{R}$-eigendivisors $D_1, \ldots, D_n$ on X such that

(1) for any $1 \leq i \leq n$ and any $g \in G$, one has $g^* D_i \equiv \chi_i(g) D_i$, which defines the group characters χ_i of G;
(2) $D := D_1 + \cdots + D_n$ is a nef and big $\mathbf{R}$-divisor on X;
(3) for any $g \in G$, there is some $1 \leq i \leq n$ such that $\chi_i(g) = \lambda_1(g)$;
(4) for any $1 \leq i \neq j \leq n$, one has $\chi_j(g_i) < 1$ and $g_i^* D_i \sim_{\mathbf{R}} \lambda_1(g_i) D_i$.

We shall stick to the above notations throughout and construct a height function $\widehat{h}_G$ associated with G as the sum of the following individual canonical heights.

Lemma 4.1. *For any $1 \leq i \leq n$ and any $x \in X(\overline{\mathbf{Q}})$, the limit*

$$\widehat{h}_{D_i, g_i}(x) := \lim_{m \to \infty} \frac{h_{D_i}(g_i^m(x))}{\lambda_1(g_i)^m}$$

exists and satisfies the following properties:

(1) $\widehat{h}_{D_i, g_i} = h_{D_i} + O(1)$.
(2) $\widehat{h}_{D_i, g_i} \circ g = \chi_i(g)\, \widehat{h}_{D_i, g_i}$ *for any $g \in G$; in particular,* $\widehat{h}_{D_i, g_i} \circ g_i = \lambda_1(g_i)\, \widehat{h}_{D_i, g_i}$.

Proof. Note that $g_i^* D_i \sim_{\mathbf{R}} \lambda_1(g_i) D_i$ by Theorem 3.6(4). Hence, the existence of each $\widehat{h}_{D_i, g_i}$ and the property (1) follow immediately from Theorem 2.10. For the property (2), fix an integer i with $1 \leq i \leq n$, an automorphism $g \in G$ of positive entropy, an ample divisor H_X on X, and a height function h_{H_X} associated with H_X. Then thanks to Matsuzawa [Mat20b, Theorem 1.7(2)], there is a constant $C_1 > 0$ such that for any rational point $x \in X(\overline{\mathbf{Q}})$ and any $m \geq 1$,

$$h_{H_X}^+(g_i^m(x)) \leq C_1 m^{\rho(X)-1} \lambda_1(g_i)^m h_{H_X}^+(x). \tag{4.1}$$

Observe that by Theorem 3.6(1), we have $g^* D_i \equiv \chi_i(g) D_i$. Then according to Theorem 2.6ii (Functoriality) and v (Algebraic equivalence), there is a constant $C_2 > 0$ such that

$$|h_{g^* D_i} - h_{D_i} \circ g| \leq C_2,$$
$$|h_{g^* D_i} - \chi_i(g)\, h_{D_i}| \leq C_2 \sqrt{h_{H_X}^+}.$$

Combining them together yields that

$$|h_{D_i} \circ g - \chi_i(g)\, h_{D_i}| \leq 2C_2 \sqrt{h_{H_X}^+}.$$

In particular, for any $m \geq 1$, one has

$$|h_{D_i}(g(g_i^m(x))) - \chi_i(g)\, h_{D_i}(g_i^m(x))| \leq 2C_2 \sqrt{h_{H_X}^+(g_i^m(x))}. \tag{4.2}$$

As G is abelian, it follows from Equations (4.1) and (4.2) that

$$\left| \frac{h_{D_i}(g_i^m(g(x)))}{\lambda_1(g_i)^m} - \frac{\chi_i(g) h_{D_i}(g_i^m(x))}{\lambda_1(g_i)^m} \right| \leq \frac{2C_2 \sqrt{h_{H_X}^+(g_i^m(x))}}{\lambda_1(g_i)^m} \leq C_3 \sqrt{\frac{m^{\rho(X)-1}}{\lambda_1(g_i)^m}},$$

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

where $C_3 > 0$ is a constant independent of $m \geq 1$. Note that both limits of the left-hand side exist (see Theorem 2.10). So taking $m \to \infty$ yields that $\widehat{h}_{D_i,g_i}(g(x)) = \chi_i(g)\,\widehat{h}_{D_i,g_i}(x)$. □

The theorem below is a precise version of our main result Theorem 1.1. Indeed, one just takes the Zariski closed proper subset Z to be the augmented base locus $\mathbf{B}_+(D)$ of the nef and big $\mathbf{R}$-divisor D constructed in Theorem 3.6(2). It extends [Sil91, Theorem 1.1] and [Kaw08, Theorem 5.2] to higher dimensions (under the maximal dynamical rank assumption).

Theorem 4.2. *Let X be a smooth projective variety of dimension $n \geq 2$ defined over $\overline{\mathbf{Q}}$ and $G \cong \mathbf{Z}^{n-1}$ a free abelian group of automorphisms of X of positive entropy. Define a function $\widehat{h}_G\colon X(\overline{\mathbf{Q}}) \to \mathbf{R}$ by*

$$\widehat{h}_G \coloneqq \sum_{i=1}^{n} \widehat{h}_{D_i,g_i},$$

where the $\widehat{h}_{D_i,g_i}$ are from Lemma 4.1. Then the following assertions hold.

(1) *The function $\widehat{h}_G$ is a Weil height corresponding to the nef and big $\mathbf{R}$-divisor D; that is,*

$$\widehat{h}_G = h_D + O(1).$$

(2) *For any $g \in G$, one has*

$$\widehat{h}_G \circ g = \sum_{i=1}^{n} \chi_i(g)\,\widehat{h}_{D_i,g_i}.$$

(3) *For any $1 \leq i \leq n$ and any $x \in (X\backslash\mathbf{B}_+(D))(\overline{\mathbf{Q}})$, one has $\widehat{h}_{D_i,g_i}(x) \geq 0$, and hence, $\widehat{h}_G(x) \geq 0$, where $\mathbf{B}_+(D)$ is the augmented base locus of D.*
(4) *The Weil height function $\widehat{h}_G$ satisfies the Northcott finiteness property on $X\backslash\mathbf{B}_+(D)$; in other words, for any positive integer d and real number T, the set*

$$\left\{x \in (X\backslash\mathbf{B}_+(D))(\overline{\mathbf{Q}}) : [K(x) : K] \leq d,\ \widehat{h}_G(x) \leq T\right\}$$

is finite, where K is a defining number field of X.
(5) *For any $x \in (X\backslash\mathbf{B}_+(D))(\overline{\mathbf{Q}})$, the following statements are equivalent.*
 (i) *$\widehat{h}_G(x) = 0$.*
 (ii) *$\widehat{h}_{D_i,g_i}(x) = 0$ for all $1 \leq i \leq n$.*
 (iii) *x is g-periodic for any $g \in G$.*
 (iv) *x is g-periodic for some $g \in G\backslash\{\mathrm{id}\}$.*
 (v) *$\widehat{h}_{D_i,g_i}(x) = 0$ for some $1 \leq i \leq n$.*
 (vi) *x is G-periodic (i.e., the G-orbit $\mathcal{O}_G(x) \coloneqq \{g(x) : g \in G\}$ of x is finite).*

We call $\widehat{h}_G$ *a canonical height function* associated with the abelian group G of maximal dynamical rank.

Proof. Assertion (1) follows immediately from Lemma 4.1(1) and Theorem 2.6iii (Additivity). Assertion (2) follows from Lemma 4.1(2). Since D is a big $\mathbf{R}$-divisor, the height function h_D satisfies the Northcott finiteness property (see Theorem 2.8), and so does $\widehat{h}_G$ by Assertion (1). We have thus proved Assertion (4).

Next, we shall show Assertion (3). Fix an index i with $1 \leq i \leq n$. According to Lemma 3.16, the augmented base locus $\mathbf{B}_+(D)$ of D is a G-invariant Zariski closed proper subset of X. Fix a point $x \in (X\backslash\mathbf{B}_+(D))(\overline{\mathbf{Q}})$; in particular, $\mathcal{O}_{g_i}(x) \cap \mathbf{B}_+(D) = \varnothing$. We notice by [LS21, Lemma 2.26] that

$h_D \geq O(1)$ outside $\mathbf{B}_+(D)$. On the other hand, by Assertion (1), we have

$$h_D = \widehat{h}_G + O(1) = \sum_{j=1}^{n} \widehat{h}_{D_j,g_j} + O(1).$$

It follows that for any $m \geq 1$,

$$O(1) \leq \sum_{j=1}^{n} \widehat{h}_{D_j,g_j}(g_i^m(x)) = \sum_{j=1}^{n} \chi_j(g_i)^m \widehat{h}_{D_j,g_j}(x), \tag{4.3}$$

where the equality is from Assertion (2). Furthermore, by Theorem 3.6(4), one has $\chi_j(g_i) < 1$ for all $j \neq i$ and $\chi_i(g_i) = \lambda_1(g_i) > 1$. Dividing (4.3) by $\chi_i(g_i)^m$ from both sides and letting m tend to infinity, it is easy to see that $\widehat{h}_{D_i,g_i}(x)$ has to be nonnegative. This thus shows Assertion (3).

At last, we prove Assertion (5). Fix a rational point $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$; in particular, $\mathcal{O}_g(x) \cap \mathbf{B}_+(D) = \emptyset$ for any $g \in G$. We shall prove the equivalence in the following order:

$$\begin{array}{ccccccc} (i) & \Longleftrightarrow & (ii) & \Longrightarrow & (iii) & \Longleftrightarrow & (vi) \\ & & \Uparrow & & \Downarrow & & \\ & & (v) & \Longleftarrow & (iv). & & \end{array}$$

By definition, 5ii $\Rightarrow$5i is trivial, while 5i $\Rightarrow$ 5ii follows from Assertion (3). The implications 5vi $\Rightarrow$ 5iii $\Rightarrow$ 5iv are also trivial.

We first show 5iii $\Rightarrow$ 5vi. Let $\{f_1, \ldots, f_{n-1}\}$ be a generating set of G and let s be the common period of x under the f_i. It follows from the commutativity of G that any point in the G-orbit $\mathcal{O}_G(x)$ of x is of the form $f_1^{a_1} \circ \cdots \circ f_{n-1}^{a_{n-1}}(x)$ such that $1 - s \leq a_i \leq s - 1$ for each $1 \leq i \leq n - 1$. In particular, the set $\mathcal{O}_G(x)$ is finite.

We next prove 5ii $\Rightarrow$ 5iii. Let $g \in G$ be fixed. By Assertion (2), for any $m \geq 1$,

$$\widehat{h}_G(g^m(x)) = \sum_{i=1}^{n} \chi_i(g)^m \widehat{h}_{D_i,g_i}(x) = 0.$$

Since these rational points $g^m(x)$ are of bounded degree over $\mathbf{Q}$, it follows from Assertion 4 that the forward g-orbit $\mathcal{O}_g(x)$ of x is finite, that is, x is g-periodic, noting that g is an automorphism.

We now show 5iv $\Rightarrow$ 5v. Suppose that x is g-periodic for some $g \in G \backslash \{\mathrm{id}\}$. Note that by Theorem 3.6(3), there is some $1 \leq i \leq n$ such that $\chi_i(g) = \lambda_1(g)$. Consider the growth of the function $\widehat{h}_G$ along the forward g-orbit $\mathcal{O}_g(x)$ of x, which is finite by assumption. In other words, we have

$$O(1) = \widehat{h}_G(g^m(x)) \geq \lambda_1(g)^m \widehat{h}_{D_i,g_i}(x),$$

where the last inequality is due to Assertion (3). As $\lambda_1(g) > 1$, we see that $\widehat{h}_{D_i,g_i}(x)$ has to be zero by letting m tend to infinity.

It remains to prove 5v $\Rightarrow$ 5ii. Without loss of generality, we may assume that $\widehat{h}_{D_1,g_1}(x) = 0$. Via a similar argument in the proof of 5ii $\Rightarrow$ 5iii, we claim that x is g_1-periodic. Indeed, by Assertion (2) and Theorem 3.6(4), we have for any $m \geq 1$,

$$\widehat{h}_G(g_1^m(x)) = \sum_{j=2}^{n} \chi_j(g_1)^m \widehat{h}_{D_j}(x) < \sum_{j=2}^{n} \widehat{h}_{D_j}(x).$$

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

Note that the $g_1^m(x)$ are of bounded degree over $\mathbf{Q}$. By Assertion 4, x is g_1-periodic. Denote the finite period by $e_1 \in \mathbf{Z}_{>0}$, i.e., $g_1^{e_1}(x) = x$. Consider the growth of the function $\widehat{h}_G$ along the orbit $\mathcal{O}_{g_1^{-e_1}}(x)$ of x under the automorphism $g_1^{-e_1}$. Precisely, for any $m \ge 1$, we have

$$\widehat{h}_G(x) = \widehat{h}_G(g_1^{-e_1}(x)) = \widehat{h}_G(g_1^{-e_1 m}(x)) = \sum_{j=2}^{n} \chi_j(g_1)^{-e_1 m}\, \widehat{h}_{D_j}(x),$$

which forces $\widehat{h}_{D_j}(x) = 0$ for all $j > 1$ by noting that $\chi_j(g_1) < 1$. This verifies 5v $\Rightarrow$ 5ii.

We thus complete the proof of Theorem 4.2. □

As a direct consequence of Theorem 4.2, we also obtain the following.

Corollary 4.3 (cf. [KS14, Proposition 7 and Proof of Theorem 2(c)]). *Under the assumption of Theorem 4.2, for any $g \in G$ and any point $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$, we have*

$$\alpha_g(x) = \begin{cases} 1 & \text{if } x \text{ is } g\text{-periodic}, \\ \lambda_1(g) & \text{if } x \text{ is not } g\text{-periodic}. \end{cases}$$

Proof. Let $g \in G$ be fixed. First, we assume that $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$ is a non-g-periodic point. As we mentioned before, the limit defining $\alpha_g(x)$ exists and is independent of the choice of the ample divisor (see [KS16a, Theorem 3] and [KS16b, Proposition 12], respectively). Choose an ample divisor H_X such that $H_X - D$ is ample (noting that the ample cone is open). It follows from Theorem 2.6iii (Additivity) and iv (Positivity) that for any $m \ge 1$,

$$h_{H_X}(g^m(x)) = h_D(g^m(x)) + h_{H_X - D}(g^m(x)) + O(1) \ge h_D(g^m(x)) + O(1).$$

On the other hand, Theorem 4.2(1) asserts that

$$h_D(g^m(x)) = \widehat{h}_G(g^m(x)) + O(1).$$

Putting them together yields that

$$h_{H_X}(g^m(x)) \ge \widehat{h}_G(g^m(x)) + O(1) = \sum_{j=1}^{n} \chi_j(g)^m\, \widehat{h}_{D_j, g_j}(x) + O(1),$$

where the equality is from Theorem 4.2(2). Furthermore, by Theorem 3.6(3), there is some $1 \le i \le n$ such that $\chi_i(g) = \lambda_1(g)$. We thus obtain that

$$h_{H_X}(g^m(x)) \ge \lambda_1(g)^m\, \widehat{h}_{D_i, g_i}(x) + O(1).$$

Note that the term $O(1)$ does not depend on x nor m. Also, according to Theorem 4.25, one has $\widehat{h}_{D_i, g_i}(x) > 0$ for all $1 \le i \le n$. Now by taking m-th roots and letting $m \to \infty$, one easily has $\alpha_g(x) \ge \lambda_1(g)$. The reverse inequality is due to [KS16b, Theorem 4] or [Mat20b, Theorem 1.4].

Secondly, we assume that x is g-periodic. Then with the ample divisor H_X chosen as above, it is clear that $(h_{H_X}(g^m(x)))_{m \in \mathbf{N}}$ is a finite set. Therefore,

$$1 \le \alpha_g(x) = \lim_{m \to \infty} (h_{H_X}^+(g^m(x)))^{1/m} \le 1.$$

We finish the proof of Corollary 4.3. □

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

Remark 4.4. Following [Sil91], we can also define a function

$$\widehat{H}_G\colon X(\overline{\mathbf{Q}}) \to \mathbf{R} \quad \text{by} \quad \widehat{H}_G(x) = \prod_{i=1}^{n} \widehat{h}_{D_i,g_i}(x).$$

Note that by the projection formula, for any $g \in G$, one has

$$0 \neq D_1 \cdots D_n = \deg(g)(D_1 \cdots D_n) = g^*D_1 \cdots g^*D_n = \chi_1(g) \cdots \chi_n(g)(D_1 \cdots D_n);$$

in particular, $\prod_{i=1}^n \chi_i(g) = 1$. It follows that

$$\widehat{H}_G \circ g = \prod_{i=1}^{n} \widehat{h}_{D_i,g_i} \circ g = \prod_{i=1}^{n} \chi_i(g)\, \widehat{h}_{D_i,g_i} = \prod_{i=1}^{n} \widehat{h}_{D_i,g_i} = \widehat{H}_G.$$

In other words, the function $\widehat{H}_G$ is G-invariant. Let $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$ be arbitrary. Then according to Theorem 4.2(3), each $\widehat{h}_{D_i,g_i}(x) \geq 0$. This yields that

$$\sqrt[n]{\widehat{H}_G(x)} \leq \frac{\widehat{h}_G(x)}{n}.$$

Moreover, by Theorem 4.25, $\widehat{H}_G(x) = 0$ if and only if x is g-periodic for any $g \in G$.

Inspired by [Sil91, Theorem 1.3(a)], we ask the following.

Question 4.5. For any point $x \in (X \backslash \mathbf{B}_+(D))(\overline{\mathbf{Q}})$ with infinite G-orbit, is there any lower bound of $\widehat{H}_G(x)$ in terms of $\widehat{h}_G(x)$?

Note that if the above question has an affirmative answer, then one could prove a similar result as [Sil91, Theorem 1.2(b)] using the Northcott property for $\widehat{h}_G$ (i.e., Theorem 4.24). Namely, there are only finitely many infinite G-orbits in $(X \backslash \mathbf{B}_+(D))(K)$, where K is any number field.

4.1. Proofs of Theorem 1.1 and Corollaries 1.3 and 1.5

Proof of Theorem 1.1. By Theorem 3.6, we take Z to be the augmented base locus $\mathbf{B}_+(D)$ of D, which is a G-invariant Zariski closed proper subset of X (see Lemma 3.16). Theorem 1.1 then follows easily from Theorem 4.2. □

Proof of Corollary 1.3. It follows readily from Lemma 3.16 and Corollary 4.3. □

Proof of Corollary 1.5. Take Z to be the augmented base locus $\mathbf{B}_+(D)$ of D as in the proof of Theorem 1.1. Fix a g-periodic point $x \in (X \backslash Z)(\overline{\mathbf{Q}})$ and an ample divisor H_X on X. By Theorem 4.2(1) and (5), we have $h_D(x) = \widehat{h}_G(x) + O(1) = O(1)$. On the other hand, according to Proposition 2.4(3), there is $\varepsilon > 0$ such that $D - \varepsilon H_X$ is an effective $\mathbf{Q}$-divisor and $\mathbf{B}_+(D) = \mathbf{B}(D - \varepsilon H_X) = \mathrm{Bs}(M(D - \varepsilon H_X))$ for some $M \geq 1$. Since $x \notin \mathbf{B}_+(D)$, by applying Theorem 2.6iii (Additivity) and iv (Positivity) to $M(D - \varepsilon H_X)$, we obtain that $h_{D-\varepsilon H_X}(x) \geq O(1)$. It thus follows that

$$O(1) = h_D(x) = h_{D-\varepsilon H_X}(x) + h_{\varepsilon H_X}(x) + O(1) \geq h_{\varepsilon H_X}(x) + O(1).$$

Therefore, $h_{\varepsilon H_X}(x)$ and $h_{H_X}(x)$ are both bounded. Assertion (1) is thus proved.

Assertion (2) follows easily from [KS16b, Proposition 3] and Corollary 4.3. □

Acknowledgements. The authors would like to express their gratitude to the organizers of the conference 'Recent Developments in Algebraic Geometry, Arithmetic and Dynamics Part 2' held in Singapore in August 2023, for providing a platform for fruitful discussions. The authors would also like to thank Jason Bell, Tien-Cuong Dinh, Shu Kawaguchi, Yohsuke Matsuzawa, Matthew Satriano, Joseph Silverman and De-Qi Zhang for their valuable suggestions and comments. The first author extends special thanks to the Fields Institute for its warm hospitality during a visit in fall 2023. The authors are grateful to the referees for their careful reading and helpful suggestions to improve the paper.

Competing interest. The authors have no competing interest to declare.

Funding statement. The first author was supported by a grant from the National Natural Science Foundation of China (No. 12371045) and a grant from Nanjing University (No. 14912209). The second author was supported by the Institute for Basic Science (IBS-R032-D1-2023-a00).

References

[Ame11] E. Amerik, 'Existence of non-preperiodic algebraic points for a rational self-map of infinite order', *Math. Res. Lett.* **18**(2) (2011), 251–256. MR2784670

[BIJ+19] R. Benedetto, P. Ingram, R. Jones, M. Manes, J. H. Silverman and T. J. Tucker, 'Current trends and open problems in arithmetic dynamics', *Bull. Amer. Math. Soc. (N.S.)* **56**(4) (2019), 611–685. MR4007163

[Bir67] G. Birkhoff, 'Linear transformations with invariant cones', *Amer. Math. Monthly* **74** (1967), 274–276. MR0214605

[Cam04] F. Campana, 'Orbifolds, special varieties and classification theory', *Ann. Inst. Fourier (Grenoble)* **54**(3) (2004), 499–630. MR2097416

[Cam11] F. Campana, 'Remarks on an example of K. Ueno', in *Classification of Algebraic Varieties* (EMS Ser. Congr. Rep.) (Eur. Math. Soc., Zürich, 2011), 115–121. MR2779469

[CS93] G. S. Call and J. H. Silverman, 'Canonical heights on varieties with morphisms', *Compos. Math.* **89**(2) (1993), 163–205. MR1255693

[CWZ14] F. Campana, F. Wang and D.-Q. Zhang, 'Automorphism groups of positive entropy on projective threefolds', *Trans. Amer. Math. Soc.* **366**(3) (2014), 1621–1638. MR3145744

[Dan20] N.-B. Dang, 'Degrees of iterates of rational maps on normal projective varieties', *Proc. Lond. Math. Soc. (3)* **121**(5) (2020), 1268–1310. MR4133708

[DGH+22] N.-B. Dang, D. Ghioca, F. Hu, J. Lesieutre and M. Satriano, 'Higher arithmetic degrees of dominant rational self-maps', *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **23**(1) (2022), 463–481. MR4407198

[DHZ15] T.-C. Dinh, F. Hu and D.-Q. Zhang, 'Compact Kähler manifolds admitting large solvable groups of automorphisms', *Adv. Math.* **281** (2015), 333–352. MR3366842

[Din05] T.-C. Dinh, 'Suites d'applications méromorphes multivaluées et courants laminaires', *J. Geom. Anal.* **15**(2) (2005), 207–227. MR2152480

[Din12] T.-C. Dinh, 'Tits alternative for automorphism groups of compact Kähler manifolds', *Acta Math. Vietnam.* **37**(4) (2012), 513–529. MR3058661

[DS04] T.-C. Dinh and N. Sibony, 'Groupes commutatifs d'automorphismes d'une variété kählérienne compacte', *Duke Math. J.* **123**(2) (2004), 311–328. MR2066940

[DS05] T.-C. Dinh and N. Sibony, 'Une borne supérieure pour l'entropie topologique d'une application rationnelle', *Ann. of Math. (2)* **161**(3) (2005), 1637–1644. MR2180409

[ELM+06] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, 'Asymptotic invariants of base loci', *Ann. Inst. Fourier (Grenoble)* **56**(6) (2006), 1701–1734. MR2282673

[ELM+09] L. Ein, R. Lazarsfeld, M. Mustaţă, M. Nakamaye and M. Popa, 'Restricted volumes and base loci of linear series', *Amer. J. Math.* **131**(3) (2009), 607–651. MR2530849

[Ful98] W. Fulton, *Intersection Theory* (Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]) vol. 2, second edn. (Springer-Verlag, Berlin, 1998). MR1644323

[Gro03] M. Gromov, 'On the entropy of holomorphic maps', *Enseign. Math. (2)* **49**(3–4) (2003), 217–235, Preprint SUNY (1977). MR2026895

[Has03] B. Hassett, 'Potential density of rational points on algebraic varieties', in *Higher Dimensional Varieties and Rational Points (Budapest, 2001)* (Bolyai Soc. Math. Stud.) vol. 12 (Springer, Berlin, 2003), 223–282. MR2011748

[HJ13] R. A. Horn and C. R. Johnson, *Matrix Analysis*, second edn. (Cambridge University Press, Cambridge, 2013). MR2978290

[HS00] M. Hindry and J. H. Silverman, *Diophantine Geometry. An Introduction* (Graduate Texts in Mathematics) vol. 201 (Springer-Verlag, New York, 2000). MR1745599

[HT21] F. Hu and T. T. Truong, 'A dynamical approach to generalized Weil's Riemann hypothesis and semisimplicity', *SCIENCE CHINA Mathematics*. https://doi.org/10.1007/s11425-024-2363-9.

[Hu20] F. Hu, 'A theorem of Tits type for automorphism groups of projective varieties in arbitrary characteristic', *Math. Ann.* **377**(3–4) (2020), 1573–1602, With an appendix by Tomohide Terasoma. MR4126902

[Hu24] F. Hu, 'Eigenvalues and dynamical degrees of self-maps on abelian varieties', *J. Algebraic Geom.* **33**(2) (2024), 265–293. MR4705374

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

[Kaw06a] S. Kawaguchi, ‘Canonical height functions for affine plane automorphisms’, *Math. Ann.* **335**(2) (2006), 285–310. MR2221115

[Kaw06b] S. Kawaguchi, ‘Canonical heights, invariant currents, and dynamical eigensystems of morphisms for line bundles’, *J. Reine Angew. Math.* **597** (2006), 135–173. MR2264317

[Kaw08] S. Kawaguchi, ‘Projective surface automorphisms of positive topological entropy from an arithmetic viewpoint’, *Amer. J. Math.* **130**(1) (2008), 159–186. MR2382145

[KK01] B. Kalinin and A. Katok, ‘Invariant measures for actions of higher rank abelian groups’, in *Smooth Ergodic Theory and Its Applications (Seattle, WA, 1999)* (Proc. Sympos. Pure Math.) vol. 69 (Amer. Math. Soc., Providence, RI, 2001), 593–637. MR1858547

[KKRH14] A. Katok, S. Katok and F. R. Hertz, ‘The Fried average entropy and slow entropy for actions of higher rank abelian groups’, *Geom. Funct. Anal.* **24**(4) (2014), 1204–1228. MR3248484

[Kle05] S. L. Kleiman, ‘The Picard scheme’, in *Fundamental Algebraic Geometry: Grothendieck’s FGA Explained* (Mathematical Surveys and Monographs) vol. 123 (American Mathematical Society, Providence, RI, 2005), 235–321. MR2223410

[KS94] A. Katok and R. J. Spatzier, ‘First cohomology of Anosov actions of higher rank abelian groups and applications to rigidity’, *Inst. Hautes Études Sci. Publ. Math.* **79** (1994), 131–156. MR1307298

[KS14] S. Kawaguchi and J. H. Silverman, ‘Examples of dynamical degree equals arithmetic degree’, *Michigan Math. J.* **63**(1) (2014), 41–63. MR3189467

[KS16a] S. Kawaguchi and J. H. Silverman, ‘Dynamical canonical heights for Jordan blocks, arithmetic degrees of orbits, and nef canonical heights on abelian varieties’, *Trans. Amer. Math. Soc.* **368**(7) (2016), 5009–5035. MR3456169

[KS16b] S. Kawaguchi and J. H. Silverman, ‘On the dynamical and arithmetic degrees of rational self-maps of algebraic varieties’, *J. Reine Angew. Math.* **713** (2016), 21–48. MR3483624

[Les18] J. Lesieutre, ‘Some constraints of positive entropy automorphisms of smooth threefolds’, *Ann. Sci. Éc. Norm. Supér. (4)* **51**(6) (2018), 1507–1547. MR3940903

[LS21] J. Lesieutre and M. Satriano, ‘Canonical heights on hyper-Kähler varieties and the Kawaguchi-Silverman conjecture’, *Int. Math. Res. Not. IMRN* **10** (2021), 7677–7714. MR4259157

[Mat20a] Y. Matsuzawa, ‘Kawaguchi–Silverman conjecture for endomorphisms on several classes of varieties’, *Adv. Math.* **366** (2020), Paper No. 107086. MR4070310

[Mat20b] Y. Matsuzawa, ‘On upper bounds of arithmetic degrees’, *Amer. J. Math.* **142**(6) (2020), 1797–1820. MR4176545

[Mat24] Y. Matsuzawa, ‘Recent advances on Kawaguchi–Silverman conjecture’, to appear in the *Proceedings of the Simons Symposia on Algebraic, Complex, and Arithmetic Dynamics* (2024+), 36 pp., https://arxiv.org/abs/2311.15489v1.

[MMS+22] Y. Matsuzawa, S. Meng, T. Shibata, D.-Q. Zhang and G. Zhong, ‘Invariant subvarieties with small dynamical degree’, *Int. Math. Res. Not. IMRN* **15** (2022), 11448–11483. MR4458556

[MSS18] Y. Matsuzawa, K. Sano and T. Shibata, ‘Arithmetic degrees and dynamical degrees of endomorphisms on surfaces’, *Algebra Number Theory* **12**(7) (2018), 1635–1657. MR3871505

[MY22] Y. Matsuzawa and S. Yoshikawa, ‘Kawaguchi-Silverman conjecture for endomorphisms on rationally connected varieties admitting an int-amplified endomorphism’, *Math. Ann.* **382**(3–4) (2022), 1681–1704. MR4403232

[MZ22] S. Meng and D.-Q. Zhang, ‘Kawaguchi-Silverman conjecture for certain surjective endomorphisms’, *Doc. Math.* **27** (2022), 1605–1642. MR4574221

[MZ23] S. Meng and D.-Q. Zhang, ‘Structures theorems and applications of non-isomorphic surjective endomorphisms of smooth projective threefolds’, Preprint, 2023, https://arxiv.org/abs/2309.07005v1.

[MZ24a] S. Meng and D.-Q. Zhang, ‘Advances in the equivariant minimal model program and their applications in complex and arithmetic dynamics’, to appear in the *Proceedings of the Simons Symposia on Algebraic, Complex, and Arithmetic Dynamics* (2024+), 26 pp., https://arxiv.org/abs/2311.16369v1.

[MZ24b] S. Meng and G. Zhong, ‘Kawaguchi-Silverman conjecture for int-amplified endomorphism’, Preprint, 2024, https://arxiv.org/abs/2408.00566v1.

[NZ10] N. Nakayama and D.-Q. Zhang, ‘Polarized endomorphisms of complex normal varieties’, *Math. Ann.* **346**(4) (2010), 991–1018. MR2587100

[Ogu14] K. Oguiso, ‘Some aspects of explicit birational geometry inspired by complex dynamics’, in *Proceedings of the International Congress of Mathematicians—Seoul 2014 . Vol. II* (Kyung Moon Sa, Seoul, 2014), 695–721. MR3728634

[Shi19] T. Shibata, ‘Ample canonical heights for endomorphisms on projective varieties’, *J. Math. Soc. Japan* **71**(2) (2019), 599–634. MR3943453

[Sil91] J. H. Silverman, ‘Rational points on $K3$ surfaces: a new canonical height’, *Invent. Math.* **105**(2) (1991), 347–373. MR1115546

[Tru20] T. T. Truong, ‘Relative dynamical degrees of correspondences over a field of arbitrary characteristic’, *J. Reine Angew. Math.* **758** (2020), 139–182. MR4048444

[Uen75] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces* (Lecture Notes in Mathematics) vol. 439 (Springer-Verlag, Berlin-New York, 1975). Notes written in collaboration with P. Cherenack. MR0506253

https://doi.org/10.1017/fms.2024.158 Published online by Cambridge University Press

[Wit18] O. Wittenberg, 'Rational points and zero-cycles on rationally connected varieties over number fields', in *Algebraic Geometry: Salt Lake City 2015* (Proc. Sympos. Pure Math.) vol. 97.2 (Amer. Math. Soc., Providence, RI, 2018), 597–635. MR3821185

[Yom87] Y. Yomdin, 'Volume growth and entropy', *Israel J. Math.* **57**(3) (1987), 285–300. MR889979

[Zha06] S.-W. Zhang, 'Distributions in algebraic dynamics', in *Surveys in Differential Geometry . Vol. X* (Surv. Differ. Geom.) vol. 10 (Int. Press, Somerville, MA, 2006), 381–430. MR2408228

[Zha09] D.-Q. Zhang, 'A theorem of Tits type for compact Kähler manifolds', *Invent. Math.* **176**(3) (2009), 449–459. MR2501294

[Zha13] D.-Q. Zhang, 'Algebraic varieties with automorphism groups of maximal rank', *Math. Ann.* **355**(1) (2013), 131–146. MR3004578

[Zha16] D.-Q. Zhang, 'n-dimensional projective varieties with the action of an abelian group of rank $n-1$', *Trans. Amer. Math. Soc.* **368**(12) (2016), 8849–8872. MR3551591

[Zho22] G. Zhong, 'Compact Kähler threefolds with the action of an abelian group of maximal rank', *Proc. Amer. Math. Soc.* **150**(1) (2022), 55–68. MR4335856

[Zho23] G. Zhong, 'Existence of the equivariant minimal model program for compact Kähler threefolds with the action of an abelian group of maximal rank', *Math. Nachr.* **296**(7) (2023), 3128–3135. MR4626875