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A PROBLEM OF HERSTEIN ON GROUP RINGS

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THEOREM. *Let F be a field of characteristic* 0 *and G a group such that each element of the group ring F*[*G*] *is either* (*right*) *invertible or a* (*left*) *zero divisor. Then G is locally finite.*

This answers a question of Herstein [1, p. 36] [2, p. 450] in the characteristic 0 case. The proof can be informally summarized as follows: Let $g_1, \ldots, g_n$ be a finite subset of G, and let

$$x = \frac{1}{n^2}(g_1+\cdots+g_n).$$

$1-x$ is not a zero divisor so it is invertible and its inverse is $1+x+x^2+\cdots$. The fact that this series converges to an element of $F[G]$ (a finite sum) forces the subgroup generated by $g_1, \ldots, g_n$ to be finite, proving the theorem. The formal proof is via epsilontics and takes place inside of $F[G]$.

Proof of the theorem. Let Q = rational numbers. $Q[G] \subset F[G]$ and by taking a basis for F over Q it is easy to see that every element of $Q[G]$ is invertible or a zero divisor in $Q[G]$. Thus we may assume that $F=Q$. We introduce a norm on $Q[G]$. If $a=a_1h_1+\cdots+a_kh_k$, ($a_i \in Q$, $h_i \in G$) we let

$$|a| = \max(|a_1|, \ldots, |a_k|).$$

Suppose $g_1, \ldots, g_n \in G$ ($n \geq 2$), and let

$$x = \frac{1}{n^2}(g_1+\cdots+g_n).$$

Consider any product

$$xa = \frac{1}{n^2}(g_1+\cdots+g_n)(a_1h_1+\cdots+a_kh_k).$$

Each coefficient occurring in xa is a sum of at most n terms

$$\frac{1}{n^2}a_{i_1}+\frac{1}{n^2}a_{i_2}+\cdots+\frac{1}{n^2}a_{i_j}$$

and hence has absolute value $\leq |a|/n$. This shows that

(*) $$|xa| \leq \frac{1}{n}|a| \text{ for any } a \in Q[G].$$

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Hence $1-x$ is not a zero divisor, for if $(1-x)a=0$, then $a=xa$, so that

$$|a| = |xa| \le \frac{1}{n}|a|, \quad \text{whence} \quad a = 0.$$

Since $1-x$ is not a zero divisor the hypothesis tells us that $1-x$ is invertible; let $(1-x)a=1$. We claim that any element of G which occurs in any of $x, x^2, x^3, \ldots$ also occurs in a. This means that $gp(g_1, \ldots, g_n)$ is finite and so proves the theorem.

To establish the claim, suppose conversely that some $g \in G$ occurs for the first time in x^m, but does not occur in a.

$$|a-(1+x+x^2+\cdots+x^{2m})| \ge \left(\frac{1}{n^2}\right)^m$$

since the coefficient of g in $1+x+x^2+\cdots+x^{2m}$ is $\ge(1/n^2)^m$. Using (*) and the triangle inequality it follows that

$$|(1-x)b| \ge |b|-|xb| \ge \left(1-\frac{1}{n}\right)|b| \quad \text{for any } b \in Q[G].$$

$$\therefore \quad |(1-x)[a-(1+x+\cdots+x^{2m})]| \ge \left(1-\frac{1}{n}\right)\left(\frac{1}{n^2}\right)^m \ge \left(\frac{1}{n}\right)^{2m+1}.$$

$$(1-x)[a-(1+x+\cdots+x^{2m})] = 1-(1-x^{2m+1}) = x^{2m+1},$$

and

$$|x^{2m+1}| \le \left(\frac{1}{n}\right)^{2m}|x| = \left(\frac{1}{n}\right)^{2m+2},$$

by repeated applications of (*). This completes the proof by virtue of the contradiction

$$|x^{2m+1}| \ge \left(\frac{1}{n}\right)^{2m+1} > \left(\frac{1}{n}\right)^{2m+2} \ge |x^{2m+1}|.$$

References

1. I. N. Herstein, *Notes from a ring theory conference*, Amer. Math. Soc. 1971.

2. I. Kaplansky, "*Problems in the theory of rings*" *revisited*, Amer. Math. Monthly **77** (1970) 445–454. MR 41 #3510.

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