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On Single-Distance Graphs on the Rational Points in Euclidean Spaces

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Abstract. For positive integers n and $d > 0$, let $G(\mathbb{Q}^n, d)$ denote the graph whose vertices are the set of rational points $\mathbb{Q}^n$, with $u, v \in \mathbb{Q}^n$ being adjacent if and only if the Euclidean distance between u and v is equal to d. Such a graph is deemed "non-trivial" if d is actually realized as a distance between points of $\mathbb{Q}^n$. In this paper, we show that a space $\mathbb{Q}^n$ has the property that all pairs of non-trivial distance graphs $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$ are isomorphic if and only if n is equal to 1, 2, or a multiple of 4. Along the way, we make a number of observations concerning the clique number of $G(\mathbb{Q}^n, d)$.

1 Introduction

Let $\mathbb{R}$, $\mathbb{Q}$ denote the fields of real and rational numbers, respectively, and let $\mathbb{Z}$ denote the ring of integers. For any $X \subseteq \mathbb{R}^n$, equip X with the usual Euclidean distance metric. That is, for $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we have $|x - y| = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$.

The central notion of this work will be that of the *Euclidean distance graph*. For $X \subseteq \mathbb{R}^n$ and $D \subset (0, \infty)$, let $G(X, D)$ designate the graph whose vertex set is X, with $x, y \in X$ being adjacent if and only if $|x - y| \in D$. If f is a graph parameter, such as the chromatic number χ, or the clique number ω, we abbreviate $f(G(X, D))$ as $f(X, D)$. In almost all cases, we will be concerned with D being a singleton, so we will also follow the convention of denoting the graph as $G(X, d)$ instead of the more awkward $G(X, \{d\})$. We refer the reader to [17] for an expansive treatment of this subject matter.

Our principal question is the following:

> "Given some $X \subseteq \mathbb{R}^n$, does there exist $d_1, d_2 > 0$ with d_1 and d_2 both realized as distances between points of X such that the graphs $G(X, d_1)$ and $G(X, d_2)$ are not isomorphic?"

Now, in the case of $X = \mathbb{R}^n$, this question is trivially resolved in the negative, as certainly any $G(\mathbb{R}^n, d_1)$ and $G(\mathbb{R}^n, d_2)$ are isomorphic by a simple scaling argument. However, in the case of $X = \mathbb{Q}^n$, such an argument does not apply when d_1 and d_2 are not rational multiples of each other. With this observation in mind, we will set

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about the business of answering the above question when $X = \mathbb{Q}^n$ for all positive integers n. We show that for n equal to 1, 2, or a multiple of 4, given $d_1, d_2 > 0$ such that both $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$ are both non-trivial, a bijection $\varphi : \mathbb{Q}^n \to \mathbb{Q}^n$ can be constructed that scales distance by a factor $\frac{d_2}{d_1}$, thus guaranteeing that the graphs $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$ are isomorphic. For all other values of n, we show that there exist particular selections of such d_1 and d_2 such that the clique numbers $\omega(\mathbb{Q}^n, d_1)$ and $\omega(\mathbb{Q}^n, d_2)$ are unequal. It follows that the corresponding $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$ are not isomorphic.

Regarding the clique number of a Euclidean distance graph, we will also recall a related parameter defined not on a graph but rather on a space $X \subseteq \mathbb{R}^n$. Originally seen in [1], let $C_n(X)$ be the n-th *clique number of* X where $C_n(X) = \max\{\omega(X, D) : D \subset (0, \infty) \text{ and } |D| = n\}$. In our work we also define $c_n(X)$ to be the n-th *lower clique number of* X where $c_n(X) = \min\{\omega(X, D) : D \subset (0, \infty), |D| = n, \text{ and each } d \in D \text{ is actually realized as a distance between points of } X\}$. In this context, our proof showing that the above question is answered in the affirmative (for those n where it does indeed have an affirmative answer) consists of verifying that $c_1(\mathbb{Q}^n) < C_1(\mathbb{Q}^n)$. We also make note of this notation, as it will also appear in the final section of this paper where we give a number of open problems for further research.

2 Preliminaries

In this section, a collection of lemmas and previous results is assembled for use in proving our main result in the section to follow. We begin with observations concerning sets of equidistant points in $\mathbb{R}^n$. We omit the proof of the following lemma as it is a standard exercise in the geometry of real inner product spaces. Guidance can be obtained in [3, 4].

Lemma 2.1 *Suppose that* $P_0, \dots, P_{n-1} \in \mathbb{R}^n$ *and for* $0 \le i < j \le n-1$, $|P_i - P_j| = \sqrt{d} > 0$. *Let* $C = \frac{1}{n}\sum_{i=0}^{n-1} P_i$, *and let*

$$S = \left\{Q \in \mathbb{R}^n : |P_0 - Q| = |P_1 - Q| = \cdots = |P_{n-1} - Q|\right\}.$$

Then each of the following is true.

(i) $|P_i - C| = \sqrt{\frac{d(n-1)}{2n}}$ *for* $i = 0, \dots, n-1$.
(ii) *S is the line $C + L$ where L is the orthogonal complement of H, the hyperplane in $\mathbb{R}^n$ spanned by the vectors $P_i - P_0$ where $i = 1, \dots, n-1$.*
(iii) *For each* $Q \in S$, *for* $i \in \{0, \dots, n-1\}$, $|P_i - Q|^2 = \frac{d(n-1)}{2n} + |C - Q|^2$.

Although we are omitting the proof, it is useful to note that the equidistance of the points P_i implies that the vectors $P_i - P_0$ for $i \in \{1, \dots, n-1\}$ are linearly independent. Also, each $P_i \in C + H$. We are, of course, primarily interested in the

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vertices of such a simplex (as in Lemma 2.1) being rational points, and in that case, we are able to give a useful characterization of distances realized between points of S.

Lemma 2.2 *Let L be a line in $\mathbb{R}^n$ and suppose that $P, Q \in L \cap \mathbb{Q}^n$, $P \neq Q$, and $P - Q = (\frac{a_1}{b}, \dots, \frac{a_n}{b})$, $a_1, \dots, a_n, b \in \mathbb{Z}$. Then there is an integer, namely, $y = a_1^2 + \dots + a_n^2$, such that for any points $P', Q' \in L \cap \mathbb{Q}^n$, $|P' - Q'| = r\sqrt{y}$ for some $r \in \mathbb{Q}$.*

Proof Let $v = P - Q$. Then $L = \{Q + tv : t \in \mathbb{R}\}$ and $L \cap \mathbb{Q}^n = \{Q + tv : t \in \mathbb{Q}\}$. Letting $P', Q' \in L \cap \mathbb{Q}^n$, for some $t_1, t_2 \in \mathbb{Q}^n$ we have $P' = Q + t_1 v$ and $Q' = Q + t_2 v$. Consequently, $|P' - Q'| = |t_1 - t_2||v| = |\frac{t_1 - t_2}{b}|\sqrt{y}$.

The following result of Schoenberg [16] is also vital to our work.

Theorem 2.3 *A regular n-simplex (that is, $n+1$ equidistant points) can be embedded in $\mathbb{Z}^n$ if and only if one of the following hold:*

(i) *n is even and $n+1$ is a perfect square;*
(ii) *$n \equiv 3 \pmod 4$;*
(iii) *$n \equiv 1 \pmod 4$ and $n+1$ is the sum of two squares.*

A regular n-simplex can be embedded in $\mathbb{Z}^n$ if and only if it is possible to embed a regular n-simplex in $\mathbb{Q}^n$, so using our notation, Theorem 2.3 gives us when $C_1(\mathbb{Q}^n) = n+1$. Furthermore, for those values of n where $C_1(\mathbb{Q}^n) \neq n+1$, certainly we have $C_1(\mathbb{Q}^n) = n$, as evidenced by the standard n orthogonal unit vectors. These observations will play a key role in several proofs presented in the next section, and for easy reference, we note them as Corollary 2.4.

Corollary 2.4 *For a positive integer n, $C_1(\mathbb{Q}^n) = n+1$ in the following cases:*

(i) *n is even and $n+1$ is a perfect square;*
(ii) *$n \equiv 3 \pmod 4$;*
(iii) *$n \equiv 1 \pmod 4$ and $n+1$ is the sum of two squares.*

Otherwise, $C_1(\mathbb{Q}^n) = n$.

We now take a moment to sample a few results from classical number theory. Theorem 2.5 summarizes Euler, Gauss, and Lagrange's well-known characterizations of integers representable as sums of two, three, and four squares, respectively. Theorem 2.6 is Legendre's result concerning the solvability of a certain type of Diophantine equation. In the next section, we will also occasionally employ a few other number-theoretic staples in the reciprocity laws, the Chinese Remainder Theorem, and Dirichlet's theorem concerning primes in arithmetic progressions. For further reference, one could consult virtually any textbook on elementary number theory, with our favorite being [12].

Theorem 2.5 *Let $n \in \mathbb{Z}^+$. There exist $a, b \in \mathbb{Z}$ such that $a^2 + b^2 = n$ if and only if the square-free part of n has no prime factor congruent to $3 \pmod 4$. There exist $a, b, c \in \mathbb{Z}$*

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such that $a^2 + b^2 + c^2 = n$ *if and only if the square-free part of n is not congruent to* $7 \pmod 8$*. For all n, there exist* $a, b, c, d \in \mathbb{Z}$ *such that* $a^2 + b^2 + c^2 + d^2 = n$*.*

Theorem 2.6 *Let* a, b, c *be non-zero integers, not each positive or each negative, and suppose that* abc *is square-free. Then the equation*

$$ax^2 + by^2 + cz^2 = 0$$

has a non-trivial integer solution (x, y, z) *if and only if each of the following are satisfied:*

(i) $-ab$ *is a quadratic residue of* c*;*
(ii) $-ac$ *is a quadratic residue of* b*;*
(iii) $-bc$ *is a quadratic residue of* a*.*

Theorem 2.5 engenders a couple of conditional ramifications to Schoenberg's Theorem 2.3. If $n \equiv 2 \pmod 4$, it must be the case that the square-free part of $n+1$ has at least one prime factor $p \equiv 3 \pmod 4$. Therefore, $n \equiv 2 \pmod 4$ implies $C_1(\mathbb{Q}^n) = n$. Also, note that for $n \equiv 0 \pmod 4$, when $C_1(\mathbb{Q}^n) = n+1$, we have $n+1$ being a perfect square, which implies $C_1(\mathbb{Q}^{n+1}) = n+2$.

We now come to what is the spiritual forebear of this work. In [4], Chilakamarri determines the clique number $\omega(\mathbb{Q}^n, 1)$ for all n. His main result is given as Theorem 2.7 below, although we have updated its statement to fit our current notation.

Theorem 2.7 *For even n, the clique number* $\omega(\mathbb{Q}^n, 1)$ *equals* $n+1$ *or* n *according to whether* $n+1$ *is or is not a perfect square. For odd n, if the Diophantine equation* $nx^2 - 2(n-1)y^2 = z^2$ *has a solution with* $x \neq 0$*, then* $\omega(\mathbb{Q}^n, 1)$ *equals* $n+1$ *or* n *according to whether* $\frac{1}{2}(n+1)$ *is or is not a perfect square. If* $nx^2 - 2(n-1)y^2 = z^2$ *is not solvable in the integers with* $x \neq 0$*, then* $\omega(\mathbb{Q}^n, 1) = n-1$*.*

As mentioned in the previous section, our primary goal is to decide for given n whether or not there are non-trivial and non-isomorphic graphs $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$. Theorem 2.7 shows that for many odd n, such graphs do indeed exist. Although Chilakamarri himself does not further address the Diophantine equation $nx^2 - 2(n-1)y^2 - z^2 = 0$ and comment on when it is and is not solvable, we remark that in [6], Elsholtz and Klotz demonstrate that for odd n, the equation is solvable if and only if the square-free part of n consists solely of factors congruent to 1 or 7 modulo 8. In conjunction with Corollary 2.4, we have, for all other odd n, $\omega(\mathbb{Q}^n, 1) < C_1(\mathbb{Q}^n)$. However, for those n where $\omega(\mathbb{Q}^n, 1) = C_1(\mathbb{Q}^n)$, which happens to include all even n, certainly more work needs to be done.

For our purposes, the method Chilakamarri uses in [4] to prove Theorem 2.7 is more useful than the theorem itself. He employs a result due to Hall and Ryser (given as Theorem 2.8 below) to show that for sets $U, V \subset \mathbb{Q}^n$ with each of U, V consisting of $n-1$ points satisfying $A, B \in U$ and $C, D \in V$ imply $|A - B| = |C - D| = 1$, there exists a rational isometry mapping U to V. Such a transformation preserves distance, and is an effective tool in proving Theorem 2.7 by the following rationale. If a specific set of $n-1$ points in $\mathbb{Q}^n$ constitute the vertices of a copy of K_{n-1} in $G(\mathbb{Q}^n, 1)$, and it can be shown that there does not exist a point in $\mathbb{Q}^n$ at distance 1 from each of

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those $n-1$ points, one immediately has that $\omega(\mathbb{Q}^n,\ 1) = n-1$. We happily add this observation to our toolbox in the form of Theorem 2.9, whose proof is a cosmetically-altered generalization of the one given in [4].

Theorem 2.8 [7] *Let A be a non-singular $n \times n$ matrix with entries from a field of characteristic not equal to 2. Suppose that $AA^T = D_1 \oplus D_2$, the direct sum of two square matrices D_1 and D_2 of orders r and s, respectively, where $r + s = n$. Let M be an arbitrary $r \times n$ matrix such that $MM^T = D_1$. Then there exists an $n \times n$ matrix Z with entries from the field and having M as its first r rows such that $ZZ^T = D_1 \oplus D_2$.*

Theorem 2.9 *Let $n, r \in \mathbb{Z}^+$ with $r \le n$. Let $U, V \in \mathbb{Q}^n$ where $|U| = |V| = r$ and both U, V each constitute the vertices of a copy of K_r appearing as a subgraph of $G(\mathbb{Q}^n,\ d)$. Then there exists a rational isometry mapping U to V.*

Proof Write $U = \{u_0, u_1, \dots, u_{r-1}\}$ and $V = \{v_0, v_1, \dots, v_{r-1}\}$ and without loss of generality, assume both u_0 and v_0 are the origin.

Let $u_r, \dots, u_n$ be independent vectors in $\mathbb{Q}^n$ that are orthogonal to all the vectors of U. Let A be the $n \times n$ matrix with rows $u_1, u_2, \dots, u_n$ and let M be the $(r-1) \times n$ matrix with rows $v_1, \dots, v_{r-1}$. Note that A is non-singular. Write $AA^T = D_1 \oplus D_2$ and $MM^T = D_1$ where D_1 is a square matrix of order $r-1$, and D_2 is a non-singular square matrix of order $n - r + 1$. By Hall and Ryser's Theorem 2.8, there exists an $n \times n$ matrix Z having M as its first $r-1$ rows such that $ZZ^T = D_1 \oplus D_2$. Let $L = Z^{-1}A$. Then L has rational entries and $v_i L = u_i$ for $i = 0, \dots, r-1$. Moreover, L is orthogonal seeing as $(Z^T)^{-1}Z^{-1}AA^T = I$ and so $LL^T = Z^{-1}AA^T(Z^{-1})^T = I$. ∎

Theorem 2.9 gives the following corollary, which will be a fundamental utility in the proofs of Theorems 3.4, 3.5, and 3.6 in the next section.

Corollary 2.10 *If $r \le \omega(\mathbb{Q}^n,\ d)$ and $P_1, \dots, P_{r-1} \in \mathbb{Q}^n$ satisfy $|P_i - P_j| = d$ for $1 \le i < j \le r-1$, then there is a point $P \in \mathbb{Q}^n$ such that $|P - P_i| = d$ for $i = 1, \dots, r-1$.*

3 Main Results

In this section we prove that a space $\mathbb{Q}^n$ has the property that any two non-trivial distance graphs $G(\mathbb{Q}^n,\ d_1)$ and $G(\mathbb{Q}^n,\ d_2)$ are isomorphic if and only if n equals 1, 2, or a multiple of 4.

Theorem 3.1 *Let n, z be positive integers where n is even and $z = a^2 + b^2$ for some $a, b \in \mathbb{Z}$. Let $d > 0$ be a distance realized between points of $\mathbb{Q}^n$. Then the graphs $G(\mathbb{Q}^n,\ d)$ and $G(\mathbb{Q}^n,\ d\sqrt{z})$ are isomorphic.*

Proof We will give two proofs of this theorem, although both are essentially different flavors of the same argument. Let $\mathbb{Q}[i]$ denote the field of Gaussian rationals; that is, all complex numbers whose real and imaginary part are both rational. We have $\mathbb{Q}^2 \simeq \mathbb{Q}[i]$. Let $\alpha = a + bi \in \mathbb{Q}[i]$. Let $n = 2k$ for some $k \in \mathbb{Z}^+$. Then $\mathbb{Q}^n \simeq (\mathbb{Q}[i])^k$ in

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an obvious way:

$$(x_1, \dots, x_n) \leftrightarrow (x_1 + x_2 i, \dots, x_{n-1} + x_n i) \in (\mathbb{Q}[i])^k.$$

Multiplication by α maps $(\mathbb{Q}[i])^k$ bijectively onto itself. For $u, v \in (\mathbb{Q}[i])^k$ with $|u - v| = d$, we have

$$|\alpha u - \alpha v| = |\alpha||u - v| = \sqrt{z}d.$$

Note that in the above discussion, $|\alpha| = \sqrt{a^2 + b^2} = \sqrt{z}$ is the modulus (absolute value) of the complex number α, while $|u - v|$ is the Euclidean distance between the rational vectors u and v.

Alternately, consider the matrix $M_0 = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. Construct an $n \times n$ matrix $M = M_0 \oplus \cdots \oplus M_0$, formed by taking the direct sum of k copies of M_0. Any two column vectors of M have dot product 0 and each has length $\sqrt{z}$. Thus, M defines a bijective transformation of $\mathbb{Q}^n$ where any two points distance d apart are mapped to points distance $d\sqrt{z}$ apart. ■

By a quick scaling argument, for any $q \in \mathbb{Q}^+$ and $d > 0$, the graphs $G(\mathbb{Q}^n, d)$ and $G(\mathbb{Q}^n, qd)$ are isomorphic. It follows that any non-trivial distance graph with vertex set $\mathbb{Q}^n$ is isomorphic to some $G(\mathbb{Q}^n, \sqrt{r})$ where $r \in \mathbb{Z}^+$. In the case of $n = 2$, any non-trivial $G(\mathbb{Q}^2, \sqrt{r})$ by necessity has $r = a^2 + b^2$ for some $a, b \in \mathbb{Z}^+$. As Theorem 3.1 implies any such graph is isomorphic to $G(\mathbb{Q}^2, 1)$, we have Corollary 3.2. We remark that this result also appeared in [11].

Corollary 3.2 *Any two non-trivial distance graphs $G(\mathbb{Q}^2, d_1)$ and $G(\mathbb{Q}^2, d_2)$ are isomorphic.*

Theorem 3.3 *Let $n \in \mathbb{Z}^+$ with $n \equiv 0 \pmod 4$. For every d realized as a distance between points of $\mathbb{Q}^n$, the graphs $G(\mathbb{Q}^n, d)$ and $G(\mathbb{Q}^n, 1)$ are isomorphic.*

Proof Let $n = 4t$, and without loss of generality, suppose $d = \sqrt{m}$ for some $m \in \mathbb{Z}^+$. By Theorem 2.5, there exist non-negative integers p, q, r, s such that $m = p^2 + q^2 + r^2 + s^2$. Denote by $\mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ the rational quaternions, and let $\alpha = p + q\mathbf{i} + r\mathbf{j} + s\mathbf{k} \in \mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}]$. We know that $\mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}] \simeq \mathbb{Q}^4$. We also have, as vector spaces of $\mathbb{Q}$,

$$\mathbb{Q}^n \simeq (\mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}])^{\frac{n}{4}} \simeq (\mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}])^t.$$

We can treat $(\mathbb{Q}[\mathbf{i}, \mathbf{j}, \mathbf{k}])^t$ as a module over the ring of rational quaternions. Multiplication by $\frac{1}{\alpha} = \frac{1}{p^2+q^2+r^2+s^2}(p - q\mathbf{i} - r\mathbf{j} - s\mathbf{k})$ maps $(\mathbb{Q}^4)^t$ bijectively onto itself, with every pair of vectors Euclidean distance d apart mapped to a pair of vectors unit distance apart. ■

The proof of the "only if" direction of our main result will be divided into three separate theorems. Although the same style of argument is presented in all three, each case has its own particular twists and turns, and this presentation is intended to heighten clarity.

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Theorem 3.4 *Let $n \equiv 2 \pmod 4$ with $n \geq 6$. There exist non-trivial distance graphs $G(\mathbb{Q}^{n+1}, d_1)$ and $G(\mathbb{Q}^{n+1}, d_2)$ that are not isomorphic.*

Proof Corollary 2.4 gives $C_1(\mathbb{Q}^{n+1}) = n+2$, so it suffices to find a positive integer d such that $\omega(\mathbb{Q}^{n+1}, \sqrt{d}) < n+2$. Let $p \equiv 1 \pmod 4$ be prime with $p > n$. By Theorem 2.5, there exist $a, b \in \mathbb{Z}$ such that $p = a^2 + b^2$. Consider points $P_1, \ldots, P_n \in \mathbb{Q}^n$ formed as follows. For odd i, P_i has its i-th coordinate equal to a, its $(i+1)$-th coordinate equal to b, and all other coordinates 0. For even i, P_i has its $(i-1)$-th coordinate equal to $-b$, its i-th coordinate equal to a, and all other coordinates 0. Note that $P_1, \ldots, P_n$ constitute the vertices of K_n appearing as a subgraph of $G(\mathbb{Q}^n, \sqrt{2p})$.

Let $P_1', \ldots, P_n' \in \mathbb{Q}^{n+1}$ be formed by placing 0 as an $(n+1)$-th coordinate for each of the corresponding P_i. Assume there exists $P \in \mathbb{Q}^{n+1}$ at distance $\sqrt{2p}$ from each of $P_1', \ldots, P_n'$ and write $P = (x_1, \ldots, x_n, x)$. As per Lemma 2.1, let $L \subset \mathbb{R}^n$ be the line of points equidistant to each of $P_1, \ldots, P_n$ and note that $(x_1, \ldots, x_n)$ must lie on L. Let $C \in \mathbb{Q}^n$ be the circumcenter of the $(n-1)$-simplex with vertices $P_1, \ldots, P_n$. We have $C = (\frac{a-b}{n}, \frac{a+b}{n}, \ldots, \frac{a-b}{n}, \frac{a+b}{n})$ and note that C lies on L.

By Lemma 2.2, there exists $z \in \mathbb{Z}^+$ such that for any two distinct rational points on L, the distance between those points is equal to $y\sqrt{z}$ for some selection of $y \in \mathbb{Q}^+$. Here, we can calculate z by finding the distance between C and $(0, \ldots, 0)$, which is also on L. The distance between these two points is

$$\sqrt{\Big(\frac{n}{2}\Big)\Big(\frac{a-b}{n}\Big)^2 + \Big(\frac{n}{2}\Big)\Big(\frac{a+b}{n}\Big)^2},$$

which simplifies to $\sqrt{\frac{p}{n}}$. So we can use $z = pn$. Note also that by Corollary 2.1, we have $|P_i - C| = \sqrt{\frac{p(n-1)}{n}}$ for each i.

In light of the above discussion, a calculation of the distance between P and any of the P_i leaves us with the following Diophantine equation:

$$\frac{p(n-1)}{n} + pny^2 + x^2 = 2p. \tag{3.1}$$

We now move to homogeneous coordinates, where equation (3.2) has a non-trivial integer solution if equation (3.1) has a non-trivial rational solution:

$$pnx^2 + y^2 - (n+1)z^2 = 0. \tag{3.2}$$

Our goal is to apply Theorem 2.6 and show the existence of p such that equation (3.2) is not solvable. We do this by way of a classical argument. Let r be the square-free part of $n+1$, and note that $r \neq 1$. For solvability, r must be a residue of p, expressed using the Legendre symbol as $(\frac{r}{p}) = 1$. Suppose r has prime factorization $r = q_1 \cdots q_s$. Let α_1 be a non-residue of q_1. For $i = 2, \ldots, s$, let α_i be a residue of q_i. Consider the system of linear congruences $t \equiv 1 \pmod 4$, $t \equiv \alpha_i \pmod{q_i}$ for $i = 1, \ldots, s$. By the Chinese Remainder Theorem, there exists a solution t to this system. Now consider the arithmetic progression $t + 4rj$ for $j \in \mathbb{Z}^+$. Dirichlet's Theorem asserts that there exists a prime in this sequence. We make this prime our choice for p. Since $p \equiv 1 \pmod 4$, by quadratic reciprocity, we have $(\frac{r}{p}) = (\frac{q_1}{p}) \cdots (\frac{q_s}{p}) = (\frac{p}{q_1}) \cdots (\frac{p}{q_s}) = -1$. This

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contradiction indicates that no such P exists to extend $P'_1, \ldots, P'_n$ to the complete graph K_{n+1} in $G(\mathbb{Q}^{n+1}, \sqrt{2p})$. Corollary 2.10 concludes that K_{n+1} is not a subgraph of $G(\mathbb{Q}^{n+1}, \sqrt{2p})$ and completes the proof of the theorem. ∎

Theorem 3.5 *Let $n \in \mathbb{Z}^+$ with $n \equiv 0 \pmod 4$. There exist non-trivial distance graphs $G(\mathbb{Q}^{n+1}, d_1)$ and $G(\mathbb{Q}^{n+1}, d_2)$ that are not isomorphic.*

Proof By Corollary 2.4, either $C_1(\mathbb{Q}^n) = n$ or $C_1(\mathbb{Q}^n) = n + 1$. However, we can address the case of $C_1(\mathbb{Q}^n) = n$ using a line-by-line replication of the argument in the proof of Theorem 3.4.

Suppose that $C_1(\mathbb{Q}^n) = n + 1$. Again considering Corollary 2.4, note that this implies that $n + 1$ is a perfect square. We then have $n + 2$ being a sum of two squares (namely, $n + 1$ and 1), and so $C_1(\mathbb{Q}^{n+1}) = n + 2$. Let $P_1, \ldots, P_{n+1} \in \mathbb{Q}^n$ constitute the vertices of a copy of K_{n+1} appearing as a subgraph of $G(\mathbb{Q}^n, \sqrt{d})$. By Theorem 3.3, such points exist for any $d \in \mathbb{Z}^+$. Without loss of generality, assume the circumcenter of the n-simplex with vertices $P_1, \ldots, P_{n+1}$ is the origin.

Let $P'_1, \ldots, P'_{n+1} \in \mathbb{Q}^{n+1}$ be formed by placing 0 as an $(n+1)$-th coordinate for each of the corresponding P_i. Assume there exists $P \in \mathbb{Q}^{n+1}$ at distance $\sqrt{d}$ from each of $P'_1, \ldots, P'_{n+1}$. Then $P = (0, \ldots, 0, x)$ for some $x \in \mathbb{Q}$. By Lemma 2.1, the distance from any of $P_1, \ldots, P_{n+1}$ to the origin is $\sqrt{\frac{dn}{2n+2}}$. Now calculating the distance from P to any P'_i, we obtain the following Diophantine equation:

$$\frac{dn}{2n+2} + x^2 = d. \tag{3.3}$$

This equation has a rational solution for x if and only if $\frac{d(n+2)}{2(n+1)}$ is a rational square. So any selection of a d (say, a prime larger than $n + 2$) that results in it not being a square contradicts the existence of P. We now apply Corollary 2.10 and conclude that $\omega(\mathbb{Q}^{n+1}, \sqrt{d}) = n + 1$ that is less than $C_1(\mathbb{Q}^{n+1})$. This completes the proof of the theorem. ∎

Theorem 3.6 *Let $n \in \mathbb{Z}^+$ with $n \equiv 0 \pmod 4$. There exist non-trivial distance graphs $G(\mathbb{Q}^{n+2}, d_1)$ and $G(\mathbb{Q}^{n+2}, d_2)$ that are not isomorphic.*

Proof By Corollary 2.4, we have $C_1(\mathbb{Q}^{n+2}) = n + 2$. First, consider the case of $C_1(\mathbb{Q}^n) = n + 1$. By Theorem 3.3, for any selection of $d \in \mathbb{Z}^+$, we can let $P_1, \ldots, P_{n+1} \in \mathbb{Q}^n$ be the vertices of a regular n-simplex of edge-length $\sqrt{d}$. Furthermore, we can assume this simplex has its circumcenter at the origin.

Create points $P'_1, \ldots, P'_{n+1} \in \mathbb{Q}^{n+2}$ by placing zeroes as the last two coordinate entries of the corresponding P_i. Assume there exists $P \in \mathbb{Q}^{n+2}$ where P is at distance $\sqrt{d}$ from each of $P'_1, \ldots, P'_{n+1}$. Then P must be of the form $(0, \ldots, 0, x, y)$ for some $x, y \in \mathbb{Q}$. Using Lemma 2.1, we calculate the distance from P to any of the P_i, and set it equal to $\sqrt{d}$ to obtain the following Diophantine equation:

$$\frac{dn}{2n+2} + x^2 + y^2 = d. \tag{3.4}$$

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Equation (3.4) can be rearranged to obtain equation (3.5).

$$x^2 + y^2 = \frac{d(n+2)}{2n+2}. \tag{3.5}$$

All we need to do to make equation (3.5) unsolvable is to select a d so that Theorem 2.5 guarantees $\frac{d(n+2)}{2n+2}$ is not a sum of two squares. As an example, $d = 3(n+2)(2n+2)$ does the work.

We now consider the case of $C_1(\mathbb{Q}^n) = n$. Recall the points $P_1, \dots, P_n$ as formed in the proof of Theorem 3.4. Since $n \equiv 0 \pmod 4$, for any prime $q \equiv 3 \pmod 4$, there exists a bijective transformation $\varphi : \mathbb{Q}^n \to \mathbb{Q}^n$ that scales distance by a factor of $\sqrt{\frac{q}{p}}$. For $i \in \{1, \dots, n\}$, let $\varphi(P_i) = Q_i$ where $Q_1, \dots, Q_n$ form the vertices of a copy of K_n appearing as a subgraph of $G(\mathbb{Q}^n, \sqrt{2q})$. Let $L \subset \mathbb{R}^n$ be the set of all points equidistant to $P_1, \dots, P_n$ and let $L' \subset \mathbb{R}^n$ be the set of all points equidistant to $Q_1, \dots, Q_n$. We have by Lemma 2.1 that L and L' are both lines and note that $\varphi(L) = L'$. By Lemma 2.2, there exist $z, z' \in \mathbb{Z}^+$ such that for any pair of rational points on L or pair of rational points on L', the distance between those points is equal to $y\sqrt{z}$ or $y'\sqrt{z'}$, respectively, for some selection of $y, y' \in \mathbb{Q}^+$. In the proof of Theorem 3.4, we explicitly found $z = pn$. Since φ scales distance by a factor of $\sqrt{\frac{q}{p}}$, we have

$$\sqrt{z'} = (\sqrt{z})\left(\sqrt{\frac{q}{p}}\right) = (\sqrt{pn})\left(\sqrt{\frac{q}{p}}\right) = \sqrt{qn}.$$

So we can use $z' = qn$.

Create points $Q'_1, \dots, Q'_n \in \mathbb{Q}^{n+2}$ by placing zeroes as the last two coordinate entries of the corresponding Q_i. Assume there exists $Q \in \mathbb{Q}^{n+2}$ at distance $\sqrt{2q}$ from each of $Q'_1, \dots, Q'_n$. Write $Q = (x_1, \dots, x_n, x, z)$ for some $x, z \in \mathbb{Q}$, and note that $(x_1, \dots, x_n)$ lies on L'. By Lemma 2.1, we have the circumradius of the $(n-1)$-simplex with vertices $Q_1, \dots, Q_n$ is $\sqrt{\frac{q(n-1)}{n}}$. We can now calculate the distance from Q to any of the Q'_i and set it equal to $\sqrt{2q}$ to obtain the following Diophantine equation:

$$\frac{q(n-1)}{n} + qny^2 + x^2 + z^2 = 2q. \tag{3.6}$$

We now combine constant terms, and move to homogeneous coordinates to produce equation (3.7), which is solvable in the integers if equation (3.6) is solvable in the rationals:

$$n(x^2 + z^2) + q\left(y^2 - (n+1)w^2\right) = 0. \tag{3.7}$$

Consider a potential solution $(x, y, z, w) \in \mathbb{Z}^4$ to equation (3.7). If $q \equiv 3 \pmod 4$ is selected so that $\gcd(n, q) = 1$, we must have $(x^2 + z^2) \equiv 0 \pmod q$. By Theorem 2.5, we would be forced to have both $x, z \equiv 0 \pmod q$. Hence, we can write $x = qx_0$ and $z = qz_0$ for some $x_0, z_0 \in \mathbb{Z}$. This results in the term $y^2 - (n+1)w^2$ necessarily being congruent to $0 \pmod q$ as well. Consider the congruence $y^2 - (n+1)w^2 \equiv 0 \pmod q$, and since $n+1$ is odd and not a perfect square, perform a similar procedure to that put forth in the proof of Theorem 3.4 where we use quadratic reciprocity along with the Chinese Remainder Theorem and Dirichlet's Theorem to find a prime $q \equiv 3$

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$(\text{mod } 4)$ with $n+1$ a non-residue of q. This results in the congruence $y^2 - (n+1)w^2 \equiv 0 \pmod{q}$ only being satisfied when y, w are both multiples of q. Combining all of these observations, we conclude that in any solution (x, y, z, w) of equation (3.7), x, y, z, w are multiples of q. This infinite descent shows that equation (3.7) is in fact not solvable for our selection of q. ∎

Finally, note that in the specific case of $\mathbb{Q}^3$ (which is not addressed by Theorem 3.4), we have $\omega(\mathbb{Q}^3, 1) = 2$ and $C_1(\mathbb{Q}^3) = 4$ by Theorem 2.7 and Corollary 2.4, respectively. Combining this observation with Corollary 3.2 and Theorems 3.3 through 3.6, we obtain the main result of this section.

Theorem 3.7 *Let $n \in \mathbb{Z}^+$. The space $\mathbb{Q}^n$ has the property that any two non-trivial distance graphs $G(\mathbb{Q}^n, d_1)$ and $G(\mathbb{Q}^n, d_2)$ are isomorphic if and only if n is equal to 1, 2, or a multiple of 4.*

4 Further Work

Open questions concerning structural properties of Euclidean distance graphs are numerous, and more are being produced at a steady clip. Indeed, the authors (particularly the third author) will freely admit to whiling away many a lazy afternoon dreaming them up. Here are a few that are ripe for further investigation.

Question 1 *For $n \in \mathbb{Z}^+$, and arbitrary d realized as a distance between points of $\mathbb{Q}^n$, determine $\omega(\mathbb{Q}^n, d)$.*

A complete generalization of Theorem 2.7 may be a little too much to ask for. As well, an answer to Question 1 may be quite unwieldy, as it would likely involve a number of conditional statements regarding solvability of Diophantine equations. For $n = 3$, however, it has been completely resolved. Theorem 4.1 is a restatement of Ionascu's main result in [8].

Theorem 4.1 *Let $z \in \mathbb{Z}^+$ be square-free, with $\sqrt{z}$ being realized as a distance between points of $\mathbb{Q}^3$.*

(i) *If $z = 2$, then $\omega(\mathbb{Q}^3, \sqrt{z}) = 4$.*
(ii) *If z is even, but has no odd factor congruent to 2 modulo 3, then $\omega(\mathbb{Q}^3, \sqrt{z}) = 3$.*
(iii) *Otherwise, $\omega(\mathbb{Q}^3, \sqrt{z}) = 2$.*

The qualification that z is square-free is not limiting at all. For any $q \in \mathbb{Q}^+$, the graphs $G(\mathbb{Q}^n, d)$ and $G(\mathbb{Q}^n, qd)$ are isomorphic, and it follows that any non-trivial distance graph $G(\mathbb{Q}^n, d)$ is isomorphic to one of the form $G(\mathbb{Q}^n, \sqrt{z})$ where z is some square-free positive integer. With Ionascu's result in mind, we give a few updates to a long-standing problem originally posed by Benda and Perles in [2]. It is Question 2 below.

Question 2 *Does there exist a graph $G(\mathbb{Q}^3, \sqrt{z})$ of chromatic number $\chi(\mathbb{Q}^3, \sqrt{z}) = 3$?*

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Chromatic numbers of graphs $G(\mathbb{Q}^n, d)$ have been well studied for small values of n (see [14, 15] for further reading). In [10], it was shown that for any $d > 0$ and $n \le 4$, $\chi(\mathbb{Q}^n, d) \le 4$. In passing, we remark that Benda and Perles proved in [2] that $\chi(\mathbb{Q}^4, 1) = 4$, and this result, taken in conjunction with Theorem 3.3, implies the main result of [10].

Let z be defined as in Theorem 4.1. It is shown in [9] that if z is odd, then $\chi(\mathbb{Q}^3, \sqrt{z}) = 2$. In [5], Chow proves that for z even, $\chi(\mathbb{Q}^3, \sqrt{z}) \ge 3$. Finally, in [13], it was shown that if $\omega(\mathbb{Q}^3, \sqrt{z}) = 3$, then $\chi(\mathbb{Q}^3, \sqrt{z}) = 4$. The author of [13] also considered the graph $G(\mathbb{Q}^3, \sqrt{10})$ and was able to show that, even though $\omega(\mathbb{Q}^3, \sqrt{10}) = 2$ by Theorem 4.1, it is nevertheless the case that $\chi(\mathbb{Q}^3, \sqrt{10}) = 4$ as well. This gives us some belief in Question 2 having a negative answer, but the problem remains open.

Also regarding Theorem 4.1, we present the following question. Define the *odd girth* of a graph G to be the minimum length of an odd cycle in G. If G is bipartite, define its odd girth to be ∞.

Question 3 *For any $d > 0$, is it true that the odd girth of $G(\mathbb{Q}^3, d)$ is equal to $3, 5,$ or ∞?*

The next few questions concern the n-th clique number and n-th lower clique number defined in Section 1.

Question 4 *Does there exist n such that $C_1(\mathbb{Q}^n) - c_1(\mathbb{Q}^n) = 4$?*

Some perspective is needed. Both $C_1(\mathbb{Q}^n)$ and $c_1(\mathbb{Q}^n)$ are non-decreasing functions of n. By Theorem 3.3, for any positive integer m, we have $C_1(\mathbb{Q}^{4m}) = c_1(\mathbb{Q}^{4m})$. As discussed in Section 2, $C_1(\mathbb{Q}^n) = n$ or $n+1$. For a space $\mathbb{Q}^n$, we may write $n = 4m + k$ where m is a non-negative integer and $k \in \{0, 1, 2, 3\}$. Then $4m \le c_1(\mathbb{Q}^n) \le C_1(\mathbb{Q}^n) \le 4m + k + 1$, which implies that $C_1(\mathbb{Q}^n) - c_1(\mathbb{Q}^n) \le 4$.

We also note that there are examples of n where $C_1(\mathbb{Q}^n) - c_1(\mathbb{Q}^n) \ge 3$. In the proof of Theorem 3.6, we let $n \equiv 0 \pmod 4$ and first considered the case of $C_1(\mathbb{Q}^n) = n + 1$. Points $P_1', \dots, P_{n+1}' \in \mathbb{Q}^{n+2}$ were selected that constitute the vertices of a copy of K_{n+1} appearing as a subgraph of $G(\mathbb{Q}^{n+2}, \sqrt{d})$. It was then shown that there exists a point $P \in \mathbb{Q}^{n+2}$ at distance $\sqrt{d}$ from each of $P_1', \dots, P_{n+1}'$ if and only if equation (3.5) had a non-trivial solution.

It did not figure into the proof of Theorem 3.6, but we may instead extend those $P_1', \dots, P_{n+1}'$ to points of $\mathbb{Q}^{n+3}$ by placing a zero as their $(n+3)$-th coordinate entry. It then follows that there exists $P \in \mathbb{Q}^{n+3}$ at distance $\sqrt{d}$ from each of those points if and only if the Diophantine equation below has a non-trivial solution:

$$x^2 + y^2 + z^2 = \frac{d(n+2)}{2n+2}. \tag{4.1}$$

Letting $d = r(n+2)(2n+2)$ where $r \in \mathbb{Z}^+$ with $r \equiv 7 \pmod 8$, by Theorem 2.5, equation (4.1) is not solvable. This results in $c_1(\mathbb{Q}^{n+3}) \le n+1$, and since Corollary 2.4 gives $C_1(\mathbb{Q}^{n+3}) = n+4$, we have $C_1(\mathbb{Q}^{n+3}) - c_1(\mathbb{Q}^{n+3}) \ge 3$.

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Question 5 *For each n, is it true that every integer in the closed interval $[c_1(\mathbb{Q}^n), C_1(\mathbb{Q}^n)]$ is realized as the clique number of some graph $G(\mathbb{Q}^n, d)$?*

Of course, Theorem 3.3 resolves the above question in the affirmative for all n equal to 1, 2, or a multiple of 4, and Theorem 4.1 gives the same answer for the specific case of $n = 3$. More work is needed for general n.

Question 6 *Is $C_2(\mathbb{Q}^n) - c_2(\mathbb{Q}^n)$ bounded over all n? If not, what about the ratio $\frac{C_2(\mathbb{Q}^n)}{c_2(\mathbb{Q}^n)}$?*

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[1] Note: This work circulated as a widely-read unpublished manuscript for over twenty years before its eventual publication.

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