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HYPERBOLIC METRIC AND MULTIPLY CONNECTED WANDERING DOMAINS OF MEROMORPHIC FUNCTIONS

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Abstract In this paper, in terms of the hyperbolic metric, we give a condition under which the image of a hyperbolic domain of an analytic function contains a round annulus centred at the origin. From this, we establish results on the multiply connected wandering domains of a meromorphic function that contain large round annuli centred at the origin. We thereby successfully extend the results of transcendental meromorphic functions with finitely many poles to those with infinitely many poles.

Keywords: wandering domain; hyperbolic metric; annulus domains; Fatou set; Julia set

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1. Introduction and main results

Let f be a meromorphic function that is not a *Möbius* transformation and let f^n, $n \in \mathbb{N}$, denote the nth iterate of f. The Fatou set $F(f)$ of f is defined to be the set of points $z \in \hat{\mathbb{C}}$ such that $\{f^n\}_{n\in\mathbb{N}}$ is well defined and forms a normal family in some neighbourhood of z. The complement $J(f)$ of $F(f)$ is called the Julia set of f. An introduction to the properties of these sets can be found in [**3**, **8**, **10**] for rational functions and in [**5**] for transcendental meromorphic functions. The Fatou set $F(f)$ is open and the Julia set is not empty and is perfect, so every component of the Fatou set is a hyperbolic domain, called a stable domain. Noting that $F(f)$ is completely invariant, i.e. $z \in F(f)$ if and only if $f(z) \in F(f)$, for a component U of $F(f)$ we always have $f^n(U)$ in a component of $F(f)$, denoted by U_n, i.e. $f^n(U) \subseteq U_n$. U is called periodic if for some n, $U = U_n$ and the least number n for the equation is called the period of the periodic domain U; U is called wandering if $U_n \neq U_m$ for $n \neq m$. A non-wandering Fatou component will fall in a periodic domain under iteration, and it is called preperiodic if it is itself not periodic.

An important difference between the dynamics of a transcendental meromorphic function and a rational function with degree at least 2 is that a rational function has neither wandering domains nor Baker domains, but a transcendental meromorphic function may have them. Here, a periodic stable domain U of period p is called a Baker domain if there exists $z_0 \in \partial U$ such that $f^{np}(z) \to z_0$ for $z \in U$ as $n \to \infty$ but f^p is not defined at z_0. Therefore, for some $0 \leqslant j < p$, $f^{np+j}(z) \to \infty$ for $z \in U$ as $n \to \infty$ and at least one of $U, f(U), \dots, f^{p-1}(U)$ is unbounded.

In [**1**] Baker proved that a multiply connected Fatou component U of a transcendental entire function must be wandering, $f^n(U) \to \infty$ $(n \to \infty)$, and, for all sufficiently large n, U_n separates 0 and U_{n+1}. In terms of the hyperbolic metric, Zheng established the following theorem, among other things, in [**17**].

Theorem A. *Let f be a transcendental meromorphic function with finitely many poles. If U is a Fatou component of f such that $f^n|_U \to \infty$ $(n \to \infty)$ and, for all sufficiently large n, U_n separates 0 and ∞, then for all $n > n_0$, U_n contains an annulus $\{z\colon r_n < |z| < R_n\}$ with $r_n \to \infty$ and $R_n/r_n \to \infty$.*

Among other things, Bergweiler *et al.* [**7**] proved the following.

Theorem B. *Let f and U be given as in Theorem A. Then for $z_0 \in U$ and an open set D in U containing z_0, there exists an $\alpha > 0$ such that for all sufficiently large n we have*

$$U_n \supset f^n(D) \supset \{z\colon |f^n(z_0)|^{1-\alpha} < |z| < |f^n(z_0)|^{1+\alpha}\}.$$

Theorem B is stated in [**7**] for a transcendental entire function, but it is available for a transcendental meromorphic function with only finitely many poles, as pointed out in [**7**]. In this paper we investigate the applicability of Theorem A and Theorem B to transcendental meromorphic functions with infinitely many poles.

In [**19**] (see also [**11**, Theorem 5]) Zheng proved that if $f^n|_U \to \infty$ for a Fatou component U, then for any compact subset W of U there exists an $M(W) > 1$ such that

$$M(W)^{-1}|f^n(z)| \leqslant |f^n(w)| \leqslant M(W)|f^n(z)| \quad \forall z, w \in W \tag{1.1}$$

provided that $\bigcup_{n=1}^{\infty} f^n(U)$ does not contain any sequence of round annuli D_m centred at 0 such that $\operatorname{dist}(0, D_m) \to \infty$ and $\operatorname{mod}(D_m) \to \infty$. We will say that $\{D_m\}$ has Property A below for convenience. Therefore, if (1.1) does not hold for some compact subset W of U, then $\bigcup_{n=1}^{\infty} f^n(U)$ contains a sequence of round annuli with Property A.

If the number of poles of f is restricted, then in terms of the hyperbolic metric, we can establish the following theorem.

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic function. Assume that there exist two points a, b in a Fatou component U such that for a sequence $\{n_k\}$ of increasing positive integers we have*

$$\frac{|f^{n_k}(a)|}{|f^{n_k}(b)|} \to \infty, \quad |f^{n_k}(b)| \to \infty \quad (k \to \infty). \tag{1.2}$$

Let D be a domain in U containing a and b.

(i) *If for all sufficiently large r,*

$$n(r, 0) > n(r, \infty) + 7\pi, \tag{1.3}$$

then for all sufficiently large n, $f^n(D)$ contains an annulus $A_n = \{z\colon r_n < |z| < R_n\}$ with $r_n \to \infty$ and $R_n/r_n \to \infty$ as $n \to \infty$ such that $A_{n+1} \subset f(A_n)$.

(ii) *If for arbitrarily large C there exists an $s(C) > 0$ such that for $r \geqslant s(C)$ we have*

$$T(Cr, f) - N(Cr, f) \geqslant T(2r, f) + 7\pi \log C, \tag{1.4}$$

then for all sufficiently large n, $f^n(D)$ contains an annulus $A_n = \{z\colon r_n < |z| < r_n^{\alpha}\}$ with $\alpha > 1$ and $r_n \to \infty$ such that $A_{n+1} \subset f(A_n)$.

Here, $n(r, 0)$ and $n(r, \infty)$ are, respectively, the number of zeros and the number of poles of f in $\{z\colon |z| < r\}$, $T(r, f)$ is the Nevanlinna characteristic of f, and $N(r, f)$ is the integrated counting function of poles of f in $\{z\colon |z| < r\}$. For details, please see § 3. In Theorem 1.1, $n(r, 0)$ can be replaced by the number $n(r, c)$ of c-points in $\{|z| < r\}$ for any finite value c.

In § 5 we will discuss the conditions and results in Theorem 1.1 via examples of transcendental meromorphic functions. We will construct an example such that U_n contains an annulus with Property A but no annulus with the form $\{z\colon r_n < |z| < r_n^{\alpha}\}$ for $\alpha > 1$, i.e. the result in case (i) holds but the result in case (ii) does not. We explain the necessity of condition (1.3) in terms of [**11**, Example 1], whose wandering domains orbit has not only infinitely many elements surrounding the origin but also infinitely many elements not surrounding the origin and, in terms of Example 5.4, every element of whose wandering domain orbit surrounds the origin. The necessity of condition (1.4) will be shown by Example 5.3.

We have a consequence of Theorem 1.1 that is a generalization of [**17**, Corollary 2 (II)].

Corollary 1.2. *Given f, a, b and U as in Theorem 1.1, if for $r \geqslant r_0 > 0$ (1.3) holds, then for any meromorphic function g satisfying, for a subset E of $[0, \infty)$ with finite logarithmic measure, i.e. $\int_E \mathrm{d}r/r < \infty$, and for $0 < c < 1$,*

$$c \log M(r, f) \geqslant \log M(r, g), \quad r \notin E,$$

we have $\delta(0, f - g) = 0$. Furthermore, $\delta(0, f - z) = 0$ and f has infinitely many repelling fixed points or indifferent fixed points with multiplier equal to 1.

Here, $M(r, f) = \max\{|f(z)|\colon |z| = r\}$, the exceptional set E depends on g, and $\delta(0, *)$ is the Nevanlinna deficiency of the function $*$ at 0. The definition of Nevanlinna deficiency will be given in § 3. A point z_0 such that $f(z_0) = z_0$ is a fixed point of f; a fixed point z_0 of f is called repelling (respectively, indifferent) if $|f'(z_0)| > 1$ (respectively, $|f'(z_0)| = 1$) and the multiplier of a fixed point z_0 is $f'(z_0)$. The existence of infinitely many repelling fixed points or indifferent fixed points with multiplier equal to 1 was proved by Bergweiler and Terglane [**6**] for the case of entire functions that have multiply connected wandering domains.

Theorem 1.3. *Let f be a transcendental meromorphic function and suppose that there exist $r_0 > 0$ and $\lambda > 1$ such that for $r \geqslant r_0$ we have*

$$T(2r, f) \geqslant \lambda T(r, f)$$

and $\delta(\infty, f) > 0$. Then for any compact subset W in a Fatou component U of f with $f^n|_U \to \infty$ $(n \to \infty)$, we have (1.1) for an $M(W) > 1$ and $\bigcup_{n=1}^{\infty} f^n(U)$ contains no sequence of annuli with Property A.

This result motivates us to raise the following problem.

Problem 1.4. *For a Fatou component U of a meoromorphic function f with $f^n|_U \to \infty$ $(n \to \infty)$, if (1.1) holds for any compact subset W of U, does $\bigcup_{n=1}^{\infty} f^n(U)$ contain no sequence of annuli with Property A?*

Obviously, the discussion of this problem only needs to focus on the case in which U and $U_n = f^n(U)$ are wandering, bounded and multiply connected. The below proof of Corollary 1.2 tells us that if (1.3) holds and f has finitely many repelling fixed points and finitely many indifferent fixed points with multiplier equal to 1, then the answer to Problem 1.4 is affirmative.

In [**7**] Bergweiler *et al*. proved that if U is a multiply connected Fatou component of an entire function f, then for $z_0 \in U$ the limit

$$h_U(z) = \lim_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|}$$

exists on U and $h_U(z)$ is a positive non-constant harmonic function on U. Therefore, the answer to Problem 1.4 for a meromorphic function with only finitely many poles is affirmative. In fact, if $\bigcup_{n=1}^{\infty} f^n(U)$ contains a sequence of annuli with Property A, then the above result is also available for such a U; that is to say, $h_U(z)$ exists on U and is not a constant, and therefore (1.1) does not hold for $W = \{z_1, z_0\}$, where $h_U(z_1) \neq 1$.

In [**7**], by using the harmonic function $h_U(z)$, Bergweiler *et al*. characterized precisely the round annuli in U_n. The establishment of Theorem B basically stems from the fact that $h_U(z)$ is non-constant. Therefore, $h_U(z)$ is an important function.

We consider a question of whether $h_U(z)$ exists for a wandering domain U of a meromorphic function f such that $f^n|_U \to \infty$ $(n \to \infty)$. To this end, for $z_0 \in U$ define

$$\bar{h}_U(z) = \limsup_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|},$$

$$\underline{h}_U(z) = \liminf_{n\to\infty} \frac{\log|f^n(z)|}{\log|f^n(z_0)|}.$$

If $\bar{h}_U(z) = \underline{h}_U(z)$, we write $h_U(z)$ for the common value. In § 5 we discuss the possibility of existence of $h_U(z)$ via examples.

In § 2, to prove Theorem 1.1 and Corollary 1.2, we establish preliminary results (i.e. Theorems 2.2 and 2.4) in view of the hyperbolic metric, which are of independent significance. We devote § 3 to a simple introduction to the Nevanlinna theory. In § 4 we show the proofs of Theorems 1.1 and 1.3 and Corollary 1.2. In § 5 we construct some examples of transcendental meromorphic functions to illustrate the necessity of the conditions in Theorem 1.1.

2. The hyperbolic metric and basic results

Let Ω be a hyperbolic domain in the complex plane, that is, $\mathbb{C}\backslash\Omega$ contains at least two points. Then there exists on Ω the hyperbolic metric $\lambda_\Omega(z)|\mathrm{d}z|$ with Gaussian curvature

-1, and by $d_\Omega(u,v)$ we denote the hyperbolic distance between two points $u, v \in \Omega$, which is defined for u and v in Ω by

$$d_\Omega(u,v) = \inf_\alpha \int_\alpha \lambda_\Omega(z)\,|\mathrm{d}z|,$$

where the infimum is taken over all piecewise-smooth paths α in Ω joining u and v.

It is well known that

$$\lambda_D(z) = \frac{1}{|z|\log|z|}, \quad D = \{z\colon |z| > 1\}. \tag{2.1}$$

Below, by $A(r,R)$ we denote the annulus $\{z \in \mathbb{C}\colon r < |z| < R\}$. Through the covering map from the unit disk $\mathbb{D}$ onto the annulus, we show that the hyperbolic density on the annulus $A = A(r,R)$ is

$$\lambda_A(z) = \frac{\pi}{|z|\,\mathrm{mod}(A)\sin(\pi\log(R/|z|)/\mathrm{mod}(A))} \quad \forall z \in A, \tag{2.2}$$

where $\mathrm{mod}(A) = \log(R/r)$ is the modulus of A.

In order to find a relation between the domain constants, Beardon and Pommerenke [**4**] introduced the notation

$$\beta_\Omega(z) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right|\colon a, b \in \partial\Omega\right\}, \quad z \in \Omega.$$

In order to consider the geometric structure of a domain with respect to a boundary point, in [**15**], for $a \notin \Omega$ and for $z \in \Omega$, Zheng defined

$$\beta_\Omega(z;a) = \inf\left\{\left|\log\frac{|z-a|}{|b-a|}\right|\colon b \in \partial\Omega\right\}.$$

If $\beta_\Omega(z_0;a) > 0$ for a $z_0 \in \Omega$, then

$$\{z\colon \mathrm{e}^{-\beta_\Omega(z_0;a)}|z_0-a| < |z-a| < \mathrm{e}^{\beta_\Omega(z_0;a)}|z_0-a|\} \subset \Omega;$$

that is to say, Ω contains a round annulus centred at a with modulus $2\beta_\Omega(z_0;a)$.

Lemma 2.1 (Beardon and Pommerenke [4], Zheng [15]). *We have*

$$\frac{1}{\beta_\Omega(z;a)+\kappa} \leqslant \lambda_\Omega(z)|z-a| \leqslant \frac{\pi}{2\beta_\Omega(z;a)}$$

for $z \in \Omega$ and $a \notin \Omega$, where $\kappa = \Gamma(\frac{1}{4})^4/(4\pi^2) = 4.3768796\ldots$.

In [**7**] Bergweiler *et al.* proved the following theorem, which gives a condition under which the image of a hyperbolic domain under an analytic map contains a definite annulus.

Theorem C. *There exists a* $\delta > 0$ *such that, for any analytic function* f *on* $U = A(r, R)$ *with* $0 \notin f(U)$*, if two points* $z_1, z_2 \in U$ *satisfy*

$$d_U(z_1, z_2) < \delta \quad \text{and} \quad |f(z_1)| \geqslant 2|f(z_2)|,$$

then we have

$$f(U) \supset A(|f(z_2)|, |f(z_1)|). \tag{2.3}$$

We take the ratio of $|f(z_1)|$ and $|f(z_2)|$ together with $d_U(z_1, z_2)$ into consideration and establish the following theorem.

Theorem 2.2. *Let* f *be analytic on a hyperbolic domain* U *with* $0 \notin f(U)$*. If there exist two distinct points* z_1 *and* z_2 *in* U *such that* $|f(z_1)| > \mathrm{e}^{\kappa\delta}|f(z_2)|$*, where* $\delta = d_U(z_1, z_2)$*, then there exists a point* $\hat{z} \in U$ *such that* $|f(z_2)| \leqslant |f(\hat{z})| \leqslant |f(z_1)|$ *and*

$$f(U) \supset A\biggl(\mathrm{e}^{\kappa}\biggl(\frac{|f(z_2)|}{|f(z_1)|}\biggr)^{1/\delta}|f(\hat{z})|, \mathrm{e}^{-\kappa}\biggl(\frac{|f(z_1)|}{|f(z_2)|}\biggr)^{1/\delta}|f(\hat{z})|\biggr); \tag{2.4}$$

if $|f(z_1)| \geqslant \exp(\kappa\delta/(1-\delta))|f(z_2)|$ *and* $0 < \delta < 1$*, then (2.3) holds. In particular, for* $\delta \leqslant \frac{1}{6}$ *and* $|f(z_1)| \geqslant \mathrm{e}|f(z_2)|$*, we have (2.3).*

Proof. We take the geodesic curve γ connecting z_1 and z_2 in U. We may assume that $|f(z_2)| \leqslant |f(z)| \leqslant |f(z_1)|$ for all $z \in \gamma$. Set $\Omega = f(U)$. In view of Lemma 2.1, we have

$$\lambda_U(z) \geqslant \lambda_\Omega(f(z))|f'(z)| \geqslant \frac{1}{\beta_\Omega(f(z); 0) + \kappa}\frac{|f'(z)|}{|f(z)|}.$$

This reduces to

$$(\beta_\Omega(f(z); 0) + \kappa)\lambda_U(z) \geqslant \frac{|f'(z)|}{|f(z)|}.$$

There exists a point $\hat{z} \in \gamma$ such that $\beta_\Omega(f(z); 0) \leqslant \beta_\Omega(f(\hat{z}); 0)$ for all $z \in \gamma$. Considering the integration along γ yields

$$(\beta_\Omega(f(\hat{z}); 0) + \kappa)d_U(z_1, z_2) \geqslant \int_\gamma \frac{|f'(z)|}{|f(z)|}\,|\mathrm{d}z| \geqslant \biggl|\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z\biggr| \geqslant \log\frac{|f(z_1)|}{|f(z_2)|}.$$

Therefore, we have

$$\beta_\Omega(f(\hat{z}); 0) \geqslant \log \mathrm{e}^{-\kappa}\biggl(\frac{|f(z_1)|}{|f(z_2)|}\biggr)^{1/\delta}.$$

In terms of the definition of β_Ω, we have

$$\begin{aligned}\Omega &\supset \{z\colon \mathrm{e}^{-\beta_\Omega(f(\hat{z});0)}|f(\hat{z})| < |z| < \mathrm{e}^{\beta_\Omega(f(\hat{z});0)}|f(\hat{z})|\}\\ &\supset A\biggl(\mathrm{e}^{\kappa}\biggl(\frac{|f(z_2)|}{|f(z_1)|}\biggr)^{1/\delta}|f(\hat{z})|, \mathrm{e}^{-\kappa}\biggl(\frac{|f(z_1)|}{|f(z_2)|}\biggr)^{1/\delta}|f(\hat{z})|\biggr).\end{aligned}$$

This is (2.4).

Suppose that $|f(z_1)| \geqslant \exp(\kappa\delta/(1-\delta))|f(z_2)|$ and $0 < \delta < 1$. Then

$$\mathrm{e}^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{(1-\delta)/\delta} \leqslant 1,$$

and we have

$$\begin{aligned}\mathrm{e}^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{1/\delta}|f(\hat{z})| \leqslant \mathrm{e}^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{1/\delta}|f(z_1)| &= \mathrm{e}^{\kappa}\left(\frac{|f(z_2)|}{|f(z_1)|}\right)^{(1-\delta)/\delta}|f(z_2)| \\ &\leqslant |f(z_2)|\end{aligned}$$

and

$$\mathrm{e}^{-\kappa}\left(\frac{|f(z_1)|}{|f(z_2)|}\right)^{1/\delta}|f(\hat{z})| \geqslant |f(z_1)|.$$

Thus, (2.3) follows from (2.4). If $\delta \leqslant \frac{1}{6}$ and $|f(z_1)| \geqslant \mathrm{e}|f(z_2)|$, then $\delta < 1/(\kappa+1)$, $\kappa\delta/(1-\delta) < 1$ and $|f(z_1)| \geqslant \mathrm{e}|f(z_2)| \geqslant \exp(\kappa\delta/(1-\delta))|f(z_2)|$. This implies (2.3). □

Let us make remarks on Theorem C and Theorem 2.2.

(A) For arbitrary $\delta = d_U(z_1, z_2)$, as long as $|f(z_1)| > \mathrm{e}^{\kappa\delta}|f(z_2)|$, Theorem 2.2 tells us that $f(U)$ contains an annulus whose modulus is

$$\frac{2}{\delta}\log\left|\frac{f(z_1)}{f(z_2)}\right| - 2\kappa.$$

However, after a restriction is imposed on $\delta = d_U(z_1, z_2)$, Theorem C can confirm the existence of an annulus contained in $f(U)$, and in Theorem 2.2 we can obtain an annulus with great modulus if either δ is small or the ratio of $|f(z_1)|$ to $|f(z_2)|$ is large.

(B) Certainly, from the proof of Theorem 2.2 we can take into account the change of argument of f along the geodesic curve γ from z_2 to z_1, denoted by $\mathrm{Arg}_\gamma(f)$. Indeed, under the assumption of Theorem 2.2 with $|f(z_1)| > \mathrm{e}^{\kappa\delta}|f(z_2)|$ replaced by $\mathrm{Arg}_\gamma(f) > \kappa\delta$, there exists a point $\hat{z} \in \gamma$ such that we have

$$f(U) \supset A\left(\exp\left(-\frac{1}{\delta}\mathrm{Arg}_\gamma(f) + \kappa\right)|f(\hat{z})|, \exp\left(\frac{1}{\delta}\mathrm{Arg}_\gamma(f) - \kappa\right)|f(\hat{z})|\right).$$

But we cannot confirm that $|f(\hat{z})| \geqslant |f(z_2)|$, and hence we do not know if (2.3) holds. However, if we assume additionally that $|f(z_2)| \leqslant |f(z)| \leqslant |f(z_1)|$ on γ and

$$|f(z_1)| < \exp\left(\frac{1}{\delta}\mathrm{Arg}_\gamma(f) - \kappa\right)|f(z_2)|,$$

we have (2.3).

In order to prove Corollary 1.2, we need the two following results, the first of which serves the second, which is essentially a general version of [**7**, Theorem 3.1].

Lemma 2.3. *We have*

$$\sin x \geqslant \frac{2}{\pi}\min\{x, \pi - x\} \quad \textit{for } 0 < x < \pi, \tag{2.5}$$

$$\exp\left(-\frac{\pi}{2}x\right) > \frac{1-x}{1+x} \quad \textit{for } 0 < x < 1. \tag{2.6}$$

Proof. To prove (2.5), by noting that $\sin x/x$ is decreasing on $(0, \pi/2]$ we have $\sin x \geqslant (2/\pi)x$ for $x \in [0, \pi/2]$; for $x \in [\pi/2, \pi]$, i.e. $0 \leqslant \pi - x \leqslant \pi/2$, we have $\sin x = \sin(\pi - x) \geqslant (2/\pi)(\pi - x)$; thus, (2.5) follows.

We rewrite (2.6) as $1 + x > (1 - x)\mathrm{e}^{\pi x/2}$ and set

$$F(x) = 1 + x - (1 - x)\mathrm{e}^{\pi x/2} \quad \text{for } 0 < x < 1.$$

Then (2.6) is equivalent to $F(x) > 0$ for $0 < x < 1$. Differentiating $F(x)$ twice, we have

$$F'(x) = 1 + \mathrm{e}^{\pi x/2} - \frac{\pi}{2}(1 - x)\mathrm{e}^{\pi x/2}, \qquad F''(x) = \pi\left[1 - \frac{\pi}{4}(1 - x)\right]\mathrm{e}^{\pi x/2} > 0$$

for $0 < x < 1$. This implies that $F'(x)$ is increasing on $[0, 1]$ and $F'(x) > F'(0) = 2 - \pi/2 > 0$ for $0 < x < 1$. Therefore, $F(x)$ is increasing on $[0, 1]$ and $F(x) > F(0) = 0$ for $0 < x < 1$. □

Theorem 2.4. *Let $h(z)$ be analytic on the annulus $B = A(r, R)$ with $0 < r < R < +\infty$ such that $|h(z)| > 1$ on B. Then*

$$\begin{aligned}\log \hat{m}(\rho, h) &\geqslant \exp\left(-\frac{\pi^2}{2}\max\left\{\frac{1}{\log(R/\rho)}, \frac{1}{\log(\rho/r)}\right\}\right)\log M(\rho, h) \\ &\geqslant \min\left\{\frac{\log(\rho/r) - \pi}{\log(\rho/r) + \pi}, \frac{\log(R/\rho) - \pi}{\log(R/\rho) + \pi}\right\}\log M(\rho, h),\end{aligned} \tag{2.7}$$

where $\rho \in (r, R)$ and $\hat{m}(\rho, h) = \min\{|h(z)|\colon |z| = \rho\}$.

Proof. Since $h(z)\colon B \to D = \{w\colon |w| > 1\}$ is analytic, we have

$$\lambda_D(h(z))|h'(z)| \leqslant \lambda_B(z) \quad \forall z \in B.$$

That is to say, from (2.1) and (2.2),

$$\frac{|h'(z)|}{|h(z)|\log|h(z)|} \leqslant \frac{\pi}{\operatorname{mod}(B)|z|\sin(\pi\log(R/|z|)/\operatorname{mod}(B))}. \tag{2.8}$$

For $\rho \in (r, R)$, take two points z_1 and z_2 such that $|z_1| = |z_2| = \rho$ and $|h(z_1)| = \hat{m}(\rho, h), |h(z_2)| = M(\rho, h)$. Let γ be the shorter arc from z_1 to z_2 on $|z| = \rho$. Then on γ we have

$$\begin{aligned}\sin\left(\pi\frac{\log(R/|z|)}{\operatorname{mod}(B)}\right) &= \sin\left(\pi\frac{\log(R/\rho)}{\operatorname{mod}(B)}\right) \\ &\geqslant \frac{2}{\pi}\min\left\{\pi\frac{\log(R/\rho)}{\log(R/r)}, \pi\left(1 - \frac{\log(R/\rho)}{\log(R/r)}\right)\right\} \\ &= \frac{2}{\operatorname{mod}(B)}\min\{\log(R/\rho), \log(\rho/r)\}.\end{aligned}$$

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Integrating both sides of (2.8) along γ yields

$$\log\frac{\log M(\rho,h)}{\log\hat{m}(\rho,h)}\leqslant\frac{\pi}{2\rho\min\{\log(R/\rho),\log(\rho/r)\}}\int_{\gamma}|\mathrm{d}z|\leqslant\frac{\pi^2}{2}\max\left\{\frac{1}{\log(R/\rho)},\frac{1}{\log(\rho/r)}\right\}.$$

And from (2.6) it follows that

$$\begin{aligned}\exp\left(-\frac{\pi^2}{2}\max\left\{\frac{1}{\log(R/\rho)},\frac{1}{\log(\rho/r)}\right\}\right)&=\exp\left(-\frac{\pi}{2}\max\left\{\frac{\pi}{\log(R/\rho)},\frac{\pi}{\log(\rho/r)}\right\}\right)\\&>\frac{\min\{1-\pi/\log(R/\rho),1-\pi/\log(\rho/r)\}}{\max\{1+\pi/\log(R/\rho),1+\pi/\log(\rho/r)\}}\\&=\min\left\{\frac{\log(R/\rho)-\pi}{\log(R/\rho)+\pi},\frac{\log(\rho/r)-\pi}{\log(\rho/r)+\pi}\right\}.\end{aligned}$$

Thus, we obtain (2.7). □

When $\min\{\log(R/\rho),\log(\rho/r)\}>\pi$, as in the proof of [**7**, Theorem 3.1], by applying the Harnack inequality we can prove that $\log\hat{m}(\rho,h)$ is not smaller than the final quantity in (2.7).

3. Nevanlinna theory of meromorphic functions

Since what we study is the dynamics of a transcendental meromorphic function, the basic notation and results of the Nevanlinna theory of meromorphic functions no doubt play a crucial role. (The reader is referred to [**18**] for more on Nevanlinna theory.) Set $\log^+ x=\log\max\{1,x\}$. Let f be a meromorphic function. Define

$$\begin{aligned}m(r,f)&:=\frac{1}{2\pi}\int_0^{2\pi}\log^+|f(re^{\mathrm{i}\theta})|\,\mathrm{d}\theta,\\N(r,f)&:=\int_0^r\frac{n(t,f)-n(0,f)}{t}\,\mathrm{d}t+n(0,f)\log r,\end{aligned}$$

where $n(t,f)$ denotes the number of poles of f counted according to their multiplicities in $\{z\colon |z|<t\}$; sometimes we write $n(t,\infty)$ for $n(t,f)$ and $n(t,0)$ for $n(t,1/f)$ when f is clear in the context, and

$$T(r,f):=m(r,f)+N(r,f).$$

$N(r,f)$ is known as the integrated counting function of poles of f, and $T(r,f)$ is known as the Nevanlinna characteristic of f. Then f is transcendental if and only if $T(r,f)/\log r\to\infty$ $(r\to\infty)$. The following is the Nevanlinna–Jensen formula (see [**18**]):

$$T(r,f)=T\left(r,\frac{1}{f}\right)+\log|c(0)|;$$

that is,

$$\frac{1}{2\pi}\int_0^{2\pi}\log|f(re^{\mathrm{i}\theta})|\,\mathrm{d}\theta=N\left(r,\frac{1}{f}\right)-N(r,f)+\log|c(0)|,$$

where $c(0)$ is the first coefficient of the Laurent series of f at 0.

For $a \in \mathbb{C}$ define

$$\delta(a, f) := \liminf_{r\to\infty} \frac{m(r, 1/(f-a))}{T(r, f)},$$

and $\delta(a, f)$ is called the Nevanlinna deficiency of f at a; if $\delta(a, f) > 0$, then a is called the Nevanlinna deficient value of f, and $\delta(\infty, f)$ is defined by the above equation with $m(r, 1/(f-a))$ replaced by $m(r, f)$. The Nevanlinna characteristic $T(r, f)$ is logarithmically convex.

When f is a transcendental entire function, $\log M(r, f)$ is logarithmically convex. In terms of the convex property, we can obtain a deeper result than Theorem A.

In fact, under the assumption of Theorem A, in [**17**] Zheng proved that given any sufficiently large s, we have a $p > 0$ such that for all n we have

$$U_{p+n} \supset A(M_n(s, f), 8M_n(2s, f)),$$

where $M_n(s, f)$ denotes the nth iterate of $M(s, f)$. We assume without loss of generality that f is analytic on $\{z\colon |z| \geqslant 1\}$. Since $\log M(r, f)$ is convex with respect to $\log r$, we have

$$\log M(Kr, f) \geqslant \left(1 + \frac{\log K}{\log r}\left(1 - \frac{\log M(1, f)}{\log M(r, f)}\right)\right) \log M(r, f) \tag{3.1}$$

for $K > 1$ and $r > 1$. Take $s > M(1, f)$ such that $M(s, f) \geqslant s^2$ and so $\log M_n(s, f) \geqslant 2^n \log s$. Inductively we have

$$\begin{aligned}
\log M_n(2s, f) &\geqslant \left(1 + \frac{\log 2}{\log s} \prod_{k=1}^{n} \left(1 - \frac{\log M(1, f)}{\log M_k(s, f)}\right)\right) \log M_n(s, f) \\
&\geqslant \left(1 + \frac{\log 2}{\log s} \prod_{k=1}^{n} \left(1 - \frac{1}{2^k}\right)\right) \log M_n(s, f) \\
&> \left(1 + \mathrm{e}^{-2} \frac{\log 2}{\log s}\right) \log M_n(s, f).
\end{aligned}$$

Therefore, we have

$$U_{p+n} \supset A(M_n(s, f), M_n(s, f)^b)$$

for some $b > 1$ in Theorem A. The result can also be obtained via Theorem B and Theorem 2.4. In fact, for all sufficiently large r one can find a $z_0 \in U$ such that for some $p > 0$, $|f^{n+p}(z_0)| \geqslant M_n(r, f)$ for all $n \in \mathbb{N}$ (see [**7**, Lemma 2.1]).

In [**7**, Theorem 2.2] a more precise inequality than (3.1) is given for a positive and convex function: given a positive and convex function $\phi(t)$ with $\phi(t)/t \to \infty$ $(t \to \infty)$, for $t \geqslant t_0 > 0$, $\phi(t)/t$ is increasing, and so for $K > 1$, $\phi(Kt) \geqslant K\phi(t)$. Furthermore, for $t_1 > t_2 \geqslant t_0$ we have $\phi(t_1) \geqslant (t_1/t_2)\phi(t_2)$. This implies that $\phi(t_1) > \phi(t_2)$ and $\phi(t_1) > t_1$; that is to say, $\phi(t)$ is increasing and $\phi(t) > t$. Therefore, we have

$$\phi_n(Kt) \geqslant K\phi_n(t), \quad t \geqslant t_0,$$

where $\phi_n(t) = \phi(\phi_{n-1}(t))$, $\phi_0(t) = t$. Indeed, from $\phi(t_1)/\phi(t_2) \geqslant t_1/t_2$ it follows that for $t \geqslant t_0$ we have

$$\phi_n(Kt) = \phi(\phi_{n-1}(Kt)) \geqslant \frac{\phi_{n-1}(Kt)}{\phi_{n-1}(t)}\phi_n(t) \geqslant \frac{\phi(Kt)}{\phi(t)}\phi_n(t) \geqslant K\phi_n(t).$$

Applying the above discussion to $T(\mathrm{e}^t, f)$ with $t = \log r$ for a transcendental meromorphic function f, we have an $r_0 > 0$ such that for $r \geqslant r_0$,

$$T(Kr, f) \geqslant \left(1 + \frac{\log K}{\log r}\right) T(r, f) \tag{3.2}$$

and

$$\hat{T}_n(Kr, f) \geqslant \hat{T}_n(r, f)^{1+\log K/\log r}, \tag{3.3}$$

where $\hat{T}(r, f) = \mathrm{e}^{T(r,f)}$ and $\hat{T}_n(r, f) = \hat{T}(\hat{T}_{n-1}(r, f), f), \hat{T}_0(r, f) = r$. These inequalities will be used in the proof of Theorem 1.1.

4. Proofs of main results

Proof of Theorem 1.1. Under the assumption of Theorem 1.1, in view of a result in [**14**], U is not a Baker domain and it is a wandering domain. Given an arbitrary $C \geqslant \mathrm{e}^2$, under the assumption of Theorem 1.1 we can take an m such that $\mathrm{e}^{-\kappa}(|f^m(a)|/|f^m(b)|)^{1/\delta} \geqslant C^d$, where $\delta = d_D(a, b)$, $d = 7\pi$ and $|f^m(b)|$ is sufficiently large such that for $r \geqslant |f^m(b)|$ inequalities (1.3) or (1.4) hold according to our discussion below. Applying Theorem 2.2 to f^m on D yields

$$U_m \supset f^m(D) \supset A(C^{-d}r_0, C^d r_0), \tag{4.1}$$

where $|f^m(b)| \leqslant r_0 \leqslant |f^m(a)|$.

Let us prove Theorem 1.1 (i). Assume that condition (1.3) of part (i) holds for $r \geqslant s_0$. Since U is wandering, we assume without loss of generality that for every n, $0 \notin f^n(U)$. In view of the Nevanlinna–Jensen formula, for $r \geqslant s_0$ we have

$$\begin{aligned} T(r, f) &= T\left(r, \frac{1}{f}\right) + \log|c(0)| \\ &\geqslant N\left(r, \frac{1}{f}\right) + \log|c(0)| \\ &= N\left(s_0, \frac{1}{f}\right) + \int_{s_0}^{r} \frac{n(t, 0)}{t}\,\mathrm{d}t + \log|c(0)| \\ &\geqslant N(r, f) + d\log r + N\left(s_0, \frac{1}{f}\right) - N(s_0, f) - d\log s_0 + \log|c(0)|. \end{aligned}$$

Therefore, for sufficiently large $r_0 \geqslant C^d s_0$ we have

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(r_0\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}\theta = m(r_0, f) = T(r_0, f) - N(r_0, f) \geqslant \log r_0.$$

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There exists a point z_0 with $|z_0| = r_0$ such that $|f(z_0)| \geqslant r_0$. Since $f(f^m(U)) \cap f^m(U) = \emptyset$, in view of (4.1) together with $|f(z_0)| \geqslant r_0$, we have $|f(z)| \geqslant C^d r_0 > 1$ on $f^m(U)$. Thus, in view of the Nevanlinna–Jensen formula, for $C^{-d} r_0 < r < C^d r_0$ we have

$$T(r,f) = T\left(r, \frac{1}{f}\right) + \log|c(0)| = N\left(r, \frac{1}{f}\right) + \log|c(0)|. \tag{4.2}$$

Set $C_n = C + n$, $n \in \mathcal{N}$, $A_0 = A(C^{-d} r_0, C^d r_0)$ and $B_0 = A(C_1^{-1} r_0, C_1 r_0)$. Since

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(C_1 r_0 \mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta = T(C_1 r_0, f) - N(C_1 r_0, f)$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(C_1^{-1} r_0 \mathrm{e}^{\mathrm{i}\theta})| \, \mathrm{d}\theta = T(C_1^{-1} r_0, f) - N(C_1^{-1} r_0, f),$$

there exist two points z_1 and z_2 with $|z_1| = C_1 r_0$ and $|z_2| = C_1^{-1} r_0$ such that

$$\log|f(z_1)| \geqslant T(C_1 r_0, f) - N(C_1 r_0, f)$$

and

$$\log|f(z_2)| \leqslant T(C_1^{-1} r_0, f) - N(C_1^{-1} r_0, f).$$

Noting that from (4.2) we have

$$\begin{aligned} T(C_1 r_0, f) &- N(C_1 r_0, f) - T(C_1^{-1} r_0, f) + N(C_1^{-1} r_0, f) \\ &= N\left(C_1 r_0, \frac{1}{f}\right) - N(C_1 r_0, f) - N\left(C_1^{-1} r_0, \frac{1}{f}\right) + N(C_1^{-1} r_0, f) \\ &= \int_{C_1^{-1} r_0}^{C_1 r_0} \frac{n(t,0) - n(t,\infty)}{t} \, \mathrm{d}t \\ &\geqslant \int_{C_1^{-1} r_0}^{C_1 r_0} \frac{d}{t} \, \mathrm{d}t \\ &= 2d \log C_1, \end{aligned}$$

we have $|f(z_1)| \geqslant C_1^{2d} |f(z_2)|$. From (2.2), we estimate $\lambda_{A_0}(z)$ for all $z \in B_0$. For $z \in B_0$,

$$\sin\left(\pi \frac{\log(C^d r_0/|z|)}{2d \log C}\right) \geqslant \sin\left(\pi \frac{\log(C^d r_0/C_1 r_0)}{2d \log C}\right) \geqslant \sin\left(\frac{\pi(d-2)}{2d}\right) = \cos\frac{\pi}{d},$$

and so $\lambda_{A_0}(z) \leqslant \pi/|z| 2d \log C \cos(\pi/d)$ for all $z \in B_0$ with $d = 7\pi$ and $C \geqslant \mathrm{e}^2$. Thus, we have

$$\begin{aligned} d_{A_0}(z_1, z_2) &\leqslant \frac{2\pi \log C_1}{2d \log C \cos(\pi/d)} + \frac{\pi^2}{2d \log C \cos(\pi/d)} \\ &\to \frac{\pi}{d \cos(\pi/d)} \\ &= \frac{1}{7\cos(1/7)} < \frac{98}{97 \times 7} < \frac{1}{6} \quad (C \to +\infty). \end{aligned}$$

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In terms of Theorem 2.2, for sufficiently large C we have

$$f(A_0) \supset A(|f(z_2)|, |f(z_1)|). \tag{4.3}$$

There exists an $r_1 > 0$ such that

$$U_{m+1} \supset f^{m+1}(D) \supset f(A_0) \supset A(C_1^{-d} r_1, C_1^d r_1) = A_1,$$

say, and $C_1^{-d} r_1 \geqslant C^d r_0$, $r_1 > C^{2d} r_0$.

We can then continue the above step and inductively we have

$$U_{m+n} \supset f^{m+n}(D) \supset f(A_{n-1}) \supset A_n,$$

$n = 1, 2, \dots$, where $A_{n-1} = A(C_{n-1}^{-d} r_{n-1}, C_{n-1}^d r_{n-1})$, $B_{n-1} = A(C_n^{-1} r_{n-1}, C_n r_{n-1})$, $A_n = A(C_n^{-d} r_n, C_n^d r_n)$, $C_0 = C$ and $r_n \geqslant C^{2nd} r_0 \to \infty$ $(n \to \infty)$. We have proved the existence of the annuli $A_n = A(r_n, R_n)$ with $R_n / r_n \to \infty$ $(n \to \infty)$ such that for all sufficiently large n, $A_n \subset f^n(D) \subset U_n$ and $A_{n+1} \subset f(A_n)$.

Now we want to prove Theorem 1.1 (ii). We have seen that (4.1) holds. Set $\hat{T}(x) = \mathrm{e}^{T(x,f)}$ and $\hat{T}_n(x) = \hat{T}(\hat{T}_{n-1}(x))$. Under assumption (1.4) of Theorem 1.1 (ii), for all sufficiently large r we have

$$T(Cr, f) - N(Cr, f) \geqslant T(2r, f) + d \log C$$

and

$$\begin{aligned} T(C^{-1}r, f) - N(C^{-1}r, f) &\leqslant T(C^{-1}r, f) \\ &\leqslant \left(1 - \frac{\log C}{\log r}\right) T(r, f) \\ &= T(r, f) - \frac{T(r, f)}{\log r} \log C \\ &\leqslant T(r, f) - d \log C, \end{aligned}$$

where (3.2) was used. Set $A_0 = A(C^{-d} r_0, C^d r_0)$ and $B_0 = A(C^{-1} r_0, C r_0)$. By the same argument as in the proof of part (i) with C_1 replaced by C, we have (4.3). It follows from (4.3) that

$$\begin{aligned} f^{m+1}(D) \supset f(A_0) &\supset A(\mathrm{e}^{T(C^{-1}r_0, f) - N(C^{-1}r_0, f)}, \mathrm{e}^{T(Cr_0, f) - N(Cr_0, f)}) \\ &\supset A(C^{-d} \hat{T}(r_0), C^d \hat{T}(2r_0)). \end{aligned}$$

For the next step, set $A_1 = A(C^{-d} r_1, C^d R_1)$ and $B_1 = A(C^{-1} r_1, C R_1)$ with $r_1 = \hat{T}(r_0)$ and $R_1 = \hat{T}(2r_0)$. Then for A_1 and B_1 we have $f(A_1) \supset A_2$, where $A_2 = A(C^{-d} r_2, C^d R_2)$, $r_2 = \hat{T}_2(r_0)$, $R_2 = \hat{T}_2(2r_0)$. Thus, inductively we have

$$f^{m+n}(D) \supset f(A_{n-1}) \supset A_n,$$

where $A_n = A(C^{-d} r_n, C^d R_n)$, $r_n = \hat{T}_n(r_0)$, $R_n = \hat{T}_n(2r_0)$.

Therefore, in view of (3.3), we have proved that for all sufficiently large n, $f^n(D)$ contains the annulus $\hat{A}_n = A(T_n, T_n^c)$ with $c \geqslant 1 + \log 2 / \log r_0 > 1$ and $T_n \to \infty$ $(n \to \infty)$ and $\hat{A}_{n+1} \subset f(\hat{A}_n)$. □

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Here, we give a condition such that (1.4) holds. If for all sufficiently large r,

$$T(r,f) \geqslant N(r,f)\log r,$$

then (1.4) holds. Indeed, in view of (3.2) we have

$$\begin{aligned} T(Cr,f) - N(Cr,f) &\geqslant \left(1 - \frac{1}{\log Cr}\right) T(Cr,f) \\ &\geqslant \left(1 - \frac{1}{\log Cr}\right)\left(1 + \frac{\log C/2}{\log 2r}\right) T(2r,f) \\ &\geqslant T(2r,f) + \tfrac{1}{2}\log \tfrac{1}{2}C \frac{T(2r,f)}{\log 2r} \\ &\geqslant T(2r,f) + d\log C. \end{aligned}$$

In the proof of Theorem 1.3, another condition such that (1.4) holds is given.

Proof of Theorem 1.3. Under the assumptions of Theorem 1.3, for sufficiently large C such that

$$\frac{\log C}{\log 2} > \frac{\log 4 - \log \hat{\delta}}{\log \lambda} + 2,$$

where $\hat{\delta} = \delta(\infty, f)$ (so $(\hat{\delta}/2)\lambda^n \geqslant 2$, $n = [\log C/\log 2] - 1$), and for $r \geqslant r_1 \geqslant r_0$ such that $T(2r,f) \geqslant d\log C$, we have

$$T(Cr,f) - N(Cr,f) = m(Cr,f) \geqslant \frac{\hat{\delta}}{2} T(Cr,f) \geqslant \frac{\hat{\delta}}{2}\lambda^n T(2r,f) \geqslant T(2r,f) + d\log C.$$

Suppose that Theorem 1.3 does not hold, so there exists a compact subset W in a Fatou component U of f such that for any sufficiently large $M > 1$ we have two points $a, b \in W$ and a positive integer m such that $M|f^m(b)| < |f^m(a)|$. Then, as in the proof of Theorem 1.1 (ii), we can prove that for all sufficiently large n we have

$$f^n(U) \supset A_n, \quad A_n = A(r_n, r_n^{\alpha}), \quad \alpha > 1.$$

Using the same argument as in the proof of [**17**, Corollary 5], we can derive a contradiction. For completeness, we give its proof. For all sufficiently large n, $|f(z)| > 1$ on $f^n(U)$, and therefore $n(r_n^{\alpha}, 0) = n(r_n, 0)$, and for $r_n < r < r_n^{\alpha}$, $m(r, 1/f) = 0$. In view of the Nevanlinna–Jensen formula, we have

$$\begin{aligned} T(r_n^{\alpha}, f) - \log|c(0)| &= T\left(r_n^{\alpha}, \frac{1}{f}\right) \\ &= N\left(r_n^{\alpha}, \frac{1}{f}\right) \\ &= N\left(r_n, \frac{1}{f}\right) + \int_{r_n}^{r_n^{\alpha}} \frac{n(t,0) - n(0,0)}{t}\,\mathrm{d}t + n(0,0)\log r_n^{\alpha-1} \end{aligned}$$

$$
\begin{aligned}
&= N\left(r_n, \frac{1}{f}\right) + (\alpha - 1) n(r_n, 0) \log r_n \\
&\leqslant N\left(r_n, \frac{1}{f}\right) + (\alpha - 1) N\left(\mathrm{e} r_n, \frac{1}{f}\right) \log r_n \\
&\leqslant (1 + (\alpha - 1)\log r_n) T\left(\mathrm{e} r_n, \frac{1}{f}\right) \\
&\leqslant (1 + (\alpha - 1)\log r_n)(T(\mathrm{e} r_n, f) - \log|c(0)|).
\end{aligned}
$$

On the other hand, we have

$$
T(r_n^{\alpha}, f) \geqslant T(2^m(\mathrm{e} r_n), f) \geqslant \lambda^m T(\mathrm{e} r_n, f) \geqslant K r_n^c T(\mathrm{e} r_n, f),
$$

where

$$
m = \left[\frac{\alpha \log r_n - 1}{\log 2}\right], \qquad K = \lambda^{-2}, \qquad c = \frac{\alpha \log \lambda}{\log 2} > 0.
$$

Thus, we have

$$
(1 + (\alpha - 1)\log r_n)(T(\mathrm{e} r_n, f) - \log|c(0)|) \geqslant K r_n^c T(\mathrm{e} r_n, f) - \log|c(0)|.
$$

This is impossible since $r_n \to \infty$ $(n \to \infty)$. Theorem 1.3 follows. □

Proof of Corollary 1.2. In view of Theorem 1.1 (i), for all sufficiently large n, $f^n(U) \supset A_n = A(r_n, R_n)$ with $R_n/r_n \to \infty$ $(n \to \infty)$. Take a $C > 1$ such that

$$
0 < c < \frac{\log C - \pi}{\log C + \pi} < 1, \quad d\,\frac{\log C - \pi}{\log C + \pi} \geqslant 3 \quad \text{and} \quad A_n \supset A(C^{-1} a_n, C a_n)
$$

for some $a_n \notin E$. Since $|f(z)| > 1$ on A_n, in view of Theorem 2.4 we have

$$
\log \hat{m}(a_n, f) \geqslant \alpha \log M(a_n, f), \quad \alpha = \frac{\log C - \pi}{\log C + \pi}.
$$

We have

$$
\begin{aligned}
\log M(a_n, f) \geqslant m(a_n, f) &= T(a_n, f) - N(a_n, f) \\
&= N\left(a_n, \frac{1}{f}\right) - N(a_n, f) + \log|c(0)| \\
&\geqslant d \log a_n + \log|c(0)| \to \infty \quad (n \to \infty).
\end{aligned}
$$

Thus, on $|z| = a_n$ we have

$$
\begin{aligned}
|f(z) - g(z)| &\geqslant |f(z)| - |g(z)| \\
&\geqslant \hat{m}(a_n, f) - M(a_n, g) \\
&\geqslant M(a_n, f)^{\alpha} - M(a_n, f)^{c} \\
&\sim M(a_n, f)^{\alpha} \to \infty \quad (n \to \infty).
\end{aligned}
$$

This implies that, for all sufficiently large n, on $|z| = a_n$ we have $|f(z) - g(z)| > 1$, and $\log^+ 1/|f(z) - g(z)| = 0$, and so

$$m\left(a_n, \frac{1}{f-g}\right) = 0 \quad \text{and} \quad \delta(0, f-g) = 0.$$

In particular, on $|z| = a_n$ we have

$$|f(z) - z| \geqslant |f(z)| - |z| \geqslant M(a_n, f)^\alpha - a_n \geqslant |c(0)|a_n^{d\alpha} - a_n \geqslant a_n^2 > 1,$$

so $\delta(0, f - z) = 0$, and basically we have proved that

$$\limsup_{r\to\infty} \frac{\hat{m}(r, f)}{r} = +\infty.$$

As pointed out in [**16**, p. 11], using the method of [**13**] and [**6**] we can prove that if

$$\limsup_{r\to\infty} \frac{\hat{m}(r, f)}{r} > 1, \tag{4.4}$$

then f has infinitely many repelling fixed points or indifferent fixed points with multiplier equal to 1.

Actually, from (4.4) we have a sequence of increasing positive numbers $\{r_n\}$ tending to ∞ such that for some $k > 0$, $\hat{m}(r_n, z - f) \geqslant kr_n$. Then

$$\left|\frac{1}{2\pi \mathrm{i}} \int_{|z|=r_n} \frac{\mathrm{d}z}{z - f(z)}\right| \leqslant \frac{1}{2\pi} \int_{|z|=r_n} \frac{|\mathrm{d}z|}{|z - f(z)|} \leqslant \frac{1}{k}.$$

Since $\delta(0, f - z) = 0$, f has infinitely many fixed points $z_1, z_2, \dots$. In view of the residue theorem we have

$$\frac{1}{2\pi \mathrm{i}} \int_{|z|=r_n} \frac{\mathrm{d}z}{z - f(z)} = \sum_{|z_j|<r_n} \operatorname{Res}\left(\frac{1}{z - f(z)}, z_j\right).$$

However, for a z_j with $|f'(z_j)| \leqslant 1$ and $f'(z_j) \neq 1$, we have

$$\operatorname{Re}\left[\operatorname{Res}\left(\frac{1}{z - f(z)}, z_j\right)\right] = \operatorname{Re}\left(\frac{1}{1 - f'(z_j)}\right) \geqslant \tfrac{1}{2}.$$

This together with the above two equations implies our desired result. □

5. Examples

In this section we construct some examples of transcendental meromorphic functions to illustrate the above results in view of the following theorem.

Theorem 5.1 (Runge's theorem (see Rudin [12])). *Let W be a compact set on the complex plane and let $h(z)$ be analytic on W. Assume that E is a set that intersects every component of $\mathbb{C} \setminus W$. Then for any $\varepsilon > 0$ there exists a rational function $R(z)$ such that all poles of $R(z)$ lie in E and*

$$|h(z) - R(z)| < \varepsilon \quad \forall z \in W.$$

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For any positive integer p, in [**2**] Baker *et al*. constructed a transcendental meromorphic function f that has a multiply connected wandering domain U of connectivity p such that every f^n is univalent on U. However, no U_n separates 0 and ∞. Dominguez [**9**] found a transcendental meromorphic function g that has a multiply connected wandering domain V such that $f^{2n}|_V \to \infty$, $f^{2n+1}|_V \to 0$ as $n \to \infty$, and every V_n separates 0 and ∞. For an open set Ω and $a \notin \Omega$ we define

$$\mathrm{Mod}_a^0(\Omega) = \sup\{\mathrm{mod}(A)\colon A \text{ is a round annulus centred at } a \text{ in } \Omega\}$$

and

$$\mathrm{Mod}_a(\Omega) = \sup\{\mathrm{mod}(A)\colon A \text{ is a doubly connected domain in } \Omega \text{ and goes around } a\}.$$

We have that $\mathrm{Mod}_a^0(\Omega) \leqslant \mathrm{Mod}_a(\Omega) \leqslant \mathrm{Mod}_a^0(\Omega) + C$, where C is an absolute constant (see [**15**]). Throughout this section, we denote by $B(a, r)$ the disk centred at a with radius r.

Example 5.2. There exists a meromorphic function f that has a multiply connected wandering domain U such that $f^n|_U \to \infty$ $(n \to \infty)$ and every U_n separates 0 and ∞ with $\sup_n \mathrm{Mod}_0(U_n) < \infty$. (Thus, (1.1) holds for any compact subset W of U.)

Proof. Take a sequence of positive numbers $\{r_n\}$ such that

$$r_1 > 10, \quad r_{n+1} > \exp r_n,$$

and a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and $\varepsilon_1 < \frac{1}{2}$. Set $\delta_n = r_{n+1}/r_n$,

$$B_n = A(\tfrac{1}{3}r_n, 3r_n), \qquad A_n = \bar{B}(0, \tfrac{1}{10}r_n)$$

and

$$C_n = \{z\colon |z| = 5r_n \text{ or } |z| = \tfrac{1}{5}r_n\}.$$

In view of Runge's theorem, we have a rational function $f_1(z)$ such that

$$|f_1(z)| < \varepsilon_1 \quad \forall z \in A_1, \qquad |f_1(z) - \delta_1 z| < \varepsilon_1 \quad \forall z \in B_1,$$
$$|f_1(z)| < \varepsilon_1 \quad \forall z \in C_1,$$

and inductively we have a rational function $f_{n+1}(z)$ such that

$$|f_{n+1}(z)| < \varepsilon_{n+1} \quad \forall z \in A_{n+1}, \qquad \left|\sum_{k=1}^{n+1} f_k(z) - \delta_{n+1} z\right| < \varepsilon_{n+1} \quad \forall z \in B_{n+1}$$

and

$$\left|\sum_{k=1}^{n+1} f_k(z)\right| < \varepsilon_{n+1} \quad \forall z \in C_{n+1}.$$

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Write $f(z)=\sum_{n=1}^{\infty} f_n(z)$. Since the series is uniformly convergent on any compact subset of $\mathbb{C}$, $f(z)$ is a meromorphic function on $\mathbb{C}$.

It is obvious that $f(B(0,1))\subset B(0,1)$, and so $B(0,1)\subset F(f)$. For $z\in B_{n+1}$, we have

$$\begin{aligned}|f(z)-\delta_{n+1}z| &\leqslant \sum_{k=n+2}^{\infty}|f_k(z)|+\left|\sum_{k=1}^{n+1} f_k(z)-\delta_{n+1}z\right|\\ &< \sum_{k=n+1}^{\infty}\varepsilon_k\\ &< 2\varepsilon_{n+1},\end{aligned}$$

so we write $f(z)=\delta_{n+1}z+\eta_{n+1}$, $z\in B_{n+1}$, with $|\eta_{n+1}|<2\varepsilon_{n+1}$. Set $D_1=A(\frac{1}{2}r_1,2r_1)$. For $z\in D_1$ we have

$$|f(z)|\geqslant \delta_1|z|-|\eta_1|>\tfrac{1}{2}\delta_1 r_1-2\varepsilon_1=\tfrac{1}{2}r_2\left(1-\frac{4\varepsilon_1}{r_2}\right)>\tfrac{1}{3}r_2$$

and

$$|f(z)|\leqslant \delta_1|z|+|\eta_1|<2\delta_1 r_1+2\varepsilon_1=2r_2\left(1+\frac{\varepsilon_1}{r_2}\right)<3r_2.$$

Therefore, $f(D_1)\subset B_2$. Inductively, suppose that $f^{n-1}(D_1)\subset B_n$. For $z\in D_1$ we have

$$|f^n(z)|\geqslant \tfrac{1}{2}r_{n+1}\left(1-\sum_{k=1}^{n}\frac{4\varepsilon_k}{r_{k+1}}\right)>\tfrac{1}{3}r_{n+1}$$

and

$$|f^n(z)|\leqslant 2r_{n+1}\left(1+\sum_{k=1}^{n}\frac{\varepsilon_k}{r_{k+1}}\right)<3r_{n+1}.$$

This implies that $f^n(D_1)\subset B_{n+1}$. Set $D_n=A(\frac{1}{2}r_n,2r_n)$. In view of the same arguments as above, we have $f^m(D_n)\subset B_{n+m}$. Therefore, D_1 is contained in a Fatou component U of f and $D_n\subset U_{n-1}$. Thus, $f^n|_U\to\infty$ $(n\to\infty)$. According to the construction of f, on C_n we have

$$|f(z)|\leqslant\left|\sum_{k=1}^{n} f_k(z)\right|+\left|\sum_{k=n+1}^{\infty} f_k(z)\right|\leqslant\sum_{k=n}^{\infty}\varepsilon_k<1$$

so that $f(C_n)\subset B(0,1)$ and obviously $B(0,1)\cap U_n=\emptyset$. Thus,

$$U_n\subset A(\tfrac{1}{5}r_{n+1},5r_{n+1})$$

and U_n is wandering with $2\log 2\leqslant \mathrm{Mod}_0(U_n)\leqslant 2\log 5$. □

Example 5.3. There exists a transcendental meromorphic function f that has a wandering domain U such that U_n contains a round annulus D_n centred at 0 and $\mathrm{mod}(D_n)\to\infty$ $(n\to\infty)$, and there are two points $a,b\in U$ such that $|f^n(a)|/|f^n(b)|\to\infty$ $(n\to\infty)$ but $h_U(z)\equiv 1$ on U.

Proof. Take a sequence of positive numbers $\{r_n\}$ such that

$$r_1 > 10^2, \quad r_{n+1} > \exp r_n,$$

and a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and $\varepsilon_1 < \frac{1}{2}$. Set $\delta_n = r_{n+1}/r_n^2$,

$$B_n = A(3^{-2^n} r_n, 3^{2^n} r_n), \qquad A_n = \bar{B}(0, 10^{-2^n} r_n)$$

and

$$C_n = \{z\colon |z| = 5^{2^n} r_n \text{ or } |z| = 5^{-2^n} r_n\}.$$

In view of Runge's theorem, we have a rational function $f_1(z)$ such that

$$|f_1(z)| < \varepsilon_1 \quad \forall z \in A_1, \qquad |f_1(z) - \delta_1 z^2| < \varepsilon_1 \quad \forall z \in B_1,$$
$$|f_1(z)| < \varepsilon_1 \quad \forall z \in C_1,$$

and inductively we have a rational function $f_{n+1}(z)$ such that

$$|f_{n+1}(z)| < \varepsilon_{n+1} \quad \forall z \in A_{n+1}, \qquad \left|\sum_{k=1}^{n+1} f_k(z) - \delta_{n+1} z^2\right| < \varepsilon_{n+1} \quad \forall z \in B_{n+1}$$

and

$$\left|\sum_{k=1}^{n+1} f_k(z)\right| < \varepsilon_{n+1} \quad \forall z \in C_{n+1}.$$

Write $f(z) = \sum_{n=1}^{\infty} f_n(z)$. Since the series is uniformly convergent on any compact subset of $\mathbb{C}$, $f(z)$ is a meromorphic function on $\mathbb{C}$.

It is obvious that $f(B(0,1)) \subset B(0,1)$, and so $B(0,1) \subset F(f)$. For $z \in B_{n+1}$ we have

$$\begin{aligned}|f(z) - \delta_{n+1} z^2| &\leqslant \sum_{k=n+2}^{\infty} |f_k(z)| + \left|\sum_{k=1}^{n+1} f_k(z) - \delta_{n+1} z^2\right| \\ &< \sum_{k=n+1}^{\infty} \varepsilon_k \\ &< 2\varepsilon_{n+1},\end{aligned}$$

so we write $f(z) = \delta_{n+1} z^2 + \eta_{n+1}$, $z \in B_{n+1}$, with $|\eta_{n+1}| < 2\varepsilon_{n+1}$. Set $D_1 = \{z\colon \frac{1}{2} r_1 < |z| < 2r_1\}$. For $z \in D_1$ we have

$$|f(z)| \geqslant \delta_1 |z|^2 - |\eta_1| > \frac{1}{2^2}\delta_1 r_1^2 - 2\varepsilon_1 = \frac{1}{2^2} r_2 \left(1 - \frac{2^3 \varepsilon_1}{r_2}\right) > \frac{1}{3^2} r_2$$

and

$$|f(z)| \leqslant \delta_1 |z|^2 + |\eta_1| < 2^2 \delta_1 r_1^2 + 2\varepsilon_1 = 2^2 r_2 \left(1 + \frac{\varepsilon_1}{2r_2}\right) < 3^2 r_2.$$

Therefore, $f(D_1) \subset B_2$. Inductively, suppose that $f^{n-1}(D_1) \subset B_n$. For $z \in D_1$ we have

$$|f^n(z)| \geqslant 2^{-2^n} r_{n+1} \prod_{k=1}^{n} \left(1 - \frac{2^{2^k+1}\varepsilon_k}{r_{k+1}}\right)^{2^{n-k}} > 3^{-2^n} r_{n+1}$$

and

$$|f^n(z)| \leqslant 2^{2^n} r_{n+1} \prod_{k=1}^{n} \left(1 + \frac{\varepsilon_k}{2^{2^{k-1}} r_{k+1}}\right)^{2^{n-k}} < 3^{2^n} r_{n+1}.$$

This implies that $f^n(D_1) \subset B_{n+1}$. Set $D_n = A(2^{-2^n} r_n, 2^{2^n} r_n)$. In view of the same arguments as above, we have $f^m(D_n) \subset B_{n+m}$. Therefore, we have a Fatou component U of f such that $D_n \subset U_{n-1}$. Thus, $f^n|_U \to \infty$ $(n \to \infty)$. According to the construction of f, on C_n we have

$$|f(z)| \leqslant \left|\sum_{k=1}^{n} f_k(z)\right| + \left|\sum_{k=n+1}^{\infty} f_k(z)\right| \leqslant \sum_{k=n}^{\infty} \varepsilon_k < 1$$

so that $f(C_n) \subset B(0,1)$ and obviously $B(0,1) \cap U_n = \emptyset$. Thus,

$$U_n \subset A(5^{-2^{n+1}} r_{n+1}, 5^{2^{n+1}} r_{n+1})$$

and U_n is wandering. A simple calculation shows that $\operatorname{mod}(D_n) = 2^{n+1}\log 2 \to \infty$ $(n \to \infty)$ and for any two points z_1 and z_2 in U we have

$$\frac{\log r_{n+1} - 2^{n+1}\log 5}{\log r_{n+1} + 2^{n+1}\log 5} \leqslant \frac{\log|f^n(z_1)|}{\log|f^n(z_2)|} \leqslant \frac{\log r_{n+1} + 2^{n+1}\log 5}{\log r_{n+1} - 2^{n+1}\log 5}.$$

This implies that $h_U(z) \equiv 1$ on U.

For a with $|a| = 2r_1$ we have

$$|f^n(a)| \geqslant 2^{2^n} r_{n+1} \prod_{k=1}^{n} \left(1 - \frac{\varepsilon_k}{2^{2^{k-1}} r_{k+1}}\right)^{2^{n-k}} \geqslant (\tfrac{3}{2})^{2^n} r_{n+1}$$

and for b with $|b| = \frac{1}{2} r_1$ we have

$$|f^n(b)| \leqslant 2^{-2^n} r_{n+1} \prod_{k=1}^{n} \left(1 + \frac{2^{2^k+1}\varepsilon_k}{r_{k+1}}\right)^{2^{n-k}} \leqslant r_{n+1}.$$

Thus,

$$\frac{|f^n(a)|}{|f^n(b)|} \geqslant (\tfrac{3}{2})^{2^n} \to \infty.$$

However, U_n contains no annulus of the form $A(T_n, T_n^{\alpha})$ for a fixed $\alpha > 1$. This shows the necessity of (1.4) in Theorem 1.1. □

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By the above method, we can construct a transcendental meromorphic function g such that g has a wandering domain U with the following properties: $g^n|_U \to \infty$ $(n \to \infty)$,

$$\sup_n \operatorname{Mod}_0(U_{2n}) < \infty \quad \text{but} \quad \sup_n \operatorname{Mod}_0(U_{2n+1}) = \infty.$$

Thus, h_U does not exist on U and, actually, $\bar{h}_U$ is a non-constant harmonic function on U and $\underline{h}_U(z) \equiv 1$ for $z \in U$.

We can also construct a transcendental meromorphic function with a wandering domain U such that for some $\beta > \alpha > 1$, $U_{2n-1} \supset A(r_n, r_n^{\alpha})$, $U_{2n-1} \not\supset A(r_n, r_n^{\beta})$ and $U_{2n} \supset A(r_{n+1}, r_{n+1}^{\beta})$.

Example 5.4. There exists a meromorphic function f that has a multiply connected wandering domain U such that $f^{2n}|_U \to \infty$, $f^{2n-1}|_U \to 0$ $(n \to \infty)$ and U_{2n} contains a round annulus $A_n = A(r_n, R_n)$ with $\operatorname{mod}(A_n) \to \infty (n \to \infty)$. Also, there exist two points $a, b \in U_{2n}$ such that $|f^{2n}(a)|/|f^{2n}(b)| \to \infty$ $(n \to \infty)$. (This shows the necessity of (1.3) in Theorem 1.1.)

Proof. Take a sequence of positive numbers $\{r_n\}$ such that

$$200 < r_1 \leqslant 250, \quad r_{n+1} > \exp r_n,$$

and a sequence of positive numbers $\{\varepsilon_n\}$ such that $\varepsilon_{n+1} < \frac{1}{2}\varepsilon_n$ and $\varepsilon_1 < \frac{1}{2}$. Set $\delta_n = r_n^2/r_{n+1}$,

$$B_n = A(3^{-2^n} r_n, 3^{2^n} r_n), \qquad A_n = \bar{B}(0, 10^{-2^n} r_n)$$

and

$$C_n = \{z \colon |z| = 5^{2^n} r_n \text{ or } |z| = 5^{-2^n} r_n\}.$$

In view of Runge's theorem, there exists a rational function $f_1(z)$ such that

$$\left|f_1(z) - \frac{1}{z}\right| < \varepsilon_1 \quad \forall z \in A_1, \qquad |f_1(z) - 5| < \varepsilon_1 \quad \forall z \in B(5,2) \cup C_1,$$

$$\left|f_1(z) - \frac{\delta_1}{z^2}\right| < \varepsilon_1 \quad \forall z \in B_1.$$

Inductively, we have a rational function $f_{n+1}(z)$ such that

$$|f_{n+1}(z)| < \varepsilon_{n+1} \quad \forall z \in A_{n+1} \cup C_{n+1},$$

$$\left|\sum_{k=1}^{n+1} f_k(z) - \delta_{n+1} z^{-2}\right| < \varepsilon_{n+1} \quad \forall z \in B_{n+1}.$$

Define $f(z) = \sum_{n=1}^{\infty} f_n(z)$; hence, $f(z)$ is a meromorphic function on $\mathbb{C}$. It is easy to see that $f(B(5,2)) \subset B(5,2)$ and $B(5,2) \subset F(f)$ so that $C_n \subset F(f)$. We also have

$$\left|f(z) - \frac{1}{z}\right| \leqslant \sum_{k=2}^{\infty} \varepsilon_k < \varepsilon_1, \qquad z \in B(0,1),$$

$$\left|f(z) - \delta_{n+1}\frac{1}{z^2}\right| \leqslant \sum_{k=n+1}^{\infty} \varepsilon_k \leqslant 2\varepsilon_{n+1}, \quad z \in B_{n+1},$$

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so we can write

$$f(z) = \frac{1}{z} + \eta, \quad |\eta| < \varepsilon_1,\ z \in B(0,1),$$

and

$$f(z) = \delta_{n+1}\frac{1}{z^2} + \eta_{n+1}, \quad |\eta_{n+1}| < 2\varepsilon_{n+1},\ z \in B_{n+1}.$$

For $z \in B_{n+1}$ we estimate

$$|f(z)| \leqslant \frac{\delta_{n+1}}{|z|^2} + |\eta_n| < \frac{3^{2^{n+2}}}{r_{n+2}} + 2\varepsilon_{n+1} < 1,$$

and so $f(B_{n+1}) \subset B(0,1)$. Thus, for $z \in B_{n+1}$ we can write

$$f^2(z) = \frac{z^2}{\delta_{n+1}}(1 + \eta_{n+1}z^2\delta_{n+1}^{-1})^{-1} + \eta.$$

Set

$$D_n = A(2^{-2^n}r_n, 2^{2^n}r_n).$$

For $z \in D_1$, we have

$$\begin{aligned}
|f^2(z)| &\geqslant \frac{|z|^2}{\delta_1}(1 + 2\varepsilon_1|z|^2\delta_1^{-1})^{-1} - \varepsilon_1 \\
&\geqslant 2^{-2^2}r_2(1 + 2\varepsilon_1 3^{2^2}r_2)^{-1} - \varepsilon_1 \\
&\geqslant 2^{-2^2}r_2\left[(1 + 2\varepsilon_1 3^{2^2}r_2)^{-1} - 2^{2^2}\frac{\varepsilon_1}{r_2}\right] \\
&> 3^{-2^2}r_2
\end{aligned}$$

and

$$\begin{aligned}
|f^2(z)| &\leqslant \frac{|z|^2}{\delta_1}(1 - 2\varepsilon_1|z|^2\delta_1^{-1})^{-1} + \varepsilon_1 \\
&\leqslant 2^{2^2}r_2(1 - 2\varepsilon_1 3^{2^2}r_2)^{-1} + \varepsilon_1 \\
&\leqslant 2^{2^2}r_2\left[(1 - 2\varepsilon_1 3^{2^2}r_2)^{-1} - 2^{-2^2}\frac{\varepsilon_1}{r_2}\right] \\
&< 3^{2^2}r_2.
\end{aligned}$$

Therefore, $f^2(D_1) \subset B_2$. Inductively, for $z \in D_1$ we have

$$\begin{aligned}
|f^{2n}(z)| &\geqslant 2^{-2^{n+1}}r_{n+1}\prod_{k=1}^{n}\left(\frac{1}{1 + 3^{2^{k+1}}\varepsilon_k r_k} - \frac{3^{2^{k+1}}\varepsilon_1}{r_k}\right)^{2^{n-k}} \\
&> 3^{-2^{n+1}}r_{n+1}
\end{aligned}$$

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and

$$|f^{2n}(z)| \leqslant 2^{2^{n+1}} r_{n+1} \prod_{k=1}^{n} \left(\frac{1}{1 - 3^{2^{k+1}} \varepsilon_k r_k} - \frac{3^{-2^{k+1}} \varepsilon_1}{r_k} \right)^{2^{n-k}}$$
$$< 3^{2^{n+1}} r_{n+1}.$$

Thus, $f^{2n}(D_1) \subset B_{n+1}$. A similar calculation yields $f^{2m}(D_n) \subset B_{n+m}$. Let U_1 be the Fatou component of f containing D_1. It is easy to see that $f^{2n}|_{U_1} \to \infty$ $(n \to \infty)$ and $f^{2n-1}|_{U_1} \to 0$ $(n \to \infty)$.

Obviously, U_1 is not a Baker domain, and it is a wandering domain of f. From the above result, we have $D_n \subset U_{2n}$ and $\mathrm{mod}(D_n) \to \infty$ $(n \to \infty)$.

A suitable calculation, like that in the construction of Example 5.3, yields that for a with $|a| = 2r_1$ and b with $|b| = 2^{-1} r_1$ we have $|f^{2n}(a)|/|f^{2n}(b)| \to \infty$ $(n \to \infty)$. □

Remark. In [**16**, Theorem 3.2.9] Zheng proved that for a wandering domain U of a transcendental meromorphic function f, one of the following cases holds:

(1) every U_n is of infinite connectivity;

(2) for all sufficiently large n, U_n is simply or doubly connected;

(3) for all sufficiently large n, U_n is of p (greater than or equal to 3) connectivity, and $f\colon U_n \to U_{n+1}$ is conformal.

Therefore, the wandering domains in Examples 5.3 and 5.4 are of infinite or double connectivity. Noting that $f(z) \approx \delta_{n+1} z^2$ in Example 5.3 and $f^2(z) \approx \delta_{n+1}^{-1} z^2$ in B_{n+1} in Example 5.4, through a complicated construction we have that the wandering domains in Examples 5.3 and 5.4 are doubly connected. However, the wandering domains in Example 5.2 can be constructed to be of any connectivity.

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