ON SUPERCRITICAL BRANCHING PROCESSES WITH EMIGRATION

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Abstract

We study supercritical branching processes under the influence of an independent and identically distributed (i.i.d.) emigration component. We provide conditions under which the lifetime of the process is finite or has a finite expectation. A theorem of Kesten–Stigum type is obtained, and the extinction probability for a large initial population size is related to the tail behaviour of the emigration.

Keywords: Extinction probabilities; random difference equation; autoregressive process; recurrence and transience; Kesten–Stigum theorem

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1. Introduction

Branching processes are a fascinating class of stochastic processes, which model the evolution of a population under the assumption that different individuals give birth to a random number of children independently of each other. Timeless monographs, which present the theory of classical branching processes, include [1], [10], and [28].

In the present article we will study the consequences of including an independent and identically distributed (i.i.d.) emigration component between consecutive generations in the supercritical regime. Intuitively speaking, the properties of this model are determined by the interplay of two opposite effects, namely the explosive nature of supercritical branching processes and the decrease in population size caused by emigration.

Formally, let $((\xi_{n,j})_{j\geq 1}, Y_n)_{n\geq 1}$ denote a sequence of i.i.d. random variables. Assume that $(\xi_{1,j})_{j\geq 1}$ is i.i.d. and let ξ be an independent copy of the family $(\xi_{n,j})_{n,j\geq 1}$. Moreover, suppose Y is an independent copy of $(Y_n)_{n\geq 1}$ and that both ξ and Y only take values in $\mathbb{N}_0$. Then we define a branching process with emigration $(Z_n)_{n\geq 0}$ by setting $Z_0 := k \in \mathbb{N}$ and recursively

$$Z_{n+1} := \left(\sum_{j=1}^{Z_n} \xi_{n+1,j} - Y_{n+1} \right)_+, \quad n \geq 0. \tag{1.1}$$

Throughout this article we will focus on the supercritical case and, more precisely, assume that

$$\lambda := \mathbb{E}[\xi] \in (1, \infty).$$

Naturally, our study will concentrate on the extinction time

$$\tau := \inf\{n \geq 1 \mid Z_n = 0\}, \quad \text{where } \inf \emptyset := \infty.$$

In Theorem 2.1, we verify that τ is almost surely finite if and only if $\mathbb{E}[\log_+ Y] = \infty$. Moreover, in Theorem 2.2, we show that $\mathbb{E}[\tau] < \infty$ if

$$0 < \sum_{n \geq 1} \prod_{m=1}^{n} \mathbb{P}\big[Y \leq r(\lambda + \varepsilon)^m\big] < \infty \quad \text{for } \varepsilon > 0 \text{ and } r \in (0, \infty), \tag{1.2}$$

and, under some additional assumptions, $\mathbb{E}[\tau] = \infty$ provided that

$$\sum_{n \geq 1} \prod_{m=1}^{n} \mathbb{P}\big[Y \leq r\lambda^m m^{-\theta}\big] = \infty \quad \text{for some } \theta \in (1, \infty), r \in (0, \infty). \tag{1.3}$$

The precise statements of all of our results will be given in Section 2. In Theorem 2.3 we relate the behaviour of the extinction probabilities

$$q_k := \mathbb{P}[\tau < \infty \mid Z_0 = k], \quad k \geq 1,$$

to the tail behaviour of Y as $k \to \infty$. We also present a strong limit theorem for the population size Z_n as $n \to \infty$; see Theorem 2.4.

Our study is motivated by a simple observation, which links our model to subcritical autoregressive processes. We will explain it in Section 3. To the best of our knowledge, this connection has not been investigated in the literature so far.

While our criteria ensuring $\mathbb{E}[\tau] < \infty$, respectively $\mathbb{E}[\tau] = \infty$, are not exact, as illustrated by the following example, the gap is quite narrow.

Example 1.1. Assume that there are $c \in (0, \infty)$ and $n_0 \in \mathbb{N}$ with

$$\mathbb{P}[Y > n] = \frac{c}{\log n} \quad \text{for all } n \geq n_0.$$

Then $\mathbb{E}[\log_+ Y] = \infty$ and hence $\tau < \infty$ almost surely. If $c > \log \lambda$, then by Raabe's test, (1.2) holds. If $c \leq \log \lambda$, then by the Gauss test, (1.3) holds.

Let us end this introduction by giving an overview of the literature dealing with branching processes with emigration.

Vatutin [31] explored the critical case $\lambda = 1$ for $Y \equiv 1$ and $\sigma^2 := \text{var}[\xi] \in (0, \infty)$. He showed that τ has a regularly varying tail with exponent $-1 - 2/\sigma^2$ and, if all moments of ξ are finite, proved that $2Z_n/n\sigma^2$, conditioned on being positive, converges weakly to an exponential distribution. These results were improved by Vinokurov [34] and Kaverin [13], and more recently by Denisov, Korshunov, and Wachtel [5]. More generally, the approach of [5] allows size-dependent offspring distribution and immigration.

More or less specific models of critical branching processes involving both immigration and emigration were studied in [19], [36], and [37].

The present article and most of the above literature deals with specific cases of controlled branching. This model was introduced by Sevastyanov and Zubkov [29], who classified eventual extinction for a control function of linear and polynomial growth. We also refer to the related work by Zubkov [39, 40]. Yanev [35] generalized this model by assuming random i.i.d.

control functions $(\varphi_n)_{n \geq 1}$. Roughly speaking, the present article assumes $\varphi_n(m) := (m - Y_n)_+$, $m \in \mathbb{N}$. For a recent monograph on controlled branching, see [32].

We also want to mention some results for models in continuous time. In this scenario the population size changes if exactly one individual gives birth to a random number of children or if an emigration event, sometimes called catastrophe, occurs. The case that each individual has either 0 or 2 children would correspond to a birth-and-death process. In [21] and [24], Pakes gave results for a catastrophe rate proportional to the population size. In [22], [23], and [25], he studied this model with a size-independent emigration rate. In the supercritical regime, he related the almost sure eventual extinction to the condition $\mathbb{E}[\log_+ Y] = \infty$; see [22, Theorem 2.1 and Corollary 3.1]. This was also verified by Grey [6], who proved the same result for the time-discrete model of our present article under the assumption that $(\xi_{1,j})_{j \geq 1}$ and Y_1 are independent. However, this independence condition can be avoided; see Theorem 2.1 of the present article.

2. Preliminaries and statement of results

In the following we usually assume $Z_0 = k \in \mathbb{N}$, $\lambda \in (1, \infty)$, as well as:

(H1) There exists a strictly increasing sequence $(k_n)_{n \geq 0}$ in $\mathbb{N}$, which satisfies $k_0 = k$ and $\mathbb{P}\big[\sum_{j=1}^{k_n} \xi_{1,k} - Y_1 = k_{n+1}\big] > 0$ for all $n \in \mathbb{N}_0$, and

(H2) $\mathbb{P}\big[\sum_{j=1}^{n} \xi_{1,j} - Y_1 \leq n - 1\big] > 0$ for all $n \geq 1$.

In some of our results, we will also need Grey's restriction:

(IND) The random variables $(\xi_{1,j})_{j \geq 1}$ and Y_1 are independent.

Conditions (H1) and (H2) ensure that neither emigration nor branching will dominate each other fully for the possibly rather small initial state, respectively for a large population size.

If $\lambda > 1$, then condition (H1) is always satisfied provided that the initial population $Z_0 = k \in \mathbb{N}$ is chosen large enough. As similar arguments will occur in many of our proofs, let us briefly explain how this can be verified. By truncation of the offspring distribution ξ (see also Observation A.1 in Appendix A), we may restrict ourselves to the case $\lambda < \infty$. Choose $\varepsilon > 0$ with $\lambda - \varepsilon > 1$. Then, by the law of large numbers,

$$\mathbb{P}\left[\sum_{j=1}^{k} \xi_{1,j} > (\lambda - \varepsilon)k\right] \to 0 \quad \text{for } k \to \infty.$$

In particular, we know that there exists $K \in \mathbb{N}$ such that this probability is smaller than $1/2$ for all $k \geq K$. Moreover, we can choose $N \in \mathbb{N}$ with $\mathbb{P}[Y_1 \geq N] < 1/2$. Then, for all $k \geq K$ with $N < \varepsilon k$, we deduce

$$\mathbb{P}[Z_1 > k \mid Z_0 = k] > 0.$$

So if $\lambda > 1$, then condition (H1) holds if $Z_0 = k$ is chosen large enough. On the other hand, we know that condition (H2) holds, for example, in the case of $\mathbb{P}[\xi = 0] > 0$ or if the emigration distribution Y is unbounded and (IND).

Note that (H1) and (H2) are preserved if the value of $Z_0 = k \geq 1$ is increased. The state space of $(Z_n)_{n \geq 0}$ may be affected by such a modification. However, this will not cause any problems in our study.

Let us give a final thought regarding (H1) and (H2). Consider a modification of the branching process $(Z_n)_{n \geq 0}$, in which the population gets revived with exactly k individuals upon extinction. Then this renewal version of $(Z_n)_{n \geq 0}$ is a Markov chain, which is irreducible and exhibits an infinite state space containing 0 if and only if both (H1) and (H2) are satisfied. In [20], Pakes investigated this revived branching process when the emigration component is absent but with a more general resetting mechanism upon extinction. Moreover, both the subcritical and critical cases are studied in [20].

Theorem 2.1. $\tau < \infty$ *almost surely if and only if* $\mathbb{E}[\log_+ Y] = \infty$.

If the process $(Z_n)_{n \geq 0}$ dies out almost surely, it is natural to ask whether its expected lifetime is finite or infinite.

Theorem 2.2. *Assume* $\mathbb{E}[\log_+ Y] = \infty$.

(I) $\mathbb{E}[\tau] < \infty$, *provided that there are* $\varepsilon > 0$ *and* $r \in (0, \infty)$ *with*

$$0 < \sum_{n \geq 1} \prod_{m=1}^{n} \mathbb{P}\big[Y \leq r(\lambda + \varepsilon)^m\big] < \infty.$$

(II) $\mathbb{E}[\tau] = \infty$, *provided that* $\mathbb{E}\big[\xi^{1+\delta}\big] < \infty$ *for* $\delta > 0$, (IND), *and*

$$\sum_{n \geq 1} \prod_{m=1}^{n} \mathbb{P}\big[Y \leq r\lambda^m m^{-\theta}\big] = \infty \quad \text{for } \theta \in (1, \infty), \, r \in (0, \infty).$$

If the process $(Z_n)_{n \geq 0}$ does not become extinct almost surely, one might try to understand the distribution of τ and the extinction probabilities $(q_k)_{k \geq 1}$ in the case of a large initial population size $k \geq 1$. For this, we use Karamata's concept of slow and regular variation and assume:

(REG) $\mathbb{P}[Y > t]$ varies regularly for $t \to \infty$ with index $\alpha \in (0, \infty)$.

For a gentle introduction to slow and regular variation, we refer the reader to [18]. A measurable function $L : [0, \infty) \to (0, \infty)$ is called slowly varying for $t \to \infty$ if for all $c \in (0, \infty)$ we have $L(ct)/L(t) \to 1$ as $t \to \infty$. Moreover, a measurable function $f : [0, \infty) \to (0, \infty)$ is regularly varying for $t \to \infty$ if there exists $\alpha \in \mathbb{R}$, $t_0 \in [0, \infty)$ and a slowly varying function L satisfying $f(t) = t^\alpha L(t)$ for all $t \geq t_0$. In this case the constant $\alpha \in \mathbb{R}$ is unique and $-\alpha$ is called the index of f.

Theorem 2.3. *Assume* (REG) *and let* $N \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$. *Then*

$$\limsup_{k \to \infty} \mathbb{P}[\tau < N \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} \leq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Furthermore, if all exponential moments of ξ *are finite, then*

$$\lim_{k \to \infty} \mathbb{P}[\tau < N \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} = \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

By choosing $N = \infty$ in Theorem 2.3, we obtain results on the extinction probabilities $(q_k)_{k \geq 1}$ for $k \to \infty$.

Besides τ and $(q_k)_{k\geq 1}$, one can also investigate the asymptotic behaviour of the process $(Z_n)_{n\geq 0}$ conditioned on its non-extinction. Observe that the sequence $W_n := \lambda^{-n} Z_n$, $n \geq 0$, is a non-negative supermartingale. Hence, as in the case without any migration, Doob's martingale convergence theorem yields the existence of the a.s. limit

$$W := \lim_{n\to\infty} W_n, \quad \text{and moreover } 0 \leq \mathbb{E}[W] \leq k.$$

Theorem 2.4.

(a) $\mathbb{P}[W > 0] > 0$ *if and only if*

$$\mathbb{E}[\xi \log_+ \xi] < \infty \quad and \quad \mathbb{E}[\log_+ Y] < \infty.$$

Furthermore, in this case, $\mathbb{P}[W > 0] = \mathbb{P}[\tau = \infty]$.

(b) *Assume $\mathbb{P}[W > 0] > 0$, $\mathbb{P}[\xi = \lambda] < 1$ and* (IND). *Then*

$$\mathbb{P}[a < W < b] > 0 \quad for\ all\ 0 \leq a < b \leq \infty.$$

The proofs of Theorem 2.1 and Theorem 2.2 are somewhat similar and are therefore addressed together in Section 4. The arguments needed for Theorem 2.3 and Theorem 2.4 are different and slightly more technical, and are thus carried out separately in Sections 5 and 6.

3. Relation to the random difference equation

In this section we always assume $\xi \equiv \lambda$. Then (1.1) simplifies to

$$Z_{n+1} = (\lambda Z_n - Y_{n+1})_+, \quad n \geq 0.$$

Consider the process $(\hat{Z}_n)_{n\geq 0}$ defined by $\hat{Z}_0 := Z_0 = k$ and

$$\hat{Z}_{n+1} := \lambda \hat{Z}_n - Y_{n+1}, \quad n \geq 0.$$

Then, by induction over $n \geq 0$, we can verify $Z_n = (\hat{Z}_n)_+$ and

$$\hat{Z}_n = \lambda^n k - \sum_{j=1}^{n} \lambda^{n-j} Y_j.$$

Let $m \in \mathbb{N}_0$. Then, for all $n \geq 0$, we know that

$$\mathbb{P}[Z_n > m] = \mathbb{P}[\hat{Z}_n > m] = \mathbb{P}\left[\lambda^{-n} m + \sum_{j=1}^{n} \lambda^{-j} Y_j < k \right] \tag{3.1}$$

$$= \mathbb{P}\left[\lambda^{-n} m + \sum_{j=1}^{n} \lambda^{-(n-j)} \lambda^{-1} Y_j < k \right] = \mathbb{P}[\hat{X}_n < k], \tag{3.2}$$

where $(\hat{X}_n)_{n\geq 0}$ is the autoregressive process defined by $\hat{X}_0 := m$ and

$$\hat{X}_{n+1} := \lambda^{-1} \hat{X}_n + \lambda^{-1} Y_{n+1}, \quad n \geq 0.$$

The study of this random difference equation was initiated by Kesten [16] in the more general random-coefficient version

$$X_{n+1} := A_{n+1}X_n + Y_{n+1}, \quad n \geq 0,$$

where the sequence $(A_n, Y_n)_{n \geq 1}$ is typically assumed to be i.i.d. and independent of X_0. The Markov chain $(X_n)_{n \geq 0}$ is sometimes called a perpetuity. For more information on this process, we also refer to the monograph [3] and the exposition in [12, Chapter 2].

Formally, (3.1) and (3.2) establish that $(Z_n)_{n \geq 0}$ and $(\hat{X}_n)_{n \geq 0}$ are dual Markov chains in the sense of Siegmund; see [30].

Let $A_1 \geq 0$ and $Y_1 \geq 0$ in the following. In the contractive case $\mathbb{E}[\log A_1] < 0$, it is well known that the condition

$$\mathbb{E}[\log_+ Y_1] < \infty$$

is related to the existence of a stationary solution for $(X_n)_{n \geq 0}$; see e.g. [33, Theorem 1.6] or [3, Theorem 2.1.3]. This can be explained in the following way. Assume that $Y_0 := X_0$ has the same distribution as Y_1. Then, for fixed $n \geq 1$, by exchangeability

$$X_n = \sum_{j=0}^{n} A_n \cdots A_{j+1} Y_j \stackrel{\mathrm{d}}{=} \sum_{j=0}^{n} A_1 \cdots A_j Y_{j+1} =: X_n',$$

and $X_n' \to X_\infty$ almost surely for $n \to \infty$, provided the limit

$$X_\infty := \sum_{n \geq 0} A_1 \cdots A_n Y_{n+1}$$

exists. If $A_1 \equiv \lambda^{-1}$ and $Y_1 \geq 0$, then the limit X_∞ is almost surely finite if and only if $\mathbb{E}[\log_+ Y] < \infty$; see e.g. Lemma 4.1 in Section 4. Inserting $m = 0$ into (3.1) and (3.2) gives

$$\mathbb{P}[\tau = \infty] = \lim_{n \to \infty} \mathbb{P}[Z_n > 0] = \lim_{n \to \infty} \mathbb{P}[\hat{X}_n < k] = \mathbb{P}[\hat{X}_\infty < k], \quad (3.3)$$

where

$$\hat{X}_\infty := \lambda^{-1} \sum_{n \geq 0} \lambda^{-n} Y_{n+1},$$

and we can recover the statement of Theorem 2.1 in this way. Moreover, consider (3.3) and the following result, which was obtained by Grincevičius in [8].

Theorem 3.1. (Grincevičius.) *Assume $\mathbb{P}[Y_1 > t]$ varies regularly for $t \to \infty$ with index $\alpha \in (0, \infty)$, $\mathbb{E}[A_1^\alpha] < 1$ and $\mathbb{E}[A_1^\beta] < \infty$ for $\beta > \alpha$. Then*

$$\lim_{k \to \infty} \mathbb{P}[X_\infty > k] \, \mathbb{P}[Y_1 > k]^{-1} = \sum_{j=0}^{\infty} \mathbb{E}[A_1^\alpha]^j.$$

In the specific case $\xi \equiv \lambda$ and $A_1 \equiv \lambda^{-1}$, we can use this result to recover the asymptotic formula for $(q_k)_{k \geq 1}$ as $k \to \infty$ given Theorem 2.3. Note that X_∞ and $\hat{X}_\infty$ differ by the constant λ^{-1}, which explains why the limit in Theorem 2.3 is $\lambda^{-\alpha}/(1 - \lambda^{-\alpha})$ rather than $1/(1 - \lambda^{-\alpha})$.

It is worth mentioning that Grey questioned some parts of the original proof of Theorem 3.1 and gave a new, improved version in [7].

Finally, note that, in our case, all random variables involved in the definition of $(\hat{X}_n)_{n\geq 0}$ are non-negative and hence Kellerer's theory of recurrence and transience of order-preserving chains is available; see [14] and [15]. By again inserting $m = 0$ into (3.1) and (3.2), we deduce that

$$\mathbb{E}[\tau] = \sum_{n\geq 0} \mathbb{P}[\tau > n] = \sum_{n\geq 0} \mathbb{P}[Z_n > 0] = \sum_{n\geq 0} \mathbb{P}[\hat{X}_n < k],$$

and hence conclude that $\mathbb{E}[\tau] = \infty$ if and only if $(\hat{X}_n)_{n\geq 0}$ is recurrent. A recent result by Zerner [38] states that this is rather generally the case if and only if there exists $b \in (0, \infty)$ with

$$\sum_{n\geq 1} \prod_{m=1}^{n} \mathbb{P}[Y_1 \leq b\lambda^m] = \infty.$$

Clearly, in the specific case $\xi \equiv \lambda$, this result characterizes the finiteness of $\mathbb{E}[\tau]$ exactly and hence more precisely than Theorem 2.2.

Interestingly, Zerner's criterion applies not only to general random-coefficient autoregressive processes but also to rather general subcritical branching processes with immigration. For many of these models, the existence of a stationary solution is related to a finite logarithmic moment of the immigration; see [11] and [26].

4. Proofs of Theorem 2.1 and Theorem 2.2

As preparation, we start with two simple lemmas.

Lemma 4.1. *Let $(U_n)_{n\geq 1}$ be i.i.d. non-negative random variables. Then*

$$\limsup_{n\to\infty} \frac{U_n}{n} = \begin{cases} 0 & \text{if } \mathbb{E}[U_1] < \infty, \\ \infty & \text{if } \mathbb{E}[U_1] = \infty. \end{cases}$$

The proof of Lemma 4.1 follows directly from the Borel–Cantelli lemma; see also [9, Chapter 6, Proposition 1.1]. Lemma 4.1 is known in the context of supercritical branching processes with immigration and can be used to obtain some of Seneta's classical results on whether immigration increases the speed of divergence; see also [27] and [4, Section 3.1.1].

We will also apply the following concentration estimate, which can be seen as a weaker but more general form of Chebyshev's inequality.

Lemma 4.2. *Let $(V_n)_{n\geq 0}$ denote a sequence of i.i.d. random variables and $S_n := \sum_{j=1}^{n} V_j$ for all $n \geq 1$. Assume $\mathbb{E}[V_1] = 0$ and that there is $\delta \in (0, 1]$ with $c := \mathbb{E}\big[|V_1|^{1+\delta}\big] < \infty$. Then*

$$\mathbb{P}[|S_n| > t] \leq 2cnt^{-1-\delta} \quad \text{for all } n \geq 1 \text{ and } t \in (0, \infty).$$

Proof. By the Marcinkiewicz–Zygmund inequality (see [9, Chapter 3, Corollary 8.2]),

$$\mathbb{E}\big[|S_n|^{1+\delta}\big] \leq 2cn \quad \text{for all } n \geq 1.$$

Thus the claim follows by applying Markov's inequality.   □

The branching process, which is obtained from $(Z_n)_{n\geq 0}$ by neglecting any emigration, will be denoted by $(Z'_n)_{n\geq 0}$. Formally, we set $Z'_0 := k \geq 1$ and

$$Z'_{n+1} := \sum_{j=1}^{Z'_n} \xi_{n+1,j}, \quad n \geq 0.$$

We will also work with the stopping time

$$\tau' := \inf\{n \geq 1 \mid Z'_{n+1} \leq Y_{n+1}\}.$$

Observe that, by definition, $Z_n \leq Z'_n$ and $\tau \leq \tau'$ almost surely.

Proof of Theorem 2.1. First, let $\mathbb{E}[\log_+ Y] = \infty$. Choose $\varepsilon > 0$ and set

$$T := \inf\{n \geq 1 \mid Z'_m \leq (\lambda + \varepsilon)^m \text{ for all } m \geq n\}.$$

Then, for fixed $n \geq 1$, Markov's inequality gives

$$\mathbb{P}\big[Z'_n > (\lambda + \varepsilon)^n\big] \leq k\left(1 + \frac{\varepsilon}{\lambda}\right)^{-n}.$$

Hence, by the Borel–Cantelli lemma, $T < \infty$ almost surely. Furthermore, applying Lemma 4.1 with $U_n := \log_+ Y_n$ gives $Y_n \geq (\lambda + \varepsilon)^n$ for infinitely many $n \geq 1$ almost surely. This yields $\tau \leq \tau' < \infty$ almost surely.

Secondly, let $\mathbb{E}[\log_+ Y] < \infty$. By truncation of the offspring distribution (see Observation A.1 from Appendix A), we can assume that the offspring distribution ξ is bounded and $\sigma^2 := \operatorname{var}[\xi] \in [0, \infty)$. Moreover, as (H1) and (H2) hold, we can choose $Z_0 = k$ large enough to ensure that these conditions remain true after truncation.

Fix $\varepsilon > 0$ with $\lambda_1 := \lambda - 2\varepsilon > 1$ and let $\lambda_0 := \lambda - \varepsilon$. Then, for all $n \geq 1$, consider the following events:

$$A_n := \left\{\sum_{j=1}^{\lfloor \lambda_0^n \rfloor} \xi_{n+1,j} \geq \left(\lambda - \frac{\varepsilon}{2}\right)\lfloor \lambda_0^n \rfloor\right\}, \quad B_n := \left\{Y_n \leq \lambda_1^n\right\}.$$

For all $n \geq 1$, we find that by Chebyshev's inequality

$$\mathbb{P}[A_n^c] \leq \mathbb{P}\left[\left|\sum_{j=1}^{\lfloor \lambda_0^n \rfloor} \xi_{1,j} - \lambda\lfloor \lambda_0^n \rfloor\right| > \frac{\varepsilon}{2}\lfloor \lambda_0^n \rfloor\right] \leq \left(\frac{\varepsilon}{2}\right)^{-2}\frac{\sigma^2}{\lfloor \lambda_0^n \rfloor}.$$

Since $\lambda_0 > 1$, due to the Borel–Cantelli lemma, almost surely only finitely many events A_n^c, $n \geq 1$, occur. On the other hand, by Lemma 4.1, we know that almost surely all but finitely many events B_n, $n \geq 1$, occur. Also note that $(A_n)_{n\geq 1}$ is a sequence of independent events, and so is $(B_n)_{n\geq 1}$. All in all, we can fix $n_0 \in \mathbb{N}$ such that

$$\left(\lambda - \frac{\varepsilon}{2}\right)\lfloor \lambda_0^n \rfloor - \lambda_1^n \geq \lfloor \lambda_0^{n+1} \rfloor \quad \text{for all } n \geq n_0 \tag{4.1}$$

and

$$\min(\mathbb{P}[A], \mathbb{P}[B]) > \frac{1}{2}, \quad \text{where } A := \bigcap_{n\geq n_0} A_n, \ B := \bigcap_{n\geq n_0} B_n. \tag{4.2}$$

Note that the value of n_0 does not depend on k. By using (4.2), we find

$$\mathbb{P}[A \cap B] = \mathbb{P}[A] + \mathbb{P}[B] - \mathbb{P}[A \cup B] \geq \mathbb{P}[A] + \mathbb{P}[B] - 1 > 0.$$

Finally, by recalling (H1) and (H2), we can increase $Z_0 = k$ to ensure

$$\mathbb{P}[C] > 0, \quad \text{where } C := \left\{ Z_{n_0} \geq \left\lfloor \lambda_0^{n_0} \right\rfloor \right\}.$$

By inserting our construction of the events A and B and using (4.1), an inductive argument yields $Z_n \geq \lfloor \lambda_0^n \rfloor$ for all $n \geq n_0$ on the event $A \cap B \cap C$. Since the events $A \cap B$ and C are independent by definition,

$$\mathbb{P}[\tau = \infty] \geq \mathbb{P}[A \cap B \cap C] = \mathbb{P}[A \cap B] \, \mathbb{P}[C] > 0. \qquad \square$$

In fact, a careful look at the second part of this proof reveals the following result, which we need in the proof of Theorem 2.4.

Proposition 4.1. *Assume* $\mathbb{E}[\log_+ Y] < \infty$. *Then* $q_k \to 0$ *for* $k \to \infty$.

The proof of this proposition is left to the reader.

Proof of Theorem 2.2. (I) Since $\tau \leq \tau'$, it suffices to verify $\mathbb{E}[\tau'] < \infty$. Fix $\varepsilon > 0$ and $r \in (0, \infty)$ according to the assumption and set

$$T := \inf\{n \geq 1 \mid Z'_m \leq r(\lambda + \varepsilon)^{m-1} \text{ for all } m \geq n\},$$
$$\hat{T} := \inf\{n > T \mid Y_n > r(\lambda + \varepsilon)^n\}.$$

Then $\tau' \leq \hat{T}$ almost surely by construction, and hence it suffices to prove $\mathbb{E}[\hat{T}] < \infty$. As in the proof of Theorem 2.1, we know $T < \infty$ almost surely. For all $n \geq 1$, by applying Markov's inequality we deduce

$$\mathbb{P}[T = n] \leq \mathbb{P}\left[Z'_{n-1} > r(\lambda + \varepsilon)^{n-2}\right] \leq \frac{\lambda k}{r} \left(1 + \frac{\varepsilon}{\lambda}\right)^{-n+2}. \tag{4.3}$$

Moreover, since $((\xi_{n,j})_{j \geq 1}), Y_n)_{n \geq 1}$ is i.i.d., we know that

$$\mathbb{E}[\hat{T}] = \sum_{n \geq 1} \mathbb{E}[\hat{T} \mid T = n] \, \mathbb{P}[T = n] = \sum_{n \geq 1} (\mathbb{E}[T_n] + n) \, \mathbb{P}[T = n], \tag{4.4}$$

where

$$T_n := \inf\left\{m \geq 1 \mid Y_{n+m} > r(\lambda + \varepsilon)^{n+m}\right\}, \quad n \geq 1.$$

For all $n \geq 1$, we have

$$\mathbb{E}[T_n] = 1 + \sum_{m \geq 1} \mathbb{P}[T_n > m] = 1 + \sum_{m \geq 1} \prod_{l=1}^{m} \mathbb{P}\left[Y \leq r(\lambda + \varepsilon)^{n+l}\right]$$

$$= 1 + \left(\sum_{m \geq n+1} \prod_{l=1}^{m} \mathbb{P}\left[Y \leq r(\lambda + \varepsilon)^l\right] \right) \left(\prod_{l=1}^{n} \mathbb{P}\left[Y \leq r(\lambda + \varepsilon)^l\right] \right)^{-1}.$$

Because of our choice of $r \in (0, \infty)$, we conclude that $1 \leq \mathbb{E}[T_n] < \infty$ for all $n \geq 1$. Observe that we can insert this formula for $\mathbb{E}[T_n]$ into (4.4). Then, by using the estimate (4.3), we can deduce $\mathbb{E}[\hat{T}] < \infty$ and the claim follows.

(II) First, note that by possibly increasing $r \in (0, \infty)$, we can guarantee that there exists $n_0 \in \mathbb{N}$ satisfying both $rn_0^{-\theta} < 1$ and

$$\mathbb{P}[Y \leq \kappa r] > 0, \quad \text{where } \kappa := \prod_{n \geq n_0} (1 - rn^{-\theta}) \in (0, 1).$$

Fix $r \in (0, \infty)$ accordingly. By possibly increasing n_0, which results in an increase of κ, and by using our assumption, we can ensure

$$\sum_{n \geq n_0} \prod_{l=n_0}^{n} \mathbb{P}[Y \leq \kappa r \lambda^l l^{-\theta}] = \infty. \tag{4.5}$$

Fix η with $(1 + \delta)^{-1} < \eta < 1$. Then, for all $n \geq n_0$, let

$$N_n := \lambda^n \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta}), \quad f_n := \kappa \lambda^{\eta n}, \quad g_n := \kappa r n^{-\theta} \lambda^n.$$

By possibly increasing $n_0 \in \mathbb{N}$ and recalling $\eta < 1$, for all $n \geq n_0$,

$$\lambda \lfloor N_n \rfloor - \lceil f_n \rceil \geq \lambda^{n+1} \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta}) - \kappa \lambda^{\eta n} n - 2$$

$$\geq \lambda^{n+1} \left(1 + \frac{1}{n}\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta}) - \lambda^{\eta n} n^2 \prod_{l=n_0}^{n} (1 - rl^{-\theta})$$

$$= \left(\lambda^{n+1} \left(1 + \frac{1}{n}\right) - \lambda^{\eta n} n^2\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta})$$

$$= \left(\lambda^{n+1} + \frac{\lambda^{n+1}}{n+1} + \frac{\lambda^{n+1}}{n(n+1)} - \lambda^{\eta n} n^2\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta})$$

$$\geq \lambda^{n+1} \left(1 + \frac{1}{n+1}\right) \prod_{l=n_0}^{n} (1 - rl^{-\theta}).$$

Moreover, by possibly increasing $n_0 \in \mathbb{N}$, we can guarantee $g_n \geq n$ for all $n \geq n_0$. So, by inserting the definition of κ, for all $n \geq n_0$ we get

$$\lceil g_{n+1} \rceil \leq g_{n+1} + 1 \leq \left(1 + \frac{1}{n+1}\right) g_{n+1}$$

$$< \lambda^{n+1} \left(1 + \frac{1}{n+1}\right) \left(\prod_{l=n_0}^{n} (1 - rl^{-\theta})\right) r(n+1)^{-\theta}.$$

By combining the previous two estimates, for all $n \geq n_0$, we directly find

$$\lambda \lfloor N_n \rfloor - \lceil f_n \rceil - \lceil g_{n+1} \rceil \geq N_{n+1}. \tag{4.6}$$

For all $n \geq n_0$, we consider the event

$$D_n := \left\{ \sum_{j=1}^{\lfloor N_n \rfloor} \xi_{n,j} \geq \lambda \lfloor N_n \rfloor - \lceil f_n \rceil \right\}.$$

Since we assume $\mathbb{E}\big[\xi^{1+\delta}\big] < \infty$, by Lemma 4.2 there is $c \in (0, \infty)$ with

$$\mathbb{P}[D_n^c] \leq \mathbb{P}\left[\left| \sum_{j=1}^{\lfloor N_n \rfloor} \xi_{1,j} - \lambda \lfloor N_n \rfloor \right| > \lceil f_n \rceil \right] \leq \frac{2c \lfloor N_n \rfloor}{\lfloor f_n \rfloor^{1+\delta}} \quad \text{for all } n \geq n_0.$$

Recalling our definition of N_n, f_n, and η, and noticing that the events $(D_n)_{n \geq n_0}$ are independent, we obtain

$$\mathbb{P}[D] > 0, \quad \text{where } D := \bigcap_{n \geq n_0} D_n.$$

Consider the stopping time

$$T := \inf\{n > n_0 \mid Y_n > g_n\}.$$

Then, by definition of T and $(g_n)_{n \geq 0}$, we deduce from (4.5)

$$\mathbb{E}[T] = \sum_{n \geq 0} \mathbb{P}[T > n] = n_0 + 1 + \sum_{n \geq n_0+1} \prod_{l=n_0+1}^{n} \mathbb{P}\big[Y \leq \kappa r l^{-\theta} \lambda^l\big] = \infty.$$

Finally, by (H1) and (H2), we may assume that the value of $Z_0 = k \geq 1$ is chosen large enough to ensure that

$$\mathbb{P}[C] > 0, \quad \text{where } C := \{Z_{n_0} \geq N_{n_0}\}.$$

Note that, by construction, C and D are independent events. All in all, by (4.6), we can deduce that $\tau \geq T$ on the event $B := C \cap D$, which occurs with a positive probability. Finally, by (IND),

$$\mathbb{E}[\tau] \geq \mathbb{E}[\tau 1_B] \geq \mathbb{E}[T 1_B] = \mathbb{E}[T \mid B] \, \mathbb{P}[B] = \mathbb{E}[T] \, \mathbb{P}[B] = \infty. \qquad \square$$

5. Proof of Theorem 2.3

For convenience, we split the proof of Theorem 2.3 into smaller parts by formulating and separately proving the following two lemmas.

Lemma 5.1. *Assume* (REG). *Then*

$$C := \limsup_{k \to \infty} q_k \, \mathbb{P}[Y > k]^{-1} \leq \frac{\lambda^{-\alpha}}{1 - \lambda^{-\alpha}}.$$

Lemma 5.2. *Assume* (REG) *and that all exponential moments of ξ are finite. Moreover, let* $N \in \mathbb{Z}_{>2} \cup \{\infty\}$. *Then*

$$\liminf_{k \to \infty} \mathbb{P}[\tau < N \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} \geq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

Proof of Lemma 5.1. By truncation of the offspring distribution (see Observation A.1 from Appendix A), we may assume that ξ is almost surely bounded and, in particular, has finite exponential moments.

Let us verify $C < \infty$ as a first step. For this, choose $\varepsilon > 0$ with $\lambda_0 := \lambda - 2\varepsilon > 1$ and set $\lambda_1 := \lambda - \varepsilon$. Then, by Lemma B.1 from Appendix B, there are $c_1, \ldots, c_N \in (0, \infty)$ such that the sequence

$$x_0 := 1, \quad x_{n+1} := \begin{cases} \lambda_1 x_n - \lambda_0^n, & n \geq N, \\ \lambda_1 x_n - c_{n+1}, & n \leq N - 1, \end{cases}$$

is strictly positive and satisfies $x_n \geq c^n$ for $c > 1$ and all $n \geq 1$. Furthermore, for all $k \geq 1$, we consider the events

$$A_{k,n} := \left\{ \sum_{j=1}^{\lfloor kx_n \rfloor} \xi_{n+1,j} \geq \lambda_1 k x_n \right\}, \quad n \geq 0, \quad A_k := \bigcap_{n \geq 0} A_{k,n}.$$

For all $k \geq 1$, we have

$$q_k = \mathbb{P}[\tau < \infty, A_k \mid Z_0 = k] + \mathbb{P}[\tau < \infty, A_k^c \mid Z_0 = k], \tag{5.1}$$

as well as

$$\mathbb{P}[\tau < \infty, A_k^c \mid Z_0 = k] \leq \sum_{n \geq 0} \mathbb{P}[A_{k,n}^c].$$

By the Cramér–Chernoff method or a sub-Gaussian concentration estimate (see [2, Section 2.1 resp. Section 2.2]), and by our knowledge concerning the sequence $(x_n)_{n \geq 0}$,

$$\mathbb{P}[\tau < \infty, A_k^c \mid Z_0 = k] \to 0 \quad \text{for } k \to \infty$$

exponentially fast. Therefore, by applying condition (REG), we deduce

$$\lim_{k \to \infty} \mathbb{P}[\tau < \infty, A_k^c \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1} = 0,$$

and, by recalling (5.1), we further conclude

$$C = \limsup_{k \to \infty} \mathbb{P}[\tau < \infty, A_k \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1}.$$

Fix $k \geq 1$ and let $Z_0 = k$. Then, by construction of A_k and $(x_n)_{n \geq 0}$,

$$\{\tau < \infty\} \cap A_k \subseteq \bigcup_{n=1}^{N-1} \{Y_n > kc_{n+1}\} \cup \bigcup_{n \geq N} \{Y_n > k\lambda_0^n\},$$

since $Z_n' \geq kc_{n+1}$ for $n < N$ and $Z_n' \geq k\lambda_0^n$ on A_k. Consequently

$$C \leq \sum_{n=1}^{N-1} \mathbb{P}[Y_n > c_{n+1}k] + \mathbb{P}\left[\sum_{n \geq N} Y_n \lambda_0^{-n} > k \right]. \tag{5.2}$$

Note that on the one hand, due to (REG),

$$\lim_{k\to\infty} \sum_{n=1}^{N-1} \mathbb{P}[Y > c_n k]\,\mathbb{P}[Y > k]^{-1} = \sum_{n=1}^{N-1} c_n^{-\alpha} < \infty,$$

and on the other hand, by applying Theorem 3.1 with $A_1 \equiv \lambda_0^{-1}$,

$$\limsup_{k\to\infty}\ \mathbb{P}\left[\sum_{n\geq N} Y_n \lambda_0^{-n} > k\right] \mathbb{P}[Y > k]^{-1} < \infty.$$

All in all, by (5.2) we conclude that $C < \infty$.

In the second step, fix $0 < \delta < \lambda_1 = \lambda - \varepsilon$ and set $\lambda_2 := \lambda_1 - \delta$. Later we will let $\varepsilon \searrow 0$ and $\delta \searrow 0$, when $\lambda_1 \nearrow \lambda$ and $\lambda_2 \nearrow \lambda$. For all $k \geq 1$, we define the events

$$D_{1,k} := \{Y_1 > \lambda_1 k\}, \quad D_{2,k} := \{\delta k \leq Y_1 \leq \lambda_1 k\} \quad \text{and} \quad D_{3,k} := \{Y_1 < \delta k\}.$$

For every $k \geq 1$, given $Z_0 = k$ we can decompose $\{\tau < \infty\}$ by using the three events $D_{1,k}$, $D_{2,k}$, and $D_{3,k}$. From this decomposition we obtain

$$q_k \leq \mathbb{P}[D_{1,k}] + \mathbb{P}[\tau < \infty, D_{2,k} \mid Z_0 = k] + \mathbb{P}[\tau < \infty,\ D_{3,k} \mid Z_0 = k].$$

Let us introduce the notation $q_r := q_{\lfloor r \rfloor}$ for $r \in (0, \infty)$. Note that the function $r \mapsto q_r$ is monotone non-increasing with respect to r, and

$$\mathbb{P}\left[\sum_{j=1}^n \xi_{1,j} \leq \left(\lambda - \frac{\varepsilon}{2}\right) n\right] \to 0 \quad \text{exponentially fast as } n \to \infty,$$

which can be verified, as in the first step, by the Cramér–Chernoff method or a sub-Gaussian concentration estimate. By combining these two remarks with our assumption (REG), for all $\varepsilon > 0$ and $\delta > 0$,

$$\begin{aligned}
\limsup_{k\to\infty}\ &\mathbb{P}[Y > k]^{-1}\,\mathbb{P}[\tau < \infty, D_{2,k} \mid Z_0 = k] \\
&= \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\,\mathbb{P}[D_{2,k}]\,\mathbb{P}[\tau < \infty \mid D_{2,k}, Z_0 = k] \\
&\leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\,\mathbb{P}[Y \geq \delta k]q_{(\varepsilon/2)k} \\
&= 0,
\end{aligned}$$

where for the last step we have also inserted our knowledge that $C < \infty$. Similarly, we can verify

$$\limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\,\mathbb{P}[\tau < \infty, D_{3,k} \mid Z_0 = k] \leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1} q_{\lambda_2 k}.$$

All in all, and again by invoking (REG),

$$\begin{aligned}
C &= \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1} q_k \\
&\leq \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\,\mathbb{P}[Y > \lambda_1 k] + \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1} q_{\lambda_2 k} \\
&= \limsup_{k\to\infty}\ \mathbb{P}[Y > k]^{-1}\,\mathbb{P}[Y > \lambda_1 k] + \limsup_{k\to\infty}\ \frac{\mathbb{P}[Y > \lambda_2 k]q_{\lambda_2 k}}{\mathbb{P}[Y > k]\,\mathbb{P}[Y > \lambda_2 k]} \\
&\leq \lambda_1^{-\alpha} + C\lambda_2^{-\alpha}.
\end{aligned}$$

Letting $\delta \searrow 0$ and $\varepsilon \searrow 0$, $C \leq \lambda^{-\alpha} + C\lambda^{-\alpha}$ and the claim follows. $\qquad\square$

Proof of Lemma 5.2. Because of monotonicity, it suffices to prove the claim for $N < \infty$. Fix $\varepsilon > 0$. For all $k \geq 1$ and $l = 1, \ldots, N - 1$, define

$$A_{k,l} := \left\{ \sum_{j=1}^{\lceil k(\lambda+\varepsilon)^l \rceil} \xi_{l,j} \leq k(\lambda+\varepsilon)^{l+1} \right\}, \quad A_k := \bigcap_{l=1}^{N-1} A_{k,l}.$$

Then, for all $l = 1, \ldots, N - 1$, $\mathbb{P}[A_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast due to the Cramér–Chernoff method. Hence, by (REG),

$$L_N^- := \liminf_{k \to \infty} \mathbb{P}[\tau < N | Z_0 = k] \, \mathbb{P}[Y > k]^{-1}$$
$$= \liminf_{k \to \infty} \mathbb{P}[\tau < N, A_k | Z_0 = k] \, \mathbb{P}[Y > k]^{-1}.$$

By inserting the definition of both A_k of $(Z_n)_{n \geq 0}$, we further obtain

$$L_N^- \geq \liminf_{k \to \infty} \mathbb{P}\big[\exists l \in \{1, \ldots, N-1\} \colon Y_l \geq k(\lambda+\varepsilon)^l, A_k \big] \, \mathbb{P}[Y > k]^{-1}.$$

Again, we combine (REG) with the fact that for all $l = 0, \ldots, N - 1$, $\mathbb{P}[A_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast. This gives us

$$L_N^- \geq \liminf_{k \to \infty} \mathbb{P}\big[\exists l \in \{1, \ldots, N-1\} \colon Y_l \geq k(\lambda+\varepsilon)^l \big] \, \mathbb{P}[Y > k]^{-1}.$$

Now, by applying the inclusion–exclusion principle, recalling that the sequence $(Y_m)_{m \geq 1}$ is i.i.d. and working with (REG), we obtain

$$L_N^- \geq \sum_{l=1}^{N-1} \lim_{k \to \infty} \mathbb{P}\big[Y_1 \geq k(\lambda+\varepsilon)^l \big] \, \mathbb{P}[Y > k]^{-1} = \sum_{l=1}^{N-1} (\lambda+\varepsilon)^{-\alpha l}.$$

The claim now follows by letting $\varepsilon \searrow 0$. $\qquad\qquad\square$

Proof of Theorem 2.3. Because of both Lemma 5.1, which covers the case $N = \infty$, and Lemma 5.2, it suffices to prove that, for fixed $2 \leq N < \infty$,

$$L_N^+ := \limsup_{k \to \infty} \mathbb{P}[\tau < N \mid Z_0 = k] \leq \sum_{l=1}^{N-1} \lambda^{-\alpha l}.$$

As in the proof of Lemma 5.1, we can assume that the offspring distribution is bounded and, in particular, all exponential moments of ξ are finite. Let us verify, for arbitrary $\varepsilon_1 \in (0, 1)$ with $\lambda_0 := \lambda - 2\varepsilon_1 > 1$,

$$L_N^+ \leq \sum_{l=1}^{N-1} \lambda_0^{-\alpha l}. \tag{5.3}$$

Let $\lambda_1 := \lambda - \varepsilon_1$ and define, for all $k \geq 1$ and $l = 1, \ldots, N - 1$, the events

$$B_{k,l} := \left\{ \sum_{j=1}^{\lfloor k\lambda_1^l \rfloor} \xi_{l,j} \geq k\left(\lambda - \frac{\varepsilon_1}{2}\right)\lambda_1^l \right\}, \quad B_k := \bigcap_{l=1}^{N-1} B_{k,l}.$$

For all $l = 1, \ldots, N-1$, the Cramér–Chernoff method or a sub-Gaussian concentration estimate implies that $\mathbb{P}[B_{k,l}^c] \to 0$ for $k \to \infty$ exponentially fast, and hence

$$L_N^+ = \limsup_{k \to \infty} \ \mathbb{P}[\tau < N, B_k \mid Z_0 = k] \, \mathbb{P}[Y > k]^{-1}.$$

Consider the events

$$C_k := \big\{ \exists l \in \{1, \ldots, N-1\} \colon Y_l > k\lambda_0^l \big\}, \quad k \geq 1.$$

Then, by (REG),

$$
\begin{aligned}
\limsup_{k \to \infty} \ \mathbb{P}[C_k] \, \mathbb{P}[Y > k]^{-1} &\leq \limsup_{k \to \infty} \sum_{l=1}^{N-1} \mathbb{P}\big[Y_l > k\lambda_0^l\big] \, \mathbb{P}[Y > k]^{-1} \\
&= \sum_{l=1}^{N-1} \lim_{k \to \infty} \mathbb{P}\big[Y > k\lambda_0^l\big] \, \mathbb{P}[Y > k]^{-1} \\
&= \sum_{l=1}^{N-1} \lambda_0^{-\alpha l},
\end{aligned}
$$

and hence, in order to obtain the inequality (5.3), it suffices to show that

$$\lim_{k \to \infty} \ \mathbb{P}\big[\tau < N, B_k, C_k^c \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1} = 0. \tag{5.4}$$

Let $\varepsilon_2 \in (0, 1)$ and introduce, for all $k \geq 1$, the random variable

$$R_k := \#\big\{ l = 1, \ldots, N-1 \mid Y_l \geq \varepsilon_2 k\lambda_0^l \big\}.$$

Then, since (REG) holds and $(Y_m)_{m \geq 1}$ is i.i.d., we easily obtain

$$\limsup_{k \to \infty} \ \mathbb{P}\big[\tau < N, \ B_k, \ C_k^c, \ R_k \geq 2 \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1} = 0. \tag{5.5}$$

For a given $\varepsilon_1 > 0$, choose $0 < \varepsilon_2 < \varepsilon_1/2$. Then, for every $k \geq 1$, by inserting the definition of B_k, R_k and $(Z_n)_{n \geq 0}$, we can deduce that

$$\mathbb{P}[\tau < N, B_k, R_k = 0 \mid Z_0 = k] = 0.$$

In particular, we obtain

$$\limsup_{k \to \infty} \ \mathbb{P}\big[\tau < N, B_k, C_k^c, R_k = 0 \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1} = 0. \tag{5.6}$$

Combining (5.5) and (5.6), in order to verify (5.4), we only need to show that

$$\limsup_{k \to \infty} \ \mathbb{P}\big[\tau < N, C_k^c, R_k = 1 \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1} = 0. \tag{5.7}$$

Let $k \geq 1$ and $l = 1, \ldots, N-1$. Then let us introduce events $B'_{k,l,1}, \ldots, B_{k,l,N-1}$ by

$$B'_{k,l,r} := \left\{ \sum_{j=1}^{\lfloor kx_{r-1} \rfloor} \xi_{r,j} \geq \lambda_1 k x_{r-1} \right\}, \quad r = 1, \ldots, N-1,$$

where $x_0 := 1$ and

$$x_{r+1} := \lambda_1 x_r - b_r, \quad b_r := \begin{cases} \lambda_0^r, & r = l, \\ \varepsilon_2 \lambda_0^r, & r \neq l. \end{cases}$$

For all $k \geq 1$, let B_k' denote the event that all events $B_{k;l,r}'$, $l, r = 1, \ldots, N-1$, occur. Then, again by the Cramér–Chernoff method or the sub-Gaussian concentration inequality, we know that

$$\limsup_{k \to \infty} \, \mathbb{P}\big[\tau < N, C_k^c, R_k = 1 \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1}$$
$$= \limsup_{k \to \infty} \, \mathbb{P}\big[\tau < N, B_k', C_k^c, R_k = 1 \mid Z_0 = k\big] \, \mathbb{P}[Y > k]^{-1}.$$

According to Lemma B.2 from Appendix B, for every $\varepsilon_1 > 0$ it is possible to choose $\varepsilon_2 > 0$ small enough to ensure

$$\mathbb{P}\big[\tau < N, B_k', C_k^c, R_k = 1 \mid Z_0 = k\big] = 0 \quad \text{for all } k \geq 1,$$

where we have inserted the definition of the events B_k' and C_k^c, as well as the definition of the random variable R_k and of the process $(Z_n)_{n \geq 0}$. In particular, by choosing $\varepsilon_2 > 0$ small enough, (5.7) follows. $\qquad\square$

6. Proof of Theorem 2.4

In the following we will again work with the branching process $(Z_n')_{n \geq 0}$, but more generally assume $Z_0' = k' \geq 1$ and possibly $k \neq k'$. Let $q' \in [0, 1)$ denote the extinction probability of $(Z_n')_{n \geq 0}$ given $k' = 1$ and recall the existence of the almost sure martingale limit

$$W' := \lim_{n \to \infty} \lambda^{-n} Z_n' \in [0, \infty).$$

By the Kesten–Stigum theorem [17], it is known that $W' = 0$ almost surely if and only if $\mathbb{E}[\xi \log_+ (\xi)] = \infty$. Moreover, if $\mathbb{E}[\xi \log_+ (\xi)] < \infty$, then, given that $(Z_n')_{n \geq 0}$ survives forever, $W' > 0$ almost surely.

Our main idea is to divide the population into two groups. Then, if the emigration is weak, it may only affect one of these groups.

Lemma 6.1. (Decomposition.) *Fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$ and $k' := k_0 - k$. Let $Z_1^{(1)} := k$, $Z_1^{(2)} := k'$, and define, for all $n \geq 1$, recursively,*

$$Z_{n+1}^{(1)} := \left(\sum_{j=1}^{Z_n^{(1)}} \xi_{n+1,j} - Y_{n+1} \right)_+, \quad Z_{n+1}^{(2)} := \sum_{j=Z_n^{(1)}+1}^{Z_n^{(1)}+Z_n^{(2)}} \xi_{n+1,j}.$$

Then

$$\big(Z_n^{(1)}\big)_{n \geq 1} \stackrel{\mathrm{d}}{=} (Z_n)_{n \geq 0}, \quad \big(Z_n^{(2)}\big)_{n \geq 1} \stackrel{\mathrm{d}}{=} (Z_n')_{n \geq 0}, \tag{6.1}$$

and if (IND) *holds, then the processes $\big(Z_n^{(1)}\big)_{n \geq 0}$ and $\big(Z_n^{(2)}\big)_{n \geq 0}$ are independent. Moreover, for all $n \geq 1$,*

$$Z_n = Z_n^{(1)} + Z_n^{(2)} \quad \text{on the event} \quad \{Z_1 = k_0\} \cap \big\{Z_n^{(1)} > 0\big\}, \tag{6.2}$$

$$Z_n \geq Z_n^{(1)} + Z_n^{(2)} \quad \text{on the event} \quad \{Z_1 \geq k_0\} \cap \big\{Z_n^{(1)} > 0\big\}. \tag{6.3}$$

Proof of Lemma 6.1. By construction, $\big(Z_n^{(1)}\big)_{n\geq 0}$ and $\big(Z_n^{(2)}\big)_{n\geq 0}$, are time-homogeneous Markov chains with the same initial state and transition probabilities as $(Z_n)_{n\geq 0}$ and $(Z_n')_{n\geq 0}$, respectively. Hence (6.1) holds.

By inserting the definitions of $Z_n^{(1)}$ and $Z_n^{(2)}$, one can straightforwardly verify both (6.2) and (6.3). The details are therefore omitted.

Finally, assume (IND) and let $a_1, a_2, b_1, b_2 \in \mathbb{N}$. Then, for all $n, m \geq 1$,

$$\mathbb{P}\big[Z_{n+1}^{(1)} = a_2, Z_{m+1}^{(2)} = b_2 \mid Z_n^{(1)} = a_1, Z_m^{(2)} = b_1\big]$$

$$= \mathbb{P}\left[\left(\sum_{j=1}^{a_1} \xi_{n+1,j} - Y_{n+1}\right)_+ = a_2, \sum_{j=a_1+1}^{a_1+b_1} \xi_{m+1,j} = b_2\right]$$

$$= \mathbb{P}\left[\left(\sum_{j=1}^{a_1} \xi_{n+1,j} - Y_{n+1}\right)_+ = a_2\right] \mathbb{P}\left[\sum_{j=a_1+1}^{a_1+b_1} \xi_{m+1,j} = b_2\right]$$

$$= \mathbb{P}\big[Z_{n+1}^{(1)} = a_2 \mid Z_n^{(1)} = a_1\big] \mathbb{P}\big[Z_{m+1}^{(2)} = b_2 \mid Z_m^{(2)} = b_1\big].$$

Hence transitions of $\big(Z_n^{(1)}\big)_{n\geq 1}$ and $\big(Z_n^{(2)}\big)_{n\geq 0}$ are independent and the claim follows by recalling that the initial states are chosen constant. □

Proof of Theorem 2.4. (a) Let $\mathbb{P}[W > 0] > 0$. Then $\mathbb{P}[\tau = \infty] > 0$, and hence by Theorem 2.1 we immediately obtain $\mathbb{E}[\log_+ Y] < \infty$. Besides, using $Z_n \leq Z_n'$ for $Z_0 = Z_0' = k$ and applying the classical Kesten–Stigum theorem (see [17]), we directly conclude $\mathbb{E}[\xi \log_+ \xi] < \infty$.

On the contrary, let us assume $\mathbb{E}[\log_+ Y] < \infty$ and $\mathbb{E}[\xi \log_+ \xi] < \infty$. Then, due to Theorem 2.1, $\mathbb{P}[\tau = \infty] > 0$, and hence it suffices to show that

$$\mathbb{P}[W > 0] = \mathbb{P}[\tau = \infty].$$

In order to obtain this claim, we note that $\{W > 0\} \subseteq \{\tau = \infty\}$ and verify

$$\mathbb{P}[W = 0, \tau = \infty] = 0. \tag{6.4}$$

By (H1) and (H2), we know that $\{\tau = \infty\} = \{Z_n \to \infty\}$ almost surely. Moreover, W is monotone with respect to $Z_0 = k$. Hence

$$\mathbb{P}[W = 0, \tau = \infty] \leq \liminf_{k\to\infty} \mathbb{P}[W = 0 \mid Z_0 = k] \tag{6.5}$$

$$= 1 - \limsup_{k\to\infty} \mathbb{P}[W > 0 \mid Z_0 = k]. \tag{6.6}$$

Choose $\varepsilon > 0$ with $\lambda - \varepsilon > 1$ and set $k_0(k) := \lfloor (\lambda - \varepsilon)k \rfloor$ for all $k \geq 1$. Then, by the strong law of large numbers,

$$\limsup_{k\to\infty} \mathbb{P}[Z_1 \geq k_0(k) \mid Z_0 = k] = 1 \tag{6.7}$$

and $k_0(k) - k \to \infty$ for $k \to \infty$. Fix $k \geq 1$, $k_0 = k_0(k)$ and assume $Z_0 = k$. Then, by making use of the notation introduced in Lemma 6.1 and (6.3),

$$\mathbb{P}[W > 0] \geq \mathbb{P}[Z_1 \geq k_0] \mathbb{P}\Big[\forall n \geq 0 \colon Z_n^{(1)} > 0, \ \lim_{n\to\infty} \lambda^{-n} Z_n^{(2)} > 0\Big]. \tag{6.8}$$

Recalling (6.1) and Proposition 4.1, we know that

$$\mathbb{P}\big[\forall n \geq 0 \colon Z_n^{(1)} > 0\big] = 1 - q_k \to 1 \quad \text{for } k \to \infty. \tag{6.9}$$

On the other hand, by (6.1) and the Kesten–Stigum theorem [17],

$$\mathbb{P}\Big[\lim_{n \to \infty} \lambda^{-n} Z_n^{(2)} > 0\Big] = \mathbb{P}[W' > 0 \mid Z_0' = k_0(k) - k] = 1 - (q')^{k_0(k)-k},$$

and, since $k_0(k) - k \to \infty$ for $k \to \infty$, we further obtain

$$\lim_{k \to \infty} \mathbb{P}\Big[\lim_{n \to \infty} \lambda^{-n} Z_n^{(2)} > 0\Big] = 1. \tag{6.10}$$

By combining (6.7), (6.9), and (6.10) with (6.8), we conclude

$$\limsup_{k \to \infty} \mathbb{P}[W > 0 \mid Z_0 = k] = 1,$$

and hence (6.4) and the claim follow by recalling (6.5) and (6.6).

(b) First, let $a = 0$. Assume that there exists $b \in (0, \infty)$ satisfying $\mathbb{P}[0 < W < b] = 0$ and $\mathbb{P}[0 < W < b + \varepsilon] > 0$ for all $\varepsilon > 0$. Then we choose $\varepsilon > 0$ and $\delta > 0$ with

$$\tilde{b} := \lambda^{-1}(b + \varepsilon) + \delta < b. \tag{6.11}$$

Also fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$ and again recall the notation from Lemma 6.1. Then, by the decomposition (6.2) and (6.11),

$$\mathbb{P}[0 < W < \tilde{b}] \geq \mathbb{P}[Z_1 = k_0] \, \mathbb{P}\Big[\lim_{n \to \infty} \lambda^{-n} Z_n^{(1)} \in \big(0, \lambda^{-1}(b + \varepsilon)\big), \ \lim_{n \to \infty} \lambda^{-n} Z_n^{(2)} < \delta\Big].$$

By using (IND) and Lemma 6.1, we further deduce that

$$\mathbb{P}[0 < W < \tilde{b}] \geq \mathbb{P}[Z_1 = k_0] \, \mathbb{P}[0 < W < b + \varepsilon] \, \mathbb{P}[0 < W' < \lambda \delta],$$

where we assume $Z_0 = k$ and $Z_0' = k_0 - k$. The first two probabilities on the right-hand side of this inequality are positive by construction. The third factor is also positive. This follows, for example, from the fact that W' has a strictly positive Lebesgue density on $(0, \infty)$; see e.g. [1, Chapter 1, Part C]. Consequently $\mathbb{P}[0 < W < \tilde{b}] > 0$, which is a contradiction to our assumptions on $b \in (0, \infty)$. Hence the claim is true if $a = 0$.

For arbitrary $a > 0$, again fix $k_0 > k$ with $\mathbb{P}[Z_1 = k_0] > 0$ and also $\varepsilon > 0$ with $\varepsilon < b - a$. Then, by the same arguments as for $a = 0$, and again assuming $Z_0 = k$ and $Z_0' = k_0 - k$, we obtain

$$\mathbb{P}[a < W < b] \geq \mathbb{P}[Z_1 = k_0] \, \mathbb{P}[0 < W < \lambda \varepsilon] \, \mathbb{P}[\lambda a < W' < \lambda(b - \varepsilon)].$$

Since we have verified the claim for $a = 0$, we know that the second factor on the right-hand side of this inequality is positive. Our choice of ε implies that the third factor is also positive. Hence, as for $a = 0$, we can indeed deduce $\mathbb{P}[a < W < b] > 0$. $\qquad\square$

Appendix A. Truncation of the offspring distribution

In some of our proofs we make use of the following observation.

Observation A.1. (*Truncation of the offspring distribution.*) Let $(Z_n)_{n \geq 0}$ denote a branching process with emigration, which is defined recursively by (1.1) under the same assumptions as

in Section 1. Moreover, let $\lambda := \mathbb{E}[\xi_{1,1}] \in (0, \infty]$ and assume that the distribution of $\xi_{1,1}$ is unbounded. Then, for every $s \in (0, \infty)$, there exists $N \in \mathbb{N}$ with the following property. If we define, for all $n, j \geq 1$,

$$\tilde{\xi}_{n,j} := \begin{cases} \xi_{n,j}, & \xi_{n,j} \leq N, \\ N, & \xi_{n,j} > N, \end{cases}$$

as well as $\tilde{Z}_0 := Z_0 := k$, and, recursively,

$$\tilde{Z}_{n+1} := \left(\sum_{j=1}^{\tilde{Z}_n} \tilde{\xi}_{n+1,j} - Y_{n+1} \right)_+, \quad n \geq 0,$$

then we arrive at a branching process with emigration $(\tilde{Z}_n)_{n\geq 0}$, which satisfies $\tilde{Z}_n \leq Z_n$ almost surely for all $n \geq 0$, has a bounded offspring distribution and, if $\lambda = \infty$, then $s < \mathbb{E}[\tilde{\xi}_{1,1}] < \infty$, respectively $0 < \lambda - \mathbb{E}[\tilde{\xi}_{1,1}] < s$, for $\lambda < \infty$. In particular, if $\tilde{\tau}$ denotes the extinction time of the process $(\tilde{Z}_n)_{n\geq 0}$, then $\tau \geq \tilde{\tau}$ almost surely.

Moreover, if condition (IND), respectively (H2), is satisfied for the process $(Z_n)_{n\geq 0}$, then, by construction, the corresponding condition also holds for $(\tilde{Z}_n)_{n\geq 0}$. Finally, if $\lambda \in (1, \infty)$ and $s < \lambda - 1$, then, as for the process $(Z_n)_{n\geq 0}$, we know that condition (H1) holds for $(\tilde{Z}_n)_{n\geq 0}$ provided $Z_0 = \tilde{Z}_0 = k$ is chosen large enough.

Appendix B. Notes on the recursion $x_{n+1} = ax_n - b_n$

The following claims can be proved by elementary arguments. We omit the details.

Lemma B.1. *Let $x_0 := 1$, $a \in (1, \infty)$ and $\varepsilon > 0$ with $a - \varepsilon_1 > 1$. Then there exists $N \in \mathbb{N}$ and $c_1, \ldots, c_N \in (0, \infty)$ such that for the sequence $(x_n)_{n\geq 0}$ defined by*

$$x_{n+1} := ax_n - b_n, \quad where \ b_n := \begin{cases} (a - \varepsilon)^n, & n \geq N, \\ c_{n+1}, & n \leq N - 1, \end{cases}$$

there exists $c > 1$ with $x_n \geq c^n > 0$ for all $n \geq 1$.

Lemma B.2. *Let $N \in \mathbb{N}$, $x_0 := 1$, $a \in (1, \infty)$ and $\varepsilon_1 > 0$ with $a - \varepsilon_1 > 1$. Then there exists $\varepsilon_2 > 0$ with the following property. For arbitrary $l \in \{1, \ldots, N - 1\}$, the recursion*

$$x_{n+1} := ax_n - b_n, \quad where \ b_n := \begin{cases} (a - \varepsilon_1)^l, & n = l, \\ \varepsilon_2(a - \varepsilon_1)^n, & n \neq l, \end{cases}$$

defines reals $x_1, \ldots, x_N$ with $x_j \geq \varepsilon_1 > 0$ for all $j = 1, \ldots, N$.

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