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On r-Cross Intersecting Families of Sets

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Let $(r-1)n \geqslant rk$ and let $\mathcal{F}_1,\dots,\mathcal{F}_r \subset \binom{[n]}{k}$. Suppose that $F_1\cap\cdots\cap F_r \neq \emptyset$ holds for all $F_i \in \mathcal{F}_i$, $1 \leqslant i \leqslant r$. Then we show that $\prod_{i=1}^{r} |\mathcal{F}_i| \leqslant \binom{n-1}{k-1}^r$.

1. Introduction

Let n,k,r be positive integers. We say that a family $\mathcal{F} \subset \binom{[n]}{k}$ is r-wise intersecting if $F_1\cap\cdots\cap F_r \neq \emptyset$ holds for all $F_i \in \mathcal{F}$, $1 \leqslant i \leqslant r$. Frankl [5] extended the Erdős–Ko–Rado theorem [4] as follows; see also [7, 9].

Theorem 1.1. *Let $(r-1)n \geqslant rk$ and let $\mathcal{F} \subset \binom{[n]}{k}$ be an r-cross intersecting family. Then we have $|\mathcal{F}| \leqslant \binom{n-1}{k-1}$.*

We say that families $\mathcal{F}_1,\dots,\mathcal{F}_r \subset \binom{[n]}{k}$ are r-cross intersecting if $F_1\cap\cdots\cap F_r \neq \emptyset$ holds for all $F_i \in \mathcal{F}_i$, $1 \leqslant i \leqslant r$. We show the following extension of Theorem 1.1.

Theorem 1.2. *Let $(r-1)n \geqslant rk$ and let $\mathcal{F}_1,\dots,\mathcal{F}_r \subset \binom{[n]}{k}$ be r-cross intersecting families. Then, we have $\prod_{i=1}^{r} |\mathcal{F}_i| \leqslant \binom{n-1}{k-1}^r$.*

We say that families $\mathcal{G}_1,\dots,\mathcal{G}_r \subset \binom{[n]}{\ell}$ are r-cross union if $G_1\cup\cdots\cup G_r \neq [n]$ holds for all $G_i \in \mathcal{G}_i$, $1 \leqslant i \leqslant r$. For $\mathcal{F} \subset \binom{[n]}{k}$ we define its complement family by $\mathcal{F}^c = \{[n]\setminus F : F \in \mathcal{F}\} \subset \binom{[n]}{\ell}$, where $\ell = n-k$. Note that $\mathcal{F}_1,\dots,\mathcal{F}_r$ are r-cross intersecting if and only if $\mathcal{F}_1^c,\dots,\mathcal{F}_r^c$ are r-cross

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union. To state our main result, we need one more definition. For $\mathcal{G} \subset \binom{[n]}{\ell}$ choose a unique real $x \geqslant \ell$ so that $|\mathcal{G}| = \binom{x}{\ell}$, and let $\|\mathcal{G}\|_\ell := x$.

Theorem 1.3. *Let $n \leqslant r\ell$ and let $\mathcal{G}_1, \ldots, \mathcal{G}_r \subset \binom{[n]}{\ell}$ be r-cross union families. Then we have the following:*

(i) $\sum_{i=1}^r \|\mathcal{G}_i\|_\ell \leqslant r(n-1)$,
(ii) $\prod_{i=1}^r |\mathcal{G}_i| \leqslant \binom{n-1}{\ell}^r$.

By considering the complement k-uniform families, where $k = n - \ell$, we get Theorem 1.2 from Theorem 1.3(ii). We will see that (i) implies (ii) easily. If $n > r\ell$, then r copies of $\binom{[n]}{\ell}$ are r-cross union families, which do not satisfy the conclusions of Theorem 1.3.

Our proof of Theorem 1.3 is very simple. In fact we only use well-known tools such as Katona's cycle method, the AM–GM inequality, and the Kruskal–Katona theorem on shadows. The novelty of the proof is to focus on inequality (i) of Theorem 1.3. There are several ways to prove Theorem 1.1, but the authors do not know any proof for Theorem 1.2 without using (i). Moreover, inequality (i) seems to be applicable for some other cases. It might be interesting to obtain the corresponding vector space version of Theorem 1.2 via (i). (See [3] for the vector space version of Theorem 1.1.)

The study of cross-intersecting families has a long history, starting with Bollobás [2]. We mention some results related to Theorem 1.2: the case $r = 2$ with different uniformity was considered by Pyber [10], Matsumoto and Tokushige [8] and Bey [1], and the non-uniform t-intersecting case was solved by Frankl [6].

2. Proof of Theorem 1.3

Let $x_i = \|\mathcal{G}_i\|_\ell$, that is, $|\mathcal{G}_i| = \binom{x_i}{\ell}$, for $1 \leqslant i \leqslant r$. First we show that (i) implies (ii).

Claim 2.1.

$$\prod_{i=1}^r |\mathcal{G}_i| = \binom{x_1}{\ell} \cdots \binom{x_r}{\ell} \leqslant \binom{\frac{x_1+\cdots+x_r}{r}}{\ell}^r \leqslant \binom{n-1}{\ell}^r.$$

Proof. The first inequality follows from the inequality of arithmetic and geometric means:

$$\text{2nd term} = \frac{1}{(\ell!)^r} \prod_{i=0}^{\ell-1} (x_1 - i) \cdots (x_r - i) \leqslant \frac{1}{(\ell!)^r} \prod_{i=0}^{\ell-1} \left(\frac{x_1 + \cdots + x_r}{r} - i \right)^r = \text{3rd term}.$$

The second inequality follows from (i). □

So all we need is to show (i). Let $s = r\ell - n$. We prove (i) by induction on s.

First we consider the initial step $s = 0$, that is, $n = r\ell$. We fix a cyclic permutation $\sigma = a_1 a_2 \cdots a_n \in S_n$, and let $\mathcal{A}^\sigma = \{A_1, A_2, \ldots, A_n\}$ be the set of arcs of length ℓ in σ, where $A_i = \{a_i, a_{i+1}, \ldots, a_{i+\ell-1}\}$ (the indices are read mod n). For $1 \leqslant i \leqslant r$, let $\mathcal{G}_i^\sigma = \mathcal{G}_i \cap \mathcal{A}^\sigma$.

Claim 2.2. *Let σ be an arbitrary cyclic permutation. Then $\sum_{i=1}^r |\mathcal{G}_i^\sigma| \leqslant r(n-\ell)$.*

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Proof. Let $\sigma = a_1 a_2 \cdots a_n \in S_n$ be given. For $1 \leqslant i \leqslant r$ and $j \in \mathbb{Z}_n$, let

$$\varepsilon_j^i = \begin{cases} 1 & \text{if } A_{j+(i-1)\ell} \in \mathcal{G}_i^\sigma, \\ 0 & \text{if } A_{j+(i-1)\ell} \notin \mathcal{G}_i^\sigma. \end{cases}$$

Then $|\mathcal{G}_i^\sigma| = \sum_{j=1}^n \varepsilon_j^i$. Note that $[n] = A_j \cup A_{j+\ell} \cup A_{j+2\ell} \cup \cdots \cup A_{j+(r-1)\ell}$ is a partition. Since $\mathcal{G}_1, \ldots, \mathcal{G}_r$ are r-cross union, we have $\#\{i : A_{j+(i-1)\ell} \in \mathcal{G}_i\} \leqslant r-1$. This gives $\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3 + \cdots + \varepsilon_j^r \leqslant r-1$ for all $j \in \mathbb{Z}_n$. Thus we have

$$\sum_{i=1}^r |\mathcal{G}_i^\sigma| = \sum_{j=1}^n (\varepsilon_j^1 + \varepsilon_j^2 + \varepsilon_j^3 + \cdots + \varepsilon_j^r) \leqslant (r-1)n = r(n-\ell),$$

where we used $n = r\ell$ in the last equality. □

Claim 2.3. *If $n = r\ell$, then we have $\sum_{i=1}^r |\mathcal{G}_i| \leqslant r\binom{n-1}{\ell}$.*

Proof. Each $G \in \mathcal{G}_i$ is counted $\ell!(n-\ell)!$ times in $\sum_{\sigma \in \mathcal{C}_n} |\mathcal{G}_i^\sigma|$, where $\mathcal{C}_n$ is the set of all cyclic permutations. This gives

$$\sum_{\sigma \in \mathcal{C}_n} \sum_{i=1}^r |\mathcal{G}_i^\sigma| = \ell!(n-\ell)! \sum_{i=1}^r |\mathcal{G}_i|.$$

On the other hand, since $|\mathcal{C}_n| = (n-1)!$, it follows from Claim 2.2 that

$$\sum_{\sigma \in \mathcal{C}_n} \sum_{i=1}^r |\mathcal{G}_i^\sigma| \leqslant (n-1)!\, r(n-\ell).$$

Thus we have

$$\sum_{i=1}^r |\mathcal{G}_i| \leqslant \frac{(n-1)!\, r(n-\ell)}{\ell!(n-\ell)!} = r\binom{n-1}{\ell},$$

as desired. □

We note that $f(x) = \binom{x}{\ell}$ is convex for $x \geqslant \ell$. In fact, one can show $f''(x) > 0$ for $x \geqslant \ell$ by a direct computation. So, we have

$$\binom{\frac{x_1+\cdots+x_r}{r}}{\ell} \leqslant \frac{1}{r}\sum_{i=1}^r \binom{x_i}{\ell} = \frac{1}{r}\sum_{i=1}^r |\mathcal{G}_i| \leqslant \binom{n-1}{\ell},$$

where we used Claim 2.3 for the last inequality. Thus we get $\frac{x_1+\cdots+x_r}{r} \leqslant n-1$, that is, part (i) of the theorem for the initial step $s = 0$.

Next we deal with the induction step. Let $s > 0$. Suppose that (i) is true for the case $r\ell - n = s$, and we will consider the case $r\ell - n = s+1$.

So, let $\mathcal{G}_1, \ldots, \mathcal{G}_r \subset \binom{[n]}{\ell}$ be r-cross union families with $r\ell - n = s+1$. Recall that $x_i = \|\mathcal{G}_i\|_\ell$ for $1 \leqslant i \leqslant r$, and we will show that

$$\sum_{i=1}^r x_i \leqslant r(n-1). \tag{2.1}$$

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Define $\mathcal{H}_i = \mathcal{G}_i \cup \mathcal{D}_i \subset \binom{[n+1]}{\ell}$ by

$$\mathcal{D}_i = \{D \cup \{n+1\} : D \in \Delta_{\ell-1}(\mathcal{G}_i)\},$$

where $\Delta_j(\mathcal{G}_i) = \{J \in \binom{[n]}{j} : J \subset \exists G \in \mathcal{G}_i\}$ is the jth shadow of $\mathcal{G}_i$. Then, by the Kruskal–Katona theorem, we have $|\mathcal{D}_i| = |\Delta_{\ell-1}(\mathcal{G}_i)| \geqslant \binom{x_i}{\ell-1}$, and

$$|\mathcal{H}_i| = |\mathcal{G}_i| + |\mathcal{D}_i| \geqslant \binom{x_i}{\ell} + \binom{x_i}{\ell-1} = \binom{x_i+1}{\ell},$$

namely,

$$z_i := \|\mathcal{H}_i\|_\ell \geqslant x_i + 1. \tag{2.2}$$

Now we note that $\mathcal{H}_1, \ldots, \mathcal{H}_r \subset \binom{[n+1]}{\ell}$ are r-cross union families. Moreover, since $r\ell - (n+1) = s$, we can apply the induction hypothesis to $\mathcal{H}_1, \ldots, \mathcal{H}_r$, and we get

$$\sum_{i=1}^{r} z_i \leqslant r((n+1)-1) = rn. \tag{2.3}$$

Finally, (2.1) follows from (2.2) and (2.3). This completes the proof of the theorem. □

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