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GRASSMANNIAN SEMIGROUPS AND THEIR REPRESENTATIONS

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Abstract

The set of row reduced matrices (and of echelon form matrices) is closed under multiplication. We show that any system of representatives for the $\mathrm{Gl}_n(\mathbb{K})$ action on the $n \times n$ matrices, which is closed under multiplication, is necessarily conjugate to one that is in simultaneous echelon form. We call such closed representative systems Grassmannian semigroups. We study internal properties of such Grassmannian semigroups and show that they are algebraic semigroups and admit gradings by the finite semigroup of partial order preserving permutations, with components that are naturally in one–one correspondence with the Schubert cells of the total Grassmannian. We show that there are infinitely many isomorphism types of such semigroups in general, and two such semigroups are isomorphic exactly when they are semiconjugate in $M_n(\mathbb{K})$. We also investigate their representation theory over an arbitrary field, and other connections with multiplicative structures on Grassmannians and Young diagrams.

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1. Introduction and preliminaries

Let $\mathbb{K}$ be an infinite field. Consider the left regular action of the general linear group $\mathrm{Gl}_n(\mathbb{K})$ on the matrices $M_n(\mathbb{K})$. A very important set of matrices is the set $\mathcal{R}$ of row reduced matrices, which is a standard system of representatives for this action. Row reduction is the basic algorithm for solving linear systems. Moreover, $\mathcal{R}$ has an additional remarkable property: it is closed under the multiplication of matrices. Strangely, this basic linear algebra fact is not as well known as one would expect; we have only noted this fact in one textbook [10, Exercise 2.19]; it is also noted in [11, page 67] (see also [1]). The fact that a product of row reduced matrices is row reduced has a geometric consequence. Consider V, the n-dimensional space over $\mathbb{K}$, with a fixed basis $\{e_1, \ldots, e_n\}$, regarded as the vector space of column vectors with n entries, and $M_n(\mathbb{K})$ acting as endomorphisms on the left. Then each $\mathrm{Gl}_n(\mathbb{K})$

orbit $\mathcal{O}$ corresponds uniquely to a subspace of $\mathbb{K}^n$, by $\mathcal{O} = \mathrm{Gl}_n(\mathbb{K}) \cdot A \to \ker(A)$ (since $\ker(A)$ is uniquely determined for $A \in \mathcal{O}$). Hence, the multiplication of row reduced matrices induces a multiplication on the total Grassmannian $\mathrm{Gr}_{\mathbb{K}}(n)$ via the bijection $\mathcal{R} \longleftrightarrow \mathrm{Gr}_{\mathbb{K}}(n)$, $A \longleftrightarrow \ker(A)$. Thus, $\mathcal{R}$ and $\mathrm{Gr}_{\mathbb{K}}(n)$ form a semigroup and, in fact, an algebraic semigroup. This motivates the introduction of the following terminology.

Definition 1.1. We say that S is a Grassmannian semigroup if S is a system of representatives for the left regular $\mathrm{Gl}_n(\mathbb{K})$ action on $M_n(\mathbb{K})$ such that S is also closed under multiplication of matrices. Equivalently, $\mathcal{S}$ is a subsemigroup of $(M_n(\mathbb{K}), \cdot)$ such that for every subspace W of V, there is a unique element $A \in S$ such that $\ker(A) = W$.

Grassmannian semigroups were also studied in [1]. On the other hand, the subject of semigroups and, more prominently, algebraic semigroups has grown a lot in recent years, and several detailed monographs or review papers have been dedicated to this [7, 13, 15, 17].

Apart from this, the starting motivating questions of this paper are: what other special systems of representatives for the $\mathrm{Gl}_n(\mathbb{K})$ action can be found or, equivalently, what other natural multiplicative structures on $\mathrm{Gr}_{\mathbb{K}}(n)$ compatible with the natural matrix multiplication can be found? What is special about $\mathcal{R}$, and is $\mathcal{R}$ somehow canonical? This is also interesting from the perspective of row reduction: what are the interesting canonical forms for row reduction, and what are all possible forms for row reduction?

Our first main result addresses this question. In Proposition 2.9 and Theorem 2.6, we show that every such Grassmannian semigroup is conjugate to a Grassmannian semigroup which is in simultaneous echelon form with all initial pivots equal to 1 (that is, a Grassmannian semigroup consisting only of row echelon form matrices), but where entries above pivots are not necessarily equal to 0. This is a consequence of a peculiarity of such semigroups, namely, that if two elements in a Grassmannian semigroup have the same rank, then they must have the same range (column space); in other words, two elements of a Grassmannian semigroup are equivalent with respect to the right Green relation [12, 13] as soon as they have the same rank. Thus, from the point of view of the right Green equivalence, such semigroups have very few equivalence classes, while the left Green equivalence (under which two elements are equivalent when they have the same kernel) is 'as large as possible': they contain exactly one element in each such left equivalence class. Beyond being conjugate to a semigroup in echelon form, we prove that there is a certain canonical form for each such semigroup, which generalizes row reduced form for matrices. In particular, we are able to describe all such Grassmannian semigroups of orders 2 and 3. We note that there are many results on simultaneous upper triangular form of sets of linear transformations (see [5, 6, 12] and references therein and the textbooks [16, 17]). Our result is of this flavor, except that it yields a more precise form which is not as common, and is more combinatorial: simultaneous echelon form. The condition we have – a system of representatives of the Gl_n action – however, is of a different type than what simultaneous triangular form results usually require.

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For each row echelon form matrix, we can define its type to be the set of positions of columns containing pivots. The set of row reduced matrices with only zeros everywhere except pivot positions forms a semigroup Π_n of 2^n elements, and any Grassmannian semigroup S is graded by Π_n. This is related to the Renner semigroup R_n – the semigroup of partial one-to-one maps on a set of n elements, in the sense that Π_n is the submonoid consisting of the order-preserving maps in R_n [17, Ch. 8]. This observation allows one to obtain a natural one–one correspondence between Schubert cells of the total Grassmannian $\mathrm{Gr}_{\mathbb{K}}(n)$, associated Young diagrams and the graded components of such a semigroup. In particular, for example, the semigroup of row reduced matrices has an algebraic semigroup structure (and, consequently, so does $\mathrm{Gr}_{\mathbb{K}}(n)$).

In Section 3, we study the algebraic structure of Grassmannian semigroups S. We show that there are several elements in such an algebraic structure that can be identified and defined intrinsically. There is a basis B in which S is echelon and, when S is in echelon form, the Jordan cell N of dimension n and eigenvalue 0 must belong to the semigroup S; also, the rank-k row reduced diagonal idempotents E_k must be in the (echelon form of the) semigroup. This element N is uniquely determined intrinsically in S by the algebraic fact that every nilpotent element of S is a left multiple of N. Moreover, the equivalence relation on the set $\mathcal{E}$ of idempotents in S defined by the well-known procedure $E \sim E'$ if $EE' = E'$ and $E'E = E$ partitions $\mathcal{E}$ into $n + 1$ equivalence classes, and allows one to define the rank, and also the type of an element of S independently of the ambient matrix algebra $M_n(\mathbb{K})$ in which S is defined. Hence, given a Grassmannian semigroup S, all this structure, including the partition (grading) of S into parts indexed by Π_n, can be recovered from internal algebraic properties of S. Hence, the algebraic structure (multiplication) of this S retains plenty of the combinatorics and geometry of $\mathrm{Gr}_{\mathbb{K}}(n)$.

In Section 4, we study the problem of isomorphisms of Grassmannian semigroups. One sees easily that isomorphisms of such algebraic structures determine inclusion-preserving bijections on $\mathrm{Gr}_{\mathbb{K}}(n)$, so that the basic fundamental theorem of projective geometry can be used. Our main result of this section and second main result of the paper is that two Grassmannian semigroups are isomorphic if and only if they are semiconjugate, that is, they are isomorphic via a *ring* automorphism of $M_n(\mathbb{K})$, which must be the composition of a conjugation by a matrix A and a ring automorphism $\overline{\sigma}$ of $M_n(\mathbb{K})$ induced by some $\sigma \in \mathrm{Aut}(\mathbb{K})$. This, together with the results of the first section, allows us to determine up to isomorphism all such algebraic Grassmannian semigroups of dimensions 2 and 3, and determine the cardinality of the set of all such isomorphism classes. For example, when $\mathbb{K} = \mathbb{R}$, we show that there are $\aleph_2 = 2^{2^{\aleph_0}}$ such isomorphism classes. This can be done in higher dimensions, but the descriptions one would obtain make such results impractical to state. In particular, this re-obtains and generalize the results of [1].

In Section 5, we further explore connections between the Grassmannian semigroups, Grassmannians and Young diagrams. We note that besides the multiplicative structures they induce on $\mathrm{Gr}_{\mathbb{K}}(n)$, one obtains a monoid structure on the set of Young diagrams. Each matrix in Π_n represents a Schubert cell in $\mathrm{Gr}_{\mathbb{K}}(n)$, and has a canonically associated Young diagram, and *vice versa*, and one can define a bijective function between the set of *all* Young diagrams and the monoid (semigroup) $\Pi = \bigcup_{n=1}^{\infty} \Pi_n$, where Π_n is regarded naturally as a submonoid of Π_{n+1} as corner matrices. This is interesting vis-a-vis the so-called plactic monoid, a monoid structure on the set of *all* Young tableaux. Motivated also by this, we study the representation theory of Grassmannian semigroups, and of the monoids Π_n and Π. We show that the semigroup algebra $\mathbb{F}[S]$ over some arbitrary possibly different field $\mathbb{F}$ of a Grassmannian semigroup S on $M_n(\mathbb{K})$ is in fact semilocal and has nilpotent Jacobson radical (although S can be quite large). We also completely determine the Ext quiver of Π_n, its left and right projective indecomposables and their dimensions; these are expressed in terms of combinatorial binomial coefficients.

Representation theory of algebraic semigroups is also a subject much studied in the past two decades or so (see [3, 13, 14, 17] and references therein). In our setup, however, not all semigroups are naturally algebraic (linear) semigroups when regarded as subsets of $M_n(\mathbb{K})$, as they may depend on some *arbitrary* functions $f : \mathbb{K} \to \mathbb{K}$ (see Section 3). Grassmannian semigroups can be endowed with a (different) algebraic variety structure as a union of affine spaces. While they are not connected as a variety, they are still interesting as combinatorial objects, via their natural grading by Young diagrams. Furthermore, we do not impose any restriction on our field (other than sometimes being infinite). Hence, this natural type of semigroup structure seems to have escaped detailed scrutiny in the abstract semigroup theory, and our results do not seem to be a direct consequence of it, but there is certain overlap of interest. At the same time, this type of multiplicative structure does not seem to have been studied before in connection to combinatorial representation theory as well.

With a general audience in mind, we take a fairly direct approach throughout the paper. We note that in our treatment and interpretation, $\mathrm{Gr}_{\mathbb{K}}(n)$ is 'naively' viewed as simply a union of affine spaces, as opposed to the usual projective space subvariety via the Plücker embedding; as noted before, this has the advantage of being of combinatorial relevance and compatible with other algebraic structures (such as the natural matrix multiplication). In algebraic terms, this translates into the difference between our semigroup algebra $\mathbb{F}[\Pi_n]$ and the exterior algebra $\Lambda(\mathbb{F}^n)$. Nevertheless, some questions regarding these structures arise on the way, such as whether there are other relevant connections of the above-mentioned product of Young diagrams with other combinatorial representation theory or algebraic combinatorics problems. We believe that another interesting question is to completely describe the semigroup algebras of Π_n as quivers with relations, and determine the properties of the semigroup algebra $\mathbb{F}[\Pi]$ of the semigroup Π of *all* Young diagrams. They are also bialgebras, and determining their representation and Grothendieck rings could be an interesting problem as well. The algebraic semigroup structures we find on Grassmannian semigroups (most importantly the semigroup of row reduced matrices) also naturally give rise dually to bialgebras (the bialgebra of algebraic representative functions). Hence, we hope that this work can be the starting point of future investigations.

The field $\mathbb{K}$ is assumed to be infinite throughout the paper; this hypothesis is used in Propositions 2.1 and 2.3, which are key observations of our investigation. As this is a standard minimal assumption in representation theory, we will adhere to it here. Some later results may still hold for finite fields, but we leave this investigation to the interested reader.

2. Simultaneous echelon form for Grassmannian semigroups

The following is an easy observation that is likely known; we include a brief argument for completeness.

PROPOSITION 2.1. *Let V be an n-dimensional vector space over an infinite field $\mathbb{K}$ and $X = \{A_i \mid i = 0, 1, \ldots, k\}$ be a finite collection of subspaces of V. Then there is a flag $0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_n = V$ on V with $\dim(B_i) = i$ such that if $\dim(A_j) = t$ then B_{n-t} is a complement for A_j.*

PROOF. Fix a basis $e_1, \ldots, e_n$ and write all vectors as column vectors with respect to this basis. Let $B = [x_{ij}]$ be a generic $n \times n$ matrix with variables as shown. Let $\dim A_i = d_i$, and let $\mathcal{B}_i$ be a basis for A_i. Replace the first d_i columns of B with the elements of the basis $\mathcal{B}_i$ and denote the resulting matrix by $M(A_i)$. Let $\Psi(A_i) = \det M(A_i)$, so $\Psi(A_i)$ is a polynomial in $\mathbb{K}[x_{ij}|i, j]$ (depending only on $n(n - d_i)$ variables). Let $\Psi = \prod \Psi(A_i)$, so Ψ is a polynomial in the n^2 variables x_{ij}. Note that Ψ is the zero polynomial only when $\det M(A_i) = 0$ for some i. But this is not possible since $\mathcal{B}_i$ can always be completed to a basis of V (so A_i has a complement), and $\det(M(A_i))$ would be nonzero at the corresponding point. Therefore, since the field $\mathbb{K}$ is infinite, Ψ has a nonzero value at some $B \in M_n(\mathbb{K})$. If B_i is be the span of the last $n - i$ columns of B, then it is clear that the B_i are the subspaces, as required. □

Common complements have been studied for a long time and, in the generality of modules over arbitrary rings, most recently by Lam and his co-authors; we refer to [2, 4, 8] also for the history of the subject. This is usually studied in the form of existence of common complements of isomorphic submodules; for vector spaces, a much more general statement is possible as above.

In what follows, S will be a Grassmannian semigroup of $n \times n$ matrices.

PROPOSITION 2.2. *Let $A \in S$ be a matrix of rank k. Then, for every subspace W of V such that $V = W \oplus \mathrm{Im}(A)$, let $E \in S$ be the unique element such that $\ker(E) = W$. Then E is an idempotent and $EA = A$. Consequently, the column spaces of A and E coincide (that is, $\mathrm{Im}(E) = \mathrm{Im}(A)$) and the columns of A span the 1-eigenspace of E.*

PROOF. Let $E \in S$ be such that $\ker(E) = W$ (this is obviously unique). Note that since $W \cap \mathrm{Im}(A) = 0$, we have that $\dim(\mathrm{Im}(EA)) = \dim(E(\mathrm{Im}(A))) = \dim(E(W + \mathrm{Im}(A))) = \dim(\mathrm{Im}(E))$, so $\mathrm{rank}(EA) = \mathrm{rank}(E) = n - \dim(W) = \mathrm{rank}(A)$. This means that $\dim(\ker(A)) = \dim(\ker(EA))$ and, since $\ker(A) \subseteq \ker(EA)$, it follows that $\ker(A) = \ker(EA)$. By the uniqueness property of the Grassmannian semigroup, it follows that $EA = A$.

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Now note that $E^2A = EA = A$ and, as above, this shows that $\mathrm{Im}\,(E^2) = \mathrm{Im}\,(E)$ and hence $\ker(E^2) = \ker(E)$. Therefore, $E^2 = E$ again by the uniqueness property. Moreover, the identity $EA = A$ shows that the columns of A are 1-eigenvectors for E and, since $\mathrm{Im}\,(E)$ has dimension k and is spanned by eigenvectors, it follows that $\mathrm{Im}\,(A) \subseteq \mathrm{Im}\,(E)$, so they coincide since they have the same dimension (the equality $EA = A$ shows this directly also). □

Proposition 2.3. *Let $A, B \in S$ be such that* $\mathrm{rank}(A) = \mathrm{rank}(B)$. *Then* $\mathrm{Im}\,(A) = \mathrm{Im}\,(B)$.

Proof. Let k be the common rank of A and B and assume that $\mathrm{Im}\,(A) \neq \mathrm{Im}\,(B)$. Note that there is a subspace W of V such that $V = \mathrm{Im}\,(A) \oplus W = \mathrm{Im}\,(B) \oplus W$. This can be seen by the remarks in the beginning of this section, or directly: take A', respectively B', to be $n \times k$ matrices formed by some vectors that span $\mathrm{Im}\,(A)$, respectively $\mathrm{Im}\,(B)$. Finding the W amounts to finding an $n \times (n-k)$ matrix U such that $\det[A'|U] \neq 0$ and $\det[B'|U] \neq 0$. This is possible, since the sets $\{U, \det[A'|U] \neq 0\}$ and $\{U, \det[B'|U] \neq 0\}$ are open subsets of the affine space $\mathbb{A}^{n(n-k)}$.
By Proposition 2.2, there are idempotents $E, F \in S$ such that $\ker(E) = \ker(F) = W$ and $\mathrm{Im}\,(E) = \mathrm{Im}\,(A)$, $\mathrm{Im}\,(F) = \mathrm{Im}\,(B)$. This shows that $E \neq F$, and this contradicts the Grassmannian semigroup property, since two elements in S have the same kernel. □

We can now note the following interesting fact about the elements of a Grassmannian semigroup.

Corollary 2.4. *There are subspaces $V_0, V_1, \ldots, V_n$ with* $\dim(V_k) = k$ *and such that:*

(i) *if $A \in S$ has* $\mathrm{rank}(A) = k$, *then* $\mathrm{Im}\,(A) = V_k$;
(ii) *for each k, there are idempotents $E \in S$ such that* $\mathrm{Im}\,(E) = V_k$.

Proposition 2.5. *With the notation of Corollary 2.4, for each $0 \leq k < n$, $V_k \subset V_{k+1}$.*

Proof. Let E_k be an idempotent with $\mathrm{Im}\,(E_k) = V_k$. If $W_k = \ker(E_k)$, obviously $\ker(E_k) \cap \mathrm{Im}\,(E_k) = 0$ since E_k is an idempotent. Let W be a subspace of codimension 1 in W_k; note that it exists since $k < n$ implies that $W_k \neq 0$. Let $A \in S$ be such that $\ker(A) = W$. We note that since $\mathrm{Im}\,(E_k) \cap \ker(A) = 0$, we have $A(\mathrm{Im}\,(E_k)) \cong \mathrm{Im}\,(E_k)$, so $\mathrm{rank}(AE_k) = \mathrm{rank}(E_k)$. As before, since $\ker(E_k) \subseteq \ker(AE_k)$, using the uniqueness property of S, this shows that $AE_k = E_k$. From this we obtain that $\mathrm{Im}\,(E_k) \subset \mathrm{Im}\,(A)$. But $\mathrm{rank}\,(A) = n - \dim(W) = k + 1$, so $\mathrm{Im}\,(A) = V_{k+1}$ and $\mathrm{Im}\,(E_k) = V_k$. Thus, $V_k \subset V_{k+1}$. □

Hence, for every element $A \in S$, we have $A(V_k) = V_{k-i}$ for some i. Using this for the flag $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$, and choosing a basis $(w_i)_{i=1,\ldots,n}$ such that $w_i \in V_i \setminus V_{i-1}$, we see that with respect to this basis every endomorphism $A \in S$ is in echelon form (here by echelon form we understand the usual one, that is, a matrix for which in every row the first nonzero element is found at least one position to the right from the first nonzero element in the previous row).

Thus, we have the following result.

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THEOREM 2.6. *Any Grassmannian semigroup of $n \times n$ matrices may be conjugated to one which is in echelon form (that is, to a semigroup consisting of matrices in echelon form; therefore, the matrices of a Grassmannian semigroup are simultaneously echelonizable).*

Let us denote by E_k the 'basic' idempotent matrices $E_k = \begin{pmatrix} 1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$, having 1 on the first k entries on the diagonal and 0 elsewhere. The following is a variant of a result present in [1]; we give there a more direct short proof.

PROPOSITION 2.7. *Let $\mathcal{S}$ be a Grassmannian semigroup in $M_n(\mathbb{K})$, which is in echelon form. Then $E_k \in \mathcal{S}$ for all $1 \le k \le n$.*

PROOF. Let $F_k \in \mathcal{S}$ be the unique matrix whose kernel is $e_{k+1}, \dots, e_n$, where e_i are the vectors of the canonical basis. As in Proposition 2.2, F_k is an idempotent and, since F_k are in echelon form, we have $F_k = \left(\begin{smallmatrix} A & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where A is a $k \times k$ upper triangular matrix. Since A is diagonalizable ($A^2 = A$) with rank $A = k$, we see that $A = I_k$, the $k \times k$ identity, and so $F_k = E_k$, and the proof is finished. □

We also note a more general easy description of the idempotents in a Grassmannian semigroup.

PROPOSITION 2.8. *Let $\mathcal{S}$ be a Grassmannian semigroup in echelon form. Then the set of idempotents of rank k in $\mathcal{S}$ consists of all the matrices of the form*

$$E = \begin{pmatrix} I_k & C \\ 0 & 0_{n-k} \end{pmatrix}$$

for arbitrary $k \times (n-k)$ matrices C with entries in $\mathbb{K}$.

PROOF. Since E has rank k and is in echelon form, in particular $E = \left(\begin{smallmatrix} H & C \\ 0 & 0_{n-k} \end{smallmatrix}\right)$. Since $E^2 = E$, we get $H^2 = H$ and $HC = C$. This shows that the columns of C are 1-eigenvectors for the idempotent H, and so they are linear combinations of the columns of H. Hence, $\operatorname{rank}(E) = \operatorname{rank}(H) = k$ and, since H is an idempotent, it follows that $H = I_k$, and so

$$E = \begin{pmatrix} I_k & C \\ 0 & 0_{n-k} \end{pmatrix}.$$

Conversely, if C is an arbitrary $k \times (n-k)$ matrix, note that the span of the columns of the $n \times (n-k)$ matrix $M(C) = \left(\begin{smallmatrix} -C \\ I_{n-k} \end{smallmatrix}\right)$ is a subspace $V(C)$ of $V = \mathbb{K}^n$ of dimension $n-k$ for which $V(C) \cap V_k = 0$. Moreover, for every such subspace W such that $W \cap V_k = 0$ there is a unique such matrix C for which $W = V(C)$. Indeed, let B be an $n \times (n-k)$ matrix whose columns span W, and column reduce this matrix. Note that rows $1, 2, \dots, k$ cannot contain a pivot, since $V_k \cap W = 0$, and the conclusion follows as $M(C)$ and $M(D)$ are not column equivalent if $C \neq D$. Finally, for every $k \times (n-k)$ matrix C, there is an element $E \in \mathcal{S}$ with $\ker(E) = V(C)$ and, since $V(C) \cap V_k = 0$, it follows that E is an idempotent by Proposition 2.2. Hence, $E = \left(\begin{smallmatrix} I_k & D \\ 0 & 0_{n-k} \end{smallmatrix}\right)$ and E annihilates $V(C)$, so $EM(C) = 0$, from which it follows that $D = C$. □

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Proposition 2.9. *Assume that $\mathcal{S}$ is a Grassmannian semigroup in echelon form with respect to a basis $e_1, \ldots, e_n$. Then, after a change of basis of the type $e_i \mapsto \lambda_i e_i$ (that is, a 'diagonal' change of basis), the matrices in S will have* 1 *on all pivot entries.*

Proof. First, we show that all rank-1 matrices with a pivot on position $(1, i)$ will have the same value of the pivot. If $A_i \in \mathcal{S}$ is a matrix having a pivot value of a_i at position $(1, i)$, $A_i = \begin{pmatrix} 0 & \ldots & a_i & a_{i+1} & \ldots & a_n \\ 0 & 0 & & \cdots & & 0 \\ \vdots & & \cdots & & \cdots & \\ 0 & 0 & & \cdots & & 0 \end{pmatrix}$, then $B_i = A_i \cdot E_i \in \mathcal{S}$ has the element $a_i \in \mathbb{K}$ as a pivot in position $(1, i)$ and 0 elsewhere. By the uniqueness condition for Grassmannian semigroups, such an element in $\mathcal{S}$ is unique, so a_i is the same for all matrices of this form.

Now consider a matrix $B \in \mathcal{S}$, with a pivot in position (i, j) equal to b_{ij}. Then it is straightforward to note that $B_i \cdot B$ is an echelon matrix in $\mathcal{S}$ having a pivot of $a_i b_{ij}$ in position $(1, j)$ and so, by the above considerations, we see that $a_j = a_i b_{ij}$. We now note that if we change bases by $e_i' = (1/a_i)e_i$, in the new basis e_i', each element of $\mathcal{S}$ will still be in echelon form, and the rank-1 matrices will have the pivots equal to 1. Moreover, by the above, the pivots in positions (i, j) will be $b_{ij} = a_j \cdot a_i^{-1} = 1$. □

It seems appropriate here to note the following remark on the structure of another set of elements which occur in every Grassmannian semigroup which in fact form the set of nilpotents of rank 1.

Remark 2.10 (Elements of shape (p), $1 \le p \le n$). If $\mathcal{S}$ is a Grassmannian semigroup in echelon form, then, for every $1 \le p \le n$, the elements of shape (p) in $\mathcal{S}$ are all the matrices $L_p(a_{p+1}, \ldots, a_n)$ for arbitrary $a_{p+1}, \ldots, a_n$, having the first row $(0, \ldots, 0, 1, a_{p+1}, \ldots, a_n)$, with the 1 on position $(1, p)$ and 0 elsewhere. Indeed, for arbitrary $a_{p+1}, \ldots, a_n$ in $\mathbb{K}$, there has to be an element $B \in \mathcal{S}$ of rank 1 which has kernel equal to the subspace given by the equation $x_p = -a_{p+1}x_{p+1} - \cdots - a_n x_n$ (such a subspace is uniquely determined by $a_{p+1}, \ldots, a_n$), and in row reduced form the matrix $B = L_p(a_{p+1}, \ldots, a_n)$ is the only such possibility. For $p \ge 2$, these are precisely the nilpotent elements of rank 1 and, for $p = 1$, these are precisely all the idempotents of rank 1 in $\mathcal{S}$.

2.1. A 'row reduced' form for Grassmannian semigroups. We give a theorem which shows that, after conjugation by a suitable element, the matrices in a Grassmannian semigroup $\mathcal{S}$ can be put (simultaneously) in a form very close to the classical row reduced echelon form.

First, we fix some notation.

Definition 2.11. Let $\tau = (k_1, k_2, \ldots, k_t)$ be a t-uplet of integers, where $1 \le k_1 < k_2 < \cdots < k_t \le n$ are integers, $1 \le t \le n$. We will say that an echelon matrix A has shape τ if it has pivots at positions (i, k_i), for $i = 1, \ldots, t$, that is, the pivots are at columns $k_1, \ldots, k_t$.

We denote by P_τ the matrix having 1 at positions (i, k_i) and 0 elsewhere. In what follows, it will be convenient to consider column reduced matrices, which means that we column reduce right to left, bottom–up.

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Definition 2.12. We say that a matrix N is right column reduced (respectively, in right column echelon form) if it is obtained from a row reduced matrix (respectively, from an echelon matrix) via reflection across the secondary diagonal; equivalently, if when the matrix under consideration is rotated ninety degrees counterclockwise and reflected across a vertical line to its left, it is row reduced.

That is, a matrix N is right column reduced if its columns, listed from left to right, are $c_1, c_2, \dots, c_n$ and these columns, when transposed and organized into rows in reverse order as ${}^tc_n, \dots, {}^tc_1$, form a row reduced matrix that we will denote by $R_c(N)$. Obviously, this operation R_c is its own inverse, so $R_c^2 = \mathrm{Id}$ on $M_n(\mathbb{K})$. Its importance is revealed when dealing with the null space of matrices, and offers a convenient way to write such null spaces.

For a shape $\tau = (k_1, \dots, k_t)$, let τ' be the shape defined as $\tau' = (l_1, \dots, l_s)$ such that $1 \le l_1 < \cdots < l_s \le n$ and $\{l_1, \dots, l_s\} \sqcup \{k_1, \dots, k_t\} = \{1, 2, \dots, n\}$ (that is, $\{l_1, \dots, l_s\}$ is the complement of $\{k_1, \dots, k_t\}$). The correspondence between τ and τ' can also be explained via conjugate (transpose) Young diagrams.

We recall a standard construction done for Schubert cells of Grassmannians. If A is a row echelon (or row reduced) matrix of shape τ, let $Y_0(A)$ be the Young diagram obtained by retaining all nonpivot positions from all rows containing pivots. Namely, if A has shape $\tau = (k_1, \dots, k_t)$, we place in row i of $Y_0(\tau)$ a number of boxes equal to the number of nonpivot positions on row i, to the right of (i, k_i). More precisely, if A has a pivot at (i, k_i), there are $n - k_i$ columns to the right of column k_i, $t - i$ of which are pivot columns. There are therefore $(n - k_i) - (t - i) = n - t + i - k_i$ nonpivot positions on row i and to the right of column k_i. Hence, we may write $Y_0(\tau) = (n - t + 1 - k_1, n - t + 2 - k_2, \dots, n - k_t)$, where the jth entry of this t-uple of integers denotes the number of boxes of $Y_0(\tau)$ on row j. This is using the French convention with rows having a nonincreasing number of boxes as we go downwards. If $\tau' = (l_1, \dots, l_s)$, let $Y_0(\tau')$ be defined similarly. Then it is not difficult to see that $Y_0(\tau)$ and $Y_0(\tau')$ are conjugate Young diagrams. We will later use further connections of row reduced and echelon form matrices and Young diagrams.

We now use the above map R_c to define a bijection from echelon matrices of shape τ to right column echelon matrices of shape τ'. It will also be useful to have some additional notation. If $\tau = (k_1, \dots, k_t)$ is a fixed shape, let us denote by W_τ the set of all matrices with entries 0 everywhere except possibly nonzero at positions (i, j) for $i \le t$ and $j > k_i$, and $j \notin \{k_1, \dots, k_t\}$. Equivalently, $A \in W_\tau$ if and only if $A + P_\tau$ is a row reduced matrix of shape τ. Obviously, W_τ is a $\mathbb{K}$-subspace of $M_n(\mathbb{K})$. Note that the dimension of this space is $\dim(W_\tau) = (n - k_1 - t + 1) + (n - t - k_2 + 2) + \cdots + (n - t - k_t + t) = t(n - t) + t(t + 1)/2 - (k_1 + \cdots + k_t)$. We will more closely investigate the relation of Grassmannian semigroups and Grassmannians later.

Remark 2.13. A t-shuffle is a permutation on n letters that preserves the order of $(1, \dots, t)$ and $(t + 1, \dots, n)$. If P is a permutation matrix, then multiplication by P on the right of a matrix A permutes the columns of A. It is easy to see that a shuffle is a permutation matrix of the form $P = \binom{R_1}{R_2}$, where R_1 and R_2 are 0, 1 echelon matrices.

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Every rank-t row reduced matrix is of the form $R = \left(\begin{smallmatrix} I_t & X \\ 0 & 0 \end{smallmatrix}\right)P$, where P is a t-shuffle. Let $M = \left(\begin{smallmatrix} I_t & X \\ 0 & 0 \end{smallmatrix}\right)$; then $\ker(M)$ is the span of the columns of $\left(\begin{smallmatrix} 0 & -X \\ 0 & I_{n-t} \end{smallmatrix}\right)$. So, the kernel (null space) of R is the span of the columns of $N(R) := P^{-1}\left(\begin{smallmatrix} 0 & -X \\ 0 & I_{n-t} \end{smallmatrix}\right) = \left(\begin{smallmatrix} R_1^t & R_2^t \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & -X \\ 0 & I_{n-t} \end{smallmatrix}\right)$. But the shape of R is the shape of R_1 and one sees that the shape of $N(R)$, according to our right–left, down–up convention, is the shape of R_2^t, which is the shape of the complement of R_1 as desired.

We have thus defined, for each row reduced matrix R as above, a right column reduced matrix $N(R)$ such that the null space of R is the span of the columns of $N(R)$; furthermore, if R has shape $\tau = (k_1, \dots, k_t)$, then $N(R)$ is a right column reduced matrix of shape $\tau' = (l_1, \dots, l_s)$ as follows with τ' being the complement of τ. Below we see an example of how this construction N works.

$$R = \begin{pmatrix} 1 & a & 0 & 0 & b & c \\ 0 & 0 & 1 & 0 & d & e \\ 0 & 0 & 0 & 1 & f & g \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow N(R) = \begin{pmatrix} 0 & 0 & 0 & -a & -b & -c \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d & -e \\ 0 & 0 & 0 & 0 & -f & -g \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We denote by S the map that does the reverse operation, so for a right column reduced matrix B of shape τ', it associates a row reduced matrix $S(B)$ of shape τ. We have that S and N are inverse maps. We summarize the properties of these in the following lemma, which is likely known, and only amounts to a careful computational observation, and therefore we omit the details of the proof.

Lemma 2.14.

(i) *Let B be a right column reduced matrix of shape τ'. Then there is a unique matrix A of shape τ such that $AB = 0$. Moreover, $A = S(B)$.*

(ii) *Let A be a row reduced matrix of shape τ. Then there is a unique right column reduced matrix B of shape τ' such that $AB = 0$. Moreover, $B = N(A)$.*

(iii) *For A, B as in (i) and (ii) above, we have $BA = 0$.*

We have the following theorem representing matrices A of shape τ which are in row reduced form, and which have the null space equal to the column space of $N(A)$. If we regard $\mathbb{K}^n$ as the space of column vectors over $\mathbb{K}$, we first note that every subspace W of $\mathbb{K}^n$ is a canonical basis which can be represented uniquely by a matrix B in right column reduced form. This is obtained by column reducing an arbitrary basis of W.

Theorem 2.15. *Let W be a subspace of $\mathbb{K}^n$. Let B be the right column reduced matrix whose columns represent a basis of W, and let τ' be the shape of B. If A is an echelon matrix of shape τ such that $AB = 0$, then there is a matrix C which has 0 entries everywhere except potentially at positions above the pivot positions of A, and such that $A = S(B) + C - CB$.*

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We saw that Grassmannian semigroups can be put into (simultaneous) echelon form. Using the previous theorem, we notice a structure statement for matrices in a Grassmannian semigroup, which will show an even closer resemblance to the semigroup of row reduced matrices. Let $\mathcal{S}$ be in echelon form with the pivots of every element of $\mathcal{S}$ equal to 1. For each shape τ, and every right column reduced matrix B of shape τ', there is a unique matrix $A \in \mathcal{S}$ with $\mathrm{Null}(A) = \mathrm{Col}(B)$ (that is, the null space of A is the column space of B), and these are all matrices of shape τ in $\mathcal{S}$, by the above remark on canonical bases in subspaces of $\mathbb{K}^n$. Hence, by the previous theorem, since $AB = 0$, there is a matrix $C = C_\tau(B)$ which has 0 entries everywhere except potentially at positions above the pivot positions from $S(B)$ such that $A = S(B) + C_\tau(B) - C_\tau(B)B$. Each such C_τ is a function of B of shape τ'. We note that in the case the functions C_τ are all 0, we obtain the semigroup $\mathcal{S}$ of row reduced $n \times n$ matrices over $\mathbb{K}$.

As corollary, for small values of n we can re-obtain a full classification of such Grassmannian semigroups as in [1].

Corollary 2.16. *If $\mathcal{S}$ is a Grassmannian semigroup in $M_2(\mathbb{K})$, then $\mathcal{S}$ is conjugate to $\mathcal{R}$, the semigroup of row reduced matrices.*

Proof. Up to conjugation we may assume that $\mathcal{S}$ is in echelon form, and with pivots equal to 1, and, therefore, the semigroup is

$$\mathcal{S} = \left\{ I_2; \begin{pmatrix} 1 & c \\ 0 & 0 \end{pmatrix}, c \in \mathbb{K}; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; 0_2 \right\},$$

which is precisely $\mathcal{R}$ (that is, there is only one Grassmannian semigroup in echelon form and with pivots equal to 1). □

Corollary 2.17. *If $\mathcal{S}$ is a Grassmannian semigroup in $M_3(\mathbb{K})$, then there is a function $f : \mathbb{K} \to \mathbb{K}$ such that $\mathcal{S}$ is conjugate to the following semigroup:*

$$\mathcal{S} = \left\{ I_3; \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 0 \end{pmatrix}, \ x, y \in \mathbb{K}; \begin{pmatrix} 1 & u & f(u) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, u \in \mathbb{K}; \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}; \right.$$
$$\left. \begin{pmatrix} 1 & z & t \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, z, t \in \mathbb{K}; \begin{pmatrix} 0 & 1 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, w \in \mathbb{K}; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; 0_3 \right\}.$$

Proof. We may use the above remark on the structure theorem of such Grassmannian semigroups and note that each of the above eight families corresponds to one of eight possible shapes, with the fourth one – $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ – of shape $(2, 3)$, or proceed as follows. Note that each of the following families of subspaces of V, which exhaust the subspaces of V, must be the kernel (null space) of a corresponding matrix, which will have dual shape (the vectors should be considered as column vectors in $V = \mathbb{K}^3$):

$$\{0;\ \mathrm{Span}(-x, -y, 1), x, y \in \mathbb{K};\ \mathrm{Span}(-u, 1, 0), u \in \mathbb{K};\ \mathrm{Span}(1, 0, 0);$$
$$\mathrm{Span}\{(-z, 1, 0), (-t, 0, 1)\}, z, t \in \mathbb{K};\ \mathrm{Span}\{(1, 0, 0), (0, -w, 1)\}, w \in \mathbb{K};$$
$$\mathrm{Span}\{(1, 0, 0), (0, 1, 0)\};\ V\}.$$

We get a semigroup in echelon form with pivots equal to 1, each corresponding to (that is, annihilating) one of these subspaces of V. In this semigroup, the elements corresponding to the second of the above types of subspaces $\mathrm{Span}(-x, -y, 1)$ would have the form $A = \left(\begin{smallmatrix} 1 & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & 0 \end{smallmatrix}\right), \alpha, \beta, \gamma \in \mathbb{K}$; nevertheless, we know that the matrix E_2 should be in $\mathcal{S}$, and so multiplying the two together we get that $AE_2 = \left(\begin{smallmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) \in \mathcal{S}$; but this and E_2 have the same kernel, and by the uniqueness property of the Grassmannian semigroup we obtain $\alpha = 0$. Using the annihilating relation $A \cdot \left(\begin{smallmatrix} -x \\ -y \\ 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}\right)$, we get $\beta = x, \gamma = z$. The third type of matrices listed – namely, those of shape $(1, 3)$ – will be of the form $A = \left(\begin{smallmatrix} 1 & s & t \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$, $s, t \in \mathbb{K}$; they have null space spanned by ${}^t(-u, 1, 0)$, so $s = u$. But, for each u, there should be a unique matrix of this type, so the corresponding t depends on u. This gives rise to a function $f : \mathbb{K} \to \mathbb{K}$, $u \mapsto f(u)$. Finally, the fourth type of matrix – that of shape $(2, 3)$ (annihilating ${}^t(1, 0, 0)$) – is $N = \left(\begin{smallmatrix} 0 & 1 & c \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and has the property $Ne_3 = e_2 + ce_1$, $Ne_2 = e_1$, $Ne_1 = 0$. After further changing the basis to $e'_3 = e_3$, $e'_2 = e_2 + ce_1$, $e'_1 = (1 + c)e_1$, we see that the semigroup becomes of the desired form in the basis $\{e'_1, e'_2, e'_3\}$ (see also Proposition 3.9).

It is easy to see that the above-described set is closed under multiplication and forms a Grassmannian semigroup. □

3. The structure of Grassmannian semigroups

We note a few basic facts on the set of shapes of matrices. We note that the set Π_n of all shapes has a monoid structure. For each shape $\tau = (k_1, \ldots, k_s)$, let P_τ be the matrix having 1 on positions (i, k_i) and 0 elsewhere. It is not difficult to see that the set of $n \times n$ matrices $\Pi_n = \{P_\tau | \tau \text{ is a shape}\}$ is closed under products. This can be used to introduce a multiplication of 'shapes': $\tau\sigma$ is such that $P_{\tau\sigma} = P_\tau P_\sigma$.

We notice also that the shape of an element in a Grassmannian semigroup can also be defined without reference to a basis for which it is in echelon form. For this, note that given the flag $0 \subset V_1 \subset \cdots \subset V_n = V$ of images of elements in $\mathcal{S}$, we have $AV_i = V_j$ for some $j \le i$, and $k_i = \min\{j \,|\, A(V_j) = V_i\}$ when $V_i = A(V_j)$ for at least one j; equivalently, $V_j \subset \mathrm{Im}(A)$. This is easy to see for $\mathcal{S}$ in any base in which it is in echelon form, so it is independent of such a basis. We will show that the shapes can be defined independently without reference to the space V on which $\mathcal{S}$ acts.

Remark 3.1. We also note here a short conceptual proof of the fact that the row reduced matrices are closed under products. Indeed, one can interpret the set of row reduced matrices $\mathcal{R}$ as the set of endomorphisms T of a finite-dimensional vector space V, which have the following properties with respect to a fixed basis $\{e_1, \ldots, e_n\}$.

(1) If $I_k = \mathrm{Span}\{e_1, \ldots, e_k\}$, then, for all $k \le n$, we have $TI_k = I_s$ for some $s \le k$.

(2) If k is such that $T(I_{k-1}) \subsetneq T(I_k)$, then $T(e_k) \in \{e_1, e_2, \ldots, e_n\}$ (more precisely, $T(e_k) = e_t$ is such that $T(I_k) = I_t$). This condition can be written equivalently as follows: if $V_s = \mathrm{Im}(T)$ and $(k_1, \ldots, k_s)$ is such that $k_i = \min\{j \,|\, T(V_j) = V_i\}$, then $T(e_{k_i}) = e_i$.

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The above two conditions make it easy to check that if two endomorphisms A, B satisfy these conditions, then AB satisfies the same conditions as well. Also, by the results of the first section, we note that we have proved that for any Grassmannian semigroup $\mathcal{S}$ there is a flag $I_1 \subset \cdots \subset I_n = \mathbb{K}^n$ on $\mathbb{K}^n$ with respect to which elements of $\mathcal{S}$ have the first property (1). The only difference to row reduced matrices is that, in general, in a Grassmannian semigroup one does not need to have property (2) hold in general.

We prove a few simple propositions on decomposition of elements in a Grassmannian semigroup. As before, $\mathcal{S}$ will denote a Grassmannian semigroup in $M_n(\mathbb{K})$. First, we note the following fact regarding solutions of equations in such semigroups.

Proposition 3.2. *Let $\mathcal{S}$ a Grassmannian semigroup and $a, b \in \mathcal{S}$. Then the equation $a = xb$ has a solution in $\mathcal{S}$ if and only if $\ker(b) \subseteq \ker(a)$. Moreover, in this case, there is a unique solution x of maximal rank, that is, with $\mathrm{rank}(x) = n - \mathrm{rank}(b) + \mathrm{rank}(a)$, and $\ker(x) = b(\ker(a))$.*

Proof. Of course, $\ker(b) \subseteq \ker(a)$ is necessary. To show that it is sufficient, note that if a solution of $a = xb$ exists then $\ker(a) = b^{-1}(\ker(x))$ (and $b(\ker(a)) \subseteq \ker(x)$). Thus, if $\ker(b) \subseteq \ker(a)$, let $W = b(\ker(a))$ and let $x \in \mathcal{S}$ be such that $\ker(x) = W$. Then $\ker(xb) = b^{-1}(\ker(x)) = b^{-1}(W) = \ker(a)$ (since $\ker(b) \subseteq \ker(a)$) and, since $xb, a \in \mathcal{S}$, by uniqueness of kernels we get $xb = a$. By Sylvester's inequality, we have $\mathrm{rank}\,(a) \geq \mathrm{rank}(x) + \mathrm{rank}(b) - n$, so $\mathrm{rank}(x) \leq n - \mathrm{rank}(b) + \mathrm{rank}(a)$. If equality is assumed, then it follows that $\dim(\ker(x)) = \dim(\ker(a)) - \dim(\ker(b))$. This shows that the linear map $b : \ker(a) = b^{-1}(\ker(x)) \to \ker(x)$ is surjective (since $\ker(b) \subseteq \ker(a)$, $\dim(\mathrm{Im}(b|_{\ker(a)})) = \dim(\ker(a)) - \dim(\ker(b)) = \dim(\ker(x))$), and so $\ker(x) = b(\ker(a))$. Therefore, x is unique as it is uniquely determined by its kernel. □

Next we give a result about unique decompositions of elements. Recall that if W is a vector space, a flag on W is a sequence $0 = W_0 \subset W_1 \subset \cdots \subset W_k = W$ with $\dim(W_i) = i$; a partial flag is a sequence $W_s \subset W_{s+1} \subset \cdots \subset W_t$ with $\dim(W_i) = i$. The next proposition shows that elements decompose uniquely along (partial) flags.

Proposition 3.3. *Let $\mathcal{S}$ be a Grassmannian semigroup and $a \in \mathcal{S}$.*

(i) *Suppose that $X_1 \subset X_2 \subset \cdots \subset X_t = \ker(a)$ is a sequence of subspaces. Then there is a unique decomposition $a = a_1 a_2 \ldots a_t$ with $a_i \in \mathcal{S}$ and $\ker(a_{i+1} \ldots a_t) = X_{t-i}$ and each a_i is of maximal rank, that is, $\mathrm{rank}(a_i) = n - \dim(X_i) + \dim(X_{i-1})$ for $i = 1, \ldots, t$, and where we set $X_0 = 0$.*

(ii) *If $0 = X_0 \subset X_1 \subset \cdots \subset X_t = \ker(a)$ is a flag on $\ker(a)$, then there is a unique decomposition $a = a_t a_{t-1} \ldots a_1$ with $a_i \in \mathcal{S}$ and $\ker(a_i a_{i-1} \ldots a_1) = X_i$ and $\mathrm{rank}\,(a_i) = n - 1$.*

Proof. (i) We apply the previous proof recursively. First, write $a = a_1 b_1$ uniquely with $\ker(b_1) = X_{t-1}$ and a_1 of maximal rank equal to $n - \dim(X_{t-1}) + \dim(X_t)$. Then repeat the procedure for b_1 and $X_{t-2} \subset \ker(b_1)$ to obtain $b_1 = a_2 b_2$ with $\ker(b_2) = X_{t-2}$ and

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a_2 has maximal possible rank, etc. To see uniqueness, if $a = a_1 \dots a_t = a'_1 \dots a'_t$ are two such decompositions, then since $\ker(a_2 \dots a_t) = \ker(a'_2 \dots a'_t)$ we have $a_2 \dots a_t = a'_2 \dots a'_t$ and, using the uniqueness of the solution x of maximal rank of the equation $a = xa_2 \dots a_t$ provided by the previous proposition, we get $a_1 = a'_1$ etc.
(ii) Follows immediately from (i). □

Proposition 3.4. *If $\mathcal{S}$ is a Grassmannian semigroup on V of dimension n, $A \in \mathcal{S}$ is of rank k and E is an idempotent of* $\operatorname{rank}(E) = s \geq k$*, then $EA = A$.*

Proof. As in the beginning, we note that $\operatorname{Im}(A) = V_k \subset V_s$, and V_s is the set of 1-eigenvectors of E since E is idempotent. Therefore, $EA(v) = A(v)$ for all $v \in V$. □

3.1. Nilpotent elements. We need one more proposition that describes the nilpotent elements in a Grassmannian semigroup. Recall that given such $\mathcal{S}$ we denoted by V_k the subspace of V which is the (common) image of the elements of rank k in $\mathcal{S}$.

Proposition 3.5. *If $a \in \mathcal{S}$, then there is k such that $a^k = a^{k+1} = \cdots$ and a^k is an idempotent.*

Proof. The ascending sequence $(\ker(a^k))_k$ of subspaces of V must stabilize

$$\ker(a^k) = \ker(a^{k+1}) = \cdots .$$

By the Grassmannian semigroup property, $a^k = a^{k+1} = \cdots a^{2k} = \cdots$, and so a^k is also an idempotent. □

Proposition 3.6. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then the following are equivalent for $x \in \mathcal{S}$:*

(i) *x is nilpotent;*
(ii) $V_1 \subseteq \ker(x)$*;*
(iii) *x is a left zero divisor in $\mathcal{S}$, that is, there is $y \in \mathcal{S}$, $y \neq 0$, such that $xy = 0$.*

Proof. (i) ⇒ (ii) If x is nilpotent, let k be such that $x^{k-1} \neq 0 = x^k$. Then $0 \neq \operatorname{Im}(x^{k-1}) \subseteq \ker(x)$. Obviously, $\operatorname{Im}(x^{k-1}) = V_i$ for some $i \geq 1$, so $V_1 \subseteq \operatorname{Im}(x^{k-1}) \subseteq \ker(x)$.

(ii) ⇒ (i) Let k be such that $x^k = x^{k+1} = \dots$, so x^k is idempotent. We claim that $x^k = 0$. Indeed, otherwise we have $V_1 \subseteq \operatorname{Im}(x^k) = V_i$ for some $i \geq 1$ and, since $V_1 \subseteq \ker(x)$, it follows that $\dim(x(V_i)) \leq \dim(V_i) - \dim(V_1) = i - 1 < i = \dim(V_i)$. So, $\operatorname{Im}(x^{k+1}) = x(V_i) \neq V_i = \operatorname{Im}(x^k)$, which contradicts $x^k = x^{k+1}$.

(ii) ⇔ (iii) Obviously, $V_1 \subseteq \ker(x)$ if and only if $V_1 = \operatorname{Im}(y) \subseteq \ker(x)$ for every $y \in \mathcal{S}$ of rank 1 (and there are such elements in $\mathcal{S}$), which is equivalent to $xy = 0$ for all such elements y. To conclude, note that $x \in \mathcal{S}$ is a left zero divisor if and only if $xy = 0$ for all $y \in \mathcal{S}$ of rank 1. □

In the proposition above we may easily see that right zero divisors are not necessarily nilpotent: if $b \in \mathcal{S}$ is such that $\operatorname{Im}(b) = V_{n-1}$ and $\ker(b) = Y \neq V_1$, and $a \in \mathcal{S}$ is such that $\ker(a) = V_{n-1}$, then $ab = 0$, so b is a right zero divisor, but b is not nilpotent since $V_1 \not\subset \ker(b)$.

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By extension of the terminology of rings, we may call a subset I of a semigroup M an ideal if for all $a \in M$ and $x \in I$, we have $ax, xa \in I$. Of course, this is not going to produce a congruence relation on M that would be suitable for doing a quotient, as it is for rings. For a Grassmannian semigroup $\mathcal{S}$, we denote by $N(\mathcal{S})$ the set of nilpotent elements of $\mathcal{S}$. This makes sense in any semigroup where there is a 'zero' element (that is, an element z such that $az = za = z$ for all a). We note the following result.

Proposition 3.7. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then the set of nilpotent elements $N(\mathcal{S})$ is a 'prime' ideal of $\mathcal{S}$, namely, if $ab \in N(\mathcal{S})$, then $a \in N(\mathcal{S})$ or $b \in N(\mathcal{S})$.*

Proof. This property is easiest visualized in matrix form. Consider a basis with respect to which the semigroup $\mathcal{S}$ is in echelon form (with pivots equaling 1). In particular, $\mathcal{S} \subset T_n$, the algebra of upper triangular matrices and, if N_n is the set of strictly upper triangular matrices, then $N(\mathcal{S}) = N_n \cap \mathcal{S}$. Since N_n is an ideal of T_n (the Jacobson radical of T_n), it follows that $N(\mathcal{S})$ is an ideal of $\mathcal{S}$.
For 'primality', let $c \notin N(\mathcal{S})$ and $a \in \mathcal{S}$. Then c must have its entry on position $(1, 1)$ equal to 1. This can be seen either from Proposition 3.6, or from the obvious fact that matrices in echelon form are nilpotent if and only if they have a 0 on position (1, 1). Now the following equalities can happen only if $a_1 = a_2 = 0$, and this shows that if either ac or ca is nilpotent, then so is a.

$$\begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ & \dots & \dots & \\ 0 & \dots & 0 & * \end{pmatrix} \cdot \begin{pmatrix} a_1 & * & \dots & * \\ 0 & * & \dots & * \\ & \dots & \dots & \\ 0 & \dots & 0 & * \end{pmatrix} = \begin{pmatrix} 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ & \dots & \dots & \\ 0 & \dots & 0 & 0 \end{pmatrix} = \begin{pmatrix} a_2 & * & \dots & * \\ 0 & * & \dots & * \\ & \dots & \dots & \\ 0 & \dots & 0 & * \end{pmatrix} \cdot \begin{pmatrix} 1 & * & \dots & * \\ 0 & * & \dots & * \\ & \dots & \dots & \\ 0 & \dots & 0 & * \end{pmatrix}. \qquad \square$$

Remark 3.8. Let $N_1 \in \mathcal{S}$ be such that $\ker(N_1) = V_1$. Since $V_1 \subseteq \ker(x)$ for all $x \in N(\mathcal{S})$, we see by Proposition 3.2 that for each such $x \in N(\mathcal{S})$ there is $a \in \mathcal{S}$ such that $x = aN_1$, that is, N_1 divides all the nilpotent elements of $\mathcal{S}$. This particular nilpotent will be important next.

We denote by N_k the nilpotent element of $\mathcal{S}$ for which $\ker(N_k) = V_k$. Let $e_n \in V_n \backslash V_{n-1}$. Since $\operatorname{rank}(N_1) = n - 1$, we have that $N_1^n = 0 \neq N_1^{n-1}$. Define the elements e_i by $e_{n-1} = N_1(e_n), \dots, e_{i-1} = N_1(e_i)$, $i \geq 1$, so $e_0 = 0$. We claim that $(e_i)_{i=1,\dots,n}$ is a basis of V and, moreover, $\{e_1, \dots, e_k\}$ is a basis of V_k for each k. For this it is enough to show that $e_i \in V_i \backslash V_{i-1}$ for all i. Suppose that $e_k \in V_{k-1}$ for some k, and let k be the largest such number; obviously, $k < n$ since $e_n \notin V_{n-1}$. We have that $N_1(V_i) = V_{i-1}$ for all i. Then $e_{k+1} \notin V_k$, so $V_{k+1} = \mathbb{K}e_{k+1} + V_k$, and therefore $N_1(V_{k+1}) = \mathbb{K}N_1(e_{k+1}) + N_1(V_k) = \mathbb{K}e_k + N_1(V_k) \subseteq V_{k-1}$. But this is a contradiction to $N_1(V_{k+1}) = V_k$, so the claim is proved. With this we have the following result.

Proposition 3.9. *With the above notations, $N_1^k = N_k$. Moreover, there is a basis of V with respect to which $\mathcal{S}$ is in echelon form with all pivots equal to 1 and, in the semigroup $\mathcal{S}$, N_k is the matrix with 1 on the kth diagonal above the main diagonal, and 0 elsewhere, so N_1 is the Jordan cell of dimension n and eigenvalue 0.*

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Proof. Consider the basis above $\{e_1, \ldots, e_n\}$. Since this basis has the property $e_i \in V_i \backslash V_{i-1}$, it follows that $\mathcal{S}$ is still in echelon form with respect to it. Obviously, in this basis N_1 is a Jordan cell of dimension n and eigenvalue 0. Moreover, $\ker(N_1^k) = \mathrm{Span}\{e_1, \ldots, e_k\} = V_k$, since $\dim(\ker(N_1^k)) = k$ and it is straightforward to note that $N_1^k(e_i) = 0$ when $i \leq k$. Hence, by the uniqueness of the elements with a given kernel in a Grassmannian semigroup, $N_1^k = N_k$. Finally, we note that, in fact, with respect to this basis all pivots of elements in $\mathcal{S}$ are 1. As in Proposition 2.9, note that $N_{k-1}E_k$ is a rank-1 matrix of shape (k) (all rows are 0 except the first) and has pivot 1 in position $(1, k)$. By the proof Proposition 2.9, it follows that all the pivots of all the matrices in $\mathcal{S}$ are 1. □

The semigroups of the form in the previous proposition will be used again, so it feels natural to introduce a definition.

Definition 3.10. We say that a Grassmannian semigroup $\mathcal{S}$ is of Jordan type if it is in echelon form, with pivots equal to 1, and contains the $n \times n$ Jordan block (cell) $J_n(0)$ of rank $n - 1$ and eigenvalue 0 (and, consequently, contains all its powers).

3.2. A few remarks on intrinsic abstract properties of Grassmannian semigroups. We note that several invariants of a Grassmannian semigroup $\mathcal{S}$ can be defined without any reference to the action of $\mathcal{S}$ on V.

(I1) First, note that the identity I and zero 0 elements of $\mathcal{S}$ are uniquely determined.

(I2) Next, the element $N_1 \in \mathcal{S}$ is uniquely determined (*as an element of* $\mathcal{S}$) by the property that for all nilpotent $x \in S$, there is $y \in \mathcal{S}$ such that $x = yN_1$ (since such an element in $\mathcal{S}$, considered as an endomorphism of V, will have its kernel contained in the kernel of all nilpotent elements in $\mathcal{S}$).

(I3) The structure of idempotents is uniquely determined. Let $\sim$ be the equivalence relation on the set $E(\mathcal{S})$ of idempotents of $\mathcal{S}$ given by $E \sim E'$ if $EE' = E'$ and $E'E = E$. If $\mathcal{S} \subset \mathrm{End}_{\mathbb{K}}(V)$ is any fixed representation of $\mathcal{S}$ as a Grassmannian semigroup on V, it is easy to see that $E \sim E'$ if and only if $\mathrm{rank}(E) = \mathrm{rank}(E')$. Thus, the number of equivalence classes is equal to $n + 1$, where n is the dimension of V. We may also introduce a quasi-ordering on $E(V)$ by setting $E' \geq E$ if and only if $EE' = E'$, which is equivalent to $\mathrm{Im}(E') \subseteq \mathrm{Im}(E)$, and further $\mathrm{rank}\,(E') \leq \mathrm{rank}(E)$. This becomes a total order on $E(\mathcal{S})/\sim$, making it a PO-set isomorphic to $\{0, 1, \ldots, n\}$. Let H_k be the equivalence class of level k (corresponding to idempotents of rank k).

(I4) The structure of idempotents determines a filtration on $\mathcal{S}$ that recovers rank. Namely, using Proposition 3.4, we see that if $A \in \mathcal{S}$, then $\mathrm{rank}(A) \leq k$ if and only if $EA = A$ for some $E \in H_k$; equivalently, for all $E \in H_k$. Hence, we may introduce the subset $R_k = R_k(\mathcal{S})$ consisting of elements A for which $EA = A$ for some (equivalently, all) $E \in H_k$. This will consist of all elements of rank $\leq k$. Now the rank can be defined abstractly as $\mathrm{rank}\,(A) = k$ if $A \in R_k \backslash R_{k-1}$. This shows that n – the dimension of the space on which $\mathcal{S}$ acts – is recovered as the cardinality of the set of equivalence classes of idempotents $E(\mathcal{S})/\sim$.

(I5) Now the shape of an element in $\mathcal{S}$ can also be defined abstractly. Fix $E_1, E_2, \ldots, E_n$, representatives of $H_1, H_2, \ldots, H_n$, respectively. Given $A \in \mathcal{S}$, consider

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the sequence of numbers $\mathrm{rank}(AE_1), \ldots, \mathrm{rank}(AE_n)$; in matrix interpretation, the images of these elements correspond to $A(V_1), \ldots, A(V_n)$. Then one defines $k_i = \min\{j \mid \mathrm{rank}\,(AV_j) = i\}$. It is easy to see that this is an equivalent reformulation of the shape of A given before in the case A is in echelon form.

(I6) Note that the shape of an element in $\mathcal{S}$ can also be recovered by using the distinguished nilpotent N_1. It is based on the following easy observation: if A is a matrix and N_1 is the Jordan cell with eigenvalue 0 of dimension n, then AN_1 is obtained by deleting the last column of A, shifting the other columns of A to the right and replacing the first one by 0. Hence, the last columns in the matrices in the sequence $A, AN_1, AN_1^2, \ldots, AN_1^{n-1}$ are all the columns of A. If A is an echelon matrix of shape $(k_1, \ldots, k_t)$, then it is easy to see that the shape of AN_1 is $(k_1 + 1, k_2 + 1, \ldots, k_t + 1)$ if $k_t < n$ and $(k_1 + 1, \ldots, k_{t-1} + 1)$ if $k_t = n$. Therefore, the nonincreasing sequence $\mathrm{rank}(A), \mathrm{rank}(AN_1), \ldots, \mathrm{rank}(AN_1^{n-1})$ will completely determine the shape of the echelon matrix A: the ranks will decrease precisely at positions $n - k_t + 1, n - k_{t-1} + 1, \ldots, n - k_1 + 1$. Hence, since rank is intrinsically determined in a Grassmannian semigroup $\mathcal{S}$, this is another way to get the shape of an element in $\mathcal{S}$ without reference to the ambient space.

While we defined the Grassmannian semigroups as systems of representatives for the left $\mathrm{Gl}_n(\mathbb{K})$ action on $M_n(\mathbb{K})$, it is natural to ask what is their relationship with the right action. This is done in the next proposition, which uses the above results on the structure of such semigroups, and shows that under the right action Grassmannian semigroups are contained in a small (finite) number of orbits.

Proposition 3.11. *Let $\mathcal{S}$ be a Grassmannian semigroup, and $E_k \in \mathcal{S}$ the previously defined basic idempotents. Then $\mathcal{S} \subset \bigcup_{i=0}^{n} E_k \mathrm{Gl}_n(\mathbb{K})$; in particular, up to equivalence under right action, there are exactly $n + 1$ classes of elements in $\mathcal{S}$.*

Proof. This is obvious, since the right $\mathrm{Gl}_n(\mathbb{K})$ action operates on columns, so preserves column space, and equivalence classes are determined precisely by column space. We have already shown that there are exactly $n + 1$ possible column spaces for elements in $\mathcal{S}$. $\square$

4. Isomorphisms of Grassmannian semigroups

In what follows, we aim to study when two Grassmannian semigroups are isomorphic. By the remarks of the previous section, we note that if two such semigroups are isomorphic, then they have the same 'dimension', that is, they are Grassmannian semigroups on the same vector space of dimension n (since n is determined by the internal structure of the semigroup as we saw in the previous section). The first step is to notice that an isomorphism of such semigroups produces an order-preserving isomorphism of the lattice of subspaces of the vector space. Denote by $\mathcal{L}(X)$ the lattice of subspaces of the vector space X. Also, if $\mathcal{S}$ is a Grassmannian semigroup on the vector space V, for each $X \in \mathcal{L}(V)$ denote by $a_X \in \mathcal{S}$ the element for which $\ker(a_X) = X$.

PROPOSITION 4.1. *Let $\mathcal{S}, \mathcal{S}'$ be Grassmannian semigroups in $M_n(\mathbb{K})$ and let $\varphi : \mathcal{S} \to \mathcal{S}'$ be an isomorphism. Let $\mathcal{L}(\mathbb{K}^n)$ be the set of all subspaces of $\mathbb{K}^n$ and let $p : \mathcal{L}(\mathbb{K}^n) \to \mathcal{L}(\mathbb{K}^n)$ be defined by $p(X) = W$ if and only if $\varphi(a_X) = a'_W$, where $a_X \in \mathcal{S}$ and $a'_W \in \mathcal{S}'$ are the unique elements with* $\ker(a_X) = X$, $\ker(a'_W) = W$. *Then p is an inclusion-preserving bijection.*

PROOF. If $X \subseteq Y \subseteq \mathbb{K}^n$ are subspaces, then there is $b \in \mathcal{S}$ such that $a_Y = ba_X$, so Proposition 3.2 implies that $\varphi(a_Y) = \varphi(b)\varphi(a_X)$. Thus, $\ker(\varphi(a_X)) \subseteq \ker(\varphi(a_Y))$, and so $p(X) \subseteq p(Y)$. □

Since the above induced map p is an inclusion-preserving bijection on $\mathcal{L}(V)$, we are in position to use the fundamental theorem of projective geometry and the Skolem–Noether theorem to characterize this map and obtain insights on the isomorphism class of a semigroup. We fix a basis $V = \mathbb{K}^n$ and identify $\mathrm{End}_{\mathbb{K}}(V) = M_n(\mathbb{K})$. For an automorphism σ of the field $\mathbb{K}$, denote $\overline{\sigma} : V \to V$ by applying σ component wise. By extension (and abuse of notation), we will also denote by $\overline{\sigma} : M_n(\mathbb{K}) \to M_n(\mathbb{K})$ the ring automorphism obtained by applying σ to each entry of a matrix. Let also $\theta \in \mathrm{Gl}_n(\mathbb{K})$. Then the composition map $\tau : A \to \theta A\theta^{-1} \to \overline{\sigma}(\theta A\theta^{-1})$ is a semilinear automorphism of the ring $M_n(A)$, and it is well known that every semilinear transformation is obtained in this way (recall that an automorphism α of the ring $M_n(\mathbb{K})$ is said to be semilinear if $\alpha(c \cdot A) = \sigma(c)\alpha(A)$ for some automorphism σ of the field $\mathbb{K}$). Then $\tau(\mathcal{S})$ is also a Grassmannian semigroup, and we introduce the following definition.

DEFINITION 4.2. We say that two Grassmannian semigroups $\mathcal{S}, \mathcal{S}'$ are semiconjugate if $\mathcal{S}' = \tau(\mathcal{S})$ for a semilinear transformation τ as above.

In what follows, we will show that if two Grassmannian semigroups are isomorphic then they are 'almost' semiconjugate, except for some trivial way of obtaining new Grassmannian semigroups by multiplying matrices by certain constants. We first observe the following result.

PROPOSITION 4.3. *Let $\mathcal{S}, \mathcal{S}'$ be two isomorphic Grassmannian semigroups. Then there exist a Grassmannian semigroup $\mathcal{S}_0$ which is semiconjugate to $\mathcal{S}$ and an isomorphism $\psi : \mathcal{S}_0 \to \mathcal{S}'$ which is kernel preserving, that is,* $\ker(\psi(x)) = \ker(x)$ *for all $x \in \mathcal{S}_0$.*

PROOF. Let $p : \mathcal{L}(\mathbb{K}^n) \to \mathcal{L}(\mathbb{K}^n)$ be the map from Proposition 4.1 induced by the isomorphism $\varphi : \mathcal{S} \to \mathcal{S}'$, which is inclusion preserving. By the fundamental theorem of projective geometry, we have that p is given by a semilinear automorphism of $M_n(\mathbb{K})$, so $p(W) = \overline{\sigma}(\theta(W))$, for all subspaces W of $\mathbb{K}^n$, where we interpret θ as an endomorphism of $\mathbb{K}^n$ via the fixed identification $M_n(\mathbb{K}) = \mathrm{End}_{\mathbb{K}}(V)$. Let $\tau(A) = \overline{\sigma}(\theta A\theta^{-1})$, where $\tau(A)$ is defined for $A \in \mathcal{S}$. Then $\mathcal{S}_0 = \tau(\mathcal{S})$ is a Grassmannian semigroup which is semiconjugate to $\mathcal{S}$. Note now that for $a_W \in \mathcal{S}$ (so $\ker(a_W) = W$), we have $\ker(\tau(a_W)) = p(W)$. Indeed, let $v \in \ker(\tau(a_W))$; this is equivalent to $\overline{\sigma}(\theta a_W\theta^{-1})v = 0$ and further to $(\overline{\sigma})^{-1}(v) \in \ker(\theta a_W\theta^{-1})$, that is, $\theta^{-1}(\overline{\sigma})^{-1}(v) \in \ker(a_W) = W$. Hence, $v \in \ker(\tau(a_W))$ if and only if $v \in \overline{\sigma}(\theta(W))$.
Lastly, if $\psi = \varphi \circ \tau^{-1}$, then the map induced by ψ on $\mathcal{L}(\mathbb{K}^n)$ takes $p(W)$ to $p(W)$ for each $W \in \mathcal{L}(\mathbb{K}^n)$, and so it is kernel preserving. □

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Next, we determine what kernel-preserving isomorphisms between Grassmannian semigroups look like. We will need the following small lemma, which may be known, but we could not find a reference.

Lemma 4.4. *Let b, c be linear transformations of V to V' vector spaces of finite dimension such that the maps b^{-1}, c^{-1} on $\mathcal{L}(V')$ are equal. Then there is $\lambda \in \mathbb{K}$, $\lambda \neq 0$ such that $b = \lambda c$.*

Proof. First, $\ker(b) = b^{-1}(0) = c^{-1}(0) = \ker(c)$, so it is easy to see that by factoring out by $\ker(b) = \ker(c)$, we may assume that b and c are injective, since if the induced maps $B = \overline{b}, C = \overline{c}$ will have $B = \lambda C$, it follows immediately that $b = \lambda c$. Also, note that $0 \neq w \in \mathrm{Im}(b)$ if and only if $b^{-1}(\mathbb{K}w) \neq 0$. Since $b^{-1}(\mathbb{K}w) = c^{-1}(\mathbb{K}w)$ for all w, this shows that $\mathrm{Im}(b) = \mathrm{Im}(c)$. Thus, we may also assume that $b \neq 0 \neq c$. Let $w_1, \ldots, w_n$ be a basis on $\mathrm{Im}(b) = \mathrm{Im}(c)$, and x_i, y_i be such that $b(x_i) = w_i = c(y_i)$. By injectivity, we have $b^{-1}(\mathbb{K}w_i) = \mathbb{K}x_i$ and $c^{-1}(\mathbb{K}w_i) = \mathbb{K}y_i$, and the hypothesis thus implies that $\mathbb{K}x_i = \mathbb{K}y_i$, so $y_i = \lambda_i x_i$. If $n = 1$, we are done. Otherwise, let $W = \mathbb{K}(w_i + w_j)$ for $i \neq j$. Since $b(x_i + x_j) = w_i + w_j = c(y_i + y_j)$, by the injectivity of b and c and hypothesis we get $\mathbb{K}(x_i + x_j) = b^{-1}(W) = c^{-1}(W) = \mathbb{K}(y_i + y_j)$, so $y_i + y_j = \lambda(x_i + x_j)$ for some λ. Hence, $\lambda_i x_i + \lambda_j x_j = \lambda x_i + \lambda x_j$ and, since x_i, x_j are linearly independent (since b is injective and w_i, w_j are independent), we get $\lambda_i = \lambda = \lambda_j$. This shows that $\lambda_1 = \cdots = \lambda_n$, so $y_i = \lambda x_i$ and therefore $b(x_i) = w_i = c(y_i) = c(\lambda x_i) = \lambda c(x_i)$, which shows that $b = \lambda c$. □

Proposition 4.5. *Let $\varphi : \mathcal{S} \to \mathcal{S}'$ be a kernel-preserving isomorphism between two Grassmannian semigroups on the n-dimensional vector space V, $n \geq 2$. Let $N(\mathcal{S})$ be the set of nilpotent elements in $\mathcal{S}$ as before. Then:*

(i) *if $n = 2$, then there is $\lambda \in \mathbb{K}$ such that $\mathcal{S}' = \mathcal{S} - \{N_1\} \cup \{\lambda N_1\}$, and*

$$\varphi(a) = \begin{cases} a & \text{if } a \notin N(\mathcal{S}), \\ \lambda \cdot a & \text{if } a = N_1; \end{cases}$$

(ii) *$\mathcal{S} = \mathcal{S}'$ and $\varphi = \mathrm{Id}$ if $n \geq 3$.*

Proof. Since φ is kernel preserving, we have $\ker(b) = \ker(\varphi(b))$ for all $b \in \mathcal{S}$. Let W be a subspace of V, let $b \in \mathcal{S}$ and let $a = a_W \in \mathcal{S}$ so that $\ker(a) = W$. Notice that

$$\begin{aligned} b^{-1}(W) &= b^{-1}(\ker(a)) \\ &= \ker(ab) = \ker(\varphi(ab)) \quad (\text{since } \varphi \text{ is kernel preserving}) \\ &= \ker(\varphi(a)\varphi(b)) \quad (\text{since } \varphi \text{ is a morphism}) \\ &= \varphi(b)^{-1}(\ker(\varphi(a))) \\ &= \varphi(b)^{-1}(\ker(a)) \quad (\text{since } \varphi \text{ is kernel preserving}) \\ &= \varphi(b)^{-1}(W). \end{aligned}$$

Therefore, b^{-1} and $\varphi(b)^{-1}$ are equal on $\mathcal{L}(V)$ and by the previous lemma $\varphi(b) = \lambda(b) \cdot b$ for some $\lambda(b) \in \mathbb{K}\backslash\{0\}$.

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Note that if $b, c \in \mathcal{S}$ are such that $bc \neq 0$, then $\lambda(bc) = \lambda(b)\lambda(c)$: indeed, $\varphi(bc) = \lambda(bc)bc = \varphi(b)\varphi(c) = \lambda(b)\lambda(c)bc$ and $bc \neq 0$. Next, if $e \in \mathcal{S}$ is a nonzero idempotent, then $\lambda(e)^2 = \lambda(e)$ in $\mathbb{K}$ and $\lambda(e) \neq 0$, so $\lambda(e) = 1$. Furthermore, if $a \notin N(\mathcal{S})$, then there is k such that $a^k = a^{k+1} \neq 0$ and a^k is an idempotent (Proposition 3.5). Hence, $\lambda(a^k) = 1$ and $\lambda(a^k) = \lambda(a \cdot a^k) = \lambda(a)\lambda(a^k)$, so $\lambda(a) = 1$.

Let x be a nilpotent element, so $V_1 \subseteq \ker(x)$ by Proposition 3.6. If x is nilpotent with $x \neq N$, then $V_1 \subsetneq \ker(x)$; let Y be a subspace of $\ker(x)$ of codimension 1 and such that $V_1 \not\subset Y$. Let $a = a_Y$ (so $\ker(a) = Y$) and let $c \in \mathcal{S}$ be such that $x = ca$ (it exists since $\ker(a) \subset \ker(x)$). Moreover, we may assume that c has maximal rank equal to $n-1$ since $\dim(\ker(x)) - \dim(\ker(a)) = 1$, and there is a unique such c by Proposition 3.2. Then a is not nilpotent since $V_1 \not\subset Y$, and so $c \in N(\mathcal{S})$ by Proposition 3.7. Thus, $V_1 \subseteq \ker(c)$, and so $V_1 = \ker(c)$ (since $\operatorname{rank}(c) = n-1$), and therefore $c = N_1$. Thus, $0 \neq x = N_1 a$, so $\lambda(x) = \lambda(N_1)\lambda(a)$ and, as $\lambda(a) = 1$ (since $a \notin N(\mathcal{S})$), we get $\lambda(x) = \lambda(N_1)$, and so λ is constant on $N(\mathcal{S})$.

Finally, if $n = 2$ there is only one nonzero nilpotent element, and the statement (i) follows. Otherwise, we have $N_1^2 \neq 0$, since $\operatorname{rank}(N_1) = n-1$, so $\operatorname{rank}(N_1^2) \geq n-1+n-1-n = n-2 \geq 1$ ($n = \dim(V) \geq 3$). Hence, $\lambda = \lambda(N_1^2) = \lambda(N_1)\lambda(N_1) = \lambda^2$ and, as $\lambda \neq 0$, we get $\lambda = 1$. The conclusion of (ii) follows. □

Since for $n = 2$ every two Grassmannian semigroups are conjugate by Corollary 2.16, we have the following result.

Corollary 4.6. *Two Grassmannian semigroups are isomorphic if and only if they are semiconjugate.*

In particular, we have the following result.

Corollary 4.7. *If $\mathbb{K}$ is such that* $\operatorname{Aut}(\mathbb{K}) = \{\mathrm{Id}_{\mathbb{K}}\}$, *then two Grassmannian semigroups are isomorphic if and only if they are conjugate. In particular, two real Grassmannian semigroups are isomorphic if and only if they are conjugate (since* $\operatorname{Aut}(\mathbb{R}) = \{\mathrm{Id}\}$*).*

4.1. Small dimensions. By the previous section, every two Grassmannian semigroups on a vector space of dimension 2 are isomorphic. We aim to study this problem in dimension 3. Since every Grassmannian semigroup is conjugate to a Grassmannian semigroup of Jordan type (Definition 3.10; see Corollary 2.17 for the $n = 3$ case and Proposition 3.9 for the general case), we will investigate when two such Grassmannian semigroups of Jordan type are isomorphic and when they are conjugate.

Proposition 4.8. *Let $\mathcal{S}_0$ and $\mathcal{S}'$ be Grassmannian semigroups of Jordan type that are conjugate by θ. Then $\theta = p(J_n(0))$, where p is a polynomial and $J_n(0) = N_1$ is the Jordan cell of dimension n and eigenvalue 0.*

Proof. Since the element N_1 is in both $\mathcal{S}_0$ and $\mathcal{S}'$, and it is uniquely defined by the internal semigroup structure and invariant properties described above – invariant property (I2), then the conjugation isomorphism $X \mapsto \theta X \theta^{-1}$ must take N_1 to N_1. Hence, $\theta N_1 = N_1 \theta$. But it is well known (and computationally straightforward to

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check) that the centralizer of the Jordan cell $J_n(0)$ consists of polynomial functions of $J_n(0)$, which ends the proof. $\square$

We apply the previous proposition to determine Grassmannian semigroups of dimension 3 up to isomorphism. As noted in [1] and Corollary 2.17 above, they are completely determined by a function f. We determine first when two such semigroups are conjugate. If $\mathcal{S}(f)$, $\mathcal{S}(g)$ are two such semigroups associated with the functions $f, g : \mathbb{K} \to \mathbb{K}$, and conjugation by θ is an isomorphism between them, then $\theta = \left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)$ as noted in the previous proposition (we may assume that the diagonal is 1, since we may always multiply the conjugation matrix by a scalar, since conjugation by scalar matrices has no effect). Conjugation will preserve the shape (since shape is an intrinsic property of the semigroup structure) and

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & w & f(w) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & w & f(w)+a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & -a & a^2-b \\ 0 & 1 & -a \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & -a+w & a^2-b-aw+f(w)+a \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore we obtain $g(w-a) = a^2 - b - aw + f(w) + a$ or, equivalently, $g(t) = f(t+a) - at + a - b$ for all $t \in \mathbb{K}$. Furthermore, it is easy to see that two Grassmannian semigroups $\mathcal{S}(f), \mathcal{S}(g)$ in echelon form are isomorphic via an isomorphism of type $\overline{\sigma}$ for $\sigma \in \mathrm{Aut}\,(\mathbb{K})$ exactly when $\sigma(f(w)) = g(\sigma(w))$ for all $w \in \mathbb{K}$. Thus, combining the two, we get the following result, which recovers in particular another result of [1].

Proposition 4.9. *Let $\mathcal{S}(f)$, $\mathcal{S}(g)$ be two Grassmannian semigroups in echelon form as in Corollary 2.17, with $f, g : \mathbb{K} \to \mathbb{K}$. Then:*

(i) *$\mathcal{S}(f), \mathcal{S}(g)$ are conjugate if and only if $g(t) = f(t+a) - at + a - b$ for all $t \in \mathbb{K}$ for some $a, b \in \mathbb{K}$;*

(ii) *$\mathcal{S}(f)$ and $\mathcal{S}(g)$ are isomorphic if and only if $g(w) = \sigma(f(\sigma^{-1}(w)) + a) - a\sigma^{-1}(w) + a - b$,* for all $w \in \mathbb{K}$ *for some $a, b \in \mathbb{K}$ and $\sigma \in$* Aut $(\mathbb{K})$.

Denote by $U_3(\mathbb{K})$ the group of unipotent upper triangular matrices of the above type $\left(\begin{smallmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{smallmatrix}\right)$, and $\mathrm{Fun}(\mathbb{K}, \mathbb{K})$ the set of maps from $\mathbb{K}$ to $\mathbb{K}$. Obviously, $U_3(\mathbb{K})$ is isomorphic to the quotient of the group of units of $\mathbb{K}[X]/(X^3)$ by the scalars $\lambda \in \mathbb{K}^\times$, and it is abelian. Also, Aut $(\mathbb{K})$ acts on this abelian group in the obvious way (acting on each entry). Thus, their semidirect product $\mathrm{Aut}\,(\mathbb{K}) \rangle U_3(\mathbb{K})$ acts on $\mathrm{Fun}(\mathbb{K}, \mathbb{K})$ by the action described in (ii) of the above proposition, and by the above remarks the orbits of this action parametrize the set of isomorphism types of Grassmannian semigroups in $M_3(\mathbb{K})$. The cardinality of the group $U_3(\mathbb{K})$ is obviously that of $\mathbb{K}$ if $\mathbb{K}$ is infinite, and the cardinality of $\mathrm{Fun}(\mathbb{K}, \mathbb{K})$ is $|\mathbb{K}|^{|\mathbb{K}|}$, which is larger than $|\mathbb{K}|$. In particular, when Aut $(\mathbb{K})$ is not too large, we can easily obtain the following result.

Corollary 4.10. *If $\mathbb{K}$ is an infinite field with* $\mathrm{Aut}(\mathbb{K}) = \{\mathrm{Id}\}$ *or, more generally, if* $|\mathrm{Aut}(\mathbb{K})| < |\mathbb{K}|$, *then there are* $|\mathbb{K}|^{|\mathbb{K}|}$ *isomorphism types of Grassmannian semigroups of dimension 3. In particular, when* $\mathbb{K} = \mathbb{R}$, *the set of isomorphism types of Grassmannian semigroups of dimension 3 has cardinality* $\aleph_2 = 2^{\aleph_1} = 2^{2^{\aleph_0}}$.

Given all the above results on the structure of Grassmannian semigroups, one may certainly ask whether there is an algebraic feature of the semigroup $\mathcal{R}$ of row reduced matrices (either an internal one or one relative to the ambient space $M_n(\mathbb{K})$ and action on $\mathcal{L}(\mathbb{K}^n)$) which distinguishes $\mathcal{R}$ from all other semigroups. Thus, we formulate the following problem.

Problem 4.11. Give a characterization of the semigroup $\mathcal{R}$ among all Grassmannian semigroups.

One such characterization is given by Remark 3.1; we mention it below without proof, which can be deduced easily from that of 3.1. The following theorem states that the semigroup of row reduced matrices is that for which the shape of an element can be read of a particular fixed basis.

Theorem 4.12. *Let $\mathcal{S}$ be a Grassmannian semigroup. Then $\mathcal{S}$ is isomorphic to $\mathcal{R}$ if and only if there is a basis $\{e_1, \dots, e_n\}$ such that for every element $A \in \mathcal{S}$, if $\tau = (k_1, \dots, k_s)$ is the shape of A, then $A(e_{k_i}) = e_i$.*

Another perhaps not so remarkable characterization that parallels Proposition 3.11 is the following: a semigroup $\mathcal{S}$ which is in echelon form equals the semigroup of row reduced matrices if for every element $A \in \mathcal{S}$, there is a permutation matrix P such that AP is an idempotent.

We note a third characterization of $\mathcal{R}$, somewhat in the same spirit as the previous two. We make the following remark: if B is an echelon matrix of shape $\tau = (k_1, \dots, k_t)$, then for $p \in \{k_1, \dots, k_t\}$, say $p = k_i$, we have that $BE_p - BE_{p-1}$ is a matrix whose only nonzero column is the pth column which equals the pth column of B. In order to have that B is in row reduced form, this column needs to have just one nonzero element equal to 1 at position (i, k_i). Since left multiplication of B by the elements E_j selects the first j lines and replaces the rest by 0, this can be tested by asking that $E_jB = 0$ for $j < i$. Hence, B is in row reduced form if $E_jBE_{k_i} = E_jBE_{k_i-1}$ for all $j < k_i$.

Proposition 4.13. *Let $\mathcal{S}$ be a Grassmannian semigroup. Let $\mathcal{S}'$ be a Grassmannian semigroup in echelon form which is conjugated to $\mathcal{S}$ and let $F_i \in \mathcal{S}$ be elements corresponding to $E_i \in \mathcal{S}'$ via this conjugation. Then $\mathcal{S}$ is conjugated to the semigroup of row reduced matrices if and only if for each $B \in \mathcal{S}$ of shape $\tau = (k_1, \dots, k_t)$, we have $E_jBE_{k_i} = E_jBE_{k_i-1}$ for all $j < k_i$.*

It is natural to ask whether it is possible to give a characterization of the semigroup of row reduced matrices, which is an intrinsic algebraic characterization, independent of the embedding into the ambient matrix algebra. The above proposition could offer some clues on the possibility of such a characterization. In fact, if the 'basic' matrix idempotents E_i that appear in any Grassmannian semigroup in echelon form could be

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characterized intrinsically only in terms of the properties of the semigroup, the above proposition would then offer such a characterization. Unfortunately, this seems to be hard to achieve. For this, note that if $\mathcal{S}$ is a Grassmannian semigroup in echelon form, then two matrices $A, B \in \mathcal{S}$ that have the same first $n-1$ columns are indistinguishable by right multiplications (except by identity). That is, if $C \in \mathcal{S}$, $C \neq I$, then the last row of C is 0, so $AC = BC$. Left multiplications on the other hand at a glance seem to be quite general. In fact, for example, as far as elements of rank 1 are concerned, left multiplication does not help either, since for any such element $A \in \mathcal{S}$ with $\mathrm{rank}(A) = 1$, we get $CA = 0$ or $CA = A$ for all $C \in \mathcal{S}$.

5. Graded algebra structure on Grassmannians and semigroups

We describe a connection between Grassmannian semigroups and a certain graded algebra structure on Grassmannians. Recall that the Grassmannian $\mathrm{Gr}_{\mathbb{K}}(k, n)$ or $\mathrm{Gr}(k, n)$ is set of subspaces of $\mathbb{K}^n$ of dimension k for $k = 1, 2, \ldots, n$. Also recall that Π_n has a monoid structure. Let $\mathrm{Gr}_{\mathbb{K}}(n)$ be the set of all subspaces of the space $\mathbb{K}^n$, that is, the 'total' Grassmannian. One can write $\mathrm{Gr}_{\mathbb{K}}(n) = \bigcup_{\tau\in\Pi_n} W_\tau$, which can be regarded as a bijection obtained by giving each subspace of $\mathbb{K}^n$ a canonical basis of column (or row) reduced vectors (the basis $e_1, \ldots, e_n$ is fixed). Recall that W_τ is the set of matrices of the form $A - P_\tau$, where A is a row reduced matrix of shape τ. Since any subspace of $\mathbb{K}^n$ regarded as line vectors has a unique row reduced basis, we see that $\mathrm{Gr}_{\mathbb{K}}(n, t) = \bigcup_{\tau=(k_1,\ldots,k_t)} W_\tau$, and each W_τ corresponds to some Schubert cell. Recall that if $\tau = (k_1, \ldots, k_t)$, then W_τ is a subspace of $M_n(\mathbb{K})$ and $\dim(W_\tau) = t(n-t) + t(t+1)/2 - (k_1 + \cdots + k_t)$, so we may view each W_τ as an affine space of appropriate dimension. If t is fixed, this is maximum when $(k_1, \ldots, k_t) = (1, \ldots, t)$. This agrees with the known fact that the dimension of the Grassmannian $G_{\mathbb{K}}(n, t)$ is $t(n-t)$.

Let $\mathcal{S}$ be a Grassmannian semigroup. By Theorem 2.15, we may assume, after possible conjugation by a matrix, that $\mathcal{S}$ is in echelon form (with pivots 1). Using either Theorem 2.15 or directly the definition of Grassmannian semigroup and the remarks of the preceding sections, the set $\mathcal{S}_\tau$ of row reduced matrices of $\mathcal{S}$ of shape τ is in one–one correspondence with the space $W_{\tau'}$ and with W_τ. Let $\psi_\tau : W_\tau \to \mathcal{S}_\tau$ be a bijection which parametrizes these matrices; this may be taken to be algebraic. It is not difficult to observe that if A is an echelon matrix of shape τ (with pivots equal to 1) and B is an echelon matrix of shape σ (with pivots equal to 1), then AB is an echelon matrix of shape $\tau\sigma$ (with pivots equal to 1). Hence, the semigroup $\mathcal{S}$ is graded by the monoid Π_n. Moreover, the maps ψ_τ can be used to introduce a multiplication on $\mathrm{Gr}_{\mathbb{K}}(n)$ in the following way.

If $A \in W_\tau$, $B \in W_\sigma$, then define $C = A * B \in W_{\tau\sigma}$ such that $\psi_{\tau\sigma}(A * B) = \psi_\tau(A)\psi_\sigma(B)$.

Since multiplication in $\mathcal{S}$ is done by algebraic equations, the set $\mathrm{Gr}_{\mathbb{K}}(n)$, viewed as an algebraic variety via the union of the maps ψ_τ, becomes an algebraic variety with a semigroup structure given by polynomial equations and thus an algebraic semigroup. However, the structure of the algebraic variety of $\mathrm{Gr}_{\mathbb{K}}(n)$ obtained in this way via the

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decomposition into affine subspaces $G_{\mathbb{K}}(n) = \bigcup_{\tau\in\Pi_n} W_\tau$ will in general differ from the one obtained via the Plücker embedding into $\Lambda(\mathbb{K}^n)$.

5.1. Representations of Π_n and Grassmannian semigroups. Recall that the set of semistandard Young tableaux admits a semigroup structure called the plactic monoid, discovered by Knuth [9], which can be defined in general independently via words in a finite alphabet modulo the Knuth relations. We note now that the set of Young diagrams also has a semigroup structure multiplication. It is not clear to us whether there is a connection between the two, but it certainly seems like an interesting question. We use here the French convention for Young diagrams, with the number of boxes in each row increasing going down.

The shapes of an echelon matrix remind one of Young diagrams. To each shape $\tau = (k_1, \ldots, k_t)$ we associate a Young diagram $Y_s(\tau)$ in a natural way by placing on each row i of $Y_s(\tau)$ a number k_i of boxes. The number of boxes in the rows of the Young diagram $Y_s(\tau)$ is strictly increasing, and one can associate a Young diagram $Y(\tau)$ having $k_i - i + 1$ boxes in its ith row (the number of boxes in the rows of $Y(\tau)$ increase nonstrictly going down). This is different than the Young diagram $Y_0(\tau)$ of Section 2 (which had the number of boxes decreasing going down as in the English convention); the number of boxes in rows i of $Y(\tau)$ and $Y_0(\tau)$ add up to $n - t$. But the definition of $Y_0(\tau)$ depends on n and, in order to define a multiplication on the set of *all* Young diagrams, we will need to define a bijection from shapes to Young diagrams in a way that does not depend on the dimension n of the 'ambient' space, as will be seen next. Note that τ is completely determined by either $Y_s(\tau)$ or $Y(\tau)$. We also note that $Y(\tau)$ is the diagram of a partition of length equal to $k_1 + k_2 - 1 + \cdots + k_t - (t-1)$. Thus, $\text{length}(Y(\tau)) \leq (n-t+1) + (n-t+2) - 1 + \cdots + (n-t+t) - (t-1) = t(n+1-t)$. Further, there is a bijection between Young diagrams (partitions) with t rows of length at most $t(n+1-t)$ and the set of shapes $\tau = (k_1, \ldots, k_t)$ with $k_t \leq n$. Let us also observe that the semigroups Π_n can be embedded in each other via the natural embedding of $M_n(\mathbb{K}) \subset M_{n+1}(\mathbb{K})$ which takes an $n \times n$ matrix and borders it with 0 down and to the right to obtain an $(n+1) \times (n+1)$ matrix. Denote $\Pi = \bigcup_n \Pi_n$; it is a semigroup (but not a monoid) since each successive embedding $\Pi_n \subset \Pi_{n+1}$ is a semigroup map. Moreover, by the above there is a bijection between Π and the set $\mathcal{Y}$ of *all* Young diagrams, and also to the set $\mathcal{Y}'$ of all strictly row increasing Young diagrams. Hence, $\mathcal{Y}$ has a semigroup structure introduced by transporting the structure of Π.

More precisely, if $y = (s_1, \ldots, s_t)$ is a partition (Young diagram with rows $s_1 \leq \cdots \leq s_t$), then $T(y) = (s_1, s_2 + 1, \ldots, s_t + t - 1) \in \Pi_n$ is a shape for any $n \geq s_t + t - 1$. If y, y' are two Young diagrams, then their multiplication is given by multiplying their associated shapes $\tau = T(y), \tau' = T(y')$ as elements of the appropriate Π_n and taking the Young diagram of the product:

$$y * y' = Y(T(y)T(y')).$$

5.2. Multiplication in the semigroup of Young diagrams. The multiplication of the Young diagrams can be described combinatorially as follows. Given Young diagrams $y = (s_1, \ldots, s_t), y' = (l_1, \ldots, l_p)$, first construct the strictly row increasing

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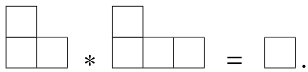
Young diagrams z, z' by adding $i-1$ boxes to the ith nonempty rows of y and y', respectively. Let z'' be the Young diagram obtained as follows: let s_i be the number of boxes on row i of z, and let $m_i = l_{s_i}$ be the number of boxes on row s_i in the second Young diagram z' (zero if that row is empty). The number of boxes on row i in z'' is m_i (it is zero if the row s_i in z' was empty). Then the product $y * y'$ is obtained by deleting appropriate boxes in z'' to revert it to a nondecreasing Young diagram, that is, delete one box in the second row of z'', two boxes in the third etc.

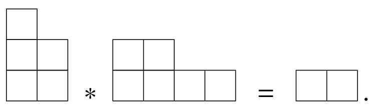
For example, we consider here the following multiplication:



This follows because the multiplication of the corresponding matrices is

$$\begin{pmatrix} 1&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} \cdot \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix}.$$

Note that, depending on the size of the Young diagrams, one needs to consider square matrices of sufficient size, namely, work inside the appropriate $\Pi_n \subset \Pi$. Note that the Young diagram with n rows and one box in each row corresponds to the rank-n idempotent E_n of Π, that is, to the identity element of the semigroup Π_n. For another example, we leave it to the reader to check that



In view of the above connections of Π to Young diagrams and Grassmannians and to better understand Grassmannian semigroups and Π_n, it is interesting to attempt to study their representation theory.

As before, let $\mathcal{S}$ be a Grassmannian semigroup, which we may (and will) assume to be in echelon form. Let $\mathbb{F}[\mathcal{S}]$ be its semigroup algebra over some field $\mathbb{F}$. In what follows, we let $R = \mathbb{F}[\mathcal{S}]$ or $R = \mathbb{F}[\Pi_n]$; the results will apply to both semigroup algebras. We denote as before the idempotents $E_i \in \mathcal{S}$ having the first i entries equal to 1 on the main diagonal and 0 elsewhere (they are also elements of Π_n). Denote by Z the zero element of $\mathcal{S}$ (in order to distinguish it from the element 0 in $\mathbb{F}[\mathcal{S}]$ and $\mathbb{F}[\Pi_n]$). We introduce some notation. For the rest of this section we will use $+$ and $-$ for the operations of R, which should be distinguished from the analogous operations on matrices inside $M_n(\mathbb{K})$. For each $1 \leq i \leq n$, let $g_i = E_i - E_{i-1} \in R$ (the elements E_i and E_{i-1} are, of course, linearly independent) and let $P_i = g_i R$. Let $P_0 = Z \cdot R = \mathrm{Span}_{\mathbb{F}}\{Z\}$. The following remark is key to the structure of the ring R.

REMARK 5.1. For the duration of this remark only, let us denote by $\oplus$ and $\ominus$ the addition and subtraction of matrices in $M_n(\mathbb{K})$ (in order to distinguish these from $+$ and $-$ in R).

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The product in R of two elements in $\mathcal{S}$ is calculated as the usual product in $M_n(\mathbb{K})$, so there is no danger of confusion there. If $A = (a_{ij})_{i,j} \in \mathcal{S}$, then $g_i A g_j = 0$ whenever $i > j$ or $a_{ij} = 0$. To see this, note that

$$g_i A g_j = E_i A E_j - E_{i-1} A E_j - E_i A E_{j-1} + E_{i-1} A E_{j-1}.$$

We now show that either:
(1) $E_i A E_j = E_{i-1} A E_j$ and $E_i A E_{j-1} = E_{i-1} A E_{j-1}$; or
(2) $E_i A E_j = E_i A E_{j-1}$ and $E_{i-1} A E_{j-1} = E_{i-1} A E_j$,
which will prove the claim. These equalities can be regarded as equalities in $M_n(\mathbb{K})$; note that we have assumed that $\mathcal{S}$ (or Π_n) is in upper triangular form, so $A = \sum^{\oplus}_{k \le l} a_{kl} e_{kl}$ (meaning a sum in $M_n(\mathbb{K})$), where e_{kl} are the standard matrix basis elements in $M_n(\mathbb{K})$. It is enough to show that either (1) or (2) holds for $A = a_{kl} e_{kl}$, $k \le l$. Now $a_{kl} E_i e_{kl} E_j = a_{kl} E_{i-1} e_{kl} E_j$ is equivalent to $a_{kl}(E_i \ominus E_{i-1}) e_{kl} E_j = 0$; but $E_i \ominus E_{i-1} = e_{ii}$, so this is further equivalent to $a_{kl} \delta_{ik} e_{il} E_j = 0$. Thus, both equalities in (1) hold if $i \ne k$ or $a_{kl} = 0$. Similarly, if $j \ne l$, one easily sees that both equalities in (2) hold. If $i = k$, $j = l$, since $k \le l$ is assumed for $A = a_{kl} e_{kl}$, then $i \le j$, and the equalities hold since $a_{ij} = 0$ is assumed in this case.

We introduce now a set of one-dimensional representations important in the study of the Jacobson radical of R. Fix i with $0 \le i \le n$. Recall that the semigroup $\mathcal{S}$ is in echelon form; hence, if $A = (a_{jk})_{j,k} \in \mathcal{S}$ is in echelon form, it is triangular, so we may write $A = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}$ with A_1 an $i \times i$ triangular matrix. Note that if $a_{ii} \ne 0$, then all the positions (k, k) with $k \le i$ are pivots, and these entries are then all $a_{kk} = 1$ (and so $a_{ii} = 1$ too). Using similar arguments as before – for example, because the sequence $(A^s)_s$ stabilizes – it follows in this case that $A_1 = I_i$ (the $i \times i$ identity matrix). We define the following map $\varphi_i : \mathcal{S} \to \mathbb{F}$. If $A = (a_{jk})_{j,k} \in \mathcal{S}$, then $\varphi_i(A) = a_{ii}$. Note that $\varphi_i(A)$ is always either 0 or 1, and it is easy to see that this is in fact a morphism of semigroups $\mathcal{S} \to (\{0, 1\}, \cdot)$. Extend this by linearity to a morphism of algebras $\varphi_i : R = \mathbb{F}[\mathcal{S}] \to \mathbb{F}$, and let $L_i = \ker(\varphi_i)$ (we use the same symbol to denote this morphism to avoid further complicating notation).

Note that $\varphi_i(E_0) = \varphi_i(E_1) = \cdots = \varphi_i(E_{i-1}) = 0$ and $\varphi_i(E_i) = \varphi_i(E_{i+1}) = \cdots = \varphi_i(E_n) = 1$ and, so, $\varphi_i(g_j) = \delta_{ij}$. Thus, $g_j \in L_i$ when $j \ne i$. Using the previous remark (or directly), it easy to see that $g_0, \ldots, g_n$ is a complete system of orthogonal idempotents: $g_j g_k = \delta_{j,k}$, and $\sum_{k=0}^n g_k = E_n = 1_R \in R$. Hence, we get $L_i = \bigoplus_{t=0}^n g_t L_i = g_i L_i \oplus \bigoplus_{t \ne i} g_t R = g_i L_i \oplus \bigoplus_{t \ne i} P_t$, because $g_t \in L_i$ when $t \ne i$, so we have $g_t L_i = g_t R = P_t$. Let $M_i = g_i L_i$; this is a maximal submodule of P_i of codimension 1 since $P_i / M_i \cong R / L_i \cong \mathbb{F}$, and we let $S_i = P_i / M_i$ be the corresponding one-dimensional simple R-module. If $A \in \mathcal{S}$, we write A_{jk} for the entries of $A = (A_{jk})_{j,k}$. With this notation,

$$M_i = \Big\{ g_i x \,\Big|\, x = \sum_t \lambda_t A_t,\ \lambda_t \in \mathbb{F},\ A_t \in \mathcal{S} \text{ and } \sum_{\{t \mid (A_t)_{ii} \ne 0\}} \lambda_t = 0 \Big\}.$$

The condition $\varphi_i(x) = 0$ reduces to the one in the set description above because those terms A_t from $x = \sum_t \lambda_t A_t$ with $(A_t)_{ii} = 0$ are automatically in $\ker \varphi_i$.

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Proposition 5.2. *Let $A = (a_{jk}) \in \mathcal{S}$. Then $g_iAg_i = 0$ if $a_{ii} = 0$, and $g_iAg_i = g_i$ if $a_{ii} \neq 0$. Furthermore, if $x \in \ker\varphi_i$, then $g_ixg_i = 0$.*

Proof. If $a_{ii} = 0$, this is already contained in the previous remark. When $a_{ii} \neq 0$, then in this case, as seen above, A has the form $A = \left(\begin{smallmatrix} I_i & A_2 \\ 0 & A_3 \end{smallmatrix}\right)$, and a straightforward computation with block matrices shows that $g_iAg_i = (E_i - E_{i-1})A(E_i - E_{i-1}) = g_i$. Finally, if $x = \sum_t \lambda_t A_t \in \ker\varphi_i$, then $g_ixg_i = g_i(\sum_t \lambda_t A_t)g_i = g_i(\sum_{\{t\,|\,(A_t)_{ii}\neq 0\}} \lambda_t A_t)g_i$, because $g_iA_tg_i = 0$ when $(A_t)_{ii} = 0$. But when $(A_t)_{ii} \neq 0$, $g_iA_tg_i = g_i$, and so

$$g_ixg_i = g_i\Big(\sum_{\{t\,|\,(A_t)_{ii}\neq 0\}} \lambda_t A_t\Big)g_i = \Big(\sum_{\{t\,|\,(A_t)_{ii}\neq 0\}} \lambda_t\Big)g_i = 0$$

because $\sum_{\{t\,|\,(A_t)_{ii}\neq 0\}} \lambda_t = 0$ when $x \in \ker\varphi_i$. □

The above computational observations can now be used to determine the Jacobson radical of R. With the notations above of P_i and M_i, we have the following result.

Proposition 5.3. *Each M_i, $i \geq 1$, is a maximal submodule of P_i, and the P_i are projective indecomposable. In particular, $(g_i)_{0\leq i\leq n}$ is a complete system of primitive orthogonal idempotents. Moreover, $J(R) = M_1 \oplus \cdots \oplus M_n$, $J(R)^{n+1} = 0$ and $S_i = P_i/M_i$, $i \geq 1$, and $S_0 = P_0$ are, up to isomorphism, the $n + 1$ types of simple right R-modules.*

Proof. Let $M = M_1 \oplus \cdots \oplus M_n$. As noted before, M_i are maximal submodules of P_i, and note that $M = \bigcap_{i=0}^n \ker\varphi_i$, an intersection of maximal right ideals, so $J(R) \subseteq M$. We now show that $M^{n+1} = 0$. It is enough to consider an element $x = x_1 \ldots x_n x_{n+1}$, with $x_t \in M_{i_t} = g_{i_t}L_{i_t}$, and show that $x = 0$, since every element in M^{n+1} is a sum of such product elements x. Since $x_t \in g_{i_t}R$, we have $x_t = g_{i_t}x_t$, and so x has the form $x = g_{i_1}x_1g_{i_2}x_2 \ldots g_{i_n}x_ng_{i_{n+1}}x_{n+1}$, with $x_t \in g_{i_t}L_{i_t} \subseteq L_{i_t}$ for all t. As the sequence $i_1, i_2, \ldots, i_{n+1}$ is contained in $\{1, 2, \ldots, n\}$, it cannot be strictly increasing, so there is some $i_s \geq i_{s+1}$. By the previous proposition and the previous remark, $g_{i_s}x_sg_{i_{s+1}} = 0$ since either $i_s > i_{s+1}$ or, if $i_s = i_{s+1}$, then $x_{i_s} \in L_{i_s} = \ker\varphi_{i_s}$, and so $g_{i_s}x_{i_s}g_{i_s} = 0$ by the last part of the previous proposition. Hence, M is a nilpotent right ideal, and so it must be contained in $J(R)$. Since $J(R) \subseteq M$, equality follows.

Finally, since $R/J(R) \cong \mathbb{F}^{n+1}$, it follows that the simple modules S_i are nonisomorphic over $R/J(R)$ and so they are neither isomorphic over R; this, in turn implies that the P_i are indecomposable, since in our case $J(P_i) = P_iJ(R) = M_i$. □

Note that the fact that the P_i are indecomposable also follows because $\mathrm{End}_R(P_i) \cong g_iRg_i = g_i(L_i \oplus \mathbb{F}\cdot 1_R)g_i = \mathbb{F}g_i$ since $g_iL_ig_i = 0$ by the previous proposition.

Corollary 5.4. *We have $\mathrm{Ext}^1_R(S_i, S_j) = 0$ if $j \geq i \geq 0$ and $\mathrm{Ext}^1(S_i, S_0) = \mathrm{Ext}^1(S_0, S_i) = 0$, so the Ext quiver is acyclic (and then, by definition, R is acyclic).*

Proof. This is standard (and known) in the representation theory of finite-dimensional (or semilocal) algebras: $g_jRg_i = 0$ for $j > i$ means that $\mathrm{Hom}_R(P_j, P_i) = 0$; if $\mathrm{Ext}^1(S_i, S_j) = 0$, then there is a nonsplit exact sequence $0 \to S_j \to V \to S_i \to 0$.

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Since P_i is local, there is an epimorphism $h : P_i \to V$. But there is a nonzero morphism $P_j \to V$, which lifts (via h) to a nonzero morphism $P_j \to P_i$, in contradiction with $\mathrm{Hom}_R(P_j, P_i) = 0$. When $i = j$, we get $\mathrm{Hom}_R(P_i, P_i) = \mathbb{F}$, by the comment preceding this corollary, and this implies that $\mathrm{Ext}^1(S_i, S_i) = 0$ by a similar standard argument. The part about the extensions with $S_0 = R \cdot Z = \mathrm{Span}_{\mathbb{F}}\{Z\}$ is obvious. □

Hence, to summarize: 0 is an isolated vertex of the Ext quiver, which is acyclic; there are exactly $n + 1$ simple modules up to isomorphism, and they are all one dimensional, and the Jacobson radical is nilpotent (thus, R is a semiprimary ring).

We now examine $R = \mathbb{F}[\Pi_n]$ more closely, where Π_n is the semigroup of the 2^n possible shapes of echelon matrices of size n. We may identify the elements P_τ of Π_n with their shape $\tau = (k_1, \ldots, k_t)$. We compute the dimensions of g_iRg_j for $i \leq j$. By Remark 5.1, we see that $g_iP_\tau g_j = 0$ if P_τ has 0 at position (i, j). This can be avoided only if $k_i = j$. In this case, note that multiplying P_τ to the left by some E_l retains the first l rows of P_τ and everything else is made 0, and multiplying it by E_p to the right retains the upper left $p \times p$ part of P_τ, and everything else is made 0. It is then not hard to notice that in R,

$$g_iP_\tau g_j = P_{(k_1,k_2,\ldots,k_i)} - P_{(k_1,k_2,\ldots,k_{i-1})}$$

if $k_i = j$. These elements span g_iRg_j and a basis for g_iRg_j is given by the set $\{P_{(k_1,k_2,\ldots,k_{i-1},j)} - P_{(k_1,k_2,\ldots,k_{i-1})} | 1 \leq k_1 < \cdots < k_{i-1} \leq j - 1\}$. Hence,

$$\dim(g_iRg_j) = \binom{j-1}{i-1}.$$

Note that these are precisely the entries in the Cartan matrix of R: $\dim(\mathrm{Hom}_R(P_i, P_j)) = \dim(\mathrm{Hom}(g_iR, g_jR)) = \dim(g_jRg_i)$. Hence, using the facts that $Rg_j = \bigoplus_{i\leq j} g_iRg_j$ and $g_iR = \bigoplus_{i\leq j} g_iRg_j$ and well-known combinatorial identities, we obtain the following result.

Corollary 5.5. *If* $R = \mathbb{F}[\Pi_n]$, *then:*
$\dim(Rg_i) = 2^{i-1}$ *if* $i \geq 1$ *and* $\dim(Rg_0) = 1$,
$\dim(g_iR) = \binom{n}{i}$ *if* $i \geq 1$ *and* $\dim(g_0R) = 1$.

Using the above, the structure of some of the algebras $\mathbb{F}[\Pi_n]$ for small n can be easily determined. We include the following result without proof, which is left to the reader.

Corollary 5.6.

(i) $\mathbb{F}[P_2] \cong \mathbb{F} \times T_2(\mathbb{F})$, *where* $T_2(\mathbb{F})$ *is the algebra of upper triangular* 2×2 *matrices over* $\mathbb{F}$.

(ii) $\mathbb{F}[P_3]$ *decomposes into indecomposable projectives of dimensions* $4, 2, 1, 1$, *as a left module and into indecomposable projectives of dimensions* $1, 3, 3, 1$ *as a right module, and it is isomorphic to the path algebra of the quiver*

$$\begin{tikzcd} \bullet & \bullet \arrow[r] & \bullet \arrow[r, bend left] \arrow[r, bend right] & \bullet \end{tikzcd}$$

with one relation that identifies the two paths of length 2.

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REMARK 5.7. We believe that it is an interesting question to determine completely the (valued) Ext quiver of R, and the structure of $\mathbb{F}[\Pi_n]$ as a quiver algebra with relations. This is perhaps further motivated by the fact that the algebra $\mathbb{F}[\Pi_n]$ has as its Cartan matrix the important combinatorial matrix $\left(\binom{j-1}{i-1}\right)_{0\le i,j\le n}$. It should not be difficult to show that $\dim(\mathrm{Ext}^1_R(S_{i+1}, S_i)) = i (= \dim(g_iRg_{i+1}))$, and it seems tempting to conjecture that $\mathbb{F}[P_n]$ is a quotient of the path algebra of the quiver Q_n with $n+1$ vertices

$$\bullet \qquad \bullet \longrightarrow \bullet \rightrightarrows \bullet \overset{\longrightarrow}{\rightrightarrows} \cdots \underset{n-2}{\overset{\cdots}{\rightrightarrows}} \bullet \underset{n-1}{\overset{\cdots}{\rightrightarrows}} \bullet\,,$$

where between vertices i and $i+1$ there are i arrows (all oriented to the right) and, hence, the above quiver would be the Ext quiver of $R = \mathbb{F}[\Pi_n]$.

For this question, one could proceed as follows. First, as we have seen before,

$$g_iP_\tau g_j = P_{(k_1,k_2,\ldots,k_i)} - P_{(k_1,k_2,\ldots,k_{i-1})}$$

and the elements $\{R_{k_1,k_2,\ldots,k_i} = P_{(k_1,k_2,\ldots,k_{i-1},j)} - P_{(k_1,k_2,\ldots,k_{i-1})} \mid 1 \le k_1 < \cdots < k_{i-1} \le j-1, k_i = j\}$ provide a basis for g_iRg_j. One can show that the multiplication on this basis of R consisting of these elements and including g_0 is done as follows. If $R_{(k(1),\ldots,k(i))} \in g_iRg_j$ is such that $k(i) = j$ and $R_{(s(1),\ldots,s(j))} \in g_jRg_k$ is such that $s(j) = k$, then

$$R_{(k(1),\ldots,k(i))} \cdot R_{(s(1),\ldots,s(j))} = R_{(sk(1),\ldots,sk(i))}; \quad sk(i) = k,$$

which makes the 'nonzero' part $\bigoplus_{1\le i\le j} g_iRg_j$ of $\mathbb{F}[\Pi_n]$ into a monoid algebra. These elements can be identified with certain paths in the path algebra of the above quiver Q_n. They can also be used to determine the 'top' of M_i and thus the dimension of $\mathrm{Ext}(S_j, S_i)$ for $j > i$.

REMARK 5.8. We also remark that $\mathbb{F}[\Pi_n]$ has a bialgebra structure, as a semigroup bialgebra, with comultiplication given by $\Delta(x) = x \otimes x$ and $\varepsilon(x) = 1$ for $x \in \Pi_n$. Thus, representations of $\mathbb{F}[\Pi_n]$ have a natural tensor product, and the free abelian group on the equivalence classes of representations of Π_n becomes a ring – the representation ring (or Green ring) of $\mathbb{F}[\Pi_n]$. It would maybe be interesting to study the structure of this representation ring or some small subrings of it.

REMARK 5.9. One may also wonder what is the relation of $\mathbb{F}[\Pi_n]$ and the Grassmann (exterior) algebra $\Lambda_n(\mathbb{F})$. Of course, they are not isomorphic: $\mathbb{F}[\Pi_n]$ has n simple one-dimensional modules, and is not Frobenius, while $\Lambda_n(\mathbb{F})$ is even a Hopf algebra (so it is Frobenius), and is local.

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