The correspondence between lines in threefold space and points of a quadric fourfold in fivefold space, established by a geometrical construction. By Mr T. L. Wren, St John's College.

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1. This correspondence has been established by Mr H. W. Turnbull, *Proc. Camb. Phil. Soc.* vol. XXII. (1925), pp. 694—9, using a construction which leaves uncorrelated the lines of a special linear complex and the points of the quadric lying in one tangent fourfold; these have to be correlated by means of an independent construction.

The geometrical construction described below gives a direct representation of the lines of a threefold space by the points of a quadric fourfold in fivefold space, with the same exception; but it will be shown that this representation involves a definite one-one correspondence between the remaining lines and points for which the construction is not directly applicable.

For this construction we assume the existence of a quadric fourfold which has a definite tangent fourfold at each of its points, and contains three real planes with a common point, no two of which meet in a line. This assumption is suggested by the fact that the equation $p_{23}p_{14} + p_{31}p_{24} + p_{12}p_{34} = 0$, connecting the six Plücker coordinates of a line in threefold space, represents a quadric fourfold containing the three real planes $p_{23} = p_{31} = p_{12} = 0$, $p_{23} = p_{24} = p_{34} = 0$, and $p_{23} = p_{31} - p_{34} = p_{12} + p_{24} = 0$, any two of which meet only in the point $p_{23} = p_{31} = p_{12} = p_{24} = p_{34} = 0$. Starting from this assumption, the existence of the two systems of generating planes on the quadric fourfold will be established with the aid of the construction in question.

2. Consider, in fivefold space, a quadric fourfold F_4^2 containing three real planes, ρ, σ, τ, of which any two meet only in the point O. Then ρ, σ, τ must lie in T_4, the tangent fourfold at O.

Let S_4 be a flat fourfold not passing through O; and Ω_3 the threefold common to S_4 and T_4. Then Ω_3 cuts F_4^2 in a quadric κ_2^2 which has three real skew generators (lying in ρ, σ, τ respectively). The section of F_4^2 by T_4 is therefore a cone generated by the two systems of real planes which project the two reguli on κ_2^2 from O; two planes of the same system meet only at O, while two planes of different systems meet in a line through O.

3. Let a, b, c be three generators of the same regulus on κ_2^2; and π, π' two planes in S_4 which pass through c but do not lie in Ω_3. The planes π, π' determine a flat threefold Λ_3 belonging to S_4.

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Any line l of Λ_3 which is skew to c will meet π, π' in distinct points P, P' not in Ω_3; the planes aP, bP' then meet in a single point M, in S_4 but not in Ω_3; and the line OM meets F_4^2 again in a single point L, not in T_4.

Conversely, any point L of F_4^2, not in T_4, is projected from O into a point M of S_4, not in Ω_3; the planes aM, bM meet π, π' respectively in distinct points P, P' which define a unique line l ($\equiv PP'$) not meeting c.

In this way *a one-one correspondence is established between all the real lines of* Λ_3 *which do not meet* c *and all the real points of* F_4^2 *which do not lie in* T_4. It remains to be proved that this correspondence involves a one-one correspondence between the lines of Λ_3 which meet c and the points of F_4^2 in T_4.

4. Consider now in Λ_3 a plane α which meets c in a single point E; and a point V of α not in π or π'.

As l describes the flat pencil through V in α, the planes aP, bP' describe projective axial pencils in two flat threefolds of S_4; these threefolds have in common a plane δ which must contain the locus of M.

The plane δ meets Ω_3 only in the line e common to the planes aE, bE; and cuts the pencils $a(P)$, $b(P')$ in two projective flat pencils in which e is self-corresponding. Hence the locus of M is a straight line x, which meets Ω_3 in a single point Z, lying on e.

If V is in π, but on c, the locus of M is a line x in the plane aV; and x still meets Ω_3 in a single point (ae): similarly for V in π'.

Thus in any case when l describes a flat pencil ϕ of which only one ray meets c, the locus of M is a line x meeting Ω_3 in a single point Z on a generator e of κ_2^2. The plane joining O to x cuts F_4^2 in a conic to which belongs the line OZ; the rest of this conic is therefore a line f on F_4^2, meeting T_4 in a single point on OZ. The line f is the locus of L.

The construction of paragraph 3 establishes a one-one correspondence between the rays of ϕ and the points of f, except for the ray meeting c and the point in T_4; these must therefore also correspond as members of the pencil ϕ and range on f, but it remains to be proved that all lines f which represent pencils ϕ containing a given line l, which meets c, must meet T_4 in the same point.

5. In Λ_3 take two fixed lines l_1, l_2 not meeting c, but having a common point V and lying in a plane α; then l_1, l_2 are represented on F_4^2 by points L_1, L_2 not in T_4.

Any flat pencil ϕ_1 to which l_1 belongs is represented on F_4^2 by a line f_1 through L_1; and any flat pencil ϕ_2 containing l_2 by a line f_2 through L_2.

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First consider only pencils ϕ_1, ϕ_2 lying in α. Since any ϕ_1 has a ray not meeting c in common with all but one ϕ_2, the corresponding line f_1 must meet all but one f_2; and *vice versa*. It follows that all these lines f_1, f_2 must lie in a single plane β. Thus all the lines of the plane α, except those meeting c, are represented on F_4^2 by all the points of a plane β, except those in T_4; and there will be one such plane β for every plane α of Λ_3 which does not contain c.

Similarly, by considering only pencils ϕ_1, ϕ_2 having V for vertex, it may be proved that all the lines of Λ_3 through V, except those meeting c, are represented on F_4^2 by all the points of a plane ω, except those in T_4; and there will be one such plane ω for every point V of Λ_3 not on c.

6. Thus *on F_4^2 there is a system of real planes β representing all planes of Λ_3 not containing c; and a second system of planes ω representing all points of Λ_3 not on c.*

Two planes β will in general have just one common point outside T_4 (since the planes they represent have a common line not in general meeting c), and therefore in general two planes β cannot meet in a line. Similarly for two planes ω.

A plane β and a plane ω will in general have no common point outside T_4; but if they represent a plane α and a point V in α they will have a common line f, which represents the flat pencil through V in α.

Through any point L of F_4^2 not in T_4 there must pass a single infinity of planes β representing the planes through the line l represented by L, and a single infinity of planes ω representing the points of l. No two of these planes β can meet in any point besides L, and similarly for two planes ω; but each plane β will have a line through L common with any plane ω. All these planes β and ω must lie in the tangent fourfold at L; and they must form two systems analogous to the two systems of planes through O, since any flat fourfold not passing through L will cut them in generators of the two systems on a ruled quadric twofold.

The argument of paragraph 4 may be reversed to show that any line on F_4^2, not wholly in T_4, represents a flat pencil (with the usual exception); it then follows that any plane on F_4^2, not wholly in T_4, represents either a plane not through c or a point not on c. Hence *the planes β and ω through L are the only planes on F_4^2 through L.*

7. It is now evident that, in the construction of paragraph 3, the point O may be replaced by any point L of F_4^2 not in T_4. If L be a point of F_4^2 in T_4 and we take L also outside the tangent fourfold at L_0, then L_0 is not in the tangent fourfold at L. Hence

through any point whatever of F_4^2 there will pass two systems of real planes on F_4^2, such that two planes of the same system have no other common point, while two of different systems have a line in common.

Thus *on F_4^2 there must be two distinct systems of real planes; two planes of the same system have always just one common point; two planes of different systems have in general no common point, but may in special cases have a common line.* One of these systems consists of all the planes β which represent planes not containing c, together with the planes of the same system in T_4; the other consists of all the planes ω representing points not on c, together with those of the same system in T_4.

All the planes of F_4^2 in T_4 must pass through O, since κ_2^2 is a proper ruled quadric.

8. Consider now the correspondence between the points of a plane ω and the lines of Λ_3 through the point V, not on c, represented by ω. Since all but one of the flat pencils through V are represented by lines f in ω, this correspondence must be a collineation.

In this collineation the pencil of planes (α) through a line l_0 meeting c must correspond to a flat pencil of lines (f) in ω; the vertex L_0 of (f) must be a point of T_4, since it cannot represent a line not meeting c.

All but one of the planes (α) through l_0 are represented by planes β on F_4^2; each of these planes β must contain the line f corresponding in the collineation to the associated plane α, and therefore passes through L_0.

Since two planes β can have but one common point, and L_0 is in the plane ω which represents V, it follows that L_0 must lie in all the planes ω which represent points of l_0.

Thus *the representation of lines of Λ_3 not meeting c by points of F_4^2 not in T_4 leads to a definite point L_0 of T_4 associated with a line l_0 meeting c;* namely the single point L_0 common to all the planes β, ω representing planes through l_0, and points on l_0, respectively.

Any point L_0 of F_4^2 in T_4, other than O, is the single point common to a set of planes β and a set of planes ω (paragraph 7); these represent respectively a set of planes α of which no two have a common line not meeting c, and a set of points V such that the join of any two meets c. From what has been said above it is evident that the line common to any two of these planes α must be common to all, and must coincide with the line joining any two of the points V

The representation of lines not meeting c by points of F_4^2 not in T_4 therefore involves a one-one correspondence between the lines meeting c and the points in T_4 other than O; leaving the single line c to correspond to the single point O.

9. *The correspondence between lines meeting c and points in T_4 is of exactly the same character as the general correspondence.*

Any plane β meets T_4 in a line f_0 not passing through O; the points of f_0 represent the lines meeting c of the plane α represented by β.

Any plane ω meets T_4 in a line whose points represent the lines meeting c which pass through the point V represented by ω.

The planes β representing all planes α through a given point V_0 of c will cut T_4 in lines f_0, any two of which must meet in the point of T_4 corresponding to the common line of the two associated planes α; these lines f_0 therefore lie in one plane. Thus the lines through V_0 are represented by the points of a plane in T_4; this plane must belong to the ω-system since it is met by certain planes β in lines.

Similarly the lines of a plane through c are represented by the points of a plane in T_4 belonging to the β-system.

Thus the planes on F_4^2 which lie in T_4 and pass through O present no exception to the general representation; those of the β-system represent planes through c, while those of the ω-system represent points of c.

The character of the representation is therefore completely independent of the special elements O, S_4, a, b, c, π, π', chosen subject to the restrictions specified in paragraphs 2, 3; and these elements provide no singularities in the correspondence between the lines of Λ_3 and the points of F_4^2, although the construction fails to determine directly the point in T_4 corresponding to a line meeting c.