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OPTIMAL FINANCING AND DIVIDEND DISTRIBUTION IN A GENERAL DIFFUSION MODEL WITH REGIME SWITCHING

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Abstract

We study the optimal financing and dividend distribution problem with restricted dividend rates in a diffusion type surplus model, where the drift and volatility coefficients are general functions of the level of surplus and the external environment regime. The environment regime is modeled by a Markov process. Both capital injection and dividend payments incur expenses. The objective is to maximize the expectation of the total discounted dividends minus the total cost of the capital injection. We prove that it is optimal to inject capital only when the surplus tends to fall below 0 and to pay out dividends at the maximal rate when the surplus is at or above the threshold, dependent on the environment regime.

Keywords: Dividend; general diffusion; optimization; optimal financing; regime-switching

2010 Mathematics Subject Classification: Primary 49L20
Secondary 91G80

1. Introduction

The optimal dividend strategy problem has gained extensive attention. In the diffusion setting, many works concerning dividend optimization use the Brownian motion model for the underlying cashflow process. In [1] the author extends the basic model by assuming that the drift coefficient is a linear function of the level of cashflow, and in [2] the authors use the mean-reverting model and solve the optimization problem. The authors in [6] considered the optimization problem under the model where the drift coefficient is proportional to the level of cashflow and the diffusion coefficient is proportional to the square root of the cashflow level. See [11], [13], [18], and the references therein for optimization problems for the general diffusion model where the drift and diffusion coefficients are general functions of the cashflow level.

An interesting and different direction of extension is to include the impact of the changing external environments/conditions (for example, macroeconomic conditions and weather conditions) into modeling of the cashflows. A continuous-time Markov chain can be used to model the state of the external environment condition, of which the use is supported by observation in financial markets. The optimal dividend problem with regular control for Markov-modulated risk processes has been investigated under a verity of assumptions. In [14] the authors solved the dividend optimization problem for a Markov-modulated Brownian motion model with both

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the drift and diffusion coefficients modulated by a two-state Markov chain. In [17] the author solved the problem for the Brownian motion model modulated by a multiple-state Markov chain.

The optimality results in all the above works imply that distributing dividends according to the optimal strategy leads almost surely to ruin. In [3] the authors proposed to include capital injections (financing) to prevent the surplus becoming negative and therefore prevent ruin. Under the Brownian motion, the authors in [10] investigated the optimal dividend and financing problem, and in [5] the authors studied the problem with risk exposure control through control of the reinsurance rate. The optimality problem with control in both capital injections and dividend distribution in a Cramér–Lundberg model was addressed in [12]. In [16] the authors solved the problem for the dual model with transaction costs.

The purpose of this paper is to investigate the optimal financing and dividend distribution problem with restricted dividend rates in a general diffusion model with regime switching. Under the model, the drift and volatility coefficients are general functions of the level of surplus and the external environment regime, which is modeled by a Markov process. Similar to the 'reflection problem', the company can control the financing/capital injection process (a deposit process) and the dividend distribution process (a 'withdrawal' process). Both capital injections and dividend payments will incur transaction costs. Sufficient capital injections must be made to keep the controlled surplus process nonnegative and the dividend payment rate is capped. This paper can be considered as an extension of the existing work on the dividend optimization problem with restricted dividend rates for the diffusion models with or without regime switching. The model considered is more general as it assumes that the drift and volatility are general functions of the cashflows, and the model risk parameters (including drift, volatility, and discount rates) are dependent on the external environment regime.

The rest of the paper is organized as follows. We formulate the optimization problem in Section 2. An auxiliary problem is introduced and solved in Section 3. In Section 4 we present the optimality results. A conclusion is provided in Section 5. Proofs are relegated to Appendix A.

2. Problem formulation

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\{W_t; t \geq 0\}$ and $\{\xi_t; t \geq 0\}$ be respectively a standard Brownian motion and a Markov chain with the finite state space $\mathcal{S}$ and the transition intensity matrix $Q = (q_{ij})_{i,j\in\mathcal{S}}$. The two stochastic processes $\{W_t; t \geq 0\}$ and $\{\xi_t; t \geq 0\}$ are independent. We use $\{\mathcal{F}_t; t \geq 0\}$ to denote the minimal complete σ-field generated by the stochastic process $\{(W_t, \xi_t); t \geq 0\}$. Let X_t denote the surplus at time t of a firm, in the absence of financing and dividend distribution. Assume that X_0 is $\mathcal{F}_0$-measurable and that X_t follows the dynamics, $\mathrm{d}X_t = \mu(X_{t-}, \xi_{t-})\,\mathrm{d}t + \sigma(X_{t-}, \xi_{s-})\,\mathrm{d}W_t$ for $t \geq 0$, where the functions $\mu(\cdot, j)$ and $\sigma(\cdot, j)$ are Lipschitz continuous, differentiable, and grow at most linearly on $[0, \infty)$ with $\mu(0, j) \geq 0$. Furthermore, the function $\mu(\cdot, j)$ is concave and the function $\sigma(\cdot, j)$ is positive and nonvanishing.

The firm must have nonnegative assets in order to continue its business. If necessary, the firm needs to raise money from the market. For each amount of money raised, it includes an amount c of transaction cost and hence leads to an increase of $1 - c$ in the surplus through capital injections. Let C_t denote the cumulative amount of capital injection up to time t. Then the total cost for capital injection up to time t is $C_t/(1 - c)$. The company can distribute part of its assets to the shareholders as dividends. For each amount of dividend received by the shareholders, there will be an amount d of cost incurred to them. Let D_t denote the cumulative amount of dividends paid out by the company up to time t. Then the total amount of dividends received by the shareholders up to time t is $D_t/(1+d)$. We consider the case where the dividend distribution rate is restricted. Let the random variable l_s denote the dividend payment rate at time s with the restriction $0 \le l_s \le \bar{l}$, where $\bar{l}(>0)$ is constant. Then $D_t = \int_0^t l_s \, \mathrm{d}s$. Both C_t and D_t are controlled by the company's decision-makers. Define $\pi = \{(C_t, D_t); t \ge 0\}$. We call π a control strategy.

Taking financing and dividend distribution into consideration, the dynamics of the (controlled) surplus process with the strategy π becomes

$$\mathrm{d}X_t^\pi = (\mu(X_{t-}^\pi, \xi_{t-}) - l_t)\,\mathrm{d}t + \sigma(X_{t-}^\pi, \xi_{t-})\,\mathrm{d}W_t + \mathrm{d}C_t, \qquad t \ge 0.$$

Define $\mathbb{P}_{(x,i)}(\cdot) = \mathbb{P}(\cdot \mid X_0 = x, \xi_0 = i), \mathbb{E}_{(x,i)}[\cdot] = \mathbb{E}[\cdot \mid X_0 = x, \xi_0 = i], \mathbb{P}_i(\cdot) = \mathbb{P}(\cdot \mid \xi_0 = i)$, and $\mathbb{E}_i[\cdot] = \mathbb{E}[\cdot \mid \xi_0 = i]$. The performance of a control strategy π is measured by its return function defined as

$$R_\pi(x,i) = \mathbb{E}_{(x,i)}\bigg[\int_0^\infty \mathrm{e}^{-\Lambda_t} \frac{l_t}{1+d}\,\mathrm{d}t - \int_0^\infty \mathrm{e}^{-\Lambda_t} \frac{1}{1-c}\,\mathrm{d}C_t\bigg], \qquad x \ge 0,\ i \in \mathcal{S}, \tag{2.1}$$

where $\Lambda_t = \int_0^t \delta_{\xi_s}\,\mathrm{d}s$ with δ_{ξ_s} representing the force of discount at time s. Assume that $\delta_i > 0, i \in \mathcal{S}$.

A strategy $\pi = \{(C_t, D_t); t \ge 0\}$ is said to be *admissible* if

(i) both $\{C_t; t \ge 0\}$ and $\{D_t; t \ge 0\}$ are nonnegative, increasing, càdlàg, and $\{\mathcal{F}_t; t \ge 0\}$-adapted processes,

(ii) there exists an $\{\mathcal{F}_t; t \ge 0\}$-adapted process $\{l_t; t \ge 0\}$ with $l_t \in [0, \bar{l}]$ such that $D_t = \int_0^t l_s\,\mathrm{d}s$, and

(iii) $X_t^\pi \ge 0$ for all $t > 0$.

We use Π to denote the class of admissible strategies.

Since $\{C_t; t \ge 0\}$ is right-continuous and increasing, we have the following decomposition: $C_t = \tilde{C}_t + C_t - C_{t-}$, where $\{\tilde{C}_t; t \ge 0\}$ represents the continuous part of $\{C_t; t \ge 0\}$.

For convenience, we use X, X^π, ξ, and (X^π, ξ) to denote the stochastic processes $\{X_t; t \ge 0\}$, $\{X_t^\pi; t \ge 0\}$, $\{\xi_t; t \ge 0\}$, and $\{(X_t^\pi, \xi_t); t \ge 0\}$, respectively. Note that for any admissible strategy π, the stochastic process X^π is right-continuous and adapted to the filtration $\{\mathcal{F}_t; t \ge 0\}$.

The objective of this paper is to study the maximal return function (value function)

$$V(x,i) = \sup_{\pi \in \Pi} R_\pi(x,i), \tag{2.2}$$

and to identify the associated optimal admissible strategy, if any. Following the standard argument in stochastic control theory [4], we can show that the value function fulfills the following dynamic programming principle:

$$V(x,i) = \sup_{\pi \in \Pi} \mathbb{E}_{(x,i)}\bigg[\int_0^\tau \frac{l_t \mathrm{e}^{-\Lambda_t}}{1+d}\,\mathrm{d}t - \int_0^\tau \frac{\mathrm{e}^{-\Lambda_t}}{1-c}\,\mathrm{d}C_t + \mathrm{e}^{-\Lambda_\tau} V(X_\tau^\pi, \xi_\tau^\pi)\bigg]$$

for any stopping time τ.

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3. An auxiliary optimization problem

Motivated by [8], which introduced an auxiliary problem where the objective functional is modified in a way such that only the 'returns' over the time period from the beginning up to the first regime switching are included plus a terminal value at the moment of the first regime switching, we start with a similar auxiliary problem. The optimality results of this problem will play an essential role in solving the original optimization problem.

Throughout the paper, we define $\boldsymbol{\delta} = \min_{j\in\mathcal{S}} \delta_j$, $q_i = -q_{ii}$, and $\sigma_1 = \inf\{t > 0: \xi_t \neq \xi_0\}$. Here, σ_1 is the first transition time of the Markov process ξ. For any function $g: \mathbb{R}^+ \times \mathcal{S} \to \mathbb{R}^+$, we use $g'(\cdot)$ and $g''(\cdot)$ to denote the first-order and second-order derivatives, respectively, with respect to the first argument. We start by introducing two special classes of functions.

Definition 3.1. (i) Let $\mathcal{C}$ denote the class of functions $g: \mathbb{R}^+ \times \mathcal{S} \to \mathbb{R}$ such that for each $j \in \mathcal{S}$, $g(\cdot, j)$ is nondecreasing and $g(\cdot, j) \le \bar{l}/\boldsymbol{\delta}(1+d)$.

(ii) Let $\mathcal{D}$ denote the class of functions $g \in \mathcal{C}$ such that for each $j \in \mathcal{S}$, $g(\cdot, j)$ is concave and $(g(x, j) - g(y, j))/(x - y) \le 1/(1-c)$ for $0 \le x < y$.

(iii) Define the distance $||\cdot||$ by $||f - g|| = \max_{x\ge 0, i\in\mathcal{S}} |f(x, i) - g(x, i)|$ for $f, g \in \mathcal{D}$.

Lemma 3.1. *The metric space* $(\mathcal{D}, ||\cdot||)$ *is complete.*

Define a modified return function and the associated optimal return function by

$$R_{f,\pi}(x, i) = \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1} \frac{l_t \mathrm{e}^{-\Lambda_t}}{1+d}\,\mathrm{d}t - \int_0^{\sigma_1} \frac{\mathrm{e}^{-\Lambda_t}}{1-c}\,\mathrm{d}C_t + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X^{\pi}_{\sigma_1}, \xi_{\sigma_1})\bigg], \qquad x \ge 0,\ i \in \mathcal{S}, \tag{3.1}$$

$$V_f(x, i) = \sup_{\pi\in\Pi} R_{f,\pi}(x, i), \qquad x \ge 0,\ i \in \mathcal{S}. \tag{3.2}$$

Lemma 3.2. *For any* $f \in \mathcal{C}$*,* $V, V_f \in \mathcal{C}$*.*

Note that the uncontrolled process (X, ξ) is a Markov process. For any $f \in \mathcal{C}$ and any $i \in \mathcal{S}$, the following Hamilton–Jacobi–Bellman equation for the modified value function $V_f(\cdot, i)$ can be obtained by using standard arguments in stochastic control: for $x \ge 0$,

$$\max\bigg\{\max_{l\in[0,\bar{l}]}\bigg(\frac{\sigma^2(x,i)}{2}V_f''(x,i) + \mu(x,i)V_f'(x,i) - \delta_i V_f(x,i) + l\bigg(\frac{1}{1+d} - V_f'(x,i)\bigg)\bigg), V_f'(x,i) - \frac{1}{1-c}\bigg\} = 0.$$

Now we define a special class of admissible strategies, which has been shown in the literature to contain the optimal strategy for the original optimization problem if there is one regime only. Since the return function of the modified optimization includes the dividends and capital injection in the first regime only, this problem can be considered as a problem to maximize the returns up to an independent exponential time for a risk model with one regime. It is worth studying the special class of strategies mentioned above to see whether the optimal strategy of the modified problem also falls into this class.

Definition 3.2. For any $b \geq 0$, define the strategy $\pi^{0,b} = \{(C_t^{0,b}, D_t^{0,b}); t \geq 0\}$ in the way such that the company pays dividends at the maximal rate $\bar{l}$ when the surplus equals or exceeds b, pays no dividends when the surplus is below b, and the company injects capital to maintain the surplus at level 0 whenever the surplus tends to go below 0 without capital injection.

We now investigate whether a strategy $\pi^{0,b}$ with an appropriate value for b is optimal or not for the modified optimization problem. We start by studying the associated return functions. For convenience, we write $X^{0,b} = X^{\pi^{0,b}}$ throughout the rest of the paper.

Remark 3.1. (i) It is not difficult to see that $\pi^{0,b}$ is admissible and that both $\pi^{0,b}$ and $X^{0,b}$ are Markov processes.

(ii) For any function $f \in \mathcal{C}$ and any $i \in \mathcal{S}$, by applying the comparison theorem used to prove the nondecreasing property of $V(\cdot, i)$ and $V_f(\cdot, i)$ in Lemma 3.2, we can show that the function $R_{f,\pi^{0,b}}(\cdot, i)$ is also nondecreasing on $[0, \infty)$.

For any $f \in \mathcal{C}$, $i \in \mathcal{S}$, and $b \geq 0$, define the operator $\mathcal{A}_{f,i,b}$ by

$$\mathcal{A}_{f,i,b}g(x) = \frac{\sigma^2(x,i)}{2}g''(x) + (\mu(x,i) - \bar{l})g'(x) - (\delta_i + q_i)g(x) + \frac{\bar{l}}{1+d} + \sum_{j\neq i} q_{ij} f(x,j)$$

The following conditions will be required for the main theorems.

Condition 3.1. *The functions $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ are the ones such that for any given function $f \in \mathcal{D}$ and any given $i \in \mathcal{S}$, the ordinary differential equation $\mathcal{A}_{f,i,b}g(x) = 0$ with any finite initial value at $x = 0$ has a bounded solution over $(0, \infty)$.*

A sufficient condition for Condition 3.1 to hold is that both the functions $\mu(\cdot, i)$ and $\sigma(\cdot, i)$ are bounded on $[0, \infty)$; see [9, Theorem 5.4.2]. However, this is far away from necessary. For example, when $\mu(\cdot, i)$ is a linear function with positive slope and $\sigma(\cdot, i)$ is a constant, Condition 3.1 also holds; see [18, Section 4.4].

Condition 3.2. *The function $\mu'(x, i) \leq \delta_i$ for all $x \geq 0$ and $i \in \mathcal{S}$.*

Define for any function $f \in \mathcal{C}$ and $i \in \mathcal{S}$,

$$A_{f,i} = \frac{\bar{l}/(1+d) + \sum_{j\neq i} q_{ij} f(\infty, j)}{q_i + \delta_i}. \tag{3.3}$$

Lemma 3.3. *Suppose that Condition 3.1 holds. For any function $f \in \mathcal{D}$, any $i \in \mathcal{S}$,*

(i) *the function $R_{f,\pi^{0,b}}(\cdot, i)$ for any $b \geq 0$, is a continuously differentiable solution on $[0, \infty)$ to*

$$\frac{\sigma^2(x,i)}{2}g''(x) + \mu(x,i)g'(x) - (\delta_i + q_i)g(x) + \sum_{j\neq i} q_{ij} f(x,j) = 0, \qquad 0 < x < b, \tag{3.4}$$

$$\frac{\sigma^2(x,i)}{2}g''(x) + (\mu(x,i) - \bar{l})g'(x) - (\delta_i + q_i)g(x) + \sum_{j\neq i} q_{ij} f(x,j) = -\frac{\bar{l}}{1+d}, \qquad x > b, \tag{3.5}$$

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and

$$g'(0+) = \frac{1}{1-c}, \qquad \lim_{x\to\infty} g(x) < \infty, \tag{3.6}$$

and is twice continuously differentiable on $(0, b) \cup (b, \infty)$*;*

(ii) *the function* $h_{f,i}(b) := R'_{f,\pi^{0,b}}(b, i)$ *is continuous with respect to b for* $0 < b < \infty$.

Throughout the paper, we use $(\mathrm{d}^-/\mathrm{d}x)g(x, i)$ and $(\mathrm{d}^+/\mathrm{d}x)g(x, i)$ to represent the derivatives of g from the left- and right-hand side, respectively, with respect to x.

Corollary 3.1. *Suppose that Condition 3.1 holds. For any* $f \in \mathcal{D}$, $i \in \mathcal{S}$, *and* $b \geq 0$,

(i) $R_{f,\pi^{0,b}}(\cdot, i)$ *is increasing, bounded, continuously differentiable on* $(0, \infty)$*, and twice continuously differentiable on* $(0, b) \cup (b, \infty)$ *with*

$$R'_{f,\pi^{0,b}}(0+, i) = \frac{1}{1-c}\left[\frac{\mathrm{d}^-}{\mathrm{d}x} R'_{f,\pi^{0,b}}(x, i)\right]_{x=b} = \lim_{x\uparrow b} R''_{f,\pi^{0,b}}(x, i)$$

and

$$\left[\frac{\mathrm{d}^+}{\mathrm{d}x} R'_{f,\pi^{0,b}}(x, i)\right]_{x=b} = \lim_{x\downarrow b} R''_{f,\pi^{0,b}}(x, i);$$

(ii) *if* $R'_{f,\pi^{0,b}}(b, i) = 1/(1+d)$ *then* $R_{f,\pi^{0,b}}(x, i)$ *is twice continuously differentiable with respect to x at* $x = b$.

We use $R'_{f,\pi^{0,b}}(0, i)$ and $R''_{f,\pi^{0,b}}(0, i)$ to denote $R'_{f,\pi^{0,b}}(0+, i)$ and $R''_{f,\pi^{0,b}}(0+, i)$, respectively.

Lemma 3.4. *Suppose that Conditions 3.1 and 3.2 hold. For any fixed* $f \in \mathcal{D}$, $i \in \mathcal{S}$, *and* $b \geq 0$, *we have* $R''_{f,\pi^{0,0}}(0+, i) \leq 0$, *and in the* $b > 0$ *case,* $R''_{f,\pi^{0,b}}(0+, i) \leq 0$ *if* $R'_{f,\pi^{0,b}}(b, i) \leq 1/(1-c)$.

Lemma 3.5. *Suppose that Conditions 3.1 and 3.2 hold. For any* $f \in \mathcal{D}$ *and* $i \in \mathcal{S}$,

(i) $R''_{f,\pi^{0,0}}(x, i) \leq 0$ *for* $x \geq 0$, *and in the* $b > 0$ *case,* $R''_{f,\pi^{0,b}}(x, i) \leq 0$ *for* $x \geq 0$ *if* $R'_{f,\pi^{0,b}}(b, i) = 1/(1+d)$*; and*

(ii) *for* $b > 0$, *if* $R'_{f,\pi^{0,b}}(b, i) > 1/(1+d)$,

$$R''_{f,\pi^{0,b}}(x, i) \leq 0 \quad \textit{for } x \in [0, b) \quad \textit{and} \quad R''_{f,\pi^{0,b}}(b-, 0) \leq 0.$$

Let $\mathbf{1}_{\{\cdot\}}$ be the indicator function. Define for any fixed $b \geq 0$ and any fixed $\pi \in \Pi$,

$$\tau_b^\pi = \inf\{t \geq 0 : X_t^\pi \geq b\}, \tag{3.7}$$

$$\begin{aligned} &W_{f,b}(x, i) \\ &= \sup_{\pi\in\Pi} \mathbb{E}_{(x,i)}\bigg[\int_0^{\tau_b^\pi \wedge \sigma_1} \mathrm{e}^{-\Lambda_s} \frac{l_s}{1+d}\,\mathrm{d}s - \int_0^{\tau_b^\pi \wedge \sigma_1} \mathrm{e}^{-\Lambda_s} \frac{1}{1-c}\,\mathrm{d}C_s \\ &\qquad + \mathrm{e}^{-\Lambda_{\tau_b^\pi}} R_{f,\pi^{0,b}}(X^\pi_{\tau_b^\pi}, \xi_0)\mathbf{1}_{\{\tau_b^\pi < \sigma_1\}} + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X^\pi_{\sigma_1}, \xi_{\sigma_1})\mathbf{1}_{\{\sigma_1 \leq \tau_b^\pi\}}\bigg]. \end{aligned} \tag{3.8}$$

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Theorem 3.1. *Suppose that Conditions 3.1 and 3.2 hold. For any* $f \in \mathcal{D}$*, any* $i \in \mathcal{S}$*, and any* $b > 0$*, if* $R'_{f,\pi^{0,b}}(b,i) > 1/(1+d)$ *then* $R'_{f,\pi^{0,b}}(x,i) > 1/(1+d)$ *for* $0 < x \le b$ *and* $R_{f,\pi^{0,b}}(x,i) = W_{f,b}(x,i)$ *for* $x \ge 0$.

We show in the following theorems that if b is chosen appropriately, the return function for the strategy $\pi^{0,b}$ coincides with the optimal return function of the modified problem.

Theorem 3.2. *Suppose that Conditions 3.1 and 3.2 hold. For any* $f \in \mathcal{D}$ *and any* $i \in \mathcal{S}$*,*

(i) *if* $R'_{f,\pi^{0,0}}(0+,i) \le 1/(1+d)$ *then* $V_f(x,i) = R_{f,\pi^{0,0}}(x,i)$ *for* $x \ge 0$*; and*

(ii) *if for a fixed* $b > 0$*,* $R'_{f,\pi^{0,b}}(b,i) = 1/(1+d)$ *then* $V_f(x,i) = R_{f,\pi^{0,b}}(x,i)$ *for* $x \ge 0$.

Lemma 3.6. *Suppose that Conditions 3.1 and 3.2 hold,* $f \in \mathcal{D}$*, and* $i \in \mathcal{S}$*. Let* $R'_{f,\pi^{0,0}}(0,i)$ *denote* $R'_{f,\pi^{0,0}}(0+,i)$*. If* $R'_{f,\pi^{0,b}}(b,i) > 1/(1+d)$ *for all* $b \ge 0$ *then*

$$V_f(x,i) = \lim_{b\to\infty} R_{f,\pi^{0,b}}(x,i) \quad \text{for } x \ge 0.$$

Again we use $R'_{f,\pi^{0,0}}(0,i)$ to denote $R'_{f,\pi^{0,0}}(0+,i)$. Define for any $f \in \mathcal{D}$ and $i \in \mathcal{S}$,

$$b_i^f = \begin{cases} \infty & \text{if } R'_{f,\pi^{0,b}}(b,i) > \dfrac{1}{1+d} \text{ for all } b \ge 0, \\ \inf\left\{b \ge 0\colon R'_{f,\pi^{0,b}}(b,i) \le \dfrac{1}{1+d}\right\} & \text{otherwise.} \end{cases} \tag{3.9}$$

We show in the following that the strategy π^{0,b_i^f} is optimal for the modified problem.

Theorem 3.3. *Suppose that Conditions 3.1 and 3.2 hold. For any* $f \in \mathcal{D}$ *and any* $i \in \mathcal{S}$*,*

(i) $0 \le b_i^f < \infty$*; and*

(ii) $V_f(x,i) = R_{f,\pi^{0,b_i^f}}(x,i)$ *for* $x \ge 0$.

4. The optimality results

We use the obtained optimality results for the modified optimization problem to address the original optimization problem. The starting point is to note that the optimal return function of the original optimization V_f, when the fixed function f is chosen to be the value function of the original optimization, coincides with the value function V.

Theorem 4.1. *If Conditions 3.1 and 3.2 hold,*

(i) $V \in \mathcal{D}$*;*

(ii) $b_i^V < \infty$ *and* $V(x,i) = R_{V,\pi^{0,b_i^V}}(x,i)$.

Theorem 4.2. *Define* π^* *to be the strategy under which the dividend pay-out rate at any time* t *is* $\bar{l}\mathbf{1}_{\{X_t^{\pi^*}\}}$*, and the company injects capital to maintain the surplus at level* 0 *whenever the surplus tends to go below* 0 *without capital injection. If Conditions 3.1 and 3.2 hold then* $V(x,i) = V^{\pi^*}(x,i)$*,* $i \in \mathcal{S}$*, and the strategy* π^* *is an optimal strategy.*

5. Conclusion

We have addressed the optimal dividend and financing problem for a regime-switching general diffusion model with restricted dividend rates. Our conclusion is that it is optimal to inject capital only when necessary and at a minimal amount sufficient for the business to continue, and to pay out dividends at the maximal rate $\bar{l}$ when the surplus exceeds the threshold dependent on the environmental state. This result is consistent with the findings for similar problems under a simpler model configuration in the literature. For example, the optimal strategy with restricted dividend rates is of the threshold type for the Brownian motion [15], the general diffusion [18], and the regime-switching Brownian motion [17].

Appendix A. Proofs for Sections 3 and 4

For any $i \in \mathcal{S}$ and $b \geq 0$, define the operator $\mathcal{B}$ by

$$\mathcal{B}g(x,i) = \frac{\sigma^2(x,i)}{2}g''(x,i) + \mu(x,i)g'(x,i) - \delta_i g(x,i).$$

Proof of Lemma 3.1. Consider any convergent sequence $\{g_n; n = 1, 2, \ldots\}$ in $\mathcal{D}$ with limit g. It is sufficient to show that $g \in \mathcal{D}$. As for any fixed i and n, $g_n(\cdot, i)$ is nondecreasing and concave, so is the function $g(\cdot, i)$. The inequality $g(\cdot, i) \leq \bar{l}/\boldsymbol{\delta}(1+d)$ follows immediately by noting that $g_n(\cdot, i) \leq \bar{l}/\boldsymbol{\delta}(1+d)$. It remains to show that $(g(x,i)-g(y,i))/(x-y) \leq 1/(1-c)$ for $0 \leq x < y$. We use proof by contradiction. Suppose that there exist x_0, y_0 with $0 \leq x_0 < y_0$, and j such that $(g(x_0, j) - g(y_0, j))/(x_0 - y_0) > 1/(1-c)$. Define

$$\varepsilon_0 := \frac{1}{2}\bigg(\frac{g(x_0,j) - g(y_0,j)}{x_0 - y_0} - \frac{1}{1-c}\bigg).$$

Clearly, $\varepsilon_0 > 0$. As g_n converges to g, we can find an $N > 0$ such that for all $n \geq N$, $||g_n - g|| \leq \varepsilon_0(y_0 - x_0)$. Therefore,

$$|g_n(y_0,j) - g(y_0,j)| \leq \varepsilon_0(y_0 - x_0) \quad \text{and} \quad |g_n(x_0,j) - g(x_0,j)| \leq \varepsilon_0(y_0 - x_0).$$

As a result,

$$\begin{aligned} g_n(y_0,j) - g_n(x_0,j) &\geq g(y_0,j) - \varepsilon_0(y_0 - x_0) - (g(x_0,j) + \varepsilon_0(y_0 - x_0)) \\ &= g(y_0,j) - g(x_0,j) - 2\varepsilon_0(y_0 - x_0) \\ &= \frac{y_0 - x_0}{1-c}. \end{aligned}$$

On the other hand, we have $(g_n(y_0, j) - g_n(x_0, j))/(y_0 - x_0) < 1/(1-c)$ (due to $g_n \in \mathcal{D}$), which is a contradiction. □

Proof of Lemma 3.2. Noting that $l_s \leq \bar{l}$ and that σ_1 is exponentially distributed with mean $1/q_i$, and $\Lambda_s = \delta_i s$ for $s \leq \sigma_1$, the upper-bounds follow easily from (2.1), (2.2), and (3.2).

Fix x and y with $y > x \geq 0$. Let $\{X_t^x; t \geq 0\}$ and $\{X_t^y; t \geq 0\}$ denote the surplus processes in the absence of control with initial surplus x and y, respectively. We use $\pi^x = \{(C_t^x, D_t^x)\colon t \geq 0\}$ with $D_t^x = \int_0^t l_s^x \,\mathrm{d}s$ to denote any admissible control strategy for the process $\{X_t^x; t \geq 0\}$. Noting that $\{C_t^x; t \geq 0\}$ is right-continuous and increasing, we have

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the following decomposition: $C^x_t = \int_0^t \mathrm{e}^x_s\,\mathrm{d}s + \sum_{0<s\le t}(C^x_s - C^x_{s-})$. Define $\zeta_0 = 0, \zeta_1 = \inf\{s > 0 \colon C^x_s - C^x_{s-} > 0 \text{ or } \xi_s \ne \xi_{s-}\}$, and $\zeta_{n+1} = \{s > \zeta_n \colon C^x_s - C^x_{s-} > 0 \text{ or } \xi_s \ne \xi_{s-}\}$ for $n = 1, 2, \ldots$. Note that $\xi_t = \xi_{\zeta_n}$ for $t \in [\zeta_n, \zeta_{n+1})$ and, hence,

$$\mathrm{d}X^{x,\pi^x}_t = (\mu(X^{x,\pi^x}_{t-}, \xi_{\zeta_n}) - l^x_t + \mathrm{e}^x_t)\,\mathrm{d}t + \sigma(X^{x,\pi^x}_{t-}, \xi_{\zeta_n})\,\mathrm{d}W_t$$

and

$$\mathrm{d}X^{y,\pi^x}_t = (\mu(X^{y,\pi^x}_{t-}, \xi_{\zeta_n}) - l^x_t + \mathrm{e}^x_t)\,\mathrm{d}t + \sigma(X^{y,\pi^x}_{t-}, \xi_{\zeta_n})\,\mathrm{d}W_t$$

for $t \in (\zeta_n, \zeta_{n+1}), n = 0, 1, \ldots$. By noting that $X^{x,\pi^x}_0 = X^x_0 = x < y = X^y_0 = X^{y,\pi^x}_0$ and applying the comparison theorem for solutions of stochastic differential equations (see [7]), we can show that with probability 1, $X^{x,\pi^x}_t \le X^{y,\pi^x}_t$ for $t \in [0, \zeta_1)$. Further, note that any discontinuity of a surplus process is caused by a jump in the associated process C^x at the same time and, hence,

$$X^{x,\pi^x}_{\zeta_1} = X^{x,\pi^x}_{\zeta_1-} + (C^x_{\zeta_1} - C^x_{\zeta_1-}) \le X^{y,\pi^x}_{\zeta_1-} + (C^x_{\zeta_1} - C^x_{\zeta_1-}) = X^{y,\pi^x}_{\zeta_1}$$

with probability 1. As a result, by applying the comparison theorem on (ζ_1, ζ_2), we can see that $X^{x,\pi^x}_t \le X^{y,\pi^x}_t$ for $t \in (\zeta_1, \zeta_2)$ with probability 1. Repeating the same procedure, we can show that $X^{x,\pi^x}_t \le X^{y,\pi^x}_t$ for $t \in (\zeta_n, \zeta_{n+1}]$ with probability 1. In conclusion, $X^{x,\pi^x}_t \le X^{y,\pi^x}_t$ for all $t \ge 0$ with probability 1. Therefore, π^x satisfies all the requirements for being an admissible strategy for the risk process X^y and, hence, $R_{f,\pi^x}(y, i) \le V_f(y, i)$ and $R_{\pi^x}(y, i) \le V(y, i)$. Using this and (3.1), we can show that $R_{f,\pi^x}(x, i) \le R_{f,\pi^x}(y, i) \le V_f(y, i)$. Similarly, we can obtain $R_{\pi^x}(x, i) \le V(y, i)$. By the arbitrariness of π^x, we conclude that $V_f(x, i) \le V_f(y, i)$ and $V(x, i) \le V(y, i)$ for $0 \le x < y$. □

Lemma A.1. *For any $f \in \mathcal{C}$ and $i \in \mathcal{S}$, suppose that the function $w_{f,i} \colon \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ with $w_{f,i}(\cdot, j) = f(\cdot, j)$ if $j \ne i$ is bounded, continuously differentiable, and piecewise twice continuously differentiable with respect to the first argument on $[0, \infty)$, and the function $w_{f,i}(\cdot, i)$ satisfies (3.4) and (3.5). Then for any $\pi \in \Pi$, there exists a positive sequence of stopping times $\{\tau_n; n = 1, 2, \ldots\}$ with $\lim_{n\to\infty} \tau_n = \infty$ such that*

$$\begin{aligned} w_{f,i}(x, i) = \mathbb{E}_{(x,i)}\bigg[&\mathrm{e}^{-\Lambda_{\tau_n\wedge\sigma_1\wedge t}} w_{f,i}(X^\pi_{\tau_n\wedge\sigma_1\wedge t}, \xi_{\tau_n\wedge\sigma_1\wedge t}) \\ &+ \int_0^{\tau_n\wedge\sigma_1\wedge t} l_s \mathrm{e}^{-\Lambda_s} w'_{f,i}(X^\pi_{\tau_n\wedge\sigma_1\wedge t}, \xi_{\tau_n\wedge\sigma_1\wedge t})\,\mathrm{d}s\bigg] \\ - \mathbb{E}_{(x,i)}\bigg[&\sum_{0<s\le\tau_n\wedge\sigma_1\wedge t} \mathrm{e}^{-\Lambda_s}(w_{f,i}(X^\pi_s, \xi_{s-}) - w_{f,i}(X^\pi_{s-}, \xi_{s-})) \\ &+ \int_0^{\tau_n\wedge\sigma_1\wedge t} \mathrm{e}^{-\Lambda_s} w'_{f,i}(X^\pi_{s-}, \xi_{s-})\,\mathrm{d}\tilde{C}_s\bigg]. \\ - \mathbb{E}_{(x,i)}\bigg[&\int_0^{\tau_n\wedge\sigma_1\wedge t} \mathrm{e}^{-\Lambda_s}\bar{l}\Big(w'_{f,i}(X^\pi_{s-}, \xi_{s-}) - \frac{1}{1+d}\Big)\mathbf{1}_{\{X^\pi_{s-}\ge b\}}\,\mathrm{d}s\bigg]. \end{aligned} \tag{A.1}$$

Proof. The result follows by applying Itô's formula to $\mathrm{e}^{-\Lambda_{\tau_n\wedge\sigma_1\wedge t}} w_{f,i}(X^\pi_{\tau_n\wedge\sigma_1\wedge t}, \xi_{\tau_n\wedge\sigma_1\wedge t})$. The full details of the proof can be found in [19]. □

Proof of Lemma 3.3. (i) The existence of a $[0,\infty)$-continuously differentiable and $[0,b)\cup(b,\infty)$ twice continuously differentiable bounded solution to (3.4), (3.5), and (3.6) can be proven by constructing a general form of the solution from the sets of independent linear solutions to (3.4) and (3.5), respectively, and then specifying the coefficients of the general form using the continuity and differentiability of the solution at b and letting the solution satisfy (3.6). Denote the solution by $g_{b,i}(x)$. It suffices to show that $R_{f,\pi^{0,b}}(x,i)=g_{b,i}(x)$ for $x\geq 0$. This can be done by defining $w_{f,i}$ by $w_{f,i}(x,j)=g_{b,i}(x)$ if $j=i$, and $w_{f,i}(x,j)=f(x,j)$ if $j\neq i$, and then applying Lemma A.1 with π there being set to $\pi^{0,b}$ and using the properties of the strategy $\pi^{0,b}$. The full details of the proof can be found in [19].

(ii) Note that, by (3.1),

$$\lim_{x\to\infty} g_{b,i}(x)=\lim_{x\to\infty} R_{f,\pi^{0,b}}(x,i)=A_{f,i},$$

where the second last equality follows by noting that given $X_0=x$, $X_s^{0,b}\to\infty$ as $x\to\infty$ and, hence, $C_s^{0,b}\to 0$ as $x\to\infty$, and the last equality follows by noting that, given $(X_0,\xi_0)=(x,i)$, σ_1 is exponentially distributed with mean $1/q_i$, and using the definition of $A_{f,i}$ in (3.3). So the constants K_1, K_2, K_3, and K_4 are solutions to

$$\begin{aligned}
K_1v_1(b;i)+K_2v_2(b;i)+B_1(b;i)&=K_3v_3(b;i)+K_4v_4(b;i)+B_2(b;i),\\
K_1v_1'(b;i)+K_2v_2'(b;i)+B_1'(b;i)&=K_3v_3'(b;i)+K_4v_4'(b;i)+B_2'(b;i),
\end{aligned}$$

$$K_1v_1'(0;i)+K_2v_2'(0;i)=\frac{1}{1-c},\qquad K_3v_3(\infty)+K_4v_4(\infty)+B_2(\infty)=A_{f,i}.$$

Note that the coefficients of the above system of equations are either constants or continuous functions of b. Hence, K_1, K_2, K_3, and K_4 are continuous functions of b, denoted by $K_1(b)$, $K_2(b)$, $K_3(b)$, and $K_4(b)$ here. As a result, the function $h_{f,i}(b)=g_{b,i}'(b)=K_1(b)v_1'(b)+K_2(b)v_2'(b)+B_1'(b;i)$ is continuous for $0<b<\infty$. □

For any $f\in\mathcal{C}$, $i\in\mathcal{S}$, and $b\geq 0$, define the functions h and $\bar{h}$ by

$$\begin{aligned}
h_{f,i,b}(x)&=(\delta_i+q_i)R_{f,\pi^{0,b}}(x,i)-\mu(x,i)R'_{f,\pi^{0,b}}(x,i)-\sum_{j\neq i}q_{ij}f(x,j)\\
&\quad-\bar{l}\bigg(\frac{1}{1+d}-R'_{f,\pi^{0,b}}(x,i)\bigg)\mathbf{1}_{\{x\geq b\}},\\
\bar{h}_{f,i,b}(x)&=(\delta_i+q_i)R_{f,\pi^{0,b}}(x,i)-\mu(x,i)R'_{f,\pi^{0,b}}(x,i)-\sum_{j\neq i}q_{ij}f(x,j)\\
&\quad-\bar{l}\bigg(\frac{1}{1+d}-R'_{f,\pi^{0,b}}(x,i)\bigg)\mathbf{1}_{\{x>b\}}.
\end{aligned}\tag{A.2}$$

Proof of Corollary 3.1. (i) The proof follows as a result of of Remark 3.1 and Lemma 3.3(i).

(ii) By Corollary 3.1(i) and Lemma 3.3(i), we have

$$\bigg[\frac{\mathrm{d}^-}{\mathrm{d}x}R'_{f,\pi^{0,b}}(x,i)\bigg]_{x=b}=\lim_{x\downarrow b}\frac{2h_{f,i,b}(b,i)}{\sigma^2(b,i)},\qquad \bigg[\frac{\mathrm{d}^+}{\mathrm{d}x}R'_{f,\pi^{0,b}}(x,i)\bigg]_{x=b}=\lim_{x\downarrow b}\frac{2h_{f,i,b}(b,i)}{\sigma^2(b,i)}.$$

Noting that $R'_{f,\pi^{0,b}}(b,i)=1/(1+d)$, we obtain

$$\bigg[\frac{\mathrm{d}^-}{\mathrm{d}x}R'_{f,\pi^{0,b}}(x,i)\bigg]_{x=b}=\bigg[\frac{\mathrm{d}^+}{\mathrm{d}x}R'_{f,\pi^{0,b}}(x,i)\bigg]_{x=b}.$$

For any sequence $\{y_n\}$, define

$$k_{f,b}(x, i; \{y_n\}) = (\delta_i + q_i - \mu'(x,i))R'_{f,\pi^{0,b}}(x,i) - \sum_{j\neq i} q_{ij} \lim_{n\to\infty} \frac{f(y_n, j) - f(x,j)}{y_n - x}. \qquad \square$$

Proof of Lemma 3.4. Throughout the proof, we assume that $f \in \mathcal{D}$, $i \in \mathcal{S}$, and $b \geq 0$, unless stated otherwise. We use proof by contradiction. Suppose that $R''_{f,\pi^{0,b}}(0+, i) > 0$.

Since $R_{f,\pi^{0,0}}(\cdot, i)$ is bounded, we can find a large enough x such that $R'_{f,\pi^{0,0}}(x,i) < 1/(1-c) = R'_{f,\pi^{0,0}}(0+, i)$, where the last equality is by Lemma 3.3(i). Hence, there exists an $x > 0$ such that $R''_{f,\pi^{0,0}}(x,i) < 0$. In the $b > 0$ case, note that $R'_{f,\pi^{0,b}}(0+, i) = 1/(1-c) \geq R'_{f,\pi^{0,b}}(b,i)$. So for $b > 0$ there exists an $x \in (0,b)$ such that $R''_{f,\pi^{0,b}}(x,i) \leq 0$. Define $x_1 = \inf\{x > 0\colon R''_{f,\pi^{0,b}}(x,i) \leq 0\}$. Then $x_1 > 0$ in the $b = 0$ case and $x_1 \in (0,b)$ in the $b > 0$ case, and for $b \geq 0$,

$$R''_{f,\pi^{0,b}}(x_1, i) = 0, \qquad R''_{f,\pi^{0,b}}(x,i) > 0 \quad \text{for } x \in [0, x_1). \tag{A.3}$$

As a result, for $b \geq 0$,

$$R'_{f,\pi^{0,b}}(x,i) > R'_{f,\pi^{0,b}}(0+, i) = \frac{1}{1-c} \quad \text{for } x \in (0, x_1]. \tag{A.4}$$

Write $R_{f,\pi^{0,b},i}(x) = R_{f,\pi^{0,b}}(x,i)$. By Lemma 3.3, it follows that, for $b \geq 0$,

$$\mathcal{A}_{f,i,b} R_{f,\pi^{0,b},i}(x) = 0 \quad \text{for } x > 0.$$

Therefore, by (A.3) and (A.2), it follows that, for $b \geq 0$,

$$h_{f,i,b}(x) = \frac{\sigma^2(x,i)}{2} R''_{f,\pi^{0,b}}(x,i) > 0 \quad \text{for } 0 < x < x_1,$$

$$h_{f,i,b}(x_1) = \frac{\sigma^2(x_1,i)}{2} R''_{f,\pi^{0,b}}(x_1,i) = 0.$$

Hence, for $b \geq 0$, we obtain

$$\frac{h_{f,i,b}(x,i) - h_{f,i,b}(x_1,i)}{x - x_1} < 0, \qquad 0 < x < x_1. \tag{A.5}$$

Note that $x_1 > b$ in the $b = 0$ case, and that $x_1 < b$ in the $b > 0$ case. Therefore, we can find a nonnegative sequence $\{x_{1n}\}$ with $b < x_{1n} \leq x_1$ in the $b = 0$ case, $x_{1n} \leq x_1 < b$ in the $b > 0$ case, and $\lim_{n\to\infty} x_{1n} = x_1$ such that $\lim_{n\to\infty}(f(x_{1n}, j) - f(x_1, j))/(x_{1n} - x_1)$ exists. By replacing x in (A.5) by x_{1n} and then letting $n \to \infty$ on both sides of (A.5), we obtain $k_{f,b}(x_1, i; \{x_{1n}\}) - (\mu(x_1,i) - \bar{l}\mathbf{1}_{\{b=0\}})R''_{f,\pi^{0,b}}(x_1,i) \leq 0$, which combined with (A.3) implies that

$$\bigg(\sum_{j\neq i} q_{ij} \lim_{n\to\infty} \frac{f(x_{1n}, j) - f(x_1, j)}{x_{1n} - x_1} - q_i R'_{f,\pi^{0,b}}(x_1,i)\bigg) + (\mu'(x_1,i) - \delta_i)R'_{f,\pi^{0,b}}(x_1,i) \geq 0.$$

By this inequality, $R'_{f,\pi^{0,b}}(x_1,i) > 1/(1-c)$ (see (A.4)), and

$$\lim_{n\to\infty} \frac{f(x_{1n}, j) - f(x_1, j)}{x_{1n} - x_1} \leq \frac{1}{1-c} \quad (\text{due to } f \in \mathcal{D})$$

it follows that $(\mu'(x_1,i) - \delta_i)R'_{f,\pi^{0,b}}(x_1,i) > 0$, which combined with (A.4) implies that $\mu'(x_1,i) - \delta_i > 0$. This contradicts the assumption that $\mu'(x_1,i) \leq \delta_i$. $\square$

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Proof of Lemma 3.5. We consider any fixed $f \in \mathcal{D}$ and $i \in \mathcal{S}$ throughout the proof. We first show that there exists a positive sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = \infty$ such that, for $b \geq 0$,

$$R''_{f,\pi^{0,b}}(x_n, i) \leq 0. \tag{A.6}$$

Suppose the contrary: for some $M > 0$, $R''_{f,\pi^{0,b}}(x, i) > 0$ for all $x \geq M$. This implies that $R'_{f,\pi^{0,b}}(x, i) > R'_{f,\pi^{0,b}}(M+1, i) > R'_{f,\pi^{0,b}}(M, i) \geq 0$ for $x > M+1$, where the last inequality follows by the increasing property of $R_{f,\pi^{0,b}}(\cdot, i)$ (see Corollary 3.1(i)). As a result, $R_{f,\pi^{0,b}}(x, i) > R_{f,\pi^{0,b}}(M+1, i) + R'_{f,\pi^{0,b}}(M+1, i)(x - M - 1)$ for $x > M+1$, which implies that $\lim_{x\to\infty} R_{f,\pi^{0,b}}(x, i) = \infty$. This contradicts the boundedness of $R_{f,\pi^{0,b}}(\cdot, i)$ (Corollary 3.1(i)).

Write $R_{f,\pi^{0,b},i}(x) = R_{f,\pi^{0,b}}(x, i)$. By Lemma 3.3, it follows that

$$\mathcal{A}_{f,i,b} R_{f,\pi^{0,b},i}(x) = 0 \quad \text{for } x > 0. \tag{A.7}$$

(i) By Lemma 3.3 and Corollary 3.1, we can see that $R_{f,\pi^{0,b},i}(\cdot)$ is twice continuously differentiable on $[0, \infty)$ with the differentiability at 0 referring to the differentiability from the right-hand side. By noting that $R'_{f,\pi^{0,b},i}(b) = R'_{f,\pi^{0,b}}(b, i) = 1/(1+d) \leq 1/(1-c)$ for $b > 0$ and Lemma 3.4, we have

$$R''_{f,\pi^{0,b},i}(0+) \leq 0 \quad \text{for } b \geq 0. \tag{A.8}$$

We use proof by contradiction to prove the statement in (i). Suppose that the statement in (i) does not hold. Then there exists a $b \geq 0$ and a $y_0 > 0$ such that $R''_{f,\pi^{0,b},i}(y_0) = R''_{f,\pi^{0,b}}(y_0, i) > 0$. Let $\{x_n\}$ be the sequence defined as before. We can find a positive integer N such that $x_N > y_0$. By noting that $R''_{f,\pi^{0,b},i}(x_N) = R''_{f,\pi^{0,b}}(x_N, i) \leq 0$ (due to (A.6)), (A.8), and the continuity of $R''_{f,\pi^{0,b},i}(\cdot)$, we can find y_1, y_2 with $0 \leq y_1 < y_0 < y_2 \leq x_N$ such that

$$R''_{f,\pi^{0,b}}(y_1, i) = 0, \qquad R''_{f,\pi^{0,b}}(y_2, i) = 0, \qquad R''_{f,\pi^{0,b}}(x, i) > 0 \quad \text{for } x \in (y_1, y_2). \tag{A.9}$$

Hence,

$$R'_{f,\pi^{0,b},i}(y_2) > R'_{f,\pi^{0,b},i}(y_1). \tag{A.10}$$

From (A.7) and (A.2), it follows that $-(\sigma^2(x, i)/2) R''_{f,\pi^{0,b},i}(x) = h_{f,b,i}(x)$ for $x > 0$. Note that for $x > 0$, $\mathbf{1}_{\{x \geq b\}} = \mathbf{1}_{\{x > b\}}$ in the $b = 0$ case, and that in the $b > 0$ case, $1/(1+d) - R'_{f,\pi^{0,b}}(b, i) = 0$ and, hence,

$$\bar{l}\bigg(\frac{1}{1+d} - R'_{f,\pi^{0,b}}(x, i)\bigg)\mathbf{1}_{\{x \geq b\}} = \bar{l}\bigg(\frac{1}{1+d} - R'_{f,\pi^{0,b}}(x, i)\bigg)\mathbf{1}_{\{x > b\}} \quad \text{for } x > 0.$$

Therefore, for $x > 0$, $(\sigma^2(x, i)/2) R''_{f,\pi^{0,b}}(x, i) = \bar{h}_{f,i,b}(x)$, which combined with (A.9) implies that, for $x \in (y_1, y_2)$,

$$\bar{h}_{f,i,b}(y_1) = \frac{\sigma^2(y_1, i)}{2} R''_{f,\pi^{0,b}}(y_1, i) = 0 < \frac{\sigma^2(x, i)}{2} R''_{f,\pi^{0,b}}(x, i) = \bar{h}_{f,i,b}(x), \tag{A.11}$$

$$\bar{h}_{f,i,b}(y_2) = \frac{\sigma^2(y_2, i)}{2} R''_{f,\pi^{0,b}}(y_2, i) = 0 < \frac{\sigma^2(x, i)}{2} R''_{f,\pi^{0,b}}(x, i) = \bar{h}_{f,i,b}(x). \tag{A.12}$$

Let $\{y_{1n}\}$ and $\{y_{2n}\}$ be two sequences with $y_{1n} \downarrow y_1$ and $y_{2n} \uparrow y_2$ as $n \to \infty$ such that

$$\lim_{n\to\infty} \frac{f(y_{1n}, j) - f(y_1, j)}{y_{1n} - y_1} \quad \text{and} \quad \lim_{n\to\infty} \frac{f(y_{2n}, j) - f(y_2, j)}{y_{2n} - y_2}$$

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exist for all $j \in \mathcal{S}$. It follows by (A.11) and (A.12) that

$$\frac{\bar{h}_{f,i,b}(y_{1n}) - \bar{h}_{f,i,b}(y_1)}{y_{1n} - y_1} > 0 > \frac{\bar{h}_{f,i,b}(y_{2n}) - \bar{h}_{f,i,b}(y_2)}{y_{2n} - y_2}.$$

By letting $n \to \infty$, we obtain

$$k_{f,b}(y_1, i; \{y_{1n}\}) - \mu(y_1, i) R''_{f,\pi^{0,b}}(y_1, i) + \bar{l} R''_{f,\pi^{0,b}}(y_1, i) \mathbf{1}_{\{y_1 > b\}} \geq 0,$$
$$k_{f,b}(y_2, i; \{y_{2n}\}) - \mu(y_2, i) R''_{f,\pi^{0,b}}(y_2, i) + \bar{l} R''_{f,\pi^{0,b}}(y_2, i) \mathbf{1}_{\{y_2 > b\}} \leq 0.$$

Therefore, by noting that $R''_{f,\pi^{0,b}}(y_1, i) = 0 = R''_{f,\pi^{0,b}}(y_2, i)$ (see (A.9)), we have

$$k_{f,b}(y_1, i; \{y_{1n}\}) \geq 0 \geq k_{f,b}(y_2, i; \{y_{2n}\}). \tag{A.13}$$

On the other hand, note that $0 < \delta_i + q_i - \mu'(y_1, i) \leq \delta_i + q_i - \mu'(y_2, i)$ (due to the concavity of $\mu(\cdot, i)$), $R'_{f,\pi^{0,b}}(y_1, i) < R'_{f,\pi^{0,b}}(y_2, i)$ (see (A.10)), and

$$\lim_{n\to\infty} \frac{f(y_{1n}, j) - f(y_1, j)}{y_{1n} - y_1} \geq \lim_{n\to\infty} \frac{f(y_{2n}, j) - f(y_2, j)}{y_{2n} - y_2}$$

(due to the concavity of $f(\cdot, j)$). As a result, $k_{f,b}(y_1, i; \{y_{1n}\}) < k_{f,b}(y_2, i; \{y_{2n}\})$, which is a contradiction to (A.13).

(ii) We distinguish two cases:

(a) $R''_{f,\pi^{0,b}}(b+, i) > 0$;

(b) $R''_{f,\pi^{0,b}}(b+, i) \leq 0$.

(a) Suppose that $R''_{f,\pi^{0,b}}(b+, i) > 0$. By (A.6), we can find $N > 0$ such that $x_N > b$ and $R''_{f,\pi^{0,b}}(x_N, i) \leq 0$. Then by the continuity of the function $R''_{f,\pi^{0,b}}(\cdot, i)$ on (b, ∞) (see Corollary 3.1(i)) we know that there exists a $y_2 \in (b, x_N]$ such that $R''_{f,\pi^{0,b}}(y_2, i) = 0$ and $R''_{f,\pi^{0,b}}(x, i) > 0$ for $x \in (b, y_2)$. We now proceed to show that $R''_{f,\pi^{0,b}}(b-, i) \leq 0$. Suppose the contrary, i.e. $R''_{f,\pi^{0,b}}(b-, i) > 0$. By noting that $R''_{f,\pi^{0,b}}(0+, i) \leq 0$ (see (A.8)), it follows that there exists a $y_1 \in (0, b)$ such that $R''_{f,\pi^{0,b}}(y_1, i) = 0$, and $R''_{f,\pi^{0,b}}(x, i) > 0$ for $x \in (y_1, b)$. In summary, (A.9) holds for $x \in (y_1, y_2) - \{b\}$. Repeating the argument immediately below (A.9) in (i), we obtain a contradiction.

(b) Suppose that $R''_{f,\pi^{0,b}}(b+, i) \leq 0$. By (A.7) and the assumption $R'_{f,\pi^{0,b}}(b, i) > 1/(1+d)$, it follows that

$$R''_{f,\pi^{0,b}}(b-, i) = \lim_{x\uparrow b} \frac{2h_{f,i,b}(x, i)}{\sigma^2(x, i)} < \lim_{x\downarrow b} \frac{2h_{f,i,b}(x, i)}{\sigma^2(x, i)} = R''_{f,\pi^{0,b}}(b+, i) \leq 0. \tag{A.14}$$

We now show that $R''_{f,\pi^{0,b}}(x, i) \leq 0$ for all $x \in [0, b)$. Suppose the contrary. That is, there exists some $x \in [0, b)$ such that $R''_{f,\pi^{0,b}}(x, i) > 0$. Then by noting that $R''_{f,\pi^{0,b}}(0+, i) \leq 0$ (see (A.8)) and $R''_{f,\pi^{0,b}}(b-, i) < 0$ (see (A.14)), we can find y_1 and y_2 with $0 \leq y_1 < y_2 < b$ such that $R''_{f,\pi^{0,b}}(y_1, i) = 0$, $R''_{f,\pi^{0,b}}(y_2, i) = 0$, and $R''_{f,\pi^{0,b}}(x, i) > 0$ for $x \in (y_1, y_2)$. Repeating again the argument immediately after (A.9) in (i), we obtain a contradiction. □

From now on, define for any $f \in \mathcal{C}$ and $i \in \mathcal{S}$ the function $w_{f,i} : \mathbb{R} \times \mathcal{S} \to \mathbb{R}$ by $w_{f,i}(\cdot, i) = R_{f,\pi^{0,b}}(\cdot, i)$ and $w_{f,i}(\cdot, j) = f(\cdot, j)$ if $j \neq i$.

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Proof of Theorem 3.1. Note that $\tau_b^\pi = 0$ given $X_0^\pi \ge b$. Hence, it follows from definition (3.8) that

$$W_{f,b}(x,i) = \sup_{\pi \in \Pi} \mathbb{E}_{(x,i)}[R_{f,\pi^{0,b}}(X_0^\pi, \xi_0)] = R_{f,\pi^{0,b}}(x,i) \quad \text{for } x \ge b,\ b = 0. \tag{A.15}$$

We consider the $b > 0$ case. By Lemma 3.5(ii) we know that $R''_{f,\pi^{0,b}}(x,i) \le 0$ for $x \in [0,b)$, and $R''_{f,\pi^{0,b}}(b-,i) \le 0$. Therefore, by Corollary 3.1(i), it follows that

$$\frac{1}{1-c} = R'_{f,\pi^{0,b}}(0+,i) \ge R'_{f,\pi^{0,b}}(x,i) \ge R'_{f,\pi^{0,b}}(b,i) > \frac{1}{1+d} \quad \text{for } 0 < x \le b. \tag{A.16}$$

By Corollary 3.1(i) and Lemma 3.3, we know that $w_i(\cdot, j)$ satisfies the conditions in Lemma A.1. Then by applying Lemma A.1 we know that for some positive sequence of stopping times $\{\tau_n; n = 1,2,\ldots\}$ with $\lim_{n\to\infty} \tau_n = \infty$, (A.1) holds. By letting t in (A.1) be $\tau_b^\pi \wedge t$, noting that $X_s^\pi - X_{s-}^\pi = C_s - C_{s-} \ge 0$, and that given $(X_0, \xi_0) = (x,i)$, $X_{s-}^\pi \in [0,b)$, and $w_i(X_{s-}^\pi, \xi_{s-}) = R_{f,\pi^{0,b}}(X_{s-}^\pi, i)$ for $s \le \sigma_1 \wedge \tau_b^\pi$, we have

$$\sum_{0<s\le \tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t} \mathrm{e}^{-\Lambda_s} \frac{X_s^\pi - X_{s-}^\pi}{1-c} + \int_0^{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}\tilde{C}_s = \int_0^{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s.$$

Using (A.16), we derive that for any $\pi \in \Pi$, $t > 0$, and $0 \le x \le b$,

$$\begin{aligned}
\mathbb{E}_{(x,i)}\bigg[\int_0^{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s \\
+ \mathrm{e}^{-\Lambda_{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t}} w_i(X^\pi_{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t}, \xi_{\tau_n\wedge\sigma_1\wedge\tau_b^\pi\wedge t})\bigg] \\
\le R_{f,\pi^{0,b}}(x,i).
\end{aligned} \tag{A.17}$$

Note that the functions $R_{f,\pi^{0,b}}(\cdot, j)$ and $f(\cdot, j)$, $j \in \mathcal{S}$ are all bounded. Hence, the functions $w_i(\cdot, j)$, $j \in \mathcal{S}$ are also bounded. By letting $\tau_n \to \infty$ and $t \to \infty$ on both sides of (A.17), using the monotone convergence theorem, the dominated convergence theorem, and noting that due to $\xi_s = \xi_0$ for $0 \le s < \sigma_1$, we have

$$\begin{aligned}
&\mathbb{E}_{(x,i)}[\mathrm{e}^{-\Lambda_{\tau_b^\pi\wedge\sigma_1}} w_{f,i}(X^\pi_{\tau_b^\pi\wedge\sigma_1}, \xi_{\tau_b^\pi\wedge\sigma_1})] \\
&\quad = \mathbb{E}_{(x,i)}[\mathrm{e}^{-\Lambda_{\tau_b^\pi}} R_{f,\pi^{0,b}}(b,\xi_0)\mathbf{1}_{\{\tau_b^\pi<\sigma_1\}} + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X_{\sigma_1}, \xi_{\sigma_1})\mathbf{1}_{\{\sigma_1 \le \tau_b^\pi\}}]
\end{aligned}$$

and that π is an arbitrary admissible strategy and (3.8), we can conclude that

$$W_{f,b}(x,i) \le R_{f,\pi^{0,b}}(x,i) \quad \text{for } 0 \le x \le b. \tag{A.18}$$

Note that $\{(X_t^{0,b}, \xi_t); t \ge 0\}$ is a strong Markov process and that by the Markov property, it follows that

$$\begin{aligned}
R_{f,\pi^{0,b}}(x,i) &= \mathbb{E}_{(x,i)}\bigg[\int_0^{\tau_b^{\pi^{0,b}}\wedge\sigma_1} \frac{\bar{l}\mathrm{e}^{-\Lambda_s}}{1+d}\mathbf{1}_{\{X_s^{0,b}\ge b\}}\,\mathrm{d}s - \int_0^{\tau_b^{\pi^{0,b}}\wedge\sigma_1} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s \\
&\qquad + \mathrm{e}^{-\delta(\tau_b^{\pi^{0,b}}\wedge\sigma_1)} R_{f,\pi^{0,b}}\Big(X^{0,b}_{\tau_b^{\pi^{0,b}}\wedge\sigma_1}, \xi_{\tau_b^{\pi^{0,b}}\wedge\sigma_1}\Big)\bigg] \\
&\le W_{f,b}(x,i) \quad \text{for } x \ge 0,
\end{aligned} \tag{A.19}$$

where the last inequality follows by noting that $\pi^{0,b} \in \Pi$ and definition (3.8). Combining (A.15), (A.18), and (A.19) completes the proof. □

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Proof of Theorem 3.2. We first show that

$$R'_{f,\pi^{0,b}}(x,i) \le R'_{f,\pi^{0,b}}(b,i) = \frac{1}{1+d} \quad \text{for } x > b,\ b \ge 0. \tag{A.20}$$

By Lemma 3.5(i), it follows that $R''_{f,\pi^{0,0}}(x,i) \le 0$ for $x \ge 0$. As a result, (A.20) holds for $b=0$. Now suppose that $b>0$. By Lemma 3.3(i), we know that $R'_{f,\pi^{0,b}}(0+,i) = 1/(1-c)$. Since $R'_{f,\pi^{0,b}}(b,i) = 1/(1+d)$, it follows by Corollary 3.1(ii) that $R_{f,\pi^{0,b}}(\cdot,i)$ is twice continuously differentiable on $[0,\infty)$ and by Lemma 3.5(i) that $R''_{f,0,b}(x,i) \le 0$ for $x \ge 0$. Hence, (A.20) also holds for $b>0$, and

$$\frac{1}{1-c} = R'_{f,\pi^{0,b}}(0+,i) \ge R'_{f,\pi^{0,b}}(x,i) \ge R'_{f,\pi^{0,b}}(b,i) = \frac{1}{1+d} \quad \text{for } x \in [0,b]. \tag{A.21}$$

Using (A.20) and (A.21), and noting that $\bar{l} \ge l_s$ for $s \ge 0$, we obtain, for $b \ge 0$,

$$\begin{aligned}
&\bar{l}\mathbf{1}_{\{X_s^\pi \ge b\}}\bigg(R'_{f,\pi^{0,b}}(X_{s-}^\pi,i) - \frac{1}{1+d}\bigg) - l_s R'_{f,\pi^{0,b}}(X_{s-}^\pi,i)\\
&\quad = (\bar{l}-l_s)\mathbf{1}_{\{X_s^\pi \ge b\}} R'_{f,\pi^{0,b}}(X_{s-}^\pi,i) - \frac{\bar{l}}{1+d}\mathbf{1}_{\{X_s^\pi \ge b\}} - l_s\mathbf{1}_{\{X_s^\pi < b\}} R'_{f,\pi^{0,b}}(X_{s-}^\pi,i)\\
&\quad \le \frac{\bar{l}-l_s}{1+d}\mathbf{1}_{\{X_s^\pi \ge b\}} - \frac{\bar{l}}{1+d}\mathbf{1}_{\{X_s^\pi \ge b\}} - \frac{l_s}{1+d}\mathbf{1}_{\{X_s^\pi < b\}}\\
&\quad = -\frac{l_s}{1+d},
\end{aligned} \tag{A.22}$$

By (A.20) again, we obtain

$$R'_{f,\pi^{0,b}}(x,i) \le \frac{1}{1-c} \quad \text{for } b \ge 0,\ x > b. \tag{A.23}$$

Further, note that for $b \ge 0$ and any $t \ge 0$,

$$\begin{aligned}
&\mathbb{E}_{(x,i)}\bigg[\int_{0<s\le\sigma_1\wedge t} \mathrm{e}^{-\Lambda_s} R'_{f,\pi^{0,b}}(X_s^\pi,\xi_{s-})\,\mathrm{d}\tilde{C}_s\\
&\qquad + \sum_{0<s\le\sigma_1\wedge t} \mathrm{e}^{-\Lambda_s}(R_{f,\pi^{0,b}}(X_s^\pi,\xi_{s-}) - R_{f,\pi^{0,b}}(X_{s-}^\pi,\xi_{s-}))\bigg]\\
&\quad \le \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}\tilde{C}_s + \sum_{0<s\le\sigma_1\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}(X_s^\pi - X_{s-}^\pi)\bigg]\\
&\quad = \mathbb{E}_{(x,i)}\bigg[\sum_{0<s\le\sigma_1\wedge t} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s\bigg],
\end{aligned} \tag{A.24}$$

where the last inequality follows from (A.21), (A.23), $\mathrm{d}\tilde{C}_s \ge 0$, $X_s^\pi - X_{s-}^\pi = C_s - C_{s-} \ge 0$, and $\mathrm{d}C_s = \mathrm{d}\tilde{C}_s + C_s - C_{s-}$.

From Corollary 3.1(i) and Lemma 3.3 we know that the conditions in Lemma 3.3 are satisfied by $w_i(\cdot,j)$. By applying Lemma A.1 we know that for some positive sequence of stopping

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times $\{\tau_n; n = 1, 2, \ldots\}$ with $\lim_{n\to\infty} \tau_n = \infty$, (A.1) holds for any $\pi \in \Pi$, any $b, t > 0$, and any $n \in \mathbb{N}$. By using (A.1), (A.22), and (A.24) (setting $t = t \wedge \tau_n$), we arrive at

$$R_{f,\pi^{0,b}}(x,i) \ge \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1\wedge t\wedge\tau_n} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\sigma_1\wedge t\wedge\tau_n} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s + \mathrm{e}^{-\Lambda_{\sigma_1\wedge t\wedge\tau_n}} w_{f,i}(X^\pi_{\sigma_1\wedge t\wedge\tau_n}, \xi_{\sigma_1\wedge t\wedge\tau_n})\bigg] \quad \text{for } b \ge 0.$$

By noting that the functions $R_{f,\pi^{0,b}}(\cdot, i)$ and $f(\cdot, j)$, $j \in \mathcal{S}$ are bounded, letting $t \to \infty$ and then $n \to \infty$, and then using the monotone convergence theorem for the first two terms inside the expectation and the dominated convergence theorem for the last term, we obtain, for $b \ge 0$,

$$R_{f,\pi^{0,b}}(x,i) \ge \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\sigma_1} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s + \mathrm{e}^{-\Lambda_{\sigma_1}} w_{f,i}(X^\pi_{\sigma_1}, \xi_{\sigma_1})\bigg].$$

By noting that $w_{f,i}(X^\pi_{\sigma_1}, \xi_{\sigma_1}) = f(X^\pi_{\sigma_1}, \xi_{\sigma_1})$ given $\xi_0 = i$, the arbitrariness of π, and the definition of V_f in (3.2), we conclude that $R_{f,\pi^{0,b}}(x,i) \ge V_f(x,i)$ for $x \ge 0$. On the other hand, $R_{f,\pi^{0,b}}(x,i) \le V_f(x,i)$ for $x \ge 0$ according to the definition (3.2). Consequently, $R_{f,\pi^{0,b}}(x,i) = V_f(x,i)$ for $x \ge 0$. □

Proof of Lemma 3.6. Recall that τ_b^π is defined in (3.7). By Theorem 3.1, it follows that for any large enough b and any $x \ge 0$,

$$\begin{aligned}
&R_{f,\pi^{0,b}}(x,i)\\
&\quad = W_{f,b}(x,i)\\
&\quad = \sup_{\pi\in\Pi} \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1\wedge\tau_b^\pi} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\sigma_1\wedge\tau_b^\pi} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s\\
&\qquad\qquad + \mathrm{e}^{-\Lambda_{\tau_b^\pi}} R_{f,\pi^{0,b}}(b,\xi_0)\mathbf{1}_{\{\tau_b^\pi<\sigma_1\}} + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X^\pi_{\sigma_1}, \xi_{\sigma_1})\mathbf{1}_{\{\sigma_1\le\tau_b^\pi\}}\bigg]\\
&\quad \ge \sup_{\pi\in\Pi} \mathbb{E}_x\bigg[\int_0^{\sigma_1\wedge\tau_b^\pi} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\sigma_1\wedge\tau_b^\pi} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X^\pi_{\sigma_1}, \xi_{\sigma_1})\mathbf{1}_{\{\sigma_1\le\tau_b^\pi\}}\bigg].
\end{aligned}$$

Note that $\lim_{b\to\infty} \tau_b^\pi = \infty$ and that f is bounded. By letting $b \to \infty$ on both sides, then using the monotone convergence theorem twice, and then the dominated convergence theorem, it follows that

$$\begin{aligned}
\liminf_{b\to\infty} R_{f,\pi^{0,b}}(x,i) &\ge \sup_{\pi\in\Pi} \mathbb{E}_{(x,i)}\bigg[\int_0^{\sigma_1} \frac{l_s \mathrm{e}^{-\Lambda_s}}{1+d}\,\mathrm{d}s - \int_0^{\sigma_1} \frac{\mathrm{e}^{-\Lambda_s}}{1-c}\,\mathrm{d}C_s + \mathrm{e}^{-\Lambda_{\sigma_1}} f(X^\pi_{\sigma_1}, \xi_{\sigma_1})\bigg]\\
&= V_f(x,i) \quad \text{for } x \ge 0.
\end{aligned}$$

This combined with the fact that $R_{f,\pi^{0,b}}(x,i) \le V_f(x,i)$ for $x \ge 0$ completes the proof. □

Proof of Theorem 3.3. (i) That $b_i^f \ge 0$ is obvious by the definition. We need just to prove that $b_i^f < \infty$. Suppose the contrary. Then by (3.9), we have $R'_{f,\pi^{0,b}}(b,i) > 1/(1+d)$ for all $b \ge 0$. Hence, it follows by Lemma 3.6 that $V_f(x,i) = \lim_{b\to\infty} R_{f,\pi^{0,b}}(x,i)$ for $x \ge 0$. For any $b \ge 0$, by Theorem 3.1, we know that $R'_{f,\pi^{0,b}}(x,i) > 1/(1+d)$ for $x \in (0,b]$, which implies that $R_{f,\pi^{0,b}}(x,i) > R_{f,\pi^{0,b}}(0,i) + x/(1+d)$ for $x \in (0,b]$. Hence, for any $x \ge 0$, we can find a $b > x$ such that $V_f(x,i) \ge R_{f,\pi^{0,b}}(x,i) > R_{f,\pi^{0,b}}(0,i) + x/(1+d)$. Hence, $\lim_{x\to\infty} V_f(x,i) = +\infty$, which contradicts $V_f(x,i) \le \bar{l}/\delta(1+d)$ for $x \ge 0$ (see Lemma 3.2).

(ii) This follows as a result of the proof of Theorem 3.3(i) and Theorem 3.2. □

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Proof of Theorem 4.1. (i) Define an operator $\mathcal{P}$ by $\mathcal{P}(f)(x,i) = R_{f,\pi^{0,b_i^f}}(x,i)$. By applying the results about $R_{f,\pi^{0,b_i^f}}$ obtained in Section 3, we can show that $\mathcal{P}$ is nondecreasing and a contraction on the complete space $(\mathcal{D}, ||\cdot||)$. Using the monotonicity of $\mathcal{P}$ and the fixed point theory, we can show that $\lim_{n\to\infty} \mathcal{P}^n(g_2) \geq V \geq \lim_{n\to\infty} \mathcal{P}^n(g_1) = \lim_{n\to\infty} \mathcal{P}^n(g_2)$, where $g_1(x,i) = 0$ and $g_2(x,i) = \bar{l}/\delta(1+d)$. As a result, $V \in \mathcal{D}$. The full details of the proof can be found in [19].

(ii) The results follow from the proof of Theorem 4.1(i) and Theorem 3.3. □

Proof of Theorem 4.2. Note that $b_i^V < \infty$ for all $i \in \mathcal{S}$. We now define an operator $\mathcal{Q}$ by $\mathcal{Q}(f)(x,i) = R_{f,\pi^{0,b_i^V}}(x,i)$. We can show that $\mathcal{Q}$ is a contraction on $(\mathcal{C}, ||\cdot||)$, and both V and $\mathcal{R}_{\pi^*}$ are fixed points in $(\mathcal{C}, ||\cdot||)$. By the fixed point theory, we conclude that $V = \mathcal{R}_{\pi^*}$. The full details of the proof can be found in [19]. □

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