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Some beta-function inequalities

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We present several new inequalities for Euler's beta function, $B(x,y)$. One of our results states that the beta function can be approximated on $(0,1]\times(0,1]$ by rational functions as follows,

$$\alpha\frac{(1-x)(1-y)}{x(1+x)y(1+y)} \leqslant \frac{1}{xy} - B(x,y) \leqslant \beta\frac{(1-x)(1-y)}{x(1+x)y(1+y)},$$

with the best possible constants $\alpha = 1$ and $\beta = \frac{2}{3}\pi^2 - 4 = 2.579\,73\ldots$.

1. Introduction

The classical beta and gamma functions, also known as Euler's integrals of the first and second kind, respectively, are defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t, \quad x,y>0,$$

and

$$\Gamma(x) = \int_0^\infty \mathrm{e}^{-t}t^{x-1}\,\mathrm{d}t, \qquad x>0.$$

They are closely connected by the elegant identity

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y>0.$$

Since both functions play an important role in various branches of mathematics and mathematical physics, they have been investigated intensively by many authors. We refer to the monographs [1, 6, 7].

In the past, numerous papers were published presenting remarkable inequalities involving the gamma function (see [11] and the extensive list of references given therein). But only few inequalities for the beta function and its relatives can be found in the literature (see [4, 9, 12, 13, 15–18]). It is the aim of this article to add to the list of beta-function inequalities.

In 2000, Dragomir *et al*. [9] proved that the beta function can be approximated by $1/(xy)$. More precisely, they proved

$$0 \leqslant \frac{1}{xy} - B(x,y) \leqslant \tfrac{1}{4}, \quad x,y \geqslant 1. \tag{1.1}$$

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In [4], it was shown that $\frac{1}{4}$ can be replaced by the best possible constant $0.087\,31\ldots$. In view of (1.1), it is natural to ask for a corresponding result, which holds for $x, y \in (0,1]$. In § 3 we present sharp upper and lower rational bounds for the difference $1/(xy) - B(x,y)$, valid for all $x, y \in (0,1]$. Moreover, we show that the beta function can be approximated on $(0,1] \times (0,1]$ by the logarithmic derivative of the gamma function. Our third result provides double inequalities for $B(x,y)$, which hold for $x, y \geqslant 1$ and $\min(x,y) \leqslant 1 \leqslant \max(x,y)$, respectively. And, finally, we prove a new functional inequality for the beta function, which is closely related to the triangle inequality

$$(B(x,z))^\alpha < (B(x,y))^\alpha + (B(y,z))^\alpha, \quad 0 < x \leqslant y \leqslant z, \quad \alpha \in \boldsymbol{R}. \tag{1.2}$$

Throughout this paper, we denote by $\gamma = 0.577\,21\ldots$ Euler's constant and by $\psi = \Gamma'/\Gamma$ the psi (or digamma) function. In order to prove our theorems, we need some lemmas. They are collected in the next section. The numerical values have been calculated by MAPLE (V Release 5.1).

2. Lemmas

In this section, we set $\beta = \frac{2}{3}\pi^2 - 4 = 2.579\,73\ldots$. The first lemma presents some basic formulae, which can be found in [1, ch. 6].

LEMMA 2.1. *For all $x > 0$, we have*

$$(-1)^{n+1}\psi^{(n)}(x) = n! \sum_{k=0}^{\infty} \frac{1}{(x+k)^{n+1}} = \int_0^\infty \mathrm{e}^{-xt} \frac{t^n}{1-\mathrm{e}^{-t}}\,\mathrm{d}t, \quad n = 1, 2, \ldots, \tag{2.1}$$

$$\Gamma(x+1) = x\Gamma(x), \quad \psi^{(n)}(x+1) = \psi^{(n)}(x) + (-1)^n \frac{n!}{x^{n+1}}, \quad n = 0, 1, 2, \ldots, \tag{2.2}$$

$$\Gamma(2x) = \frac{4^x}{2\sqrt{\pi}}\Gamma(x)\Gamma(x+\tfrac{1}{2}), \quad \psi(2x) = \tfrac{1}{2}\psi(x) + \tfrac{1}{2}\psi(x+\tfrac{1}{2}) + \log 2, \tag{2.3}$$

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} \sim x^{a-b}, \quad \psi(x) \sim \log x, \quad x \to \infty. \tag{2.4}$$

The following monotonicity properties are proved in [2, 3].

LEMMA 2.2.

(i) *Let $n \geqslant 1$ be an integer and $c \geqslant n+1$. The functions*

$$x \mapsto x\frac{\psi^{(n+1)}(x)}{\psi^{(n)}(x)} \quad \textit{and} \quad x \mapsto x^c(-1)^{n+1}\psi^{(n)}(x)$$

are strictly increasing on $(0,\infty)$.

(ii) *The function $p(x) = x\psi(x)$ is negative and strictly decreasing on $(0, x_0)$, where $x_0 = 0.216\ldots$. Also, p is strictly convex on $(0,\infty)$.*

Lemma 2.3. *Let*

$$\begin{aligned} u_1(x) &= \psi(x+2) - \psi(2x), \\ u_2(x) &= \psi'(x+2) - 2\psi'(2x), \\ u_3(x) &= \psi''(x+2) - 4\psi''(2x), \\ v_1(x) &= 2(1+x) + 2\beta(1-x), \\ v_2(x) &= (1+x)^2 - \beta(1-x)^2. \end{aligned}$$

Then u_1, $-u_2$, u_3 and v_1 are positive and decreasing on $(0,1)$. The function v_2 is increasing on $(0,1)$ and has precisely one zero, namely $x_1 = 0.232\ldots$.

Proof. It suffices to consider u_1, u_2 and u_3. Let $0 < x < 1$. Applying (2.1), we get

$$u_3'(x) = \psi'''(x+2) - 8\psi'''(2x) = 6\sum_{k=0}^{\infty}\left(\frac{1}{(x+2+k)^4} - \frac{8}{(2x+k)^4}\right).$$

Since

$$\frac{1}{(x+2+k)^4} \leqslant \frac{8}{(2x+k)^4} \quad \text{for } k \geqslant 0,$$

we obtain $u_3'(x) \leqslant 0$ and $u_3(x) \geqslant u_3(1) = 1.46\ldots$. This implies $u_2'(x) = u_3(x) > 0$ and $u_2(x) \leqslant u_2(1) = -0.89\ldots$. Hence $u_1'(x) = u_2(x) < 0$ and $u_1(x) \geqslant u_1(1) = \frac{1}{2}$. □

Lemma 2.4. *For all $x \in (0,1]$, we have*

$$0 \leqslant \frac{(\Gamma(x+2))^2}{\Gamma(2x)} + \beta(1-x)^2 - (1+x)^2.$$

Proof. Let u_1, u_2, u_3, v_1, v_2 and x_1 be defined in lemma 2.3. Since $v_2(x) \leqslant 0$ for $0 < x \leqslant x_1$, we may assume that $x \in (x_1, 1]$. Let

$$f(x) = 2\log\Gamma(x+2) - \log\Gamma(2x) - \log((1+x)^2 - \beta(1-x)^2).$$

Then we obtain

$$\tfrac{1}{2}v_2(x)f'(x) = u_1(x)v_2(x) - \tfrac{1}{2}v_1(x) = g(x), \quad \text{say.}$$

Further, we have

$$g'(x) = u_1(x)v_1(x) + u_2(x)v_2(x) + \beta - 1$$

and

$$g''(x) = 2(1-\beta)u_1(x) + 2u_2(x)v_1(x) + u_3(x)v_2(x).$$

Let $x_1 \leqslant r \leqslant x \leqslant s \leqslant 1$. From lemma 2.3 we get

$$g''(x) \leqslant 2(1-\beta)u_1(s) + 2u_2(s)v_1(s) + u_3(r)v_2(s) = h(r,s), \quad \text{say.}$$

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The numerical values

$$\begin{aligned} h(x_1, 0.35) \leqslant h(0.232, 0.35) &= -7.03\ldots, \\ h(0.35, 0.48) &= -0.04\ldots, \\ h(0.48, 0.61) &= -1.17\ldots, \\ h(0.61, 0.75) &= -0.38\ldots, \\ h(0.75, 0.89) &= -0.26\ldots, \\ h(0.89, 1) &= -0.79\ldots \end{aligned}$$

imply that $g''(x) < 0$ for $x \in [x_1, 1]$. Since $g(1) = g'(1) = 0$, we conclude that $g(x) \leqslant 0$. This leads to $f(x) \geqslant f(1) = 0$ for $x_1 \leqslant x \leqslant 1$. □

Lemma 2.5. *Let* $0 < x \leqslant y \leqslant 1$. *Then we have*

$$(x+y)[\psi(x+y) - \psi(x+2)] \leqslant -1.$$

Proof. Let $0 < x \leqslant y \leqslant 1$ and

$$f(x,y) = (x+y)[\psi(x+y) - \psi(x+2)].$$

We have

$$\frac{\partial f(x,y)}{\partial y} = \psi(x+y) - \psi(x+2) + (x+y)\psi'(x+y)$$

and

$$\frac{\partial^2 f(x,y)}{\partial y^2} = 2\psi'(x+y) + (x+y)\psi''(x+y) = 2\sum_{k=1}^{\infty} \frac{k}{(x+y+k)^3} > 0.$$

This implies

$$f(x,y) \leqslant \max(f(x,x), f(x,1)). \tag{2.5}$$

Let

$$g(x) = \psi(x+2) - \psi(2x) - \frac{1}{2x}.$$

Using (2.1) and (2.3), we get

$$2g'(x) = 2\psi'(x+2) - \psi'(x+\tfrac{1}{2}) - \psi'(x) + \frac{1}{x^2} = \int_0^{\infty} \mathrm{e}^{-xt}\Delta(t)\,\mathrm{d}t,$$

where

$$\Delta(t) = \frac{t\mathrm{e}^{-2t}}{1-\mathrm{e}^{-t}}[2 - \mathrm{e}^{3t/2} - \mathrm{e}^{t}] < 0 \quad \text{for } t > 0.$$

Thus g is strictly decreasing on $(0, \infty)$, and we obtain

$$g(x) \geqslant g(1) = 0 \quad \text{for } 0 < x \leqslant 1. \tag{2.6}$$

Since $f(x,x) = -2xg(x) - 1$ and $f(x,1) = -1$, we conclude from (2.5) and (2.6) that $f(x,y) \leqslant -1$. □

LEMMA 2.6. *Let $0 < x \leqslant y \leqslant 1$. Then we have*

$$0 \leqslant [\psi(x+2) - \psi(x+y)]^2 + \psi'(x+2) - \psi'(x+y).$$

Proof. Let $0 < x \leqslant y \leqslant 1$ and

$$f(x,y) = [\psi(x+2) - \psi(x+y)]^2 + \psi'(x+2) - \psi'(x+y).$$

Then

$$\frac{x+y}{\psi'(x+y)} \frac{\partial f(x,y)}{\partial y} = -(x+y)\frac{\psi''(x+y)}{\psi'(x+y)} + 2(x+y)[\psi(x+y) - \psi(x+2)]. \quad (2.7)$$

Applying (2.2), we get

$$z\frac{\psi''(z)}{\psi'(z)} = \frac{-2 + z^3\psi''(z+1)}{1 + z^2\psi'(z+1)}, \quad z > 0. \quad (2.8)$$

From (2.8) and lemma 2.2 (i), we obtain

$$-(x+y)\frac{\psi''(x+y)}{\psi'(x+y)} \leqslant 2, \quad (2.9)$$

so that (2.7), (2.9) and lemma 2.5 yield $\partial f(x,y)/\partial y \leqslant 0$. Thus $f(x,y) \geqslant f(x,1) = 0$. □

LEMMA 2.7. *For all $x \in (0,1]$, we have*

$$\psi'(x+2)\frac{(\Gamma(x+2))^2}{\Gamma(2x)} \leqslant \beta - 1. \quad (2.10)$$

Proof. We define, for $x \in (0,1]$,

$$f(x) = \frac{\Gamma(x+\frac{1}{2})}{\Gamma(x+1)} \quad \text{and} \quad g(x) = -x(x+1)^2 4^{-x}\psi'(x+2).$$

Using (2.2) and (2.3), we conclude that (2.10) is equivalent to

$$0 \leqslant (\beta-1)(4\pi)^{-1/2} f(x) + g(x) = h(x), \quad \text{say.}$$

Differentiation gives

$$f'(x) = f(x)[\psi(x+\tfrac{1}{2}) - \psi(x+1)] < 0$$

and

$$(x+2)\frac{g'(x)}{g(x)} = \delta(x) + (x+2)\frac{\psi''(x+2)}{\psi'(x+2)}, \quad (2.11)$$

where

$$\delta(x) = (x+2)\left(\frac{1}{x} + \frac{2}{x+1} - \log 4\right).$$

We have

$$-\tfrac{1}{2}\delta'(x) = \frac{x^2 + (x+1)^2(1 + x^2\log 2)}{x^2(x+1)^2} > 0,$$

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so that (2.11) and lemma 2.2 (i) yield

$$(x+2)\frac{g'(x)}{g(x)} \geqslant \delta(1) + 2\frac{\psi''(2)}{\psi'(2)} = 0.58\ldots.$$

Hence f and g are decreasing on $(0,1]$. This leads to $h(x) \geqslant h(1) = 0$. □

LEMMA 2.8. *For all* $x \in (0,1]$, *we have*

$$[\psi(x+2) - \psi(2x)]\frac{(\Gamma(x+2))^2}{\Gamma(2x)} + x(\beta - 1) - (\beta + 1) \leqslant 0.$$

Proof. Let $0 < x \leqslant 1$ and

$$f(x) = [\psi(x+2) - \psi(2x)]\frac{(\Gamma(x+2))^2}{\Gamma(2x)} + x(\beta - 1) - (\beta + 1).$$

We obtain

$$f'(x) = (2[\psi(x+2) - \psi(2x)]^2 + \psi'(x+2) - 2\psi'(2x))\frac{(\Gamma(x+2))^2}{\Gamma(2x)} + \beta - 1. \quad (2.12)$$

Applying lemma 2.6, we get

$$2[\psi(x+2) - \psi(2x)]^2 + \psi'(x+2) - 2\psi'(2x) \geqslant -\psi'(x+2). \quad (2.13)$$

From (2.12), (2.13) and lemma 2.7, we obtain

$$f'(x) \geqslant \beta - 1 - \psi'(x+2)\frac{(\Gamma(x+2))^2}{\Gamma(2x)} \geqslant 0.$$

Thus $f(x) \leqslant f(1) = 0$. □

LEMMA 2.9. *Let*

$$\begin{aligned} w_1(x) &= \psi(2x) - \psi(x), \\ w_2(x) &= \psi'(x) - 2\psi'(2x), \\ w_3(x) &= x[\psi(2x) - \psi(x)]. \end{aligned}$$

The functions w_1 *and* w_2 *are positive and decreasing and* $(0,\infty)$, *whereas* w_3 *is positive and increasing on* $(0,\infty)$.

Proof. Let $x > 0$. We have

$$-w_1'(x) = w_2(x) = \tfrac{1}{2}[\psi'(x) - \psi'(x+\tfrac{1}{2})] > 0$$

and

$$w_2'(x) = \tfrac{1}{2}[\psi''(x) - \psi''(x+\tfrac{1}{2})] < 0.$$

Further,

$$\begin{aligned} 2w_3(x) &= x[\psi(x+\tfrac{1}{2}) - \psi(x) + \log 4] > 0, \\ 2w_3'(x) &= \psi(x+\tfrac{1}{2}) - \psi(x) + \log 4 + x[\psi'(x+\tfrac{1}{2}) - \psi'(x)], \\ \frac{2}{x}w_3''(x) &= \frac{2}{x}[\psi'(x+\tfrac{1}{2}) - \psi'(x)] + \psi''(x+\tfrac{1}{2}) - \psi''(x). \end{aligned}$$

Using the integral formula given in (2.1) and the convolution theorem for Laplace transforms, we get

$$\frac{2}{x}w_3''(x) = \int_0^\infty \mathrm{e}^{-xt}\Delta(t)\,\mathrm{d}t,$$

where

$$\Delta(t) = \frac{t^2}{1+\mathrm{e}^{-t/2}} - 2\int_0^t \frac{s}{1+\mathrm{e}^{-s/2}}\,\mathrm{d}s.$$

We have

$$\Delta(0) = 0 \quad \text{and} \quad \Delta'(t) = \frac{t^2\mathrm{e}^{-t/2}}{2(1+\mathrm{e}^{-t/2})^2} > 0 \quad \text{for } t > 0,$$

so that we obtain $w_3''(x) > 0$ for $x > 0$. Since $\lim_{x\to 0} w_3'(x) = 0$, we conclude that w_3' is positive on $(0, \infty)$. □

Lemma 2.10. *For all* $x \in (0, 1]$, *we have*

$$\frac{6\gamma}{\pi^2} \leqslant [\psi(x) - \psi(2x)]\frac{\psi(x)}{\psi'(x)}.$$

Proof. Let $a = 6\gamma/\pi^2$ and let p, x_0, w_1, w_2, w_3 be defined in lemmas 2.2 and 2.9, respectively. Further, let

$$w_4(x) = x^2\psi'(x) \quad \text{and} \quad f(x) = [\psi(x) - \psi(2x)]\psi(x) - a\psi'(x).$$

Lemma 2.2 yields that $-p$ is positive and increasing on $(0, x_0)$ and that w_4 is increasing on $(0, \infty)$. Thus we get, for $0 < x < x_0$,

$$x^2 f(x) = w_3(x)(-p(x)) - aw_4(x) \geqslant w_3(0)(-p(0)) - aw_4(0.22) = 0.12\ldots.$$

Differentiation gives

$$-f'(x) = w_2(x)(-\psi(x)) + w_1(x)\psi'(x) + a\psi''(x).$$

Applying lemma 2.9, we get, for $0.21 \leqslant r \leqslant x \leqslant s \leqslant 1$,

$$-f'(x) \geqslant w_2(s)(-\psi(s)) + w_1(s)\psi'(s) + a\psi''(r) = g(r, s), \quad \text{say.}$$

We have

$$\begin{aligned}
g(0,21, 0.24) &= 2.37\ldots,\\
g(0.24, 0.27) &= 4.77\ldots,\\
g(0.27, 0.31) &= 1.64\ldots,\\
g(0,31, 0.36) &= 0.78\ldots,\\
g(0.36, 0.42) &= 0.73\ldots,\\
g(0.42, 0.50) &= 0.26\ldots,\\
g(0.50, 0.61) &= 0.08\ldots,\\
g(0,61, 0.77) &= 0.02\ldots,\\
g(0.77, 1) &= 0.12\ldots.
\end{aligned}$$

This implies that $f'(x) < 0$ for $x \in [0.21, 1]$. Since $f(1) = 0$, we conclude that f is positive on $[0.21, 1)$. □

Further, we need the following integral inequalities, which are due to Steffensen. A proof is given in [16, § 2.16].

LEMMA 2.11. *Let f and g be integrable functions on $[a,b]$. If f is strictly decreasing on (a,b) and $0<g<1$ on (a,b), then*

$$\int_{b-\lambda}^{b} f(t)\,\mathrm{d}t < \int_a^b f(t)g(t)\,\mathrm{d}t < \int_a^{a+\lambda} f(t)\,\mathrm{d}t,$$

where $\lambda = \int_a^b g(t)\,\mathrm{d}t$.

3. Main results

First, we present sharp rational bounds for $B(x,y)$, which are valid for $x,y\in(0,1]$.

THEOREM 3.1. *For all real numbers $x,y\in(0,1]$, we have*

$$\alpha\frac{(1-x)(1-y)}{x(1+x)y(1+y)} \leqslant \frac{1}{xy} - B(x,y) \leqslant \beta\frac{(1-x)(1-y)}{x(1+x)y(1+y)}, \tag{3.1}$$

with the best possible constants

$$\alpha = 1 \quad \textit{and} \quad \beta = \tfrac{2}{3}\pi^2 - 4 = 2.579\,73\ldots.$$

Proof. Let $0<x\leqslant y\leqslant 1$ and

$$f(x,y) = \log\Gamma(x+y+1) - \log\Gamma(x+2) - \log\Gamma(y+2) + \log 2.$$

Differentiation gives

$$\frac{\partial f(x,y)}{\partial x} = \psi(x+y+1) - \psi(x+2) \leqslant 0,$$

which implies

$$f(x,y) \geqslant f(y,y) = \log\Gamma(2y+1) - 2\log\Gamma(y+2) + \log 2 = g(y), \quad \text{say.}$$

Since

$$\tfrac{1}{2}g'(y) = \psi(2y+1) - \psi(y+2) \leqslant 0,$$

we get

$$g(y) \geqslant g(1) = 0.$$

Hence $f(x,y)\geqslant 0$, which is equivalent to the left-hand side of (3.1) with $\alpha=1$.

To prove the second inequality of (3.1) with $\beta=\frac{2}{3}\pi^2-4$, we show that

$$h(x,y) = \frac{\Gamma(x+2)\Gamma(y+2)}{\Gamma(x+y)} + (xy+1)(\beta-1) - (x+y)(\beta+1) \geqslant 0 \quad \text{for } 0<x\leqslant y\leqslant 1.$$

We have

$$\frac{\partial h(x,y)}{\partial x} = [\psi(x+2)-\psi(x+y)]\frac{\Gamma(x+2)\Gamma(y+2)}{\Gamma(x+y)} + y(\beta-1) - (\beta+1)$$

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and

$$\frac{\partial^2 h(x,y)}{\partial x^2} = ([\psi(x+2)-\psi(x+y)]^2 + \psi'(x+2) - \psi'(x+y))\frac{\Gamma(x+2)\Gamma(y+2)}{\Gamma(x+y)}.$$

Applying lemma 2.6, we get $\partial^2 h(x,y)/\partial x^2 \geqslant 0$, which leads to

$$\frac{\partial h(x,y)}{\partial x} \leqslant \frac{\partial h(x,y)}{\partial x}\bigg|_{x=y} = [\psi(y+2)-\psi(2y)]\frac{(\Gamma(y+2))^2}{\Gamma(2y)} + y(\beta-1)-(\beta+1). \quad (3.2)$$

From lemma 2.8, we conclude that $\partial h(x,y)/\partial x \leqslant 0$, so that lemma 2.4 gives

$$h(x,y) \geqslant h(y,y) = \frac{(\Gamma(y+2))^2}{\Gamma(2y)} + \beta(1-y)^2 - (1+y)^2 \geqslant 0.$$

It remains to show that in (3.1) the constants $\alpha = 1$ and $\beta = \frac{2}{3}\pi^2 - 4$ are sharp. Double-inequality (3.1) is equivalent to

$$\alpha \leqslant \frac{(x+1)(y+1)}{(1-x)(1-y)}\left(1 - \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y)}\right) \leqslant \beta. \quad (3.3)$$

If we let x tend to 0, then (3.3) gives $\alpha \leqslant y+1$, which implies $\alpha \leqslant 1$.

Next, we set $q(x,y) = \Gamma(x+1)\Gamma(y+1)/\Gamma(x+y)$. Then the right-hand side of (3.3) yields

$$\frac{(x+1)(y+1)}{1-y}\frac{q(x,y)-q(1,y)}{x-1} \leqslant \beta.$$

We let x tend to 1 and obtain

$$\frac{2(y+1)}{1-y}\frac{\partial q(x,y)}{\partial x}\bigg|_{x=1} = 2(y+1)\frac{\psi(y+1)-\psi(2)}{y-1} \leqslant \beta.$$

And, if y tends to 1, then $4\psi'(2) = \frac{2}{3}\pi^2 - 4 \leqslant \beta$. This completes the proof of theorem 3.1. □

The psi function and its derivatives have a number of interesting applications. For example, certain trigonometric integrals can be expressed in a closed form in terms of ψ (see [10]). A close relationship between $\psi^{(n)}(x)$ (with $x = \frac{1}{4}$ and $x = \frac{3}{4}$) and Bernoulli and Euler numbers is presented in [14]. And, in a recently published article [5], it is proved that the constants of Landau and Lebesgue can be approximated by the psi function. We now provide sharp upper and lower bounds for $B(x,y)$ in terms of $\psi(x)\psi(y)$.

Theorem 3.2. *For all real numbers $x, y \in (0,1]$, we have*

$$\left(\frac{\psi(x)\psi(y)}{\gamma^2}\right)^{\alpha} \leqslant B(x,y) \leqslant \left(\frac{\psi(x)\psi(y)}{\gamma^2}\right)^{\beta}, \quad (3.4)$$

with the best possible constants

$$\alpha = \frac{6\gamma}{\pi^2} = 0.350\,90\ldots \quad \textit{and} \quad \beta = 1.$$

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Proof. To establish the left-hand side of (3.4) with $\alpha = 6\gamma/\pi^2$, we define, for $0 < x \leqslant y \leqslant 1$,

$$f(x,y) = \log\Gamma(x) + \log\Gamma(y) - \log\Gamma(x+y) \\ - \alpha\log(-\psi(x)) - \alpha\log(-\psi(y)) + 2\alpha\log\gamma.$$

Applying lemma 2.10, we get

$$\frac{\partial f(x,y)}{\partial x} = \psi(x) - \psi(x+y) - \alpha\frac{\psi'(x)}{\psi(x)} \\ \leqslant \psi(x) - \psi(2x) - \alpha\frac{\psi'(x)}{\psi(x)} \leqslant 0.$$

This leads to

$$f(x,y) \geqslant f(y,y) = 2\log\Gamma(y) - \log\Gamma(2y) - 2\alpha\log(-\psi(y)) + 2\alpha\log\gamma.$$

Applying lemma 2.10 again, we obtain

$$\frac{1}{2}\frac{\mathrm{d}f(y,y)}{\mathrm{d}y} = \psi(y) - \psi(2y) - \alpha\frac{\psi'(y)}{\psi(y)} \leqslant 0.$$

Thus $f(y,y) \geqslant f(1,1) = 0$.

Since $p(x) = x\psi(x)$ is convex on $(0,\infty)$ with $p(0) = -1 < -\gamma = p(1)$, we get

$$\frac{1}{x} \leqslant -\frac{\psi(x)}{\gamma} \quad \text{for } 0 < x \leqslant 1.$$

Theorem 3.1 implies

$$B(x,y) \leqslant \frac{1}{xy} \quad \text{for } 0 < x, y \leqslant 1,$$

so that we obtain

$$B(x,y) \leqslant \frac{\psi(x)\psi(y)}{\gamma^2} \quad \text{for } 0 < x, y \leqslant 1.$$

Finally, we show that the best possible constants in (3.4) are given by $\alpha = 6\gamma/\pi^2$ and $\beta = 1$. Setting $x = y$, the left-hand side of (3.4) yields, for $x \in (0,1)$,

$$\alpha \leqslant \frac{2\log\Gamma(x) - \log\Gamma(2x)}{2\log(-\psi(x)) - 2\log\gamma} = Q(x), \quad \text{say.}$$

We apply l'Hospital's rule and get

$$\alpha \leqslant \lim_{x\to 1} Q(x) = \lim_{x\to 1}\frac{\psi(x) - \psi(2x)}{\psi'(x)/\psi(x)} = \frac{6\gamma}{\pi^2}.$$

And, if $y = 1$, then the right-hand side of (3.4) gives, for $x \in (0,1)$,

$$\frac{-\log x}{\log(1 - x\psi(x+1)) - \log x - \log\gamma} \leqslant \beta.$$

We let x tend to 0 and obtain $\beta \geqslant 1$. □

Remark 3.3. A simple calculation shows that, for all $x, y \in (0, 1)$, the upper bound for $B(x, y)$ given in (3.1) is better than the upper bound in (3.4), whereas the lower bounds given in (3.1) and (3.4) cannot be compared. This means that the function

$$(x,y) \mapsto \left(\frac{\psi(x)\psi(y)}{\gamma^2}\right)^{\alpha} + \beta\frac{(1-x)(1-y)}{x(1+x)y(1+y)} - \frac{1}{xy}, \quad \alpha = \frac{6\gamma}{\pi^2}, \quad \beta = \tfrac{2}{3}\pi^2 - 4,$$

attains positive and negative values on $(0, 1) \times (0, 1)$.

The next theorem presents inequalities for the beta function, which are valid on $[1, \infty) \times [1, \infty)$ and $(0, 1] \times [1, \infty)$, respectively.

Theorem 3.4. *For all real numbers $x, y \geqslant 1$, we have*

$$\left(1 - \frac{1}{\max(x,y)}\right)^{\min(x,y)} \leqslant 1 - \min(x,y)B(x,y) \leqslant 1 - \left(\frac{1}{\max(x,y)}\right)^{\min(x,y)}. \quad (3.5)$$

If $0 < \min(x, y) \leqslant 1 \leqslant \max(x, y)$, then (3.5) holds with the inequalities reversed. In both cases, the sign of equality is valid if and only if $x = 1$ or $y = 1$.

Proof. We consider two cases.

Case 1 ($1 < x \leqslant y$). We define $f(t) = (1-t)^{x-1}$ and $g(t) = t^{y-1}$. Then f is strictly decreasing on $(0, 1)$ and $0 < g(t) < 1$ for $t \in (0, 1)$. Applying lemma 2.11 with $a = 0$, $b = 1$, $\lambda = 1/y$, we get

$$\int_{1-1/y}^{1} (1-t)^{x-1}\,\mathrm{d}t < B(x,y) < \int_0^{1/y} (1-t)^{x-1}\,\mathrm{d}t,$$

which leads to (3.5) with '$<$' instead of '$\leqslant$'.

Case 2 ($0 < x < 1 < y$). We set $f(t) = -(1-t)^{x-1}$ and $g(t) = t^{y-1}$. Then lemma 2.11 with $a = 0$, $b = 1$, $\lambda = 1/y$ gives

$$-\int_{1-1/y}^{1} (1-t)^{x-1}\,\mathrm{d}t < -B(x,y) < -\int_0^{1/y} (1-t)^{x-1}\,\mathrm{d}t.$$

This yields (3.5) with '$>$' instead of '$\leqslant$'. □

The well-known inequality

$$0 \leqslant (y-x)(z-x)x^{\alpha} + (x-y)(z-y)y^{\alpha} + (x-z)(y-z)z^{\alpha}, \quad x, y, z > 0, \quad \alpha \in \boldsymbol{R}, \quad (3.6)$$

is due to Schur. A proof and an extension of (3.6) can be found in [16, § 2.17]. The following result, which is also valid for all $x, y, z > 0$ and $\alpha \in \boldsymbol{R}$, presents a beta-function variant of Schur's inequality:

$$0 \leqslant (y-x)(z-x)(B(y,z))^{\alpha} + (x-y)(z-y)(B(x,z))^{\alpha} + (x-z)(y-z)(B(x,y))^{\alpha}. \quad (3.7)$$

The sign of equality holds in (3.7) (and also in (3.6)) if and only if $x = y = z$. Inequality (3.7) and its counterpart (1.2) both follow easily from the fact that $x \mapsto B(x, y)$ is strictly decreasing on $(0, \infty)$. We now provide a companion of (3.7) involving two parameters.

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THEOREM 3.5. *Let $\alpha \neq 0$ and $\beta > 0$ be real numbers. The inequality*

$$(z-x)^{\beta}(B(x,z))^{\alpha} \leqslant (y-x)^{\beta}(B(x,y))^{\alpha} + (z-y)^{\beta}(B(y,z))^{\alpha} \tag{3.8}$$

holds for all real numbers x, y, z with $0 < x \leqslant y \leqslant z$ if and only if $\alpha < 0 < \beta \leqslant 1$.

Proof. Let $\alpha < 0 < \beta \leqslant 1$. If $x = y$ or $y = z$, then equality is valid in (3.8). Hence we may assume that $0 < x < y < z$. Then (3.8) can be written as

$$1 \leqslant \left(\frac{y-x}{z-x}\right)^{\beta}\left(\frac{B(x,y)}{B(x,z)}\right)^{\alpha} + \left(\frac{z-y}{z-x}\right)^{\beta}\left(\frac{B(y,z)}{B(x,z)}\right)^{\alpha}. \tag{3.9}$$

Since

$$0 < \frac{y-x}{z-x} < 1 \quad \text{and} \quad 0 < \frac{z-y}{z-x} < 1,$$

we conclude that the expression on the right-hand side of (3.9) is decreasing with respect to β. This implies that it suffices to prove (3.8) for $\alpha < 0$ and $\beta = 1$. We define

$$f(x,y,z;\alpha,\beta) = (y-x)^{\beta}(B(x,y))^{\alpha} + (z-y)^{\beta}(B(y,z))^{\alpha} - (z-x)^{\beta}(B(x,z))^{\alpha}. \tag{3.10}$$

Differentiation yields

$$\frac{\partial f(x,y,z;\alpha,1)}{\partial x} = \Delta(x,z;\alpha) - \Delta(x,y;\alpha), \tag{3.11}$$

where

$$\Delta(x,z;\alpha) = (B(x,z))^{\alpha}[1 - \alpha(z-x)(\psi(x) - \psi(x+z))].$$

Further, we have

$$\begin{aligned}\frac{1}{\alpha(B(x,z))^{\alpha}}\frac{\partial \Delta(x,z;\alpha)}{\partial z} &= [\psi(z) - \psi(x+z)][1 - \alpha(z-x)(\psi(x) - \psi(x+z))] \\ &\qquad - \psi(x) + \psi(x+z) + (z-x)\psi'(x+z).\end{aligned} \tag{3.12}$$

Since the expression on the right-hand side of (3.12) is strictly decreasing with respect to α, we obtain

$$\frac{1}{\alpha(B(x,z))^{\alpha}}\frac{\partial \Delta(x,z;\alpha)}{\partial z} > \psi(z) - \psi(x) + (z-x)\psi'(x+z) > 0.$$

Hence $z \mapsto \Delta(x,z;\alpha)$ is strictly decreasing on $[y,\infty)$, so that (3.11) implies

$$\frac{\partial f(x,y,z;\alpha,1)}{\partial x} < 0.$$

This leads to $f(x,y,z;\alpha,1) > f(y,y,z;\alpha,1) = 0$.

Next, let (3.8) be valid for all x, y, z with $0 < x \leqslant y \leqslant z$. We assume that $\beta > 1$. Then we get

$$f(x,y,z;\alpha,\beta) \geqslant 0 = f(y,y,z;\alpha,\beta) \quad \text{for } 0 < x \leqslant y < z.$$

This implies

$$\left.\frac{\partial f(x,y,z;\alpha,\beta)}{\partial x}\right|_{x=y} = (z-y)^{\beta-1}(B(y,z))^{\alpha}[\beta - \alpha(z-y)(\psi(y)-\psi(y+z))] \leqslant 0.$$

Hence $\beta \leqslant \alpha(z-y)(\psi(y)-\psi(y+z))$, which is false for all z, which are sufficiently close to y. Thus $0 < \beta \leqslant 1$. We suppose that $\alpha > 0$ and $0 < y < z$. Then we obtain

$$0 \leqslant \lim_{x\to 0} \frac{f(x,y,z;\alpha,\beta)}{(\Gamma(x))^{\alpha}} = y^{\beta} - z^{\beta}.$$

Hence $\alpha < 0$. □

Remark 3.6. The proof of theorem 3.5 reveals that if $\alpha < 0 < \beta \leqslant 1$, then equality holds in (3.8) if and only if $x = y$ or $y = z$.

Remark 3.7. There do not exist real parameters $\alpha \neq 0$ and $\beta > 0$ such that the converse of (3.8) is valid for all x, y, z with $0 < x \leqslant y \leqslant z$. To prove this, we denote by f the function defined in (3.10) and assume that

$$f(x,y,z;\alpha,\beta) \leqslant 0 \quad \text{for } 0 < x \leqslant y \leqslant z. \tag{3.13}$$

If $\alpha > 0$, then (3.13) gives, for $\beta/\alpha < x < y \leqslant z$,

$$0 < (y-x)^{\beta}(B(x,y))^{\alpha} \leqslant \left(\frac{z-x}{z}\right)^{\beta}(\Gamma(x))^{\alpha}\left(z^{x}\frac{\Gamma(z)}{\Gamma(x+z)}\right)^{\alpha} z^{\beta-\alpha x}. \tag{3.14}$$

Applying (2.4), we conclude that the product on the right-hand side of (3.14) converges to 0 if z tends to ∞. Thus $\alpha < 0$. From theorem 3.5, we obtain $\beta > 1$. Further,

$$f(x,y,z;\alpha,\beta) \leqslant f(y,y,z;\alpha,\beta) \quad \text{and} \quad \left.\frac{\partial f(x,y,z;\alpha,\beta)}{\partial x}\right|_{x=y} \geqslant 0.$$

This leads to $\alpha(z-y)(\psi(y)-\psi(y+z)) \leqslant \beta$, which is false for all sufficiently large z.

Remark 3.8. In [9], it is proved that, for $x, y > 0$,

$$\sqrt{B(x,x)B(y,y)} \leqslant B(x,y). \tag{3.15}$$

Using the integral formula

$$B(x,y) = \frac{1}{2}\int_0^{\infty} \frac{t^{x-1}+t^{y-1}}{(t+1)^{x+y}}\,\mathrm{d}t$$

(see [1, p. 258]), we get, for $x, y > 0$,

$$\begin{aligned} B(x,x) &+ B(y,y) - 2B(x,y) \\ &= \int_0^{\infty} \frac{t^{x-1}}{(t+1)^{2x}}\left[1 - \left(\frac{t}{t+1}\right)^{y-x}\right]\left[1 - \frac{1}{(t+1)^{y-x}}\right]\mathrm{d}t \\ &\geqslant 0. \end{aligned} \tag{3.16}$$

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Thus $B(x,y)$ separates the arithmetic and geometric means of $B(x,x)$ and $B(y,y)$. Inequalities (3.15) and (3.16) are the best possible in the following sense. Let

$$M_r(a,b) = (\tfrac{1}{2}(a^r + b^r))^{1/r} \quad (r \neq 0), \quad M_0(a,b) = \sqrt{ab},$$

be the power mean of order r of $a, b > 0$.

The double inequality

$$M_u(B(x,x), B(y,y)) \leqslant B(x,y) \leqslant M_v(B(x,x), B(y,y)) \tag{3.17}$$

holds for all $x, y > 0$ if and only if $u \leqslant 0$ and $v \geqslant 1$.

Since the power mean is increasing on $\boldsymbol{R}$ with respect to its order (see [8, p. 159]), we conclude from (3.15) and (3.16) that if $u \leqslant 0$ and $v \geqslant 1$, then (3.17) is valid for all $x, y > 0$. Conversely, we assume that there exist parameters u, v with $0 < u < v < 1$ such that (3.17) holds for all $x, y > 0$. Setting $y = 1$, the left-hand side yields

$$(\tfrac{1}{2}((B(x,x))^u + 1))^{1/u} \leqslant B(x,1) = \frac{1}{x},$$

which is false for all sufficiently large x. Further, the right-hand side of (3.17) gives

$$xB(x,1) \leqslant x(\tfrac{1}{2}((B(x,x))^v + 1))^{1/v}.$$

We let x tend to 0 and obtain $1 \leqslant 2^{1-1/v}$. This leads to $v \geqslant 1$.

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