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ARTICLES

CONSISTENT EXPECTATIONS EQUILIBRIA

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We consider a class of nonlinear dynamic economic models in which the actual realizations of a certain variable (e.g., price) depend on the agents' expectations about this variable. We define a consistent expectations equilibrium (CEE) by the property that the sample average and the sample autocorrelations of the realizations of the actual law of motion equal the average and the autocorrelations of the perceived law of motion. Along a CEE agent's expectations are thus self-fulfilling in terms of the observable sample average and sample autocorrelations. Restricting ourselves to the case in which beliefs are described by an AR(1) process, we study existence and stability of three different types of CEE: steady-state, two-cycle, and chaotic. We illustrate how these equilibria can emerge in the nonlinear cobweb model. Learning dynamics based on sample average and sample autocorrelations are introduced and stability of CEE under this learning process is investigated.

Keywords: Expectations, Nonlinear Dynamics, Learning

1. INTRODUCTION

Most models in economics and other social sciences involve a description of human behavior. In economics, one usually postulates that human beings (or, in more general terms, economic agents) behave in a rational way whereby rationality is meant to cover two different aspects. The first one is that agents behave optimally in any given situation. For example, they maximize their utility or their profit. The second aspect of rationality is that agents form expectations about the future in a way that is not systematically wrong. Most economists seem to agree on how to formulate optimizing behavior, but there are various opinions on how one should model the second aspect of rationality. This paper suggests that agents form expectations about future variables in such a way that their beliefs are *consistent*

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with the observed realizations in a linear statistical sense. In other words, it is supposed that agents act like econometricians using linear statistical techniques and, in doing so, they do not make systematic forecasting errors.

A dynamic economic model is always an expectations feedback system: expectations affect actual dynamics and actual dynamics feed back into the expectations scheme. Since its introduction by Muth (1961), and its application to macroeconomics by Lucas (1971), the Rational Expectations Hypothesis (REH) has become the predominant paradigm in expectation formation in economics. The REH assumes that an agent's subjective expectation of a future variable equals the objective expectation of that variable conditional on the information available to the agent at the time the expectation is formed. In applications of the REH, it usually is assumed that the information available to the agents includes the market equilibrium equations. A rational expectations equilibrium (REE) is a fixed point of this expectations feedback system. Despite its undeniable appeal as a normative model of expectation formation, the REH has been criticized for various reasons. For example, some scholars have pointed out that the REH assumes unrealistic computing power for the agents: In most nonlinear market equilibrium models, even if agents knew all equilibrium equations, it would be impossible to compute the REE analytically and it would require quite an effort to do it numerically. Others have argued that knowledge about the model equations is already too extreme an assumption. In the recent literature on bounded rationality it has been claimed that the REH in fact assumes that the agents in the model know more about the model than the model builders themselves [Sargent (1993, p. 21)]. As an alternative, it has been suggested that it be assumed that the agents do not necessarily know the market equilibrium equations, but that they base their beliefs only on observations of actual time series. For example, the agents might behave like econometricians and compute their expectations from actual time-series observations in the past, e.g., by ordinary least-squares regressions [see, e.g., Bray (1982), Bray and Savin (1986), Marcet and Sargent (1989), Woodford (1990), Bullard (1994), or Evans and Honkapohja (1995)]. Under this assumption it may or may not be the case that a REE is the asymptotic outcome of such a learning process.

The present paper introduces the notion of *consistent expectations equilibrium* (CEE) in nonlinear dynamic economic models. The key feature of a CEE is that agents' expectations of a certain variable are consistent with the realizations of that variable in the sense that their sample average and their sample autocorrelations are the same. For example, suppose that the agents believe that prices follow a stochastic low-order $\mathrm{AR}(k)$ process and thus predict that tomorrow's price will be some linear combination of past prices. Given this belief, a certain time path of actual prices will be realized through market clearing. We call this time path of equilibrium prices a CEE if its sample average and its sample autocorrelation function equal the average and the autocorrelation function, respectively, of the $\mathrm{AR}(k)$ belief process. Stated differently, a CEE is a fixed point of the expectations feedback system in terms of the *observable* sample average and sample autocorrelations. CEE is thus an equilibrium concept for which beliefs are self-fulfilling in a linear statistical sense.

This paper deals with the existence, the possible structures, and the stability of CEE. We show that in the simple class of nonlinear dynamic models with a cobweb demand-supply structure, at least three types of CEE are possible given an AR(1) belief: steady-state, two-cycle, and chaotic. The simple steady-state and the two-cycle CEE are asymptotically equal to REE. In our framework, however, the strong conclusions drawn from the REH can be obtained without assuming that the agents have any knowledge about the underlying market equilibrium equations. Instead, these conclusions follow simply from the assumption that the agents have an AR(1) belief that is consistent with actual observations. The more complicated chaotic CEE are not REE, not even in the long run. In those equilibria, agents do make mistakes, but these mistakes are not systematic: The sample average and sample autocorrelations of the chaotic equilibrium prices and of the belief coincide. Thus, if the agents use linear statistical tests, they are not able to distinguish between the actual chaotic price sequence and their stochastic AR(1) belief. As a consequence, they do not have any reason for deviating from their simple predictor and the situation qualifies as an equilibrium. The most important difference between the notion of CEE and REE is that, in the former, the agents do not need any knowledge about the underlying market equilibrium equations. This is also the case, e.g., for adaptive expectations. Contrary to applications of adaptive expectations, however, we assume consistency between the beliefs and the actual observations. In a CEE, agents would not be able to detect any discrepancy between their model [the AR(1) belief] and reality (the actual price sequence) if they only use linear statistical techniques.

We consider two different types of stability notions that are relevant for CEE. The first one captures the robustness of a CEE with respect to small perturbations of the initial state variable (e.g., the price), but does not allow for perturbations of the parameters describing the belief process. The second stability notion involves a learning process that is based upon sample average and sample autocorrelations. In the learning process, the parameters describing the AR(1) belief are updated as additional observations become available. The learning scheme is closely related to, but not identical to ordinary least-squares learning. For each of the three types of CEE (steady-state, two-cycle, and chaotic), we provide examples in which the CEE is stable in the learning dynamics. We also provide examples with multiple stable CEE. In that case, the initial state variable together with the initial belief parameters determine which CEE eventually will be learned.

Finally, we relate the present work to some other recent contributions in expectations formation and learning. Our approach fits well into the bounded rationality literature because the agents base their expectations upon time-series observations and adapt their beliefs accordingly. In particular, the concept of consistent expectations originates from three closely related papers. Grandmont (1994) introduces the notion of a *self-fulfilling mistake*, in which agents incorrectly believe that prices follow a stochastic process whereas the actual dynamics are generated by a deterministic chaotic process that is indistinguishable from white noise by linear statistical tests. Sorger (1998) constructs an explicit example of such a self-fulfilling mistake in an overlapping-generations model with physical capital. In that example, the agents believe that the interest rates form a sequence of independent and identically distributed random variables (i.i.d. r.v.'s), whereas the actual dynamics of the interest rates are described by a chaotic tent map with zero autocorrelations at all lags. The example is a special case of our more general CEE concept, which applies to nonzero autocorrelations as well. Hommes (1998) investigates autocorrelation functions of forecasting errors in the cobweb model with naive, adaptive, and low-order AR(k) expectations when price fluctuations are chaotic, and calls a forecasting rule consistent when these (chaotic) errors have zero autocorrelations. The present paper can be viewed as an attempt to build the results of Hommes (1998) and Sorger (1998) into a more general framework.

Our approach, in fact, is somewhat similar in spirit to some of the early rational expectations literature, in which attempts have been made to put restrictions on the parameters of simple forecasting rules, as in Muth (1960) and Sargent (1971); see also the discussion by Sargent (1993, pp. 18–19). In particular, our approach is related to the concept of quasi-rational expectations introduced by Nerlove et al. (1979, Ch. XIII), in which the expectations about both exogenous and endogenous variables are given by those predictors that minimize the mean squared prediction errors in an ARIMA model. However, our focus is on the deterministic feedback between (linear) expectations and (nonlinear) actual dynamics. More recent related work on bounded rationality and expectation formation includes the rational belief equilibria of Kurz (1994), the pseudo rational learning of Marcet and Nicolini (1995), the expectational stability and adaptive learning rules of Evans and Honkapohja (1994, 1995), the perfect predictors of Böhm and Wenzelburger (1996), and the adaptive rational equilibrium dynamics of Brock and Hommes (1997a, b). Stability and instability of adaptive learning processes have been investigated by Grandmont and Laroque (1991), Bullard (1994), Grandmont (1994), Chatterji and Chattopadhyay (1996), and Schönhofer (1996). See Marimon (1996) for an overview of the recent learning literature.

The paper is organized as follows: Section 2 presents a general definition of CEE's. Section 3 focuses on existence and stability of simple, that is, steady-state or two-cycle CEE's, whereas Section 4 deals with complicated, chaotic CEE. A learning process based on sample average and sample autocorrelations is introduced in Section 5. Section 6 applies the CEE concept to nonlinear versions of the familiar cobweb model. Section 7 concludes.

2. DEFINITION OF CEE

In this section we introduce the notion of CEE for a simple, general class of models,

$$p_t = F\left(p_t^e\right), \quad (1)$$

where p_t represents the endogenous variable (henceforth referred to as the price) at date t and p_t^e represents agents' expected price for period t, formed at date $t-1$. The familiar cobweb demand-supply model is of this type. For example, Bray and Savin (1986) and Fourgeaud et al. (1986) have investigated the dynamics of the linear cobweb model with boundedly rational agents using an OLS learning scheme. In the present paper, the map F in (1) is nonlinear.

To explain how the agents form the price expectation p_t^e, we specify the information that is available to them at the time that this expectations is formed. We assume that agents do *not* know market equilibrium equations, and thus are not able to use them in forming their expectations. Instead, agents form expectations based only on time-series observations. We assume that the agents know all past prices $p_0, p_1, \ldots, p_{t-1}$, that they believe that prices follow a simple linear stochastic process, and that expectations are homogeneous across agents. More specifically, we assume that all agents believe that prices are generated by a stochastic AR(1) process, $p_t = \alpha + \beta(p_{t-1} - \alpha) + \epsilon_t$, where α and β are real numbers, $\beta \in [-1, 1]$, and ϵ_t an i.i.d. process with zero mean. At this point, we assume that the belief parameters α and β are fixed over time and known to (or believed to be known by) the agents. In Section 5, we consider the more general case in which the belief parameters are unknown to the agents and change over time according to a learning process as additional observations become available. Note that the parameter α is the limit, as t approaches infinity, of $E(p_t)$, i.e., α is the long-run average of the belief process. Given this perceived law of motion, the unique predictor or forecasting rule for p_t that minimizes the squared prediction errors is given by

$$p_t^e = \alpha + \beta(p_{t-1} - \alpha). \tag{2}$$

The expected price is therefore the constant α (the long-run average price) plus the constant β (the first-order autocorrelation coefficient) times the deviation of the previous price from the long-run average.

Assuming that agents use the linear predictor (2), the *implied actual law of motion* for model (1) is

$$p_t = F_{\alpha,\beta}(p_{t-1}) := F(\alpha + \beta(p_{t-1} - \alpha)). \tag{3}$$

Now recall that the empirical or sample average of a time series $(p_t)_{t=0}^{\infty}$ is defined as [see, e.g., Box et al. (1994)]

$$\bar{p} = \lim_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} p_t$$

and the empirical or sample autocorrelation coefficients are given by

$$\rho_j = \lim_{T\to\infty} \frac{c_{j,T}}{c_{0,T}}, \qquad j \geq 1,$$

where

$$c_{j,T} = \frac{1}{T+1} \sum_{t=0}^{T-j} (p_t - \bar{p})(p_{t+j} - \bar{p}), \qquad j \geq 0.$$

In the special case in which the time series is constant, the definition of ρ_j involves an indeterminate expression and all sample autocorrelations can be defined as β^j for some $\beta \in [-1, 1]$.[1] We are now ready for the definition of a CEE.

DEFINITION 1. *A triple* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$, *where* $(p_t)_{t=0}^{\infty}$ *is a sequence of prices and* α *and* β *are real numbers,* $\beta \in [-1, 1]$, *is called a consistent expectations equilibrium if*

(i) *The sequence* $(p_t)_{t=0}^{\infty}$ *satisfies the implied actual law of motion* (3).
(ii) *The sample average* $\bar{p}$ *is equal to* α.
(iii) *For the sample autocorrelation coefficients* ρ_j, *the following is true:*
   a. *If* $(p_t)_{t=0}^{\infty}$ *is a convergent sequence, then* $\text{sgn}(\rho_j) = \text{sgn}(\beta^j)$, $j \geq 1$.
   b. *If* $(p_t)_{t=0}^{\infty}$ *is not convergent, then* $\rho_j = \beta^j$, $j \geq 1$.

Property (i) in the definition of CEE simply states that the sequence $(p_t)_{t=0}^{\infty}$ is the price sequence generated by the implied actual law of motion, provided that agents use the forecasting rule (2). Condition (ii) requires that the sample average of the actual time-series equals the expected average of the stochastic AR(1) belief. In formulating the equivalence of autocorrelation coefficients of actual and anticipated prices, we have to distinguish between two cases: (a) the price sequence converges to some steady state or (b) it does not. In the second case, condition (iii) (b) states that the sample autocorrelation coefficients of the actual realizations are exactly the same as the autocorrelation coefficients of the perceived law of motion, i.e., the AR(1) belief process. In the case in which the price sequence converges to a steady state, we make the weaker requirement that autocorrelation coefficients of observations and beliefs have the same sign. The reason for weakening the condition is related to the indeterminacy of the autocorrelation coefficients of a time series converging to a constant.[2]

Summarizing, a CEE is a price sequence and an AR(1) belief process such that the expectations are self-fulfilling in terms of the observable sample average and sample autocorrelations. Stated differently, along a CEE, expectations are not systematically wrong in a linear statistical sense and, using time-series observations only, the agents would have no reason to deviate from their belief.

The definition of CEE can be generalized easily to higher-order belief processes, e.g., AR(k) processes with $k \geq 2$. Here we restrict ourselves to AR(1) beliefs, because this allows already for rich dynamical behavior. We show that, given an AR(1) belief, there are at least three possible types of CEE:

- steady-state, in which the price sequence $(p_t)_{t=0}^{\infty}$ converges to a steady-state price p^*;
- two-cycle, in which the price sequence $(p_t)_{t=0}^{\infty}$ converges to a period-2 cycle $\{p_1^*, p_2^*\}$ with $p_1^* \neq p_2^*$;[3]
- chaotic, in which the price sequence $(p_t)_{t=0}^{\infty}$ is chaotic.

Which of these cases occurs in a particular model depends on the mapping F. We refer to steady-state and two-cycle CEE as simple CEE.

3. SIMPLE CEE

In this section we first study which belief parameters α and β correspond to simple CEE, i.e., to steady-state and two-cycle CEE. Next, we explore the relation between simple CEE and fixed points of the function F. We prove that, in the case in which the function F is monotonic, the only possible bounded CEE are simple CEE. Finally, we define stability of simple CEE and derive stability conditions.

THEOREM 1.

(i) *If* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ *is a steady-state CEE converging to* p^*, *then* $\alpha = p^*$ *and* p^* *is a fixed point of* F.

(ii) *If* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ *is a two-cycle CEE converging to* $\{p_1^*, p_2^*\}$, *then* $\alpha = (p_1^* + p_2^*)/2$ *and* $\beta = -1$. *Furthermore, both* p_1^* *and* p_2^* *are fixed points of* F.

Proof.

(i) Whenever $\lim_{t\to\infty} p_t$ exists, the sample average $\bar{p}$ must be equal to this limit. Together with condition (ii) of Definition 1, this implies that $\alpha = p^*$. The actual price dynamics (3) therefore is $p_t = F(p^* + \beta(p_{t-1} - p^*))$. Because p^* must be a steady state of this difference equation it follows that p^* is a fixed point of F.

(ii) Analogously to case 1, it can be seen that $\alpha = \bar{p} = (p_1^* + p_2^*)/2$. Using this it is also straightforward to show that $\lim_{T\to\infty} c_{0,T} = -\lim_{T\to\infty} c_{1,T} = (p_1^* - p_2^*)^2/4$. This implies that $\rho_1 = -1$ and it follows from condition (iii)(a) of Definition 1 that $\beta = -1$. The actual dynamics therefore are given by $p_t = F(p_1^* + p_2^* - p_{t-1})$. It is easy to see that $\{p_1^*, p_2^*\}$ is a two-cycle of this difference equation if and only if both p_1^* and p_2^* are fixed points of F. This concludes the proof of the theorem. ■

Theorem 1 shows that, along a simple CEE, in the long run, prices will be close to the set of fixed points of F. In this regard it is worth mentioning that a perfect-foresight equilibrium in model (1) is any price sequence $(p_t)_{t=0}^{\infty}$ such that, at each date t, p_t equals one of the fixed points of the map F. We therefore conclude that the long-run behavior of simple CEE is the same as the long-run behavior of some REE. The following result proves a partial converse to this statement and Theorem 1, in showing that we can always construct CEE when we know the fixed points of F.

THEOREM 2.

(i) *Assume that* p^* *is a fixed point of the map* F *and define* $p_t = p^* = \alpha$ *for all* t. *For any* $\beta \in [-1, 1]$ *it follows that* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ *is a steady-state CEE.*

(ii) *Assume that* p_1^* *and* p_2^* *are two different fixed points of the map* F *and define* $p_{2t} = p_1^*$ *and* $p_{2t+1} = p_2^*$ *for all* t, $\alpha = (p_1^* + p_2^*)/2$, *and* $\beta = -1$. *Then* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ *is a two-cycle CEE.*

Proof.

(i) If $\alpha = p^*$ is a fixed point of F, then the constant sequence $(p^*, p^*, p^*, \ldots)$ satisfies the actual dynamics (3) so that condition (i) in Definition 1 is satisfied. Conditions (ii) and (iii) of Definition 1 hold trivially.

(ii) If $\alpha = (p_1^* + p_2^*)/2$ and $\beta = -1$, where p_1^* and p_2^* are fixed points of F, then the periodic sequence $(p_1^*, p_2^*, p_1^*, p_2^*, \ldots)$ satisfies the actual dynamics (3) so that condition (i) of Definition 1 is satisfied. Conditions (ii) and (iii) of Definition 1 again are verified easily. ■

We call a CEE $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ bounded if the sequence $(p_t)_{t=0}^{\infty}$ is bounded. Theorem 3 proves that the only bounded CEE in model (3) with a monotonic function F are simple CEE.

THEOREM 3. *Let $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ be a bounded CEE.*

(i) *If the map F is increasing, then $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ is either a steady-state CEE or a two-cycle CEE.*
(ii) *If the map F is decreasing, then $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ is a steady-state CEE. Moreover, it holds that $\beta = 0$.*

Proof.

(i) Assume that F is increasing and let $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ be a bounded CEE. First, consider the case $0 \leq \beta \leq 1$. In that case the actual law of motion, $F_{\alpha,\beta}$, in (3) is nondecreasing. Hence, $(p_t)_{t=0}^{\infty}$ must be monotonic and, because it is bounded, it must converge to a steady-state p^*. Thus, the CEE is a steady-state CEE. Next, consider the case $-1 \leq \beta \leq 0$. In this case the actual law of motion, $F_{\alpha,\beta}$, is nonincreasing. Together with the boundedness of $(p_t)_{t=0}^{\infty}$, this implies that the price sequence either converges to a steady-state p^* or to a two-cycle $\{p_1^*, p_2^*\}$ with $p_1^* \neq p_2^*$. This concludes the proof of part (i).
(ii) Suppose that F is decreasing and consider the bounded CEE $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$. First assume $0 \leq \beta \leq 1$. In that case, the actual law of motion $F_{\alpha,\beta}$ is nonincreasing. As before, this implies that prices either converge to a steady state or to a two-cycle. The latter cannot be a CEE, however, because the sample autocorrelation coefficient at the first lag, ρ_1, would be -1, which leads to a contradiction in condition (iii)(b) of Definition 1 (note that β was assumed to be nonnegative). Therefore, the CEE must be a steady-state CEE. Because the actual law of motion is nonincreasing, the sample autocorrelation of $\{p_t\}_{t=0}^{\infty}$ at the first lag cannot be positive, i.e., it holds that $\rho_1 \leq 0$. According to property (iii)(a) in Definition 1, we have $\mathrm{sgn}(\beta) = \mathrm{sgn}(\rho_1) \leq 0$. Because we have assumed that $\beta \geq 0$, it follows that $\beta = 0$.

Next, assume that $-1 \leq \beta \leq 0$, in which case $F_{\alpha,\beta}$ is nondecreasing. As before, we conclude that the price sequence $(p_t)_{t=0}^{\infty}$ must converge so that the CEE is a steady-state CEE. Furthermore, because $F_{\alpha,\beta}$ is nondecreasing the price sequence must be monotonic and the sample autocorrelation coefficient at the first lag, ρ_1, must be nonnegative. According to property (iii)(a) of Definition 1, $\mathrm{sgn}(\beta) = \mathrm{sgn}(\rho_1) \geq 0$. Because we have assumed that $\beta \leq 0$, we conclude that $\beta = 0$. This concludes the proof of part (ii) of the theorem. ■

The long-run behavior of prices in a steady-state CEE is characterized by the limit price p^*. We write $\{(p^*); \beta\}$ for the steady-state CEE in which the price sequence is constant and equal to p^*. Notice that, for any steady-state CEE converging to p^*, the belief parameter α coincides with p^* and therefore can be dropped from the notation. The long-run behavior of prices in a two-cycle CEE is characterized by the two different limit prices p_1^* and p_2^*. We write $\{(p_1^*, p_2^*)\}$ for the two-cycle CEE in which the price sequence is exactly periodic and alternates between the two values p_1^* and p_2^*. Because the corresponding belief parameters are uniquely determined by $\alpha = (p_1^* + p_2^*)/2$ and $\beta = -1$, they have been dropped from the notation. Note that both $\{(p^*); \beta\}$ and $\{(p_1^*, p_2^*)\}$ are (perfect-foresight) REE. We now introduce a stability notion for these simple CEE.

DEFINITION 2.

(i) $\{(p^*); \beta\}$ *is said to be a* locally stable *steady-state CEE if there exists an open neighborhood* $U \subset \mathbf{R}$ *of* p^* *such that, for all initial prices* $p_0 \in U$, *the triple* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$, *where* $\alpha = p^*$ *and* $(p_t)_{t=0}^{\infty}$ *is the unique price sequence satisfying the implied actual dynamics* (3), *is a CEE and* $\lim_{t\to\infty} p_t = p^*$.

(ii) $\{(p_1^*, p_2^*)\}$ *is said to be a* locally stable *two-cycle CEE if there exists an open neighborhood* $U \subset \mathbf{R}$ *of* $\{p_1^*, p_2^*\}$ *such that, for all initial prices* $p_0 \in U$, *the triple* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$, *where* $\alpha = (p_1^* + p_2^*)/2$, $\beta = -1$, *and* $(p_t)_{t=0}^{\infty}$ *is the unique price sequence satisfying the implied actual dynamics* (3), *is a two-cycle CEE converging to* $\{p_1^*, p_2^*\}$ (*orbital convergence; see note* 3).

Note that, in the stability definition above, the belief parameters α and β are fixed. Local stability of a steady-state or a two-cycle CEE therefore means that the common AR(1) belief described by the parameters α and β is self-fulfilling (in terms of sample average and sample autocorrelation) for an open set of initial prices p_0. In Section 5, we discuss a different and more general stability concept, which we call learnability, dealing with the robustness of CEE with respect to perturbations of both the state variables and the belief parameters. The following result gives conditions for the stability of steady-state and two-cycle CEE.

THEOREM 4. *Assume that* F *is continuously differentiable.*

(i) *Let* $\{(p^*); \beta\}$ *be a steady-state CEE.*
 (a) *If* $\beta = 0$, *then* $\{(p^*); \beta\}$ *is locally stable.*
 (b) *If* $F'(p^*) > 0$ *and* $|\beta| < F'(p^*)^{-1}$, *then* $\{(p^*); \beta\}$ *is locally stable.*
 (c) *If* $F'(p^*) < 0$, *then* $\{(p^*); \beta\}$ *is locally stable if and only if* $\beta = 0$.

(ii) *A two-cycle CEE* $\{p_1^*, p_2^*\}$ *is locally stable if* $|F'(p_1^*)F'(p_2^*)| < 1$.

Proof.

(i)(a) Let U be any open neighborhood of p^*. The actual dynamics with $\alpha = p^*$ and $\beta = 0$ are given by $p_t = F(p^*) = p^*$. The unique price sequence of this difference equation emanating from p_0 is $(p_0, p^*, p^*, \ldots)$. It is straightforward to see that the sample average of this sequence $(p_t)_{t=0}^{\infty}$ is p^* and that all sample autocorrelation coefficients of $(p_t)_{t=0}^{\infty}$ are equal to 0. Consequently, $\{(p_t)_{t=0}^{\infty}; p^*, 0\}$ is a steady-state CEE converging to p^*.

(i)(b) Assume that $F'(p^*) > 0$ and $|\beta| < F'(p^*)^{-1}$. The actual dynamics with $\alpha = p^*$ are given by $p_t = F(p^* + \beta(p_{t-1} - p^*)) = F_{p^*,\beta}(p_{t-1})$. The map $F_{p^*,\beta}$ has the fixed point p^* and $F'_{p^*,\beta}(p^*) = \beta F'(p^*)$. Let U be an open neighborhood of p^*

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such that $|F'_{p^*,\beta}(p)| < 1$ for all $p \in U$. If $\beta > 0$, then $F_{p^*,\beta}$ is increasing on U and $0 < F'_{p^*,\beta}(p) < 1$ for all $p \in U$. Consequently, for each initial state $p_0 \in U$, the unique time series $(p_t)_{t=0}^{\infty}$ satisfying the actual dynamics converges monotonically to p^*. This implies that all sample autocorrelation coefficients are positive and it follows that $\{(p_t)_{t=0}^{\infty}; p^*, \beta\}$ is a steady-state CEE. Alternatively, if $\beta < 0$, then $F_{p^*,\beta}$ is decreasing on U and $-1 < F'_{p^*,\beta}(p) < 0$ for all $p \in U$. For each initial state $p_0 \in U$, the unique time series $(p_t)_{t=0}^{\infty}$ satisfying the actual dynamics converges to p^* in the form of damped oscillations. That is, $p_t - p^*$ changes its sign in every period. This implies that all sample autocorrelation coefficients ρ_j satisfy $\operatorname{sgn}(\rho_j) = \beta^j$ and it follows that $\{(p_t)_{t=0}^{\infty}; p^*, \beta\}$ is a steady-state CEE.

(i)(c) If $F'(p^*) < 0$, it follows as in the proof of Theorem 3(ii) that a price sequence with initial state p_0 close to p^* that converges to p^* can only be a CEE when $\beta = 0$. This proves the necessity of $\beta = 0$. Sufficiency follows from part (i)(a).

(ii) For $\alpha = (p_1^* + p_2^*)/2$ and $\beta = -1$ the actual dynamics (3) are given by $p_t = F(p_1^* + p_2^* - p_{t-1})$. According to Theorem 1, both p_1^* and p_2^* are fixed points of F, and $\{p_1^*, p_2^*\}$ is a two-cycle of the actual dynamics. The condition $|F'(p_1^*)F'(p_2^*)| < 1$ implies that this periodic solution locally asymptotically stable. Consequently, there exists an open neighborhood U of $\{p_1^*, p_2^*\}$ such that, for all initial prices $p_0 \in U$, the unique solution $(p_t)_{t=0}^{\infty}$ of (3) converges to the two-cycle $\{p_1^*, p_2^*\}$. This, in turn, implies that the sample average of $(p_t)_{t=0}^{\infty}$ is α and that the sample autocorrelation coefficient at lag j is equal to $(-1)^j$. Thus, $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ is a two-cycle CEE and it follows that the two-cycle CEE $\{(p_1^*, p_2^*)\}$ is locally stable.

■

4. CHAOTIC CEE

In addition to the simple steady-state and two-cycle CEE discussed in Section 3, much more complicated CEE can arise even if the agents have a simple AR(1) belief. In fact, an important motivation for the present paper was the question whether chaotic equilibria without systematic forecasting errors can arise in dynamic economic models with simple (linear) forecasting rules. It is well known that simple nonlinear deterministic models can generate chaotic time paths with random-looking behavior. In particular, chaotic time series may have zero autocorrelations at all lags and, therefore, from a linear statistical point of view, chaos may be indistinguishable from pure white noise. On the other hand, not every chaotic time series has zero autocorrelations. Properties of the typical sample autocorrelation coefficients of chaotic processes are far from being well understood and only recently have statisticians started exploring autocorrelation functions of chaotic processes. For many chaotic time series, the sample autocorrelation coefficients are exponentially decaying, very much like the autocorrelation coefficients of a weakly stochastic process [Bunow and Weiss (1979) and Hall and Wolff (1993); see Hommes (1998) and Hommes and van Eekelen (1996) for autocorrelation coefficients of chaotic price fluctuations in the cobweb model]. There seems to be only one class of chaotic processes in the literature, in which the autocorrelation structure is completely understood: For so-called asymmetric tent maps, the sample autocorrelation coefficients of chaotic time paths coincide exactly with the autocorrelation functions of stochastic AR(1) processes [Sakai and Tokumaru (1980)]. We briefly discuss this example here because it will be used several times throughout the paper.

The *asymmetric tent map* is the continuous, piecewise linear map $T_{\beta,[a,b]}:[a,b] \mapsto [a,b]$ defined as

$$T_{\beta,[a,b]}(x) = \begin{cases} \dfrac{2}{1+\beta}(x-a)+a & \text{if} \quad a \le x \le a + \dfrac{1+\beta}{2}(b-a) \\ \dfrac{2}{1-\beta}(b-x)+a & \text{if} \quad a + \dfrac{1+\beta}{2}(b-a) < x \le b, \end{cases}$$

where $-1 < \beta < +1$. The map is expanding; that is, in absolute value, its slope is everywhere greater than 1. The properties of the piecewise linear difference equation

$$p_t = T_{\beta,[a,b]}(p_{t-1}) \tag{4}$$

are well understood. In particular, the following properties are known:

1. For any integer $j \ge 1$, equation (4) has a periodic orbit of period j. All periodic orbits are unstable. The union of all periodic orbits is dense in $[a,b]$.
2. For Lebesgue almost all initial states $p_0 \in [a,b]$, the trajectory of (4) is aperiodic and dense in the interval $[a,b]$.
3. The uniform distribution on the interval $[a,b]$ is ergodic and invariant under $T_{\beta,[a,b]}$.
4. For Lebesgue almost all initial states $p_0 \in [a,b]$, the sample average of the trajectory of (4) is $\bar{p} = \lim_{T\to\infty} \frac{1}{T+1}\sum_{t=0}^{T} p_t = (a+b)/2$.
5. For Lebesgue almost all initial states $p_0 \in [a,b]$, the sample autocorrelation coefficient at lag j of the corresponding trajectory of (4) is $\rho_j = \beta^j$.

Property 1 follows easily by studying the graph of the map $T^{(j)}_{\beta,[a,b]}$, i.e., the map $T_{\beta,[a,b]}$ composed with itself j times. Property 2 can be shown by using symbolic dynamics [see, e.g., Devaney (1989)]. Property 3 is well known from ergodic theory [see, e.g., Lasota and McKay (1985)]. Property 4 is an immediate consequence of property 3 and the ergodic theorem, stating that time averages equal space averages. Property 5 has been proven by Sakai and Tokumaru (1980). Notice that, according to properties 1–4, both from a topological and a measure theoretic point of view, the dynamics of $T_{\beta,[a,b]}$ and $T_{\beta',[a,b]}$, $\beta \ne \beta'$, are equivalent.[4] However, property 5 shows that the sample autocorrelation coefficients depend upon the parameter β. Figure 1 shows the graphs of $T_{\beta,[0,1]}$, for $\beta = -0.7$, $\beta = 0$, and $\beta = 0.7$. From a graphical analysis, it should be intuitively clear that a chaotic trajectory generated by the tent map with $\beta = 0.7$ must have a positive first-order autocorrelation, whereas for a chaotic trajectory of (4) with $\beta = -0.7$, the first-order autocorrelation coefficient must be negative.

The following lemma is used several times throughout the paper to construct chaotic CEE.

LEMMA 1. *Let real numbers* a, b, γ, *and* δ *be given such that* $a < b, \gamma \ne 0$, *and* $\delta \in (-1, 1)$. *Furthermore, define* $\alpha = (a+b)/2$. *Then there exist real numbers*

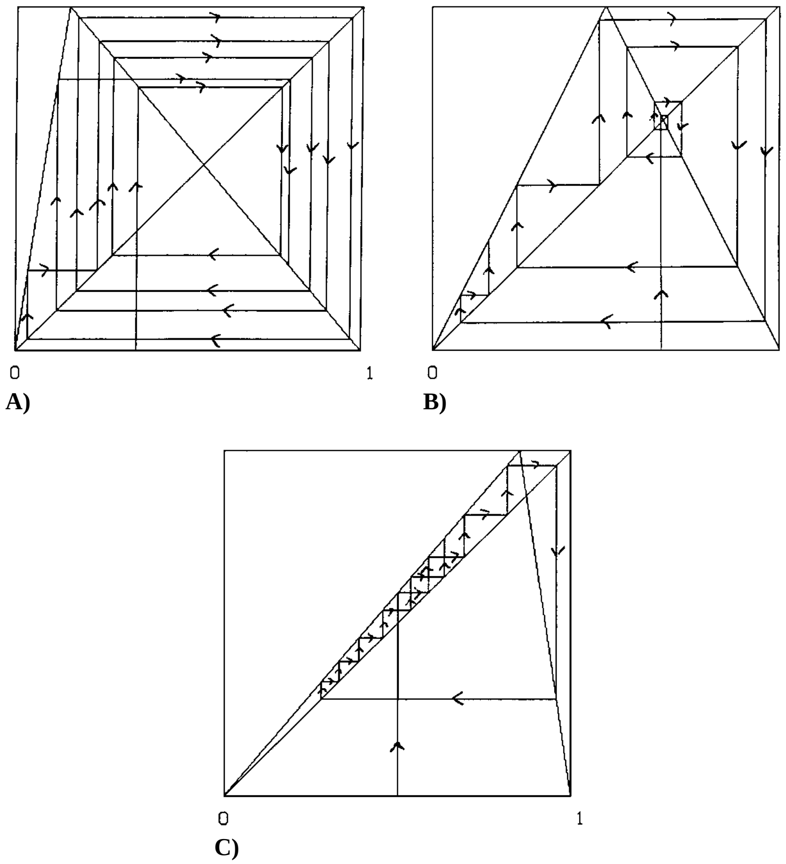

FIGURE 1. Chaotic fluctuations for asymmetric tent maps: (A) $\beta = -0.7$, (B) $\beta = 0$ (no autocorrelations), and (C) $\beta = 0.7$.

A, B, C, D with $A > 0$ and $C > 0$ such that the continuous and piecewise linear map $G : \mathbf{R} \to \mathbf{R}$ defined by

$$G(x) = \begin{cases} Ax - B & \text{if} \quad x \leq (B + D)/(A + C) \\ -Cx + D & \text{if} \quad x \geq (B + D)/(A + C) \end{cases}$$

satisfies $G(\alpha + \gamma(x - \alpha)) = T_{\delta,[a,b]}(x)$ for all $x \in [a, b]$.

Proof. First, assume that $\gamma > 0$. In that case we choose

$$A = \frac{2}{\gamma(1 + \delta)}, \qquad C = \frac{2}{\gamma(1 - \delta)},$$

$$B = \frac{2}{\gamma(1 + \delta)}[\alpha + \gamma(a - \alpha)] - a,$$

$$D = \frac{2}{\gamma(1 - \delta)}[\alpha + \gamma(b - \alpha)] + a.$$

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With these definitions, verification that $\alpha + \gamma(a - \alpha) < (B + D)/(A + C)$ and $\alpha + \gamma(b - \alpha) > (B + D)/(A + C)$ is straightforward. Therefore, we have $G(\alpha + \gamma(a - \alpha)) = A[\alpha + \gamma(a - \alpha)] - B = a$ and $G(\alpha + \gamma(b - \alpha)) = -C[\alpha + \gamma(b - \alpha)] + D = a$. Finally, $(d/dx)G(\alpha + \gamma(x - \alpha))|_{x=a} = A\gamma = 2/(1 + \delta)$ and $(d/dx)G(\alpha+\gamma(x-\alpha))|_{x=b} = -C_\gamma = -2/(1-\delta)$. Thus, we have shown that both the values and the slopes of the function $G(\alpha + \gamma(x - \alpha))$ at the points $x = a$ and $x = b$ coincide with the corresponding values and slopes of the function $T_{\delta,[a,b]}$. Because both functions are piecewise linear, it follows that the function coincide for all $x \in [a, b]$.

In the case in which $\gamma < 0$, a completely analogous argument can be used by choosing

$$A = -\frac{2}{\gamma(1-\delta)}, \qquad C = -\frac{2}{\gamma(1+\delta)},$$

$$B = -\frac{2}{\gamma(1-\delta)}[\alpha + \gamma(b - \alpha)] - a,$$

$$D = -\frac{2}{\gamma(1+\delta)}[\alpha + \gamma(a - \alpha)] + a.$$

This completes the proof of the lemma. ■

As an immediate consequence of this lemma, we obtain the following result.

THEOREM 5. *Let real numbers α and β be given such that $\beta \in (-1, 1)$ and $\beta \neq 0$. There exists a continuous and piecewise linear function F such that* (3) *has uncountably many chaotic CEE* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$.

Proof. Choose any numbers a and b such that $a < b$ and $(a+b)/2 = \alpha$. Applying Lemma 1 with $\gamma = \delta = \beta$, we see that there exists a piecewise linear function F such that $F_{\alpha,\beta}(p) = F(\alpha + \beta(p - \alpha)) = T_{\beta,[a,b]}(p)$ holds for all $p \in [a, b]$. The implied actual law of motion (3) coincides therefore with the tent map dynamics (4). From properties 1–5 of the tent map dynamics, it follows that, for Lebesgue almost all initial states $p_0 \in [a, b]$, the actual price sequence is chaotic, its sample average coincides with $(a+b)/2 = \alpha$, and its autocorrelation coefficients are given by $\rho_j = \beta^j$. Thus, all conditions of Definition 1 are satisfied and the theorem is proved. ■

In contrast to a steady-state or a two-cycle CEE, a chaotic CEE is not a REE, not even in the long run. The existence of chaotic CEE is caused by the similarity of certain chaotic processes to stochastic processes. Because the basic rationality assumption of the CEE concept implies that agents using linear statistical techniques cannot distinguish between these processes, there is no way they could ever detect that they are constantly making forecasting mistakes. Those mistakes are self-fulfilling and fully consistent with their own model of the world [see Grandmont (1994)]. The stability definition that we introduced for the simple steady-state and the two-cycle CEE cannot be generalized directly to the case of chaotic CEE. For example, in the case of chaotic dynamics on a compact set A, the set of unstable periodic points is typically dense in A, so that in general there does not exist an open set of initial states exhibiting chaotic dynamics. For chaotic CEE, we therefore present a slightly different stability notion, which we call observability. The definition of observability uses the concept of a chaotic attractor. For the sake of completeness, we first define chaotic attractors and then observability of chaotic CEE.

In the literature, there exist several definitions of a (chaotic or strange) attractor. See Milnor (1985) for a general discussion of the concept of an attractor, and Guckenheimer and Holmes (1983, pp. 256–259) or Palis and Takens (1993, pp. 138–148) for a discussion of strange or chaotic attractors. We use the following definition.

DEFINITION 3. *Let* $f : X \mapsto X$ *be a continuous mapping where* $X \subseteq \mathbf{R}^k$. *A non-empty and compact set* $A \subseteq X$ *is called an* attractor *of the dynamical system* $x_{t+1} = f(x_t)$ *if the set A is* invariant *under* f, *that is,* $f(A) \subseteq A$, *and if every open neighborhood* $U \subset \mathbf{R}^k$ *of A contains a set B with positive Lebesgue measure such that* $B \cap A = \emptyset$ *and such that*

(i) *for all initial states* $p_0 \in B$, *it holds that* $\lim_{t\to\infty} \inf\{\|p_t - p\| \,|\, p \in A\} = 0$ *and*
(ii) *there exists an initial state* $p_0 \in B$ *such that the orbit* $(p_t)_{t=0}^{\infty}$ *is dense in A.*

An attractor A is called a chaotic *attractor if, in addition,*

(iii) *for any pair of different initial states* p_0 *and* q_0 *in B with corresponding trajectories* $(p_t)_{t=0}^{\infty}$ *and* $(q_t)_{t=0}^{\infty}$, *it holds that* $\liminf_{t\to\infty} \|p_t - q_t\| = 0$ *and* $\limsup_{t\to\infty} \|p_t - q_t\| > 0$.

The invariance property implies that, for an initial state on the attractor A, the entire orbit is contained in the attractor. Property (i) means that every neighborhood of the attractor contains a large set (in the sense that it has positive Lebesgue measure) of initial states converging to the attractor. The existence of a dense trajectory as specified in (ii) is also called topological transitivity and ensures that the attractor is indecomposable. Simple examples of attractors satisfying properties (i) and (ii) are a stable steady state, a stable periodic orbit, or a quasiperiodic attractor. The additional condition (iii) is the key feature of a chaotic attractor and can be interpreted as sensitive dependence on initial conditions for a large set of initial states. We are now ready to state the definition of observability of chaotic CEE.

DEFINITION 4. *Let* $A \subset \mathbf{R}$ *be a non-empty and compact set. The triple* $\{A; \alpha, \beta\}$ *is called an* observable *chaotic CEE if A is a chaotic attractor for the implied actual law of motion* (3) *and, if for all initial states* $p_0 \in B$ (*where B has the same meaning as in Definition* 3), *the triple* $\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$ *is a CEE.*

An observable chaotic CEE $\{A; \alpha, \beta\}$ thus occurs with positive probability with respect to the set of initial states, given that agents share the common AR(1) belief with parameters α and β.

In Sections 3 and 4, we have shown that for the simple class of models (1), there exist steady-state CEE, two-cycle CEE, and chaotic CEE. We believe that

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these are the only possible CEE for the general class of dynamic models (1), as long as we restrict ourselves to AR(1) beliefs. In particular, it is quite clear that, given an AR(1) belief, there cannot exist a periodic CEE with period different from 1 or 2. This is so because such a periodic price sequence cannot have sample autocorrelation coefficients that are identical to those of an AR(1) process. It is worth mentioning, however, that periodic CEE with periods larger than two may be obtained in the case of an AR(2) belief.

5. SAMPLE AUTOCORRELATION (SAC) LEARNING

The definition of a CEE involves a fixed AR(1) belief described by the parameters α and β. The agents are supposed to adhere to this belief over the entire time horizon and the consistency of the implied actual dynamics with the belief can be verified only if the entire price sequence is known. In the present section, we consider the more flexible situation in which agents change their forecasting function within the class of AR(1) beliefs, and update their belief parameters α_t and β_t as additional observations become available. This leads to a natural learning scheme that is based on sample average and sample autocorrelation coefficients and that fits our framework of CEE.

For any finite set of observations $\{p_0, p_1, \ldots, p_t\}$, the finite sample average is given by

$$\alpha_t = \frac{1}{t+1}\sum_{i=0}^{t} p_i, \qquad t \geq 1, \tag{5}$$

and the finite-sample first-order autocorrelation coefficient is given by [see Box et al. (1994)]

$$\beta_t = \frac{\sum_{i=0}^{t-1}(p_i - \alpha_t)(p_{i+1} - \alpha_t)}{\sum_{i=0}^{t}(p_i - \alpha_t)^2}, \qquad t \geq 1. \tag{6}$$

When, in each period, the belief parameters are updated according to their sample average and sample first-order autocorrelation, the (temporary) law of motion (1) becomes

$$p_{t+1} = F_{\alpha_t,\beta_t}(p_t) = F(\alpha_t + \beta_t(p_t - \alpha_t)), \qquad t \geq 0. \tag{7}$$

We call the dynamical system (5)–(7) the actual dynamics with *sample autocorrelation learning* (SAC learning). The SAC-learning process is related to, but not identical to, OLS learning.[5] The initial state for the system (5)–(7) can be any triple (p_0, α_0, β_0) with $\beta_0 \in [-1, 1]$. It is straightforward to check that, independently of the choice of these initial values, it always holds that $\beta_1 = -1/2$ and that the first-order sample autocorrelation $\beta_t \in [-1, 1]$ for all $t \geq 2$.

The SAC-learning dynamics (5)–(7) is a high-dimensional, nonlinear, and non-autonomous system.[6] Few results concerning the stability of such systems are available. Therefore, we investigate the stability of the learning dynamics mainly numerically.[7] In the application of the cobweb model in Section 6, we support our numerical findings with graphical analysis.

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We are now ready to introduce a second stability notion for the long-run outcomes of simple CEE, which we call learnability. In the definition of local stability (Section 3, Definition 2) the belief parameters α and β were fixed over time. Learnability is an extension of this stability notion to the SAC-learning dynamics, that is, to the case in which the belief parameters are continually updated. Thus, learnability encompasses robustness with respect to perturbations of the state variable as well as the belief parameters.

DEFINITION 5.

(i) *A steady-state CEE* $\{(p^*); \beta\}$ *is* learnable, *if there exists an open neighborhood* $U \subset \mathbf{R}^3$ *of* (p^*, p^*, β) *such that, for all initial states* $(p_0, \alpha_0, \beta_0) \in U$, *the corresponding unique trajectory of the SAC-learning dynamics* (5)–(7) *satisfies* $\lim_{t\to\infty} p_t = \lim_{t\to\infty} \alpha_t = p^*$.

(ii) *A two-cycle CEE* $\{(p_1^*, p_2^*)\}$ *is called* learnable, *if there exists an open neighborhood* $U \subset \mathbf{R}^3$ *of* $\{(p_i^*, \alpha, -1) \mid i = 1, 2\}$ *with* $\alpha = (p_1^* + p_2^*)/2$ *such that, for all initial states* $(p_0, \alpha_0, \beta_0) \in U$, *the corresponding unique trajectory of the SAC-learning dynamics* (5)–(7) *satisfies* $\lim_{t\to\infty} p_{2t} = p_1^*$, $\lim_{t\to\infty} p_{2t+1} = p_2^*$ (*or vice versa*), $\lim_{t\to\infty} \alpha_t = (p_1^* + p_2^*)/2$, *and* $\lim_{t\to\infty} \beta_t = -1$.

The definition says that a steady-state or a two-cycle CEE is learnable if there exists an open set of initial states p_0 and initial belief parameters α_0 and β_0, for which the SAC-learning dynamics converge to that steady-state or two-cycle. Notice that the definition of learnability of a steady-state CEE does not require that the sample autocorrelation coefficient β_t converge; this is in accordance with the definition of a steady-state CEE for which the coefficient β is arbitrary. In numerical simulations, we typically observed that, if the SAC-learning dynamics converge to a steady state, the variable β_t also converges but the limit depends on the initial state.

For the same reasons already mentioned in the context of stability, a formal definition of learnability in the case of chaotic CEE is more complicated.

DEFINITION 6. *Let* $A \subset \mathbf{R}$ *be a compact set. The triple* $\{A; \alpha, \beta\}$ *is called a* learnable *chaotic CEE if A is a chaotic attractor of the implied actual law of motion* (3) *and if, for every open neighborhood* $U \subset \mathbf{R}^3$ *of* $\{(p, \alpha, \beta) | p \in A\}$, *there exists a set* $B \subset U$ *with positive Lebesgue measure such that, for all initial states* $(p_0, \alpha_0, \beta_0) \in B$, *the corresponding unique trajectory of the SAC-learning dynamics* (5)–(7) *satisfies*

(i) $\lim_{t\to\infty} \alpha_t = \alpha$ *and* $\lim_{t\to\infty} \beta_t = \beta$,
(ii) $\lim_{t\to\infty} \inf\{|p_t - p| | p \in A\} = 0$,
(iii) $(p_t)_{t=0}^{\infty}$ *is aperiodic and dense in A.*

Stated informally, a chaotic CEE is learnable if, in the SAC-learning process, the belief parameters α_t and β_t converge to constants and the price p_t converges to a chaotic attractor whenever the initial belief parameters and the initial state are chosen from a certain set with positive measure. At first sight, it may seem special to have a dynamic system in which certain variables do converge to constants, whereas other variables do not converge to a constant. We emphasize, however, that in the SAC-learning dynamics, the role of the state variable p_t is quite different from the role of the belief parameters α_t and β_t, both mathematically and economically. In particular, if the price p_t remains bounded, then both the changes in the average α_t and the changes in the first-order sample autocorrelation β_t tend to zero as time t goes to infinity. This implies that, in the case of bounded prices, the belief parameters (α_t, β_t) are slow variables in the SAC-learning dynamics. On the other hand, at date t the state variable p_t is generated by the temporary law of motion F_{α_t,β_t} in (7). After some transient phase, the SAC-learning process thus may be viewed as a price-generating system with a slowly changing law of motion. A CEE arises when the belief parameters converge to constants, that is, when the temporary law of motion F_{α_t,β_t} converges to a limiting actual law of motion $F_{\alpha,\beta}$. This limiting law of motion $F_{\alpha,\beta}$ remains unknown to agents and it does *not* coincide with the limiting perceived law of motion and its corresponding linear forecasting rule $p^e_{t+1} = \alpha + \beta(p_t - \alpha)$. We see that there are at least three possible outcomes of the learning process with converging belief parameters, namely that the corresponding limiting law of motion $F_{\alpha,\beta}$ has a stable steady state, a stable two-cycle, or a chaotic attractor. In the case of a steady state or a two-cycle, the limiting perceived law of motion and its corresponding linear forecasting rule perfectly predict the steady state or the two-cycle, respectively. In the case of a chaotic CEE, the limiting linear forecasting rule for the chaotic price series is correct in terms of sample average and sample autocorrelations.

At this point, a few remarks seem to be in order concerning the (in)stability of the related OLS-learning scheme in a deterministic expectations feedback context. First, in most of the related literature on instability of OLS learning [e.g., Grandmont and Laroque (1991), Grandmont (1994), Bullard (1994), Chatterji and Chattopadhyay (1996), and Schönhofer (1996)], the average α_t is excluded from the learning dynamics. It seems that this is mainly done for analytical tractability. In particular, by excluding the sample average α_t from the updating scheme, both for OLS- and SAC-learning, it is easy to write down a set of three autonomous difference equations that describe the learning dynamics (cf. note 6), and the analysis would be simplified considerably. However, we believe that, in real markets, the average of the state variables may be at least as important for expectation formation as the autocorrelations or other regression coefficients. Therefore, at the expense of analytical tractability, we have deliberately chosen to include α_t in the SAC-learning dynamics. As a consequence, we have to investigate the learning dynamics mainly by numerical tools. Using the graphs of the temporary law-of-motion mappings F_{α_t,β_t}, however, it is possible to obtain some theoretical insight into the SAC-learning dynamics, especially concerning possible long-run outcomes.

Second, the stability or instability of the OLS-learning process seems to be related to the range of allowable β-values, or, as Grandmont (1994) puts it, to the range of real trends that traders are willing to extrapolate out of past deviations from equilibrium. Stated differently, the (in)stability of OLS-learning depends upon the range of the projection facility, that is, the interval $[-\mu_1, \mu_2]$ of allowable β_t-values

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[see Marcet and Sargent (1989)]. In this respect, there is an important difference between OLS and SAC learning. For SAC learning, $\beta_t \in [-1, 1]$ holds for all t because, for any time series, the (first-order) sample autocorrelation is between -1 and 1. In contrast, for OLS learning, β_t may be outside the interval $[-1, 1]$. Grandmont and Laroque (1991) and Grandmont (1994) show that local instability of a steady state may arise when the range of trends that traders are willing to extrapolate is sufficiently large. Moreover, in the global OLS-learning dynamics, the periodic learning equilibria detected by Bullard (1994) and the chaotic learning equilibria found by Schönhofer (1996) are characterized by switching between a stable phase with $|\beta_t| < 1$ and an unstable phase with $|\beta_t| > 1$. Along these learning equilibria, prices are unbounded and agents keep running OLS regression on an unbounded price series. In all cases that we consider, prices remain bounded. In fact, SAC learning may be seen as OLS learning with the projection facility $[-1, 1]$ and thus may be interpreted in the sense that traders believe in a stationary (or at least nonexplosive) AR(1) process. In particular, our focus is on unstable cases in which the bounded prices do not converge to a steady state but to a cycle or a strange attractor, whereas the belief parameters do converge to constant values. Such an outcome of the SAC-learning dynamics represents a situation in which agents believe that prices follow a stationary AR(1) process, whereas prices fluctuate endogenously on a bounded attractor.

The numerical simulations presented in Section 6 indicate that all three types of long-run CEE behavior (i.e., steady states, two-cycles, and chaos) can be learnable. Multiple learnable CEE may even coexist in simple examples.

6. COBWEB MODEL

In this section we investigate existence and learnability of CEE for the nonlinear cobweb model. The cobweb model has played an important role in economic dynamics, in particular with respect to the role of expectation formation. For example, Ezekiel (1938) investigated the cobweb model with naive expectations, Nerlove (1958) with adaptive expectations, and Muth (1961) used the model for introducing rational expectations. The cobweb model describes market equilibrium prices for a single commodity that is produced with a fixed production lag of one period. Market equilibrium price p_t is determined by

$$D(p_t) = S\left(p_t^e\right), \tag{8}$$

where D is the demand curve, S the supply curve, and p_t^e the producers' price expectation formed in period $t-1$. As usual, we assume that the demand function D is strictly decreasing and, hence, invertible. Denoting by D^{-1} the inverse demand function and rewriting (8) as $p_t = D^{-1}[S(p_t^e)]$, one sees that the cobweb model is of type (1).

6.1. Monotonic Case

In this subsection we consider the case in which the supply curve is monotonically increasing. When demand is decreasing and supply is increasing, there exists a

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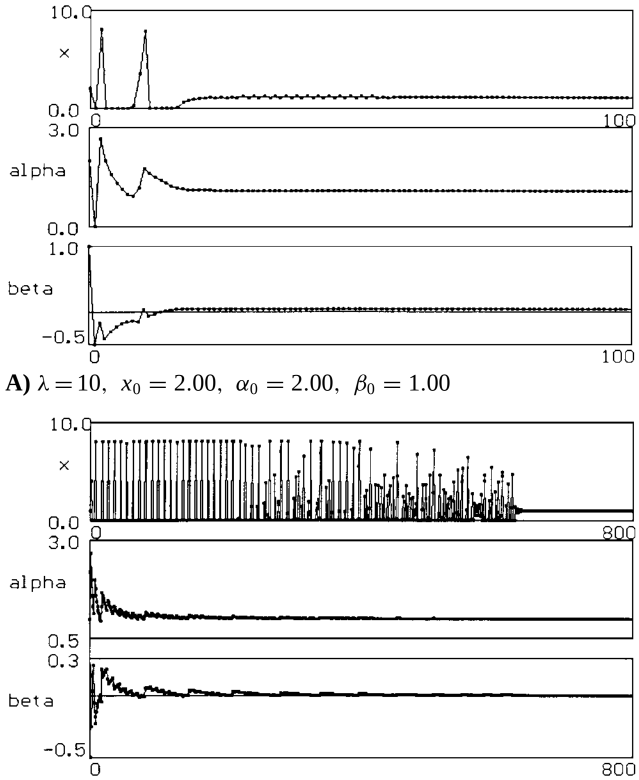
unique market-clearing price p^* defined by the intersection of the demand and supply curves. Because in this case the map $F(p) = D^{-1}[S(p)]$ is decreasing, it follows from Theorems 1 and 3 in Section 3, that the only bounded CEE are steady-state CEE converging to p^* and that the corresponding belief parameters must equal $\alpha = p^*$ and $\beta = 0$. Hence, in the cobweb model with monotonic demand and supply, the only AR(1) belief that is not systematically wrong is $p_t = p^* + \epsilon_t$. All other AR(1) beliefs are inconsistent with the actual price sequences they generate in the sense that the sample average and sample autocorrelation coefficients of the belief process are different from those of the actual realizations. Such beliefs would be abandoned even by boundedly rational agents observing only the sample average and sample autocorrelations. In other words, in the cobweb model with monotonic demand and supply, the only bonded CEE are price series converging to the unique REE.

Let us now investigate the SAC-learning process in the nonlinear cobweb model to see whether the unique steady-state p^* (i.e., the REE) is learnable. We would like to emphasize that this is a nontrivial problem because simple expectation schemes may lead to complicated, chaotic price fluctuations even if both demand and supply are monotonic functions. For example, Hommes (1991, 1994, 1998) shows that, in the case of adaptive expectations[8] of the form $p_t^e = (1-w)\,p_{t-1}^e + wp_{t-1}$, and in the case of simple linear AR(2) or AR(3) expectation rules, chaotic price fluctuations can arise even when demand and supply are monotonic.

In all of our simulations of the SAC-learning process in the cobweb model with decreasing demand and increasing supply, all bounded price series converge to the unique steady-state price p^*. In most cases, the convergence is so fast that the prices get close to the steady-state p^* within five or even fewer iterations. Only in cases in which the market is very unstable, i.e., $S'(p^*)/D'(p^*) \ll -1$, convergence to the steady state may be slow and may occur only after a long, possibly erratic transient. Figure 2 shows two typical time series of prices and updated belief parameters in the SAC-learning dynamics in the cobweb model with a linear demand curve $D(p) = 2 - p/4$ and a nonlinear (but monotonic) supply curve $S_\lambda(p^e) = 1 + \tanh(\lambda p^e)$. In Hommes (1994) it has been shown that, for sufficiently high values of λ, the corresponding cobweb model with adaptive expectations leads to chaotic price fluctuations. Figure 2 shows that, in the long run, in the SAC-learning dynamics such fluctuations do not occur. Even for a strongly nonlinear supply function with $\lambda = 10$ (Figure 2A), convergence to the steady state p^* is fairly fast. For $\lambda = 1000$, the supply curve is almost vertical at the steady state. In that case, convergence to the steady state still occurs but only after a long (chaotic) transient phase (see Figure 2B).

The intuition behind the stability of the SAC-learning process is as follows: When prices fluctuate around the steady state, the belief parameter α_t will not be too far from p^* after a few time periods. Moreover, as long as the first-order sample autocorrelation β_t is positive, actual prices exhibit (up and down) oscillations around the steady-state p^*, which leads to negative first-order autocorrelation and a decrease in β_t. On the other hand, if $\beta_t < 0$, actual prices exhibit a monotonic movement toward (or away from) the steady-state p^* such that the first-order



A) $\lambda = 10$, $x_0 = 2.00$, $\alpha_0 = 2.00$, $\beta_0 = 1.00$

B) $\lambda = 1000$, $x_0 = 1.00$, $\alpha_0 = 1.00$, $\beta_0 = 1.00$

Figure 2. SAC-learning dynamics in cobweb model, with nonlinear, monotonic supply curve $S_\lambda(x) = 1 + \tanh(\lambda x)$: (A) strong nonlinearity ($\lambda = 10$), (B) very strong nonlinearity ($\lambda = 1000$).

autocorrelation tends to become positive and β_t increases. As t becomes large, only small changes in β_t can arise. This suggests that β_t eventually will converge with a limit close to zero. Once β_t is sufficiently close to zero (with α_t close to p^*), such that $|\beta_t S'(p^*)/D'(p^*)| \leq 1$, prices rapidly converge to p^*.[9]

In summary, we conclude that, in the nonlinear cobweb model with decreasing demand and increasing supply, the unique steady-state p^* is learnable. In the case of a bounded supply curve, p^* seems to be globally stable in the SAC-learning dynamics in all of our simulations (usually with fast convergence). For the nonlinear, monotonic cobweb model, we thus arrive at the same conclusion as Bray and Savin (1986), who studied a linear cobweb model with OLS learning where agents regress prices on an exogenous random variable. In our CEE approach, agents employ an average as well as the first-order autocorrelation in predicting prices. Our numerical results show that the unique steady-state REE is stable under learning. If demand is decreasing and supply is increasing, the constant belief $p_t = p^* + \epsilon_t$ is the only AR(1) belief for which expectational errors are not systematically wrong in terms of average and sample autocorrelation. Thus, in a cobweb world with monotonic demand and supply, agents are able to learn the unique REE even without any knowledge of the underlying market equilibrium equations, simply by observing the sample average and sample autocorrelations.

6.2. Nonmonotonic Example

Next, we construct an example of a cobweb model with chaotic CEE. From the results in Section 6.1, it follows that this requires a nonmonotonic supply function (we continue to assume that the demand function is invertible). We first construct an example in which we can show by analytical means that chaotic CEE exist and then we simulate the SAC-learning process in this example. We consider this nonmonotonic cobweb model mainly as a didactical toy model to illustrate the intuition behind CEE and to investigate convergence to different types of CEE in the SAC-learning process.

THEOREM 6. *Let α and β be real numbers such that $\alpha > 0$, $\beta \in (-1, 1)$, $\beta \neq 0$. There exists a monotonic demand function D:$[0, \infty) \mapsto [0, \infty)$ and a piecewise linear supply function S:$[0, \infty) \mapsto [0, \infty)$ such that the cobweb model* (8) *has uncountably many chaotic CEE*$\{(p_t)_{t=0}^{\infty}; \alpha, \beta\}$.

To prove the theorem, we use the following simple lemma.

LEMMA 2. *Let a, b, β, x_0 be real numbers such that $a < b$, $\beta \in (-1, 1)$, and $x_0 \in [a, b]$. Consider the unique trajectory $(x_t)_{t=0}^{\infty}$ satisfying the tent map dynamics $x_{t+1} = T_{\beta,[a,b]}(x_t)$ with initial condition x_0 and the unique trajectory $(y_t)_{t=0}^{\infty}$ of the upside-down tent map dynamics $y_{t+1} = a + b - T_{-\beta,[a,b]}(y_t)$ with initial condition $y_0 = a + b - x_0$. Then it holds that the sample averages $\bar{x}$ and $\bar{y}$ satisfy $\bar{y} = a + b - \bar{x}$ and the sample autocorrelation coefficients of the two trajectories coincide.*

Proof. We claim that $y_t = a + b - x_t$ holds for all t. For $t = 0$, this follows directly from the assumptions. Now assume that it holds for some t. Then, we have

$$y_{t+1} = a + b - T_{-\beta,[a,b]}(y_t) = a + b - T_{-\beta,[a,b]}(a + b - x_t)$$
$$= a + b - T_{\beta,[a,b]}(x_t) = a + b - x_{t+1}.$$

The claim follows therefore by induction. It is obvious that the property $y_t = a+b-x_t$ implies $\bar{y} = a + b - \bar{x}$. Therefore, we obtain $(y_t - \bar{y})(y_{t+j} - \bar{y}) = (x_t - \bar{x}) \times (x_{t+j} - \bar{x})$, which apparently implies that the autocorrelation coefficients of $(x_t)_{t=0}^{\infty}$ and $(y_t)_{t=0}^{\infty}$ coincide. This concludes the proof of the lemma. ■

Proof of Theorem 6. Applying Lemma 1 in Section 4 with $a = 0, b = 2\alpha, \gamma = \beta$, and $\delta = -\beta$, it follows that there exists a piecewise linear map $G : \mathbf{R} \mapsto \mathbf{R}$ such that

$$G(\alpha + \beta(x - \alpha)) = T_{-\beta,[0,b]}(x) \qquad (9)$$

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holds for all $x \in [0, b]$. Now, define $p_l = \min\{\alpha + \beta(x - \alpha) \mid x \in [0, b]\}$ and $p_u = \max\{\alpha + \beta(x - \alpha) \mid x \in [0, b]\}$. Note that $0 < p_l < p_u < b$ and that (9) implies that $G(p_l) = G(p_u) = 0$. Because G is strictly increasing to the right of its kink, we must have $G(0) < 0$. We specify

$$D(p) = \begin{cases} b - G(0) - p & \text{if} \quad 0 \le p \le b - G(0), \\ 0 & \text{if} \quad p \ge b - G(0), \end{cases}$$

and

$$S(p) = \begin{cases} G(p) - G(0) & \text{if} \quad 0 \le p \le p_u, \\ \epsilon(p - p_u) - G(0) & \text{if} \quad p \ge p_u, \end{cases}$$

where ϵ is an arbitrary nonnegative number. Note that D is strictly decreasing whenever it is strictly positive and that S is piecewise linear with three linear segments. If $p_{t-1} \in [0, b]$ then it follows that $\alpha + \beta(p_{t-1} - \alpha) \in [p_l, p_u]$ and, therefore,

$$S(\alpha + \beta(p_{t-1} - \alpha)) = G(\alpha + \beta(p_{t-1} - \alpha)) - G(0) \in [-G(0), b - G(0)].$$

Thus, defining $F = D^{-1} \circ S$, we obtain

$$p_t = F(\alpha + \beta(p_{t-1} - \alpha)) = b - G(\alpha + \beta(p_{t-1} - \alpha)) = b - T_{-\beta,[0,b]}(p_{t-1}).$$

The actual dynamics under the belief parameters α and β therefore are given by an upside-down tent map. From Lemma 2 and the properties of the tent map dynamics, it follows that, for Lebesgue almost all initial prices $p_0 \in [0, b]$, the sample average of actual prices is α and the sample autocorrelation coefficient at lag j is given by β^j. For any of those initial prices, therefore, we have a chaotic CEE and the proof is complete. ■

Example

Choosing belief parameters $\alpha = 2$ and $\beta = 4/5$ and following the construction used in the proof of Theorem 6, one obtains

$$D(p) = \begin{cases} 9 - p & \text{if} \quad 0 \le p \le 9, \\ 0 & \text{if} \quad p \ge 9, \end{cases}$$

and

$$S(p) = \begin{cases} (25/2)p & \text{if} \quad 0 \le p \le 18/25, \\ 10 - (25/18)p & \text{if} \quad 18/25 \le p \le 18/5, \\ 5 + \epsilon[p - (18/5)] & \text{if} \quad p \ge 18/5. \end{cases}$$

The function $p \mapsto F(p) = D^{-1}(S(p))$ is given by

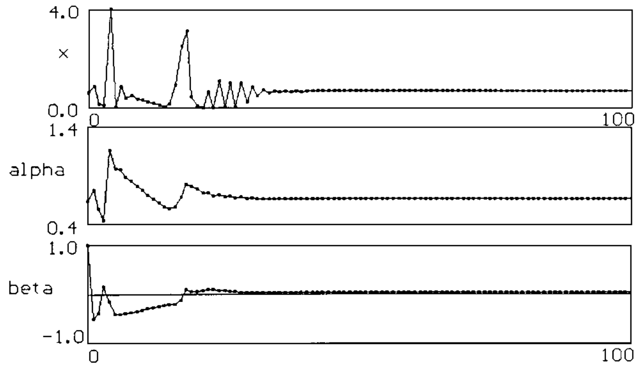
$$F(p) = \begin{cases} 9 - (25/2)p & \text{if} \quad 0 \le p \le 18/25, \\ (25/18)p - 1 & \text{if} \quad 18/25 \le p \le 18/5, \\ 4 - \epsilon[p - (18/5)] & \text{if} \quad 18/5 \le p \le 18/5 + 4/\epsilon, \\ 0 & \text{if} \quad p \ge 18/5 + 4/\epsilon. \end{cases}$$

By construction, this example has uncountable many chaotic CEE with belief parameters $\alpha = 2$ and $\beta = 4/5$. Let us explore the existence of other, simple CEE first. It is straightforward to check that the map F has three fixed points at $p_1^* = 2/3$, $p_2^* = 18/7$, and $p_3^* = (20 + 18\epsilon)/(5 + 5\epsilon)$. Applying Theorem 2, we conclude that there exist three steady-state CEE corresponding to the three fixed points p_1^*, p_2^*, and p_3^*, and three two-cycle CEE corresponding to the three pairs of different fixed points $\{p_1^*, p_2^*\}$, $\{p_1^*, p_3^*\}$, and $\{p_2^*, p_3^*\}$. We have $F'(p_1^*) = -25/2$, $F'(p_2^*) = 25/18$, and $F'(p_3^*) = -\epsilon$. According to Theorem 4, the steady-state CEE $\{(p_1^*); \beta\}$ and $\{(p_3^*); \beta\}$ are locally stable if and only if the belief parameter β is equal to zero, whereas the steady-state CEE $\{(p_2^*); \beta\}$ is locally stable if $|\beta| < 18/25$. The two-cycle CEE $\{(p_1^*, p_2^*)\}$, on the other hand, is unstable because $|F'(p_1^*)F'(p_2^*)| > 1$, whereas the two-cycle CEE $\{(p_1^*, p_3^*)\}$ and $\{(p_2^*, p_3^*)\}$ are locally stable if ϵ is sufficiently small. Thus, in this example, there exist (stable) steady-state CEE, (stable) two-cycle CEE, and infinitely many chaotic CEE at the same time.

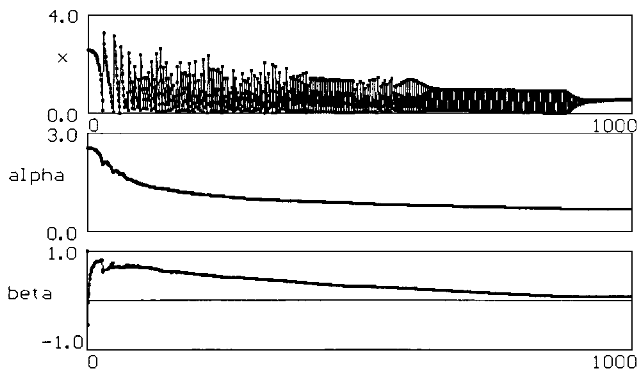
To which of these CEE will the SAC-learning process converge? Figure 3 illustrates the typical behavior observed in simulations of the SAC-learning dynamics for small ϵ. The results can be summarized as follows:

- For initial states with $p_0 = \alpha_0 \approx p_1^*$, convergence to the steady-state p_1^* occurs (Figure 3A).
- For initial states with $p_0 = \alpha_0 \approx p_2^*$, convergence to the steady-state p_1^* occurs after a (possibly long) transient phase with price fluctuations similar to those of the chaotic CEE with $\alpha = 2$ and $\beta = 4/5$ (Figure 3B).
- For initial states with $p_2^* < p_0 \neq \alpha_0$ and $\beta_0 = -1$, two possibilities have been observed:
  —there is a first transient phase with prices close to the steady-state p_2^*, followed by a second transient phase with price fluctuations similar to those of the chaotic CEE with $\alpha = 2$ and $\beta = 4/5$, and finally convergence to the steady-state p_3^* (Figure 3C);
  —there is a transient phase with prices close to the steady-state p_2^* but, eventually, convergence to the two-cycle $\{(p_1^*, p_3^*)\}$ (Figure 3D).

These numerical simulations suggest that the steady-state CEE $\{(p_1^*); \beta\}$ and $\{(p_3^*); \beta\}$ as well as the two-cycle CEE $\{(p_1^*, p_3^*)\}$ are learnable. On the other hand, the steady-state CEE $\{(p_2^*); \beta\}$, the two-cycle CEE $\{(p_1^*, p_2^*)\}$ and $\{(p_2^*, p_3^*)\}$, and the chaotic CEE with $\alpha = 2$ and $\beta = 4/5$ all seem to be unstable in the learning dynamics; in none of our simulations of the SAC-learning process have we observed convergence to one of these CEE.



A) $x_0 = 0.65,\ \alpha_0 = 0.65,\ \beta_0 = 1.00$



B) $x_0 = 2.55,\ \alpha_0 = 2.55,\ \beta_0 = 1.00$

Figure 3. Typical outcomes of SAC-learning dynamics in cobweb model, with nonmonotonic supply curve, for different initial states (p_0, α_0, β_0): (A) convergence to steady-state CEE p_1^*; (B) convergence to steady-state CEE p_1^*, after long chaotic transient; (C) unstable steady-state CEE p_2^*, chaotic transient, and convergence to steady-state CEE p_3^*; (D) learning to believe in the two-cycle CEE $\{(p_1^*, p_3^*)\}$.

Figure 4 shows graphs of the implied actual law of motion mapping $F_{\alpha,\beta}$ or its second iterate $F_{\alpha,\beta}^{(2)}$ for different values of the belief parameters α and β. These graphs suggest why certain CEE are learnable whereas others are not and thus provide some intuition for the highly nonlinear, nonautonomous SAC-learning dynamics. Recall that, if $\alpha_t = \alpha$ and $\beta_t = \beta$, the next price p_{t+1} is generated by the temporary law of motion $F_{\alpha,\beta}$, that is, $p_{t+1} = F_{\alpha,\beta}(p_t)$. Because in the long run, the belief parameters α_t and β_t are slow variables, the graphs of the maps $F_{\alpha,\beta}$ are useful in understanding the price dynamics in the SAC-learning process during phases in which $(\alpha_t, \beta_t) \approx (\alpha, \beta)$.

Figure 4A shows the graph of the second iterate $F_{\alpha,\beta}^{(2)}$ for $\alpha = (p_1^* + p_3^*)/2$ and $\beta = -1$ and illustrates that the two-cycle CEE $\{(p_1^*, p_3^*)\}$ is stable. There is a (large) open set of initial states p_0 from which convergence to the two-cycle occurs. If

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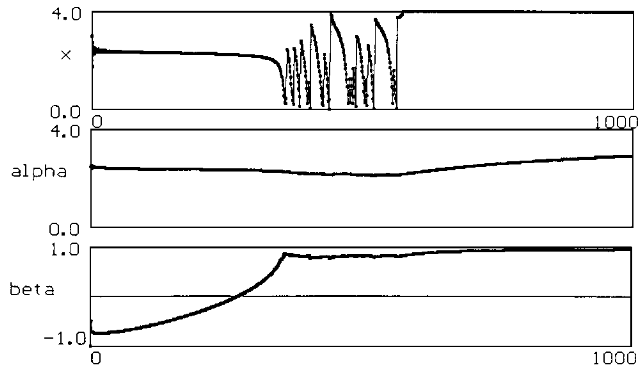
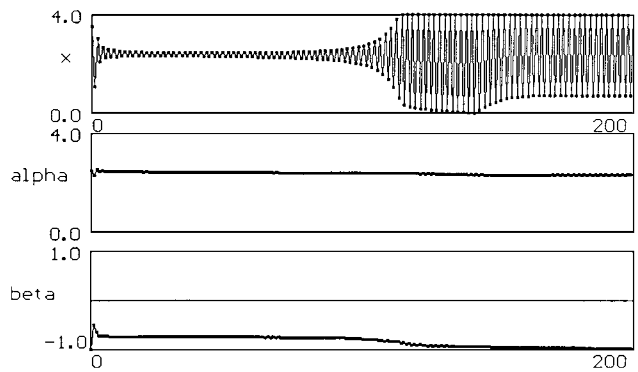


C) $x_0 = 3.00,\ \alpha_0 = 2.50,\ \beta_0 = -1.00$



D) $x_0 = 3.50,\ \alpha_0 = 2.50,\ \beta_0 = -1.00$

FIGURE 3. (Continued.)

the belief parameters α_t and β_t are different from, but close to, $\alpha = (p_1^* + p_3^*)/2$ and $\beta = -1$, respectively, then the temporary map F_{α_t,β_t} still has a stable two-cycle. This suggests that, once (α_t, β_t) is sufficiently close to $[(p_1^* + p_3^*)/2, -1]$, the SAC-learning dynamics will lock into the two-cycle $\{p_1^*, p_3^*\}$ (see also the corresponding time series in Figure 3D). Figure 4B shows the graph of the second iterate $F_{\alpha,\beta}^{(2)}$ with $\alpha = (p_2^* + p_3^*)/2$ and $\beta = -1$ or $\beta = -1/2$, respectively. Although the two-cycle CEE $\{p_2^*, p_3^*\}$ is stable according to Theorem 2, it does not seem to be learnable. In none of our simulations of the SAC-learning dynamics have we ever observed convergence to $\{p_2^*, p_3^*\}$. A possible explanation for this is that, in the SAC-learning dynamics, one always has $\beta_1 = -1/2$. Figure 4B shows that, for $\alpha = (p_2^* + p_3^*)/2$ and $\beta = -1/2$, the implied actual law of motion does not have a stable two-cycle, but a stable steady state instead. Apparently, in the SAC-learning process, β_t does not get close enough to -1 to lock into the self-fulfilling two-cycle $\{p_2^*, p_3^*\}$. We expect that in a slightly different learning scheme, in which the updating of β_t would occur on a slower timescale or where more weight would be given to the old belief parameter β_{t-1} (especially in the initial stage of the learning

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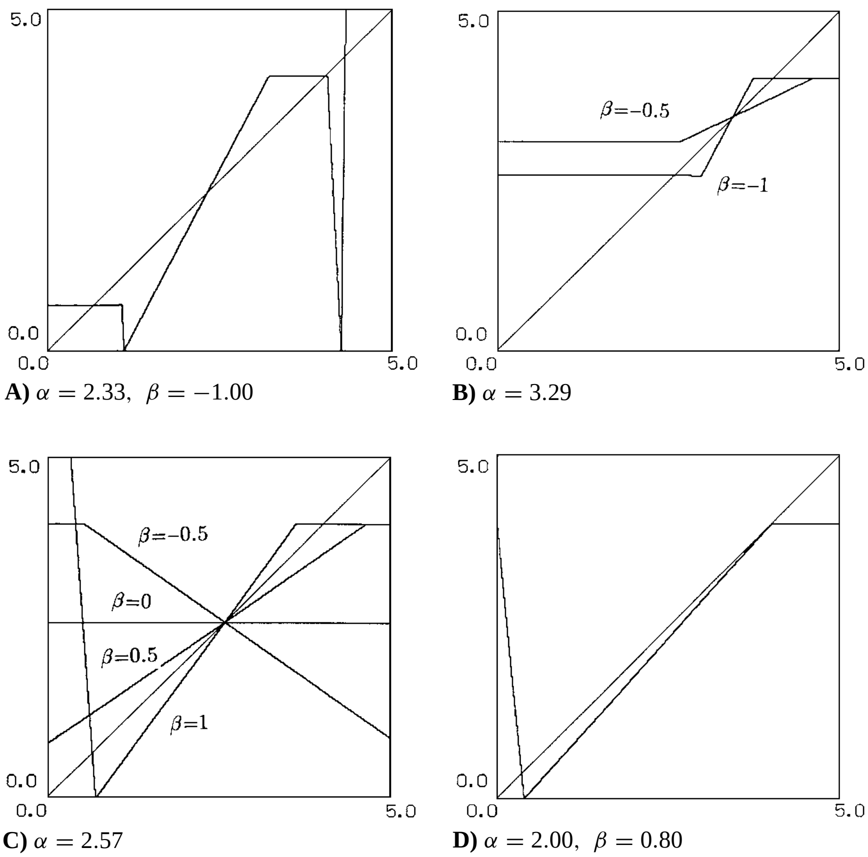


FIGURE 4. Implied actual laws of motion for nonmonotonic cobweb model, for different belief parameters α and β: (A) $\alpha = (p_1^* + p_3^*)/2$ and $\beta = -1$, stable two-cycle with large basin of attraction; (B) $\alpha = (p_2^* + p_3^*)/2$ and $\beta = -1$, two-cycle with small basin of attraction; (C) $\alpha = p_2^*$ and different values of β, $F'(p_2^*) > 1$, and the steady-state CEE is not learnable. (D) $\alpha = 2$, $\beta = 0.8$, chaotic CEE not learnable due to one-sided stable steady-state $p^* = 4$.

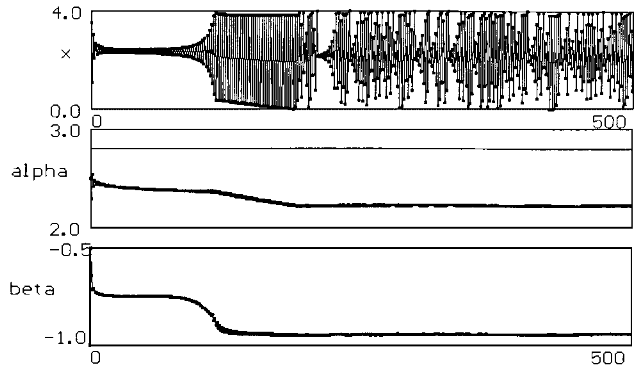
process), the first-order sample autocorrelation would gain enough momentum to move toward -1, thus triggering convergence to the two-cycle CEE $\{p_2^*, p_3^*\}$.

Figure 4C shows the graphs of $F_{\alpha,\beta}$ for $\alpha = p_2^*$ and different β-values and suggests why the steady-state p_2^* is not learnable. Suppose that $\alpha_t \approx p_2^*$ and we take an initial state p_0 close to p_2^*. If β_t is negative, up-and-down price oscillations around p_2^* would arise, the stability of which is determined by the absolute value of $\beta_t F'(p_2^*)$. Because $F'(p_2^*) > 1$, the corresponding first-order autocorrelation would become smaller than β_t so that β_t would decrease. As β_t decreases, the instability of the up-and-down price fluctuations around p_2^* becomes stronger. When β_t gets close enough to -1 and α_t is still close to p_2^*, the SAC-learning process is likely to settle down to the two-cycle $\{p_1^*, p_3^*\}$ as in Figure 3D. On the other hand, if β_t would be a small positive value, then monotonic convergence to p_2^* would arise. Because $F'(p_2^*) > 1$, the corresponding first-order autocorrelation would become larger than β_t, so that β_t would increase. As β_t increases beyond $1/F'(p_2^*) = 18/25$, the steady-state p_2^* becomes unstable and prices move away from p_2^*. They can either increase and eventually converge to the stable steady-state p_3^*, or they could decrease for some time until they enter an erratic phase in which prices fluctuate irregularly in the interval $[0, p_2^*]$, and finally escape to the stable steady-state p_3^* as in Figure 3C.

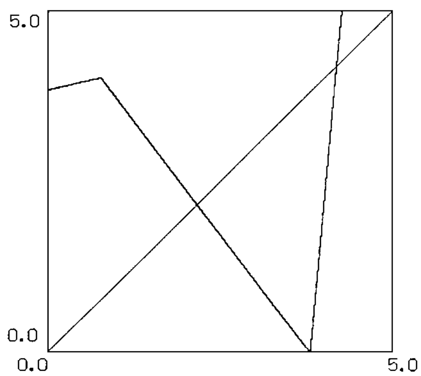
Let us finally discuss learnability of chaotic CEE. Figure 4D shows the graph of the implied actual law of motion $F_{\alpha,\beta}$ for $\alpha = 2$ and $\beta = \frac{4}{5}$. Notice that, in this case, the map $F_{\alpha,\beta}$ is exactly an upside-down asymmetric tent map on the interval $[0, p_2^*]$ and, by construction, there exists an uncountable set of chaotic CEE. The graph in Figure 4D suggests why these chaotic CEE are not learnable: The map $F_{2,4/5}$ has a one-sided stable steady-state $p^* = 4$. If, at some date t, $\alpha_t > 2$ and/or $\beta_t > \frac{4}{5}$, the corresponding temporary law of motion F_{α_t,β_t} has a stable fixed point $p^* > 4$. As a consequence, the SAC-learning dynamics may easily lock into the steady-state $p_3^* = (20 + 18\epsilon)/(5 + 5\epsilon)$.

The chaotic CEE constructed in Theorem 6 therefore are not learnable. The question arises whether agents can learn a chaotic CEE. Figure 5A illustrates that, for a somewhat larger parameter value $\epsilon = 1/4$, convergence of the SAC-learning process to a chaotic CEE can arise. For this larger ϵ-value, the two-cycle CEE $\{(p_1^*, p_3^*)\}$ is not stable anymore because $|F'(p_1^*)F'(p_3^*)| = 25/8 > 1$. In the SAC-learning dynamics, the belief paramers (α_t, β_t) seem to converge to $(2.22, -0.94)$ whereas price fluctuations remain chaotic. Notice that this chaotic CEE is different from the one that we constructed in Theorem 6. Figure 5B shows the graph of the law of motion $F_{\alpha,\beta}$ for $\alpha = 2.22$ and $\beta = -0.94$. It is a piecewise linear map with two critical points. Figure 5C shows that $F_{\alpha,\beta}^{(8)}$ (i.e., the map $F_{\alpha,\beta}$ iterated eight times) is uniformly expanding. Therefore, the results of Lasota and Yorke (1973) apply and it follows that the map $F_{\alpha,\beta}$ has a unique ergodic invariant measure that is absolutely continuous with respect to Lebesgue measure. Consequently, if $\alpha_t = 2.22$ and $\beta_t = -0.94$ were fixed over time, the sample average and sample autocorrelations would exist and would be the same for Lebesgue almost all initial states. Apparently, the SAC-learning dynamics have settled down to a learnable chaotic CEE.

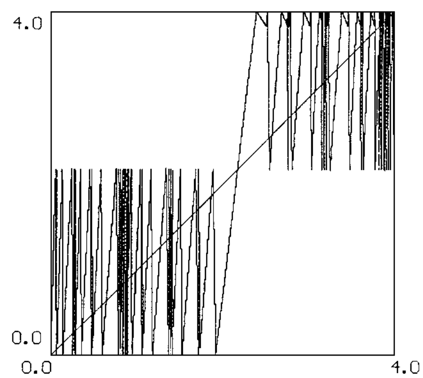
In summary, our numerical simulations as well as the graphical analysis show that steady-state, two-cycle, and chaotic CEE can be learned in the SAC-learning process. Our results suggest that, in nonlinear cobweb-type models, the learnability condition for a steady-state CEE $\{(p^*); \beta\}$ is $F'(p^*) < 1$. This condition coincides with the stability condition of Bray and Savin (1986) for the linear cobweb model under OLS learning. Our results also suggest that the learnability condition for a two-cycle CEE $\{(p_1^*, p_2^*)\}$ is $|F'(p_1^*)F'(p_2^*)| < 1$.[10] The precise conditions under which chaotic CEE are learnable remain unclear, but our numerical simulations



A) $x_0 = 3.50$ $\alpha_0 = 2.50$ $\beta_0 = -1.00$



B) $\alpha = 2.22$ $\beta = -0.94$ eps $= 0.25$



C) $\alpha = 2.22$ $\beta = -0.94$ eps $= 0.25$

FIGURE 5. Learning to believe in chaos in nonmonotonic cobweb model, with $\epsilon = 0.25$: (A) $(\alpha_t, \beta_t) \to (2.22, -0.94)$, whereas price fluctuations are chaotic; (B) implied actual law of motion for $\alpha = 2.22$ and $\beta = -0.94$; (C) eight iterations of implied actual law of motion for $\alpha = 2.22$ and $\beta = -0.94$; the map $F^8_{2.22,-0.94}$ is expanding.

suggest that learning to believe in chaos can occur with positive probability, that is, from a set of initial states of positive measure.

7. CONCLUDING REMARKS

There seems to be a growing consensus that the rational expectations hypothesis is an extremely strong rationality assumption concerning expectation formation of economic agents, which may at best hold only in the long run. As a consequence, the literature on alternative *bounded rationality* models for expectation formation and learning processes in economic modeling is growing rapidly. As Sargent (1993) emphasizes, one of the key features of this new approach is that boundedly rational agents base their expectations on time-series observations and not on (unknown) market equilibrium equations. Boundedly rational agents behave like econometricians: They learn, adapt, and update parameters in their perceived law of motion in accordance with observed realizations. We would like to make some final remarks on how the CEE concept fits into the bounded-rationality literature.

A large part of this literature [e.g., Bray (1982), Bray and Savin (1986), Marcet and Sargent (1989)] deals with *linear stochastic* models. Both the perceived law of motion and the implied actual law of motion are linear, and fluctuations are driven by stochastic exogenous shocks. The parameters of the implied actual law of motion are a (possibly nonlinear) function of the parameters of the perceived law of motion. A REE is a fixed point of the mapping from the perceived law of motion to the actual law of motion. In a REE the actual law of motion thus coincides exactly with the perceived law of motion. A similar remark applies to the nonlinear stochastic models investigated, for example, by Evans and Honkapohja (1994, 1995), and to the sunspot literature [e.g., Azariadis and Guesnerie (1986), or Woodford (1990)]. Attention generally is focused on cases in which the beliefs are self-fulfilling in the sense that the perceived law of motion exactly coincides with the implied actual law of motion.

Our approach deviates in an important aspect from the (linear) stochastic approach. Our perceived law of motion is linear, but our actual law of motion is nonlinear. In this way we attempt to model (a significant part of) economic fluctuations by *endogenous nonlinear dynamics*, in which agents form expectations in a simple way based on *linear forecasting rules*. Obviously, in general, there cannot exist parameter values for which the linear perceived law of motion and the implied nonlinear actual law of motion coincide. Hence, the perceived law of motion always will be *different* from the (unobserved) actual law of motion. However, we have shown that, given a linear perceived law of motion, there can exist a large set of actual realizations of the implied nonlinear actual law of motion such that, for the agents who only observe time series, the actual realizations are consistent with their linear stochastic perceived law of motion. Along a CEE, both sample average and sample autocorrelations of the realizations of the nonlinear actual law of motion coincide with the average and the autocorrelation of the linear stochastic perceived law of motion. The linear forecasting rules thus are self-fulfilling in terms of average and autocorrelations. In such a case, boundedly rational agents who use linear statistical techniques would have no reason to deviate from their simple linear forecasting rule. Obviously, one might impose additional consistency requirements using other observable time-series characteristics, but the sample average and sample autocorrelation should give a useful start and may be seen as important observable characteristics of first-order approximate forecasting rule of an irregularly fluctuating variable.

In an economic model with rational expectations, the starting point often is the extreme assumption that the agents have perfect knowledge about the underlying market equilibrium equations. In contrast, consistent expectations are based on another perhaps extreme assumption, namely that the agents use very simple (linear) forecasting rules. It is rather surprising that for a general class of nonlinear dynamic models, these simple linear forecasting rules need *not* be systematically wrong and can be correct in terms of average and autocorrelation.

We have restricted attention to the case in which the perceived law of motion is a linear stochastic AR(1) process. In that case, we have shown existence of three different types of CEE: steady states, two-cycles, and chaotic CEE. We also have presented numerical simulations, supported by graphical analysis, that the SAC-learning process can converge to each of these types of CEE. Multiple CEE may coexist, and the learning process can exhibit path dependence, i.e., the final outcome of the learning process may depend upon initial belief parameters.

In the case of a steady-state or a two-cycle CEE, the long-run outcome of the implied actual nonlinear law of motion coincides exactly with the linear forecasts made by the agents. These simple CEE thus are long-run REE that are triggered by a simple AR(1) belief. Moreover, the simple steady-state and two-cycle CEE are learnable in a large number of cases. In the (unstable) cobweb model with monotonic demand and supply, in all of our simulations of the SAC-learning dynamics, convergence to the steady-state CEE occurred.

In addition to the simple CEE, we also have shown the existence of complicated chaotic CEE. Chaotic CEE are not REE, not even in the long run, but chaotic equilibrium paths have exactly the same sample average and sample autocorrelation function as the linear stochastic perceived law of motion. These chaotic CEE arise because, from a linear statistical viewpoint, the chaotic realizations of the nonlinear actual law of motion are indistinguishable from the linear stochastic perceived law of motion. Along a chaotic CEE, agents do make forecasting errors, but the forecasting errors are not systematic, because the linear forecasting rules are correct in terms of sample average and autocorrelations.

Endogenous dynamics and the instability of learning also has been studied by Grandmont and Laroque (1991), Grandmont (1994), and Chatterji and Chattopadhyay (1996). These studies focus on local instability however, whereas we have focused on global instability. The global instability of learning processes also has been studied by Bullard (1994) and Schönhofer (1996). In these studies the OLS-learning process does *not* converge but, instead, learning equilibria arise in which the belief parameters exhibit periodic or chaotic fluctuations.[11] In contrast, along chaotic CEE the learning process does converge, that is, in the SAC dynamics the parameters of the perceived law of motion do converge to constants, whereas the state variable keeps fluctuating. In particular, learning to believe in chaos occurs where the parameters of the perceived law of motion converge and the corresponding limiting actual law of motion has a chaotic attractor.

A number of extensions of our CEE concept seem to be worth pursuing. First, we apply the CEE concept to other general classes of dynamic models, for example, $p_t = F(p^e_{t+1})$ and $p_t = F(p_{t-1}, p^e_t)$ [Hommes and Sorger (1998)]. The first class includes the overlapping-generations (OLG) model (without capital). The only difference with the cobweb-type models is the timing of expectation formation, but this different timing leads to a number of different results. The steady-state and the two-cycle CEE seem to be the most likely outcomes of the SAC-learning dynamics in the OLG model (without capital). Even in the presence of a strong income effect, and chaotic perfect-foresight cycles as in Grandmont (1985), convergence of the

SAC-learning process to a steady-state or a two-cycle CEE seems to occur in all simulations. The steady-state and the two-cycle REE are, in fact, special cases of two-state stationary sunspot equilibria in which some of the transition probabilities equal zero or one [see, e.g., Azariadis and Guesnerie (1986)]. Self-fulfilling sunspot equilibria occur if all agents coordinate, or learn to coordinate, their beliefs on the same stochastic sunspot variable. Woodford (1990) has shown that learning to believe in sunspots can occur; that is, for an open set of initial states, learning processes such as OLS may converge to sunspot equilibria. Because of randomness, the self-fulfilling sunspot equilibria may be very complicated and coordination on an erratic sunspot variable, even if possible with positive probability, may be viewed as unlikely in a many-agent world. The CEE concept with a simple AR(1) predictor may be a theoretical explanation of coordination on *simple* perfect foresight outcomes in a many-agent world.

In the second class of dynamic models, $p_t = F(p_{t-1}, p_t^e)$, there is dynamic feedback from both the previous actual price and the expected price. An application is, for example, an OLG model with capital. An important question is whether, in this more general class of models, learning to believe in chaos, that is, convergence to a chaotic CEE, is more likely.

Another topic for future work would be an extension of CEE to a linear AR(2) perceived law of motion, which may lead to periodic CEE with any period $q > 2$ and also to quasiperiodic CEE. Dynamic economic models in which traders use AR(2) forecasting rules thus may lead to all possible attractors in nonlinear dynamics: stable steady states, stable cycles of arbitrary period, quasiperiodic as well as strange attractors. Another possible extension may be to consider simple *nonlinear* forecasting rules. For example, piecewise linear predictors, where traders believe prices to go up for N consecutive periods before dropping by a certain amount, may play a role in financial markets.

One might argue against our CEE concept, by saying that there exist simple *nonlinear* time-series techniques that could distinguish easily between the realizations of the nonlinear actual law of motion and the linear stochastic perceived law of motion. In fact, for all examples presented in this paper, a simple phase-space plot (p_t, p_{t+1}) would reveal immediately that prices are generated by a one-dimensional (chaotic) map and not by a stochastic AR(1) model. However, in more complicated two- and higher-dimensional models, this simple phase-space reconstruction would not work. One could argue that, in higher-dimensional cases, one might still use Takens' embedding theorem and try to compute characteristics such as the correlation dimension or the Lyapunov exponents of a possibly underlying strange attractor of the unknown law of motion. Unfortunately, all of these techniques are very sensitive to (dynamic) noise, sample size, and method of aggregation; see Brock and Dechert (1991), Brock et al. (1991), and Barnett et al. (1997) for a discussion. We have focused for analytical tractability on extremely simple examples that indeed could be detected by these nonlinear techniques even in the presence of a small amount of noise. However, in more realistic cases, the methods become very sensitive to noise. In practice it may be impossible to distinguish between a noisy chaotic and a (linear) stochastic time series, and prediction based on sample average and sample autocorrelations may be the most reasonable thing to do.

NOTES

1. Alternatively, one might define the autocorrelation coefficients of a constant sequence to be zero at all lags. However, when we introduce the stability notion for a steady-state CEE in Section 3, it will be more convenient to allow the autocorrelation coefficients of a constant sequence to be of the more general form β^j for some $\beta \in [-1, 1]$.

2. Assume, for example, a linear model $F(p) = ap$, $0 < a < 1$, and consider belief parameters $\alpha = 0$ and $\beta \in [-1, 1]$. It is easy to see that the actual dynamics (3) generates a time series $(p_t)_{t=0}^{\infty}$ with $p_t \to 0$ and sample autocorrelation coefficient $(a\beta)^j$ at lag j. Imposing the equality of sample autocorrelation coefficients as in condition 3b of the definition would imply $a\beta = \beta$ and, hence, $\beta = 0$. However, when F is nonlinear, the sample autocorrelation coefficients of a convergent trajectory of (3) depend, in general, on the initial value p_0 and may be consistent with other β values between -1 and 1 or with no such β value at all.

3. Here we mean orbital convergence, that is, $p_{2t} \to p_1^*$ and $p_{2t+1} \to p_2^*$, or vice versa. Because $p_1^* \neq p_2^*$, the sequence $(p_t)_{t=0}^{\infty}$ is not convergent in the usual sense.

4. More specifically, $T_{\beta,[a,b]}$ and $T_{\beta',[c,d]}$ are *topologically equivalent*; that is, there exists a homeomorphism $h : [a, b] \mapsto [c, d]$ such that $h \circ T_{\beta,[a,b]} = T_{\beta',[c,d]} \circ h$. The homeomorphism h is a one-to-one mapping from orbits generated by $T_{\beta,[a,b]}$ onto orbits generated by $T_{\beta',[c,d]}$.

5. The OLS estimate for α is identical to (5). The OLS estimate for β is slightly different from (6), namely $\beta_t = [\sum_{i=0}^{t-1}(p_i - \bar{p}_t^-)(p_{i+1} - \bar{p}_t^+)]/[\sum_{i=0}^{t-1}(p_i - \bar{p}_t^-)^2]$ for $t \geq 2$, where $\bar{p}_t^- = (1/t)\sum_{i=0}^{t-1} p_i$ and $\bar{p}_t^+ = (1/t)\sum_{i=1}^{t} p_i$. See also the discussion on differences between SAC and OLS learning at the end of this section.

6. The way we stated the SAC-learning dynamics, it is not an ordinary difference equation because α_t and β_t depend on all previous prices. It is shown in the Appendix that one can rewrite the system in a recursive form $(p_{t+1}, \alpha_{t+1}, n_{t+1}, z_{t+1}) = \Phi(p_t, \alpha_t, n_t, z_t, t, p_0)$, such that $n_t/z_t = \beta_t$ for all t. The SAC-learning dynamics therefore can be considered as a four-dimensional nonautonomous system.

7. Marcet and Sargent (1989) and Evans and Honkapohja (1995) analyze the stability of OLS learning in a class of linear and nonlinear stochastic models by relating the stability of the OLS dynamics to the stability of an associated ordinary differential equation. However, their approach cannot be applied here, because we deal with expectations and learning feedback in a nonlinear deterministic framework.

8. Adaptive expectations are sometimes referred to as fixed coefficient learning, because at each date the expected price is updated by a fixed proportion of the forecasting error, into the direction of the most recently observed actual price.

9. β_t remains in a small neighborhood of zero but does not necessarily converge to zero; see note 2.

10. For the steady-state CEE, this condition coincides with the weak E-stability condition for a stationary sunspot equilibrium (SSE) near a steady state, whereas for the two-cycle CEE, this condition coincides with the strong E-stability condition for an SSE near a two-cycle; see Evans and Honkapohja (1994, 1995).

11. In both cases, along the learning equilibria the price sequence is unbounded, since inflation arises because of a constant growth of the money supply. Agents run OLS regression on these unbounded price sequences. To our best knowledge there are no global results concerning periodic or chaotic dynamics in the OLS-learning scheme with regression on bounded variables.

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APPENDIX

Define $n_t = \sum_{i=0}^{t}(p_i - \alpha_t)^2$ and $z_t = \sum_{i=0}^{t-1}(p_i - \alpha_t)(p_{i+1} - \alpha_t)$. Obviously, we have $\beta_t = z_t/n_t$ for all t. The actual dynamics (7) therefore can be written as

$$p_{t+1} = \Phi_1(p_t, \alpha_t, n_t, z_t) := F(\alpha_t + (z_t/n_t)(p_t - \alpha_t)).$$

Together with (5), this implies that

$$\alpha_{t+1} = \Phi_2(p_t, \alpha_t, n_t, z_t, t) := \frac{t+1}{t+2}\alpha_t + \frac{1}{t+2}\Phi_1(p_t, \alpha_t, n_t, z_t).$$

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Finally, it is straightforward to verify that

$$n_{t+1} = \Phi_3(p_t, \alpha_t, n_t, z_t, t) := n_t + \frac{t+1}{t+2}[\alpha_t - \Phi_1(p_t, \alpha_t, n_t, z_t)]^2$$

and

$$\begin{aligned} z_{t+1} &= \Phi_3(p_t, \alpha_t, n_t, z_t, t, p_0) \\ &:= z_t + [\Phi_1(p_t, \alpha_t, n_t, z_t) - \alpha_t]\left[p_t + \frac{p_0}{t+2} - \frac{t^2+5t+5}{(t+2)^2}\alpha_t \right. \\ &\quad \left. - \frac{1}{(t+2)^2}\Phi_1(p_t, \alpha_t, n_t, z_t)\right]. \end{aligned}$$

The above equations provide a recursive form of the SAC-learning dynamics.

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